Symmetry, Integrability and Geometry: Methods and Applications SIGMA **7** (2011), 060, 18 pages

The BGG Complex on Projective Space*

Michael G. EASTWOOD † *and A. Rod GOVER* †‡

† *Mathematical Sciences Institute, Australian National University, ACT 0200, Australia*
E-mail: *meastwoo@member.ams.org*

‡ *Department of Mathematics, University of Auckland,*
Private Bag 92019, Auckland 1142, New Zealand
E-mail: *r.gover@auckland.ac.nz*

Received January 30, 2011, in final form June 18, 2011; Published online June 23, 2011
doi:10.3842/SIGMA.2011.060

Abstract. We give a complete construction of the Bernstein–Gelfand–Gelfand complex on real or complex projective space using minimal ingredients.

Key words: differential complex; BGG complex; projective space; Lie algebra cohomology; parabolic geometry

2010 Mathematics Subject Classification: 53A20; 17B56; 58D19; 58J70

Dedicated to Willard Miller on his retirement
from the University of Minnesota.

1 Introduction

This article is concerned with differential calculus on real or complex projective space, invariant under projective transformations. These transformations constitute a semisimple Lie group and projective space is hence a homogeneous space of the form G/P for G semisimple. The subgroup P is parabolic and, more generally, differential geometries modelled on homogeneous spaces of this form are known as 'parabolic' [5]. Projective space gives rise to projective differential geometry in this sense. Conformal and CR geometry are included amongst other examples of parabolic differential geometry. The interplay between the symmetries of projective space and its invariant differential operators is mediated by representation theory. In this article we present the 'BGG complex' on projective space as perhaps the simplest of these constructions in one of the simplest of settings. We anticipate that our approach will extend to G/P in general and perhaps to other homogeneous spaces. In [19], Olver introduced complexes of differential operators on Euclidean space acting between 'hyperforms'. These are BGG complexes constructed directly, employing only affine invariance and the associated Schur functors. He constructs some 'easy' examples and observes that "Other examples, of greater complexity, can of course be constructed at will, but the expressions rapidly get out of hand, even in low dimensional spaces". We maintain that the extra symmetry that the operators exhibit under projective transformations allows one to control these expressions more effectively. The fascinating combination of symmetries and differential operators is a defining feature in the work of Willard Miller to whom we dedicate this article.

The Bernstein–Gelfand–Gelfand (BGG) complexes on $\mathbb{RP}_n$ or $\mathbb{CP}_n$ are, by now, well-known complexes of vector bundles and differential operators between them generalising the de Rham complex on $\mathbb{RP}_n$ and the holomorphic de Rham complex on $\mathbb{CP}_n$, respectively. An introduction to such complexes is given in [11] and the particular case of projective space is discussed in [12].

*This paper is a contribution to the Special Issue "Symmetry, Separation, Super-integrability and Special Functions (S^4)". The full collection is available at http://www.emis.de/journals/SIGMA/S4.html

Usually, their construction involves choosing so-called 'splitting operators' [3, 6] constructed from Kostant's Laplacian [17] or 'quabla operator' [3] or from the Jantzen–Zuckerman translation functor [24]. Here, we avoid the direct use of splitting operators, instead relying only on diagram chasing, as is already done in [11, 12] in simple cases. In fact, the BGG complex of holomorphic differential operators on $\mathbb{CP}_2$ was already constructed in this manner [10, p. 351] before it was realised by John Rice [21] that complexes like this were dual to Lepowsky's construction [18] on the level of Verma modules. (On $\mathbb{CP}_1$ there is just one family of BGG operators, already singled out for their invariance in [13, Proposition 2.1].) In this article, for simplicity and cleanliness we employ a spectral sequence to effect the diagram chasing. We employ projective invariance to derive explicit formulæ for the operators in the projective BGG complex on $\mathbb{RP}_n$ in terms of the usual round metric on the sphere.

As is often done in differential geometry, when it is necessary to write out tensors and their natural operations and we shall adorn them with upper or lower indices corresponding to the tangent or cotangent bundle respectively. For example, a vector field can be written as X^a, a one-form as ω_a, and the natural pairing between them as $X^a\omega_a$ in accordance with the 'Einstein summation convention'. For any tensor ϕ_{abc} we shall write its symmetric part as $\phi_{(abc)}$ and its skew part as $\phi_{[abc]}$. For example, to say that ω_{ab} is a two-form is to say that

$$\omega_{ab} = -\omega_{ba} \qquad \text{or, equivalently,} \qquad \omega_{ab} = \omega_{[ab]} \qquad \text{or, equivalently,} \qquad \omega_{(ab)} = 0$$

and then

$$\nabla_{[a}\omega_{bc]} \qquad \text{and} \qquad X^a\nabla_a\omega_{bc} - 2(\nabla_{[b}X^a)\omega_{c]a},$$

for any torsion-free connection ∇_a, are the exterior derivative of ω_{ab} and the Lie derivative of ω_{ab} in the direction of the vector field X^a, respectively. Such formulæ are not meant to imply any choice of local coördinates. More precisely, this is Penrose's 'abstract index notation' [20] and it formalises the conventions used by many classical authors – see, for example, the discussion of projective differential geometry by Schouten [22].

We shall view $\mathbb{RP}_n$ as a homogeneous space

$$\mathbb{RP}_n = \mathrm{SL}(n+1,\mathbb{R})/P = G/P, \qquad \text{where} \quad P = \left\{ \left(\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \right\}$$

or as a quotient of the 'projective sphere'

$$S^n = \mathrm{SL}(n+1,\mathbb{R})/P = G/P, \qquad \text{where} \quad P = \left\{ \left(\begin{array}{c|ccc} \lambda & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \text{ s.t. } \lambda > 0 \right\}$$

under the antipodal map. In either case, it is convenient to write the associated Lie algebra as

$$\mathfrak{sl}(n+1,\mathbb{R}) = \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \tag{1}$$

where

$$\mathfrak{g}_- = \left\{ \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & 0 & \\ * & & & \end{array} \right) \right\}, \quad \mathfrak{g}_0 = \left\{ \left(\begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \right\}, \quad \mathfrak{g}_+ = \left\{ \left(\begin{array}{c|ccc} 0 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array} \right) \right\}$$

and then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

From now on we shall discuss only real projective space $\mathbb{RP}_n$ or its double cover, the sphere S^n. The complex case is completely parallel with real numbers being replaced by complex numbers everywhere and by working in the holomorphic category rather than the smooth.

2 An outline of the construction

If $\mathbb{V}$ is a finite-dimensional representation of P, we shall denote by V the induced homogeneous vector bundle on G/P constructed as

$$V = G \times_P \mathbb{V} = G \times \mathbb{V}/\sim, \qquad \text{where } (g, v) \sim \big(gp, p^{-1}v\big), \quad \forall\, p \in P.$$

Notice that if $\mathbb{V}$ is actually a G-module restricted to P, then V is canonically trivialised

$$V = G \times_P \mathbb{V} \cong G/P \times \mathbb{V} \qquad \text{by} \quad (g, v) \mapsto (gP, gv) \tag{2}$$

as a vector bundle (but not as a homogeneous vector bundle). Hence, in this case V is naturally equipped with a G-equivariant flat connection ∇ obtained by transporting to V the exterior derivative $d : \Lambda^0 \otimes \mathbb{V} \to \Lambda^1 \otimes \mathbb{V}$ with values in $\mathbb{V}$. More generally, the coupled de Rham sequence

$$V \xrightarrow{\nabla} \Lambda^1 \otimes V \xrightarrow{\nabla} \Lambda^2 \otimes V \xrightarrow{\nabla} \Lambda^3 \otimes V \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-1} \otimes V \xrightarrow{\nabla} \Lambda^n \otimes V \to 0 \tag{3}$$

is exact on the level of germs and provides a resolution of $\mathbb{V}$ as a locally constant sheaf on G/P.

This general reasoning holds on any homogeneous space G/P but the following discussion is specific to $\mathbb{RP}_n$ or S^n. Suppose $\mathbb{V}$ is irreducible as a G-module. In this case we shall see that as a P-module $\mathbb{V}$ is filtered

$$\mathbb{V} = \mathbb{V}_0 + \mathbb{V}_1 + \mathbb{V}_2 + \cdots + \mathbb{V}_{N-1} + \mathbb{V}_N \tag{4}$$

meaning that these are the subquotients listed in a natural order, starting on the left with the smallest quotient of $\mathbb{V}$. (In other words $\mathbb{V}_N$ is the smallest P-submodule in the filtration, the quotient $\mathbb{V}/\mathbb{V}_N$ has a filtration $\mathbb{V}_0 + \mathbb{V}_1 + \mathbb{V}_2 + \cdots + \mathbb{V}_{N-1}$, and the meaning is now clear by induction.) It follows that the bundle V is correspondingly filtered

$$V = V_0 + V_1 + V_2 + \cdots + V_{N-1} + V_N \tag{5}$$

and now we claim that the connection $\nabla : V \to \Lambda^1 \otimes V$, and consequently the whole complex (3), is compatible with this filtration (as detailed in Theorem 1 below). The spectral sequence of a filtered complex [9] now comes into play, having as its E_0-level the following.

$$\begin{array}{c|cccccccc}
 & V_0 & \Lambda^1 \otimes V_0 & \Lambda^2 \otimes V_0 & \Lambda^3 \otimes V_0 & \Lambda^4 \otimes V_0 & \Lambda^5 \otimes V_0 & \cdots \\
 & & \uparrow\partial & \uparrow\partial & \uparrow\partial & \uparrow\partial & \uparrow\partial & \\
 & & V_1 & \Lambda^1 \otimes V_1 & \Lambda^2 \otimes V_1 & \Lambda^3 \otimes V_1 & \Lambda^4 \otimes V_1 & \cdots \\
 & & & \uparrow\partial & \uparrow\partial & \uparrow\partial & \uparrow\partial & \\
q & & & V_2 & \Lambda^1 \otimes V_2 & \Lambda^2 \otimes V_2 & \Lambda^3 \otimes V_2 & \cdots \\
 & & & & \uparrow\partial & \uparrow\partial & \uparrow\partial & \\
 & & & & V_3 & \Lambda^1 \otimes V_3 & \Lambda^2 \otimes V_3 & \cdots \\
 & & & & & \uparrow\partial & \uparrow\partial & \\
 & & & & & V_4 & \Lambda^1 \otimes V_4 & \cdots \\
\hline
 & & & p & & & &
\end{array} \tag{6}$$

Here, the precise positioning of the coördinate axes is a matter of convention. The important property of the E_0-level is that the differentials ∂ are simply homomorphisms of vector bundles and we shall show that they are induced by a complex of G_0-modules

$$\mathbb{V} \xrightarrow{\partial} \mathfrak{g}_-^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2\mathfrak{g}_-^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^3\mathfrak{g}_-^* \otimes \mathbb{V} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^{n-1}\mathfrak{g}_-^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^n\mathfrak{g}_-^* \otimes \mathbb{V},$$

where

$$G_0 = \left\{ \left(\begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \ast & \\ 0 & & & \end{array} \right) \in \mathrm{SL}(n+1,\mathbb{R}) \right\}.$$

Furthermore, we shall show that this complex defines the Lie algebra cohomology $H^r(\mathfrak{g}_-, \mathbb{V})$, which in turn has been computed by Kostant [17]. It follows that the E_1-level of the spectral sequence is rather sparse, typically

$$\begin{array}{ccccccc} H^0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 0 & H^1 & H^2 & 0 & 0 & \cdots \\ & & 0 & 0 & 0 & H^3 & \cdots \\ & & & 0 & 0 & 0 & \cdots \\ & & & & 0 & 0 & \cdots \end{array} \qquad q \uparrow \ \ \xrightarrow{\ p\ } \tag{7}$$

where $H^0 = V_0$ and, in particular, there is precisely one irreducible bundle in each diagonal $E_1^{p,d-p}$ for d fixed. Since (3) resolves $\mathbb{V}$, we know that this spectral sequence is also converging to $\mathbb{V}$ and the only way that this can happen is if the differentials fit together as a resolution

$$0 \to \mathbb{V} \to H^0 \to H^1 \to H^2 \to H^3 \to \cdots \to H^{n-1} \to H^n \to 0.$$

This is the required BGG resolution. The rest of the article is devoted to filling in the details of this argument.

3 The filtering of $\mathbb{V}$ as a P-module

Recall the decomposition (1) of $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ and consider the element

$$H = \frac{1}{n+1} \left(\begin{array}{c|ccc} n & 0 & \cdots & 0 \\ \hline 0 & -1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & -1 \end{array} \right) \in \mathfrak{g}_0.$$

uniquely characterised as lying in the centre of $\mathfrak{g}_0$ with $[H, X] = X$, for $X \in \mathfrak{g}_+$. It is called the *grading element* [5] of the $|1|$-graded Lie algebra (1). If the G-module $\mathbb{V}$ is restricted to G_0, then it splits into eigenspaces under H. For the standard representation by matrix multiplication on column vectors for example,

$$H \begin{pmatrix} 0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix} = -\frac{1}{n+1} \begin{pmatrix} 0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \text{and} \qquad H \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{n}{n+1} \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8}$$

Rather than use the actual eigenvalues, which are rational in general, let us subtract the lowest eigenvalue and write

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots \oplus \mathbb{V}_{N-1} \oplus \mathbb{V}_N \tag{9}$$

for the eigenspace decomposition, noting that $\mathfrak{g}_+$ acts by $\mathbb{V}_j \to \mathbb{V}_{j+1}$ for all j. It follows that

$$\mathbb{V}^j \equiv \mathbb{V}_j \oplus \mathbb{V}_{j+1} \oplus \cdots \oplus \mathbb{V}_N \tag{10}$$

are P-submodules of $\mathbb{V}$ for all j and we have our filtration (4).

4 The filtered complex $\Lambda^\bullet \otimes V$ and its spectral sequence

The filtration (10) of $\mathbb{V}$ as a P-module certainly induces a filtration

$$V = V^0 \supseteq V^1 \supseteq V^2 \supseteq \cdots \supseteq V^{N-1} \supset V^N$$

of V by G-homogeneous vector bundles on G/P.

Theorem 1. *The connection $\nabla : V \to \Lambda^1 \otimes V$ is compatible with this filtration in the sense that there is a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{\nabla} & \Lambda^1 \otimes V \\ \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\ V^k & \xrightarrow{\nabla} & \Lambda^1 \otimes V^{k-1} \end{array}$$

for all $k = 1, 2, \ldots, N$.

Proof. Because the assertion is local, without loss of generality it suffices to prove it on the standard affine coördinate patch, namely

$$\mathbb{R}^n = Q/G_0 \subset G/P = \mathbb{RP}_n, \qquad \text{where} \quad Q = \left\{ \left(\begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & \huge{*} & \\ * & & & \end{array} \right) \in \mathrm{SL}(n+1, \mathbb{R}) \right\}. \tag{11}$$

Recall (9) that $\mathbb{V}$ splits as a G_0-module. Hence the same is true of V restricted to this patch:

$$V|_{\mathbb{R}^n} = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_{N-1} \oplus V_N. \tag{12}$$

To proceed we need a formula for ∇. The appendix discusses various natural constructions on a general Lie group G, which we now specialise to be the Abelian Lie group

$$\mathbb{R}^n = G_- = \left\{ \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & \mathrm{Id} & \\ * & & & \end{array} \right) \in \mathrm{SL}(n+1, \mathbb{R}) \right\}.$$

and according to (23) we find that $\nabla = d + \theta$, where

- $V|_{\mathbb{R}^n}$ is trivialised by $\begin{array}{ccc} G_- \times \mathbb{V} & \xrightarrow{\cong} & Q \times_{G_0} \mathbb{V} \\ \cap & \nearrow & \| \\ Q \times \mathbb{V} & & V|_{\mathbb{R}^n} \end{array}$ to define $d : V|_{\mathbb{R}^n} \to \Lambda^1 \otimes V|_{\mathbb{R}^n}$,

- $\theta : V|_{\mathbb{R}^n} \to \Lambda^1 \otimes V|_{\mathbb{R}^n}$ is defined by the same trivialisation together with the Maurer–Cartan form θ on G_-; more specifically,

$$V|_{\mathbb{R}^n} \cong G_- \times \mathbb{V} \xrightarrow{\theta\otimes\mathrm{Id}} \Lambda^1 \otimes \mathfrak{g}_- \otimes \mathbb{V} \xrightarrow{\mathrm{Id}\otimes\rho} \Lambda^1 \otimes \mathbb{V} \cong \Lambda^1 \otimes V|_{\mathbb{R}^n},$$

where $\rho : \mathfrak{g}_- \otimes \mathbb{V} \to \mathbb{V}$ is the representation of $\mathfrak{g}$ on $\mathbb{V}$ restricted to $\mathfrak{g}_-$.

Evidently, d preserves the splitting (12) whilst θ sends V_k to V_{k-1} for all $k = 1, 2, \ldots, N$. In particular, ∇ sends $V^k = V_k \oplus \cdots$ to $\Lambda^1 \otimes V^{k-1} = \Lambda^1 \otimes V_{k-1} \oplus \cdots$, as required. ■

Corollary 1. *The complex* (3) *is compatible with the filtration* (5)*, i.e.* ∇ *sends* $\Lambda^p \otimes V^k$ *to* $\Lambda^{p+1} \otimes V^{k-1}$.

Proof. In fact, in the trivialisation $V|_{\mathbb{R}^n} \cong G_- \times \mathbb{V}$ employed in the proof of Theorem 1

$$\Lambda^p \otimes V|_{\mathbb{R}^n} \cong \Lambda^p \otimes \mathbb{V} \ni \omega \otimes v \stackrel{\nabla}{\longmapsto} d\omega \otimes v + (-1)^p \omega \wedge (\theta \mathbin{\lrcorner} \rho) v \in \Lambda^{p+1} \otimes \mathbb{V} \cong \Lambda^{p+1} \otimes V|_{\mathbb{R}^n}$$

and the conclusion is manifest. ■

According to this corollary, we may now consider the spectral sequence of the filtered complex $\nabla : \Lambda^\bullet \otimes V$ on G/P, the E_0-level of which is

$$E_0^{p,q} = \Lambda^{p+q} \otimes V_{-q} \qquad \text{with differential} \quad \partial : \ E_0^{p,q} \to E_0^{p,q+1}, \tag{13}$$

where we have chosen to normalise (6) by placing V_0 at the origin. By construction, the bundle V_k on G/P is the homogeneous bundle induced from $\mathbb{V}_k = \mathbb{V}^k/\mathbb{V}^{k+1}$ as a P-module (cf. (10)). We already know that, as an eigenspace for the grading element H in the centre of $\mathfrak{g}_0$, the vector space $\mathbb{V}_k$ is a G_0-module. By regarding it as the quotient $\mathbb{V}^k/\mathbb{V}^{k+1}$ we are equivalently making $\mathbb{V}_k$ into a P-module by decreeing that G_+ act trivially. By construction, the E_0-differential is G-equivariant. Furthermore, its definition

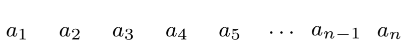
$$\begin{array}{ccccccccc}
0 & \to & \Lambda^{p+1} \otimes V^{k-2} & \to & \Lambda^{p+1} \otimes V^{k-1} & \to & \Lambda^{p+1} \otimes V_{k-1} & \to & 0 \\
 & & \nabla \uparrow & & \nabla \uparrow & & \partial \uparrow & & \\
0 & \to & \Lambda^{p} \otimes V^{k-1} & \to & \Lambda^{p} \otimes V^{k} & \to & \Lambda^{p} \otimes V_{k} & \to & 0
\end{array}$$

and the Leibniz rule ensure that it is linear over the functions. In other words ∂ is a G-equivariant homomorphism of homogeneous bundles and, as such, must be induced by a homomorphism of P-modules $\Lambda^p(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V}_k \to \Lambda^{p+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V}_{k-1}$, which we shall also denote by ∂. In fact, from the formula for ∇ displayed in the proof of Corollary 1, we see that ∂ is induced by

$$\Lambda^p(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V} = \Lambda^p \mathfrak{g}_-^* \otimes \mathbb{V} \ni v_{\beta\gamma\cdots\delta} \stackrel{\partial}{\longmapsto} \rho_{[\alpha} v_{\beta\gamma\cdots\delta]} \in \Lambda^{p+1}\mathfrak{g}_-^* \otimes \mathbb{V} = \Lambda^{p+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V}, \tag{14}$$

where $\mathbb{V}$ is regarded as a P-module by restricting the G action to G_0 and decreeing that G_+ act trivially. Since $\mathfrak{g}_-$ is Abelian, this formula agrees with (26) for the Koszul differential (used in [8] to define Lie algebra cohomology). As observed in the appendix, this is a complex of G_0-modules. Alternatively, we could come to the same conclusion by restricting attention to a standard affine coördinate chart $G_- \cong \mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ and noticing that the kernel of $d : \Lambda^p \otimes \mathbb{V} \to \Lambda^{p+1} \otimes \mathbb{V}$ consists precisely of the left-invariant $\mathbb{V}$-valued p-forms under the action of G_- on itself. Since $\nabla = d + \partial$, we may now invoke the second realisation of $H^r(\mathfrak{g}_-, \mathbb{V})$ from Theorem 5 in the Appendix.

To proceed we need to be explicit concerning the irreducible representation $\mathbb{V}$ of $\mathrm{SL}(n+1, \mathbb{R})$. In fact, one usually deals with complex representations of $\mathrm{SL}(n+1, \mathbb{C})$ (starting with $\mathfrak{sl}(n+1, \mathbb{C})$) but here there is no real difference and we shall adopt the notation from [2] in denoting such representations by attaching non-negative integers to its Dynkin diagram

$\mathbb{V} =$ 

(meaning that $-[a_1, a_2, \ldots, a_n]$ is the lowest weight of this representation with respect to the standard basis of fundamental weights). With this notation, here is the conclusion of Kostant's computation [17] of Lie algebra cohomology. We describe the result as an $\mathrm{SL}(n, \mathbb{R})$-module (by attaching non-negative integers to an A-series Dynkin diagram with one fewer nodes) and fix the action of G_0 by specifying how the grading element $H \in \mathfrak{g}_0$ acts.

$$
\begin{array}{rcll}
H^0(\mathfrak{g}_-, \mathbb{V}) & = & \overset{a_2}{\bullet}\!-\!\overset{a_3}{\bullet}\!-\!\overset{a_4}{\bullet}\!-\!\overset{a_5}{\bullet}\!-\cdots-\!\overset{a_{n-1}}{\bullet}\!-\!\overset{a_n}{\bullet} & H \rightsquigarrow -c \\
H^1(\mathfrak{g}_-, \mathbb{V}) & = & \overset{a_1+a_2+1}{\bullet}\!-\!\overset{a_3}{\bullet}\!-\!\overset{a_4}{\bullet}\!-\!\overset{a_5}{\bullet}\!-\cdots-\!\overset{a_{n-1}}{\bullet}\!-\!\overset{a_n}{\bullet} & H \rightsquigarrow -c + a_1 + 1 \\
H^2(\mathfrak{g}_-, \mathbb{V}) & = & \overset{a_1}{\bullet}\!-\!\overset{a_2+a_3+1}{\bullet}\!-\!\overset{a_4}{\bullet}\!-\!\overset{a_5}{\bullet}\!-\cdots-\!\overset{a_{n-1}}{\bullet}\!-\!\overset{a_n}{\bullet} & H \rightsquigarrow -c + a_1 + a_2 + 2 \\
H^3(\mathfrak{g}_-, \mathbb{V}) & = & \overset{a_1}{\bullet}\!-\!\overset{a_2}{\bullet}\!-\!\overset{a_3+a_4+1}{\bullet}\!-\!\overset{a_5}{\bullet}\!-\cdots-\!\overset{a_{n-1}}{\bullet}\!-\!\overset{a_n}{\bullet} & H \rightsquigarrow -c + a_1 + a_2 + a_3 + 3 \\
\vdots & \vdots & \vdots & \\
H^{n-1}(\mathfrak{g}_-, \mathbb{V}) & = & \overset{a_1}{\bullet}\!-\!\overset{a_2}{\bullet}\!-\!\overset{a_3}{\bullet}\!-\!\overset{a_4}{\bullet}\!-\cdots-\!\overset{a_{n-2}}{\bullet}\!-\!\overset{a_{n-1}+a_n+1}{\bullet} & H \rightsquigarrow -c + a_1 + \cdots + a_{n-1} + n - 1 \\
H^n(\mathfrak{g}_-, \mathbb{V}) & = & \overset{a_1}{\bullet}\!-\!\overset{a_2}{\bullet}\!-\!\overset{a_3}{\bullet}\!-\!\overset{a_4}{\bullet}\!-\cdots-\!\overset{a_{n-2}}{\bullet}\!-\!\overset{a_{n-1}}{\bullet} & H \rightsquigarrow -c + a_1 + \cdots + a_{n-1} + a_n + n
\end{array}
\tag{15}
$$

where

$$c = \frac{na_1 + (n-1)a_2 + (n-2)a_3 + \cdots + 2a_{n-1} + a_n}{n+1}$$

(obtained by acting on $[a_1, a_2, \ldots, a_n]$ with the first column of the inverse Cartan matrix for $\mathfrak{sl}(n+1)$). Notice that each cohomology is an irreducible representation of G_0.

Theorem 2. *The E_1-level of the spectral sequence of the filtered complex $\nabla : \Lambda^\bullet \otimes V$ consists of irreducible homogeneous vector bundles on $\mathbb{RP}_n$ under the action of $G = \mathrm{SL}(n+1, \mathbb{R})$. Only the following terms are non-zero*

$$
\begin{array}{rcl}
E_1^{0,0} & \leftrightsquigarrow & H^0(\mathfrak{g}_-, \mathbb{V}) \\
E_1^{a_1+1, -a_1} & \leftrightsquigarrow & H^1(\mathfrak{g}_-, \mathbb{V}) \\
E_1^{a_1+a_2+2, -a_1-a_2} & \leftrightsquigarrow & H^2(\mathfrak{g}_-, \mathbb{V}) \\
E_1^{a_1+a_2+a_3+3, -a_1-a_2-a_3} & \leftrightsquigarrow & H^3(\mathfrak{g}_-, \mathbb{V}) \\
\vdots & \vdots & \vdots \\
E_1^{N+n-1-a_n, -N+a_n} = E_1^{a_1+a_2+a_3+\cdots+a_{n-1}+n-1, -a_1-a_2-a_3-\cdots-a_{n-1}} & \leftrightsquigarrow & H^{n-1}(\mathfrak{g}_-, \mathbb{V}) \\
E_1^{N+n, -N} = E_1^{a_1+a_2+a_3+\cdots+a_{n-1}+a_n+n, -a_1-a_2-a_3-\cdots-a_{n-1}-a_n} & \leftrightsquigarrow & H^n(\mathfrak{g}_-, \mathbb{V})
\end{array}
$$

meaning that the bundle in question is induced by the G_0-module as listed and extended trivially as a P-module.

Proof. According to (13), we already know that $\Lambda^r \otimes V$ is spread along the r^{th} diagonal

$$\Lambda^r \otimes V = \Lambda^r \otimes V_0 + \Lambda^r \otimes V_1 + \Lambda^r \otimes V_2 + \cdots = E_0^{r,0} + E_0^{r+1,-1} + E_0^{r+2,-2} + \cdots$$

of the E_0-level of the spectral sequence and that the E_0-differential is induced by the Koszul differential (14) defining the Lie algebra cohomology $H^r(\mathfrak{g}_-, \mathbb{V})$. We see from (15) that each of these $H^r(\mathfrak{g}_-, \mathbb{V})$ is irreducible and so it follows that the r^{th} diagonal of the E_1-level consists of a single irreducible homogeneous vector bundle. To complete the proof it suffices to locate the position of this bundle along this particular diagonal and, to do this, the action of the grading element H turns out to be sufficient. More precisely,

$$
\begin{array}{llllllll}
H \text{ acts by} & -c & \text{on } V_0 & \therefore & H \text{ acts by} & -c + k & \text{on } V_k, \\
H \text{ acts by} & -1 & \text{on } \mathfrak{g}_- & \therefore & H \text{ acts by} & p & \text{on } \Lambda^p \mathfrak{g}_-^*, \\
 & & & \therefore & H \text{ acts by} & -c + k + p & \text{on } \Lambda^p \mathfrak{g}_-^* \otimes \mathbb{V}_k, \\
 & & & \therefore & H \text{ acts by} & -c + p & \text{on } \Lambda^{p+q} \mathfrak{g}_-^* \otimes \mathbb{V}_{-q} \leftrightsquigarrow E_0^{p,q}.
\end{array}
$$

Therefore H acts by $-c+p$ on the G_0-module inducing $E_1^{p,q}$ and so, from the action of H in table (15), the bundle induced by $H^r(\mathfrak{g}_-,\mathbb{V})$ is located at $E_1^{a_1+a_2+\cdots+a_r+r,-a_1-a_2-\cdots-a_r}$.

Finally, we are claiming that $N=a_1+a_2+\cdots+a_n$. To see this we note that, since H acts on V_0 by $-c$, it acts on V_N by $N-c$. On the other hand, in accordance with the action of the longest element of the Weyl group,

$$\mathbb{V}^* = \overset{a_n}{\bullet}\!\!-\!\!\!-\!\!\overset{a_{n-1}}{\bullet}\!\!-\!\!\!-\!\cdots\cdots\!-\!\!\!-\!\!\overset{a_5}{\bullet}\!\!-\!\!\!-\!\!\overset{a_4}{\bullet}\!\!-\!\!\!-\!\!\overset{a_3}{\bullet}\!\!-\!\!\!-\!\!\overset{a_2}{\bullet}\!\!-\!\!\!-\!\!\overset{a_1}{\bullet} = \mathbb{V}_N^* + \mathbb{V}_{N-1}^* + \cdots + \mathbb{V}_2^* + \mathbb{V}_1^* + \mathbb{V}_0^*$$

and so H acts on $\mathbb{V}_N^*$ by $-c'$, where

$$c' = \frac{na_n + (n-1)a_{n-1} + \cdots + 3a_3 + 2a_2 + a_1}{n+1}.$$

We conclude that $N = c + c' = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$, as required. ■

As outlined in Section 2, the E_1-level of this spectral sequence is rather sparse (7) and Theorem 2 says exactly how sparse. In particular, since there is only one non-zero bundle on each diagonal of the E_1-level, the general theory of spectral sequences provides a complex of differential operators

$$H^0 \to H^1 \to H^2 \to H^3 \to \cdots \to H^{n-1} \to H^n \to 0$$

whose cohomology on the level of sheaves coincides with that of (3), namely $\mathbb{V}$ for $\ker : H^0 \to H^1$ and otherwise zero. The bundle H^r is induced on $\mathbb{RP}_n = G/P$ from the G_0-module $H^r(\mathfrak{g}_-,\mathbb{V})$ extended trivially as as P-module. This is the BGG resolution (constructed on a general G/P by Lepowsky [18] on the level of generalised Verma modules). More explicitly, in the discussion following Corollary 1, in any affine coördinate patch we identified $\nabla : \Lambda^p \otimes V \to \Lambda^{p+1} \otimes V$ as $d + \partial$ where $V|_{\mathbb{R}^n}$ is trivialised as $\mathbb{R}^n \times \mathbb{V}$ and

$$E_0^{p,q} = \Lambda^{p+q} \otimes V_{-q} \cong \Lambda^{p+q} \otimes \mathbb{V}_{-q} \xrightarrow{d} \Lambda^{p+q+1} \otimes \mathbb{V}_{-q} \cong \Lambda^{p+q+1} \otimes V_{-q} = E_0^{p+1,q}$$

is the exterior derivative with values in $\mathbb{V}_{-q}$. Adding these operators to the diagram (6) gives a double complex as employed by Baston [1, p. 120]. We conclude that in any affine coördinate patch, our spectral sequence (of a filtered complex) coincides with Baston's spectral sequence (of a double complex). In particular, the operators $H^k \to H^{k+1}$ are obtained as zigzag compositions of the first order differential operators d together with choices of algebraic splittings of the operators ∂ (Baston and others use the algebraic adjoint ∂^* introduced by Kostant [17]). This confirms that the resulting operators $H^k \to H^{k+1}$ are differential. By using the spectral sequence of a filtered complex as we have done, it is manifest that the differential operators $H^k \to H^{k+1}$ are independent of any choice of splittings and that the whole construction and resulting BGG complex is G-equivariant. It is also clear that the first steps in this approach can be taken on an arbitrary homogeneous space G/P.

5 Formulæ for the BGG operators

Already, the affine invariance of the operators occurring in the BGG resolution on $\mathbb{RP}_n$ fixes their formulæ with respect to the flat connection on $\mathbb{R}^n \subset \mathbb{RP}_n$ as follows (affine invariance being ensured by noting that Q as in (11) is acting on $\mathbb{R}^n = Q/G_0$ by affine transformations). Firstly, consider the symbol of the BGG differential operator $H^r \to H^{r+1}$. As a homomorphism of homogeneous bundles $\bigodot^s \Lambda^1 \otimes H^r \to H^{r+1}$ for some s, it is induced by a homomorphism of G_0-modules

$$\textstyle\bigodot^s \mathfrak{g}_-^* \otimes H^r(\mathfrak{g}_-,\mathbb{V}) \to H^{r+1}(\mathfrak{g}_-,\mathbb{V})$$

for some s, which we can determine just from the action of the grading element. Specifically, we know that

$$\begin{array}{llll} & H \text{ acts by} & -c+a_1+a_2+\cdots+a_r+r & \text{on } H^r(\mathfrak{g}_-,\mathbb{V}) \\ & H \text{ acts by} & -1 & \text{on } \mathfrak{g}_- \\ \therefore & H \text{ acts by} & s & \text{on } \bigodot^s \mathfrak{g}_-^*. \end{array}$$

It is immediate that $s = a_{r+1}+1$. Furthermore, from the Littlewood–Richardson rules (e.g., [14]), as an $\mathfrak{sl}(n,\mathbb{R})$-module $H^{r+1}(\mathfrak{g}_-,\mathbb{V})$ occurs with multiplicity one in the following decomposition

$$\begin{array}{ccc} \bigodot^{a_{r+1}+1}\mathfrak{g}_-^* \otimes H^r(\mathfrak{g}_-,\mathbb{V}) & = & \cdots \oplus H^{r+1}(\mathfrak{g}_-,\mathbb{V}) \oplus \cdots \\ \| & & \| \\ \overset{a_{r+1}+1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet} \otimes \overset{a_1}{\bullet}\!-\cdots-\!\overset{a_r+a_{r+1}+1}{\bullet}\!\!-\!\!\overset{a_{r+2}}{\bullet}\!-\cdots-\!\overset{a_n}{\bullet} & = & \oplus \overset{a_1}{\bullet}\!-\cdots-\!\overset{a_r}{\bullet}\!\!-\!\!\overset{a_{r+1}+a_{r+2}+1}{\bullet}\!-\cdots-\!\overset{a_n}{\bullet} \oplus \end{array}$$

The symbol of the BGG operator $H^r \to H^{r+1}$ is thus determined uniquely (up to scale) as induced by the projection onto this summand. Similarly, there are no lower order terms since no appropriate invariant homomorphisms are available. We record this conclusion as follows.

Theorem 3. *Each bundle H^r in the BGG complex on $\mathbb{RP}_n$ is an irreducible tensor bundle and the differential operator $\nabla^{(s)} : H^r \to H^{r+1}$ is given by $\phi \mapsto \pi(\nabla^s\phi)$ where ∇ is the flat affine connection and $\pi : \bigodot^s \Lambda^1 \otimes H^r \to H^{r+1}$ is the unique projection onto this irreducible summand.*

Equivalently, the bundles H^r and the operators between them are exhibited as Young tableau in [12]. It is often useful, however, to be able to write the BGG complex globally on $\mathbb{RP}_n$ without recourse to affine coördinates and projective invariance. For this, we shall use the round metric on the sphere and the corresponding Levi Civita connection on S^n or $\mathbb{RP}_n$. By way of normalisation, if g_{ab} denotes the round metric and $\delta_a{}^b$ the identity endomorphism on the tangent bundle, let us choose the radius of the sphere so that

$$R_{ab}{}^c{}_d = \delta_a{}^c g_{bd} - \delta_b{}^c g_{ad} \qquad \text{where} \quad (\nabla_a\nabla_b - \nabla_b\nabla_a)X^d = R_{ab}{}^c{}_d X^d \tag{16}$$

defines the Riemann curvature tensor. For any irreducible covariant tensor bundle E on $\mathbb{RP}_n$, let us write ∇^2 for the composition

$$\bigodot^r \Lambda^1 \otimes E \xrightarrow{\nabla\circ\nabla} \Lambda^1 \otimes \Lambda^1 \otimes \bigodot^r \Lambda^1 \otimes E \to \bigodot^{r+2}\Lambda^1 \otimes E$$

and g for the composition

$$\bigodot^r \Lambda^1 \otimes E \xrightarrow{g\otimes \mathrm{Id}} \bigodot^2 \Lambda^1 \otimes \bigodot^r \Lambda^1 \otimes E \to \bigodot^{r+2}\Lambda^1 \otimes E.$$

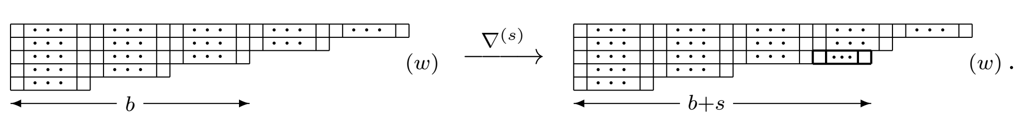
Theorem 4. *If s is odd, the operator $\nabla^{(s)} : H^r \to H^{r+1}$ is given by*

$$\pi\big((\nabla^2 + (s-1)^2 g)\cdots(\nabla^2+16g)(\nabla^2+4g)\nabla\big)$$

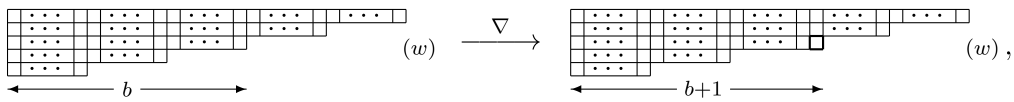
and, if s is even,

$$\pi\big((\nabla^2 + (s-1)^2 g)\cdots(\nabla^2+9g)(\nabla^2+g)\big).$$

Proof. It is convenient to use some notation from [12] where the projective BGG operators are written as acting between covariant tensor bundles specified by weighted Young tableau:



Here, if it is the r^{th} row to which the boxes on the right hand side are being added and there are a total of v boxes on the left hand side (so that v is the valence of the corresponding tensor), then $w + r = s + v + b$ (see [12]). In particular, if we consider only first order operators



(17)

then $w = v + b + 1 - r$. More generally, taking conventions from [12], if ∇ and $\hat{\nabla}$ are two torsion-free connections in the same projective class

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$$

for some 1-form Υ_a, then for $\phi_{bc\cdots d}$ having symmetries and projective weight specified by the left hand side of (17),

$$\pi(\hat{\nabla}_a \phi_{bc\cdots d}) = \pi\big(\nabla_a \phi_{bc\cdots d} + (w - (v + b + 1 - r))\Upsilon_a \phi_{bc\cdots d}\big),$$

where π is the Young projector corresponding to the right hand side of (17). We may iterate this formula, adding more boxes to the r^{th} row (assuming that there is room to do so) and, each time, both b and v increase by one. Suppressing indices, after s iterations the result is that

$$\pi(\hat{\nabla}^s \phi) = \pi\big((\nabla + (k - 2s + 2)\Upsilon\phi) \cdots (\nabla + (k-4)\Upsilon\phi)(\nabla + (k-2)\Upsilon\phi)(\nabla + k\Upsilon)\phi\big),$$

where we are writing $k = w - (v + b + 1 - r)$. In particular, if $s = w + r - v - b$ as it is in the case of a BGG operator, then this iteration reads

$$\pi(\hat{\nabla}^s \phi) = \pi\big((\nabla - k\Upsilon\phi) \cdots (\nabla + (k-4)\Upsilon\phi)(\nabla + (k-2)\Upsilon\phi)(\nabla + k\Upsilon)\phi\big),$$

where $k = s - 1$. So far, this conclusion holds under any projective change of connection but now we specialise to the case of the round connection ∇ on the sphere, being projectively equivalent to the flat connection $\hat{\nabla}$ (under gnomonic projection). The general formula [12, (3.4)] for the change in the Ricci tensor specialises to

$$0 = g_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b \qquad \text{or, suppressing indices,} \qquad \nabla\Upsilon = g + \Upsilon^2.$$

In this equation Υ is viewed as a tensor but if it is viewed as an operator $\phi \xmapsto{\Upsilon} \Upsilon\phi$, then we should write $\nabla\Upsilon = \Upsilon\nabla + g + \Upsilon^2$. This equation allows us to deal with the iterated formula above. For example, when $k = 2$, also bearing in mind that as operators ∇ and g commute,

$$\begin{aligned}
\underline{\pi\big(\hat{\nabla}^3\phi\big)} &= \pi\big((\nabla - 2\Upsilon)\nabla(\nabla + 2\Upsilon)\big) \\
&= \pi\big(\nabla^3 - 2\Upsilon\nabla^2 + 2\nabla(\nabla\Upsilon) - 4\Upsilon(\nabla\Upsilon)\big) \\
&= \pi\big(\nabla^3 - 2\Upsilon\nabla^2 + 2\nabla\big(\Upsilon\nabla + g + \Upsilon^2\big) - 4\Upsilon\big(\Upsilon\nabla + g + \Upsilon^2\big)\big) \\
&= \pi\big(\nabla^3 - 2\Upsilon\nabla^2 + 2(\nabla\Upsilon)\nabla + 2g\nabla + 2(\nabla\Upsilon)\Upsilon - 4\Upsilon^2\nabla - 4g\Upsilon - 4\Upsilon^3\big) \\
&= \pi\big(\nabla^3 - 2\Upsilon\nabla^2 + 2\big(\Upsilon\nabla + g + \Upsilon^2\big)\nabla + 2g\nabla + 2(\nabla\Upsilon)\Upsilon - 4\Upsilon^2\nabla - 4g\Upsilon - 4\Upsilon^3\big) \\
&= \pi\big(\nabla^3 + 4g\nabla + 2\big(\Upsilon\nabla + g + \Upsilon^2\big)\Upsilon - 2\Upsilon^2\nabla - 4g\Upsilon - 4\Upsilon^3\big) \\
&= \pi\big(\nabla^3 + 4g\nabla + 2\Upsilon(\nabla\Upsilon) - 2\Upsilon^2\nabla - 2g\Upsilon - 2\Upsilon^3\big) \\
&= \pi\big(\nabla^3 + 4g\nabla + 2\Upsilon\big(\Upsilon\nabla + g + \Upsilon^2\big) - 2\Upsilon^2\nabla - 2g\Upsilon - 2\Upsilon^3\big) \\
&= \pi(\nabla^3 + 4g\nabla) = \underline{\pi\big((\nabla^2 + 4g)\nabla\big)},
\end{aligned}$$

as advertised in the statement of the theorem. For higher k, direct calculations rapidly get out of hand. Instead, it suffices to prove the following lemma in which we have isolated the required algebra (and then we prove the lemma by indirect means). ■

Remark 1. Our curvature normalisation (16) implies that the Ricci tensor on our round sphere is given by

$$R_{ab} \equiv R_{ca}{}^{c}{}_{b} = (n-1)g_{ab}.$$

Thus, the metric g_{ab} coincides with $\frac{1}{n-1}R_{ab}$, a tensor generally known in projective differential geometry [12] as the *Schouten tensor* or *Rho-tensor* P_{ab}. Replacing g by P in the formulæ of Theorem 4 gives expressions that are valid on any space of constant curvature. More generally, there is a Rho-tensor that arises in similar contexts [4] within parabolic differential geometry [5].

Lemma 1. *Define an associative algebra $\mathcal{R} = \mathbb{R}\langle\nabla, \Upsilon\rangle$ with generators subject to the 'Riccati relation' $\nabla\Upsilon = \Upsilon\nabla + 1 + \Upsilon^2$. Then the following identities hold in $\mathcal{R}$. If k is even, then*

$$(\nabla - k\Upsilon)(\nabla - (k-2)\Upsilon)\cdots(\nabla + (k-2)\Upsilon)(\nabla + k\Upsilon) = (\nabla^2 + k^2)\cdots(\nabla^2 + 16)(\nabla^2 + 4)\nabla$$

and, if k is odd, then

$$(\nabla - k\Upsilon)(\nabla - (k-2)\Upsilon)\cdots(\nabla + (k-2)\Upsilon)(\nabla + k\Upsilon) = (\nabla^2 + k^2)\cdots(\nabla^2 + 9)(\nabla^2 + 1).$$

Proof. The algebra $\mathcal{R}$ may be realised by the following differential operators on the circle

$$f(\theta) \stackrel{\nabla}{\longmapsto} df(\theta)/d\theta, \qquad f(\theta) \stackrel{\Upsilon}{\longmapsto} (\tan\theta)f(\theta).$$

To see this, note that these operators certainly satisfy the Riccati relation and we are required, therefore, to show that they satisfy no further relations. Within $\mathcal{R}$ we may normalise any element as follows. By induction, the Riccati relation extends to

$$\nabla\Upsilon^\ell = \Upsilon^\ell\nabla + \ell\big(\Upsilon^{\ell-1} + \Upsilon^{\ell+1}\big), \qquad \forall\, \ell \geq 1$$

whence

$$\nabla^k\Upsilon^\ell = \nabla^{k-1}\big(\Upsilon^\ell\nabla + \ell\big(\Upsilon^{\ell-1} + \Upsilon^{\ell+1}\big)\big) = \big(\nabla^{k-1}\Upsilon^\ell\big)\nabla + \ell\big(\nabla^{k-1}\Upsilon^{\ell-1}\big) + \ell\big(\nabla^{k-1}\Upsilon^{\ell+1}\big)$$

and it follows by induction on k that

$$\begin{aligned}\nabla^k\Upsilon^\ell = {}& \Upsilon^\ell\nabla^k + k\ell\big(\Upsilon^{\ell-1} + \Upsilon^{\ell+1}\big)\nabla^{k-1} \\ & + \tfrac{1}{2}k(k-1)\ell\big((\ell-1)\Upsilon^{\ell-2} + 2\ell\Upsilon^\ell + (\ell+1)\Upsilon^{\ell+2}\big)\nabla^{k-2} + \cdots,\end{aligned}$$

where the ellipsis $\cdots$ denotes terms of lower order in ∇ with coefficients that are real polynomial in Υ. It follows that every element of $\mathcal{R}$ can be written uniquely in the form

$$\sum_{p=0}^{k} A_p(\Upsilon)\,\nabla^p$$

for suitable real polynomials $A_p(\Upsilon)$. In our claimed realisation, such an element is represented by the differential operator

$$\sum_{p=0}^{k} A_p(\tan\theta)\,d^p/d\theta^p$$

and now it suffices to observe (by acting on $1, \theta, \theta^2, \ldots, \theta^k$ near 0) that such a differential operator vanishes if and only if all the polynomials $A_p(\Upsilon)$ are zero. (More precisely, we should restrict the action of such operators to smooth functions on $(-\pi/2, \pi/2)$ or some other suitable function space.)

Having realised $\mathcal{R}$ by differential operators, we are reduced to proving identities amongst these operators. This is accomplished in the following lemma. ■

Lemma 2. *Let D denote the differential operator $f(\theta)\mapsto(\cos^2\theta)(df(\theta)/d\theta)$ on the circle. Then the following identities hold. If k is even, then*

$$\frac{1}{\cos^k\theta}\frac{d}{d\theta}\left(D^k\left(\frac{f(\theta)}{\cos^k\theta}\right)\right)=\left(\frac{d^2}{d\theta^2}+k^2\right)\cdots\left(\frac{d^2}{d\theta^2}+16\right)\left(\frac{d^2}{d\theta^2}+4\right)\frac{d}{d\theta}f(\theta)$$

and, if k is odd, then

$$\frac{1}{\cos^k\theta}\frac{d}{d\theta}\left(D^k\left(\frac{f(\theta)}{\cos^k\theta}\right)\right)=\left(\frac{d^2}{d\theta^2}+k^2\right)\cdots\left(\frac{d^2}{d\theta^2}+9\right)\left(\frac{d^2}{d\theta^2}+1\right)f(\theta).$$

Proof. Writing D_{k+1} for the operators on the left-hand-sides of the displays in this lemma, the following identity is easily verified

$$D_{k+3}=\left(\frac{d}{d\theta}-(k+2)\tan\theta\right)D_{k+1}\left(\frac{d}{d\theta}+(k+2)\tan\theta\right).$$

It follows that, in our realisation of the algebra $\mathcal{R}$, we obtain the expressions on the left-hand-sides of the displays in the Lemma 1.

On the other hand, the operators on the right-hand-sides of the claimed identities, in the current lemma, are characterised up to scale as annihilating the functions

$$\cos(k\theta),\quad \sin(k\theta),\quad \cos((k-2)\theta),\quad \sin((k-2)\theta),\quad \cos((k-4)\theta),\quad \ldots.$$

Since all operators have $d^{k+1}/d\theta^{k+1}$ as leading term, it is therefore sufficient to show that the left hand sides of these purported identities have the same property. Notice that there is an invertible relationship

$$\cos(m\theta)=2^{m-1}\cos^m\theta+\cdots=T_m(\cos\theta),$$

where T_m is the m^{th} Chebyshev polynomial of the first kind and a similar invertible relationship

$$\sin(m\theta)=(\sin\theta)\big(2^{m-2}\cos^{m-1}\theta+\cdots\big)=(\sin\theta)U_{m-1}(\cos\theta)$$

where U_{m-1} is the $(m-1)^{\text{st}}$ Chebyshev polynomial of the second kind. Only the degree of these Chebyshev polynomials concerns us and it now suffices to show that D^{k+1} annihilates the first $k+1$ of

$$1,\quad \frac{\sin\theta}{\cos\theta},\quad \frac{1}{\cos^2\theta},\quad \frac{\sin\theta}{\cos^3\theta},\quad \frac{1}{\cos^4\theta},\quad \frac{\sin\theta}{\cos^5\theta},\quad \frac{1}{\cos^6\theta},\quad \frac{\sin\theta}{\cos^7\theta},\quad \ldots.$$

Since

$$D\frac{1}{\cos^{2\ell}\theta}=2\ell\frac{\sin\theta}{\cos^{2\ell-1}\theta}\qquad\text{and}\qquad D\frac{\sin\theta}{\cos^{2\ell-1}\theta}=(2\ell-1)\frac{1}{\cos^{2(\ell-1)}\theta}-2(\ell-1)\frac{1}{\cos^{2(\ell-2)}\theta},$$

this follows easily by induction. ∎

Remark 2. More generally, the formula

$$D_{k+1}f=u^{-k-2}\left(u^2\frac{d}{dx}\right)^{k+1}u^{-k}f,\qquad\text{where}\qquad\left(\frac{d^2}{dx^2}+\Phi\right)u=0$$

is used in [15] to derive expressions for BGG operators in conformal geometry and in [4] these expressions are extended to parabolic geometries in general. Lemma 2 concerns the case $\Phi=1$.

6 An example

The following is amongst the simplest of non-trivial examples. Let us take $n = 2$ so $G = \mathrm{SL}(3, \mathbb{R})$ and let $\mathbb{V} = \mathbb{R}^3$, regarded as column vectors with $\mathrm{SL}(3, \mathbb{R})$ acting by left matrix multiplication. We saw in (8) how H splits $\mathbb{R}^3$ and in Section 3 that this results in the filtering

$$\mathbb{R}^3 = \mathbb{R}^2 + \mathbb{R} \quad \text{as a } P\text{-module}$$

where $\mathrm{SL}(2, \mathbb{R}) \subset P$ acts on $\mathbb{R}^2$ as the standard representation and acts trivially on $\mathbb{R}$. Dropping projective weights, the corresponding bundle V on $\mathbb{RP}_2$ is filtered

$$V = V_0 + V_1 = T + \Lambda^0,$$

where T is the tangent bundle and Λ^0 is the trivial bundle. The E_0-level (6) of our spectral sequence becomes

$$\begin{array}{c|cccc} q & & & & \\ & T & \Lambda^1 \otimes T & \Lambda^2 \otimes T & 0 \\ & & \uparrow\partial & \uparrow\partial & \\ & 0 & \Lambda^0 & \Lambda^1 & \Lambda^2 \\ \hline & & & & p \end{array}$$

and one checks that $\partial : \Lambda^0 \to \Lambda^1 \otimes T$ and $\partial : \Lambda^1 \to \Lambda^2 \otimes T$ are given by

$$\mu \mapsto \delta_b{}^c \mu \qquad \text{and} \qquad \mu_b \mapsto \delta_{[a}{}^c \mu_{b]},$$

respectively. In this simple case, we do not need Kostant's Theorem to see that

$$\begin{aligned} &\partial : \ \Lambda^0 \to \Lambda^1 \otimes T \ \text{ is injective with cokernel} = (\Lambda^1 \otimes T)_\circ, \\ &\partial : \ \Lambda^1 \to \Lambda^2 \otimes T \ \text{ is an isomorphism,} \end{aligned}$$

where $(\Lambda^1 \otimes T)_\circ$ denotes the trace-free part of $\Lambda^1 \otimes T$. Therefore, the E_1-level (7) is

$$\begin{array}{c|cccc} q & & & & \\ & T \ \to & (\Lambda^1 \otimes T)_\circ & 0 & 0 \\ & & & & \\ & 0 & 0 & 0 & \Lambda^2 \\ \hline & & & & p \end{array}$$

and we obtain

$$0 \to \mathbb{R}^3 \to T \to (\Lambda^1 \otimes T)_\circ \to \Lambda^2 \to 0$$

as the resulting BGG complex. Writing out the flat connection ∇ on V in terms of the round connection gives

$$\begin{pmatrix} \sigma^c \\ \mu \end{pmatrix} \stackrel{\nabla}{\longmapsto} \begin{pmatrix} \nabla_b \sigma^c + \delta_b{}^c \mu \\ \nabla_b \mu - \sigma_b \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sigma_b{}^c \\ \mu_b \end{pmatrix} \stackrel{\nabla}{\longmapsto} \begin{pmatrix} \nabla_{[a} \sigma_{b]}{}^c + \delta_{[a}{}^c \mu_{b]} \\ \nabla_{[a} \mu_{b]} + \sigma_{[ab]} \end{pmatrix}$$

for the two operators in $V \to \Lambda^1 \otimes V \to \Lambda^2 \otimes V$ and, noting that

$$\Lambda^1 \otimes V \ni \begin{pmatrix} \sigma_b{}^c \\ -\nabla_c \sigma_b{}^c \end{pmatrix} \stackrel{\nabla}{\longmapsto} - \begin{pmatrix} 0 \\ \nabla_c \nabla_{[a} \sigma_{b]}{}^c + g_{c[a} \sigma_{b]}{}^c \end{pmatrix},$$

we obtain a formula for the projectively invariant operator $(\Lambda^1 \otimes T)_\circ \xrightarrow{\nabla^{(2)}} \Lambda^2$ in agreement with Theorem 4.

Appendix: Lie algebra cohomology as a geometrical construction

Although in this article we shall need only the cohomology of an Abelian Lie algebra, we take the opportunity here to describe the cohomology of a general Lie algebra $\mathfrak{g}$ in terms of differential geometry on G, a Lie group whose Lie algebra is $\mathfrak{g}$. We believe that for a general parabolic geometry, we shall need this geometric interpretation for a nilpotent Lie algebra. Suppose $\mathbb{V}$ is a G-module and use the same notation for the corresponding representation of $\mathfrak{g}$. Following but adapting [8], we are going to present the Lie algebra cohomology $H^r(\mathfrak{g},\mathbb{V})$ as a geometrical construction on G. Beware that G is no longer the Lie group $\mathrm{SL}(n+1,\mathbb{R})$ as it was until now. This section is written to be self-contained with the aim of being useful elsewhere. This material is well-known to experts and implicit in [8] but we believe it worthwhile laying out the details.

We shall view G as a homogeneous space under its own action on the left. Its tangent bundle TG is then regarded as a homogeneous bundle and can be identified as $G\times\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. It is convenient to write this isomorphism as

$$X \mapsto X \lrcorner\, \theta \qquad \text{for vector fields } X \text{ on } G, \tag{18}$$

where θ is a 1-form on G with values in $\mathfrak{g}$ known as the *Maurer–Cartan* form [23]. To compute with θ it is convenient to write functions on G with values in $\mathfrak{g}$ as X^α and then (18) becomes

$$X^a \mapsto X^\alpha \equiv \theta_a^\alpha X^a \qquad \text{with inverse} \quad X^\alpha \mapsto X^a \equiv \phi_\alpha^a X^\alpha,$$

where $\phi_\alpha^a\theta_a^\beta = \delta_\alpha{}^\beta$ and $\theta_a^\alpha\phi_\alpha^b = \delta_a{}^b$. A vector field X^a on G is *left-invariant* if and only if the corresponding function $X^\alpha : G \to \mathfrak{g}$ is constant. Choosing any torsion-free affine connection ∇_a on G and expanding the definition $\nabla_a(\theta_b^\alpha X^b) = 0$, the left-invariant vector fields are those that satisfy

$$\theta_b^\alpha \nabla_a X^b - X^b \nabla_a \theta_b^\alpha = 0$$

from which it follows easily that the Lie bracket of two left-invariant vector fields is again left-invariant. Since the left-invariant vector fields on G are of the form $\phi(X)$ for $X\in\mathfrak{g}$ we may define the Lie bracket on $\mathfrak{g}$ by transportation:

$$\phi([X,Y]) = [\phi(X),\phi(Y)] \qquad \text{for} \quad X,Y\in\mathfrak{g}. \tag{19}$$

For computational purposes, let us write $[X,Y]^\gamma = \Gamma_{\alpha\beta}{}^\gamma X^\alpha Y^\beta$ for the Lie bracket on $\mathfrak{g}$. Then we can write out (19) explicitly as

$$\begin{aligned}\phi_\gamma^c\Gamma_{\alpha\beta}{}^\gamma X^\alpha Y^\beta = [\phi_\alpha^a X^\alpha, \phi_\beta^b Y^\beta]^c &= \phi_\alpha^a X^\alpha \nabla_a(\phi_\beta^c Y^\beta) - \phi_\alpha^b Y^\alpha \nabla_b(\phi_\beta^c X^\beta)\\ &= (\phi_\alpha^a\nabla_a\phi_\beta^c - \phi_\beta^a\nabla_a\phi_\alpha^c)X^\alpha Y^\beta\end{aligned}$$

or, in other words, as

$$\Gamma_{\alpha\beta}{}^\gamma\phi_\gamma^c = \phi_\alpha^a\nabla_a\phi_\beta^c - \phi_\beta^a\nabla_a\phi_\alpha^c.$$

Bearing in mind that $\phi_\beta^b\theta_b^\gamma = \delta_\beta{}^\gamma$ whence $\theta_b^\gamma\nabla_a\phi_\beta^b + \phi_\beta^b\nabla_a\theta_b^\gamma = 0$, we may rewrite this as

$$\Gamma_{\alpha\beta}{}^\gamma = -\phi_\alpha^a\phi_\beta^b\nabla_a\theta_b^\gamma + \phi_\beta^a\phi_\alpha^b\nabla_a\theta_b^\gamma = -\phi_\alpha^a\phi_\beta^b(\nabla_a\theta_b^\gamma - \nabla_b\theta_a^\gamma)$$

and finally as

$$\nabla_a\theta_b^\gamma - \nabla_b\theta_a^\gamma + \Gamma_{\alpha\beta}{}^\gamma\theta_a^\alpha\theta_b^\beta = 0. \tag{20}$$

This formula employs an arbitrary torsion-free connection. Without indices and without this connection it is more usually written as

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0 \qquad \text{or} \qquad d\theta + \theta\wedge\theta = 0. \tag{21}$$

In other words, the definition (19) of the Lie bracket on $\mathfrak{g}$ is equivalent to (20) or (21), usually known as the *Maurer–Cartan* equation [23].

Now suppose $\mathbb{V}$ is a G-module and use the same notation for the corresponding $\mathfrak{g}$-module. There are two canonically defined connections on the vector bundle $V = G\times\mathbb{V}$ over G. One of them is the evident flat connection, ignoring the action of G on $\mathbb{V}$. We shall denote it by d since it is the exterior derivative acting on functions with values in $\mathbb{V}$. The other one takes the isomorphism from (2)

$$V = G\times\mathbb{V} \cong G\times\mathbb{V} \qquad \text{by} \quad (g,v)\mapsto(g,gv) \tag{22}$$

and pulls back the evident flat connection on the right hand side as was done in Section 2 and we shall denote this one by ∇ as was done there. To relate these two connections more explicitly suppose $f : G\to\mathbb{V}$ is a section of V that is constant after the twisting (22). It means that the function $g\mapsto gf(g)$ is constant. If so, then for fixed $g\in G$ and $X\in\mathfrak{g}$, the function

$$\mathbb{R}\ni t\mapsto ge^{tX}f\big(ge^{tX}\big)\in\mathbb{V}$$

is constant. Equivalently, the function $t\mapsto e^{tX}f(ge^{tX})$ is constant and so

$$0 = \frac{d}{dt}\big(e^{tX}f\big(ge^{tX}\big)\big)\Big|_{t=0} = \frac{d}{dt}f\big(ge^{tX}\big)\big|_{t=0} + Xf(g) = (\phi(X)f + Xf)(g),$$

where this last equality is due to the flow of the left-invariant vector field $\phi(X)$ being the one-parameter subgroup of right-translations $g\mapsto ge^{tX}$ (see, e.g. [25]). For computational purposes, let us write $X^\alpha\mapsto X^\alpha\rho_\alpha$ where $\rho_\alpha\in\mathfrak{g}^*\otimes\operatorname{End}(\mathbb{V})$ for the action of $\mathfrak{g}$ on $\mathbb{V}$. Then $g\mapsto gf(g)$ is constant if and only if

$$0 = \phi(X)f + Xf = X^\alpha\phi_\alpha^a d_af + X^\alpha\rho_\alpha f = X^a(d_af + \theta_a^\alpha\rho_\alpha f)$$

for all left-invariant vector fields X^a. It follows that the connection ∇_a on V is given by

$$f\mapsto\nabla_af = d_af + \theta_a^\alpha\rho_\alpha f \qquad \text{or, without indices, as} \quad f\mapsto\nabla f = df + \theta f, \tag{23}$$

where $\theta\in\Lambda^1\otimes\mathfrak{g}$ is the Maurer–Cartan form. As a check, the differential in the coupled de Rham sequence (3) is

$$\Lambda^p\otimes\mathbb{V}\ni\omega\overset{\nabla}{\longmapsto}d\omega + \theta\wedge\omega\in\Lambda^{p+1}\otimes\mathbb{V}$$

and the Maurer–Cartan equation (21) shows that the composition $\mathbb{V}\xrightarrow{\nabla}\Lambda^1\otimes\mathbb{V}\xrightarrow{\nabla}\Lambda^2\otimes\mathbb{V}$ is given by

$$\nabla^2 f = d(df + \theta f) + \theta\wedge(df + \theta f) = (d\theta + \theta\wedge\theta)f = 0$$

and the connection ∇ is flat, as expected. More generally, the whole sequence

$$0\to\mathbb{V}\xrightarrow{\nabla}\Lambda^1\otimes\mathbb{V}\xrightarrow{\nabla}\Lambda^2\otimes\mathbb{V}\xrightarrow{\nabla}\cdots\xrightarrow{\nabla}\Lambda^p\otimes\mathbb{V}\xrightarrow{\nabla}\Lambda^{p+1}\otimes\mathbb{V}\xrightarrow{\nabla}\cdots \tag{24}$$

is a complex.

Consider the linear mapping

$$\Lambda^p\mathfrak{g}^*\otimes\mathbb{V}\ni v_{\alpha\beta\cdots\gamma}\overset{\theta}{\longmapsto}v_{ab\cdots c}\equiv\theta_a^\alpha\theta_b^\beta\cdots\theta_c^\gamma v_{\alpha\beta\cdots\gamma}\in\Gamma(G,\Lambda^p\otimes\mathbb{V})$$

in which $\mathbb{V}$ is just a passenger (i.e. plays no rôle). We shall refer to the resulting $\mathbb{V}$-valued p-form as *left-invariant* just as we would if $\mathbb{V}$ were absent.

Lemma 3. *The connection $\nabla : \Lambda^p \otimes \mathbb{V} \to \Lambda^{p+1} \otimes \mathbb{V}$ preserves left-invariance.*

Proof. We use the Maurer–Cartan equation (20) to compute

$$\begin{aligned}
\nabla_{[a} v_{bc\cdots d]} &= d_{[a}(\theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta v_{\beta\gamma\cdots\delta}) + \theta_{[a}^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta \rho_\alpha v_{\beta\gamma\cdots\delta} \\
&= p(d_{[a}\theta_b^\epsilon)\theta_c^\gamma \cdots \theta_{d]}^\delta v_{\epsilon\gamma\cdots\delta} + \theta_{[a}^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta \rho_\alpha v_{\beta\gamma\cdots\delta} \\
&= -(p/2)\Gamma_{\alpha\beta}{}^\epsilon \theta_{[a}^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta v_{\epsilon\gamma\cdots\delta} + \theta_{[a}^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta \rho_\alpha v_{\beta\gamma\cdots\delta} \\
&= \theta_{[a}^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta \rho_\alpha v_{\beta\gamma\cdots\delta} + (-1)^p (p/2) \theta_{[a}^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_{d]}^\delta \Gamma_{\alpha\beta}{}^\epsilon v_{\gamma\cdots\delta\epsilon} \\
&= \theta_a^\alpha \theta_b^\beta \theta_c^\gamma \cdots \theta_d^\delta \left(\rho_{[\alpha} v_{\beta\gamma\cdots\delta]} + (-1)^p (p/2)\Gamma_{[\alpha\beta}{}^\epsilon v_{\gamma\cdots\delta]\epsilon}\right),
\end{aligned} \tag{25}$$

as required. ■

In fact (25) shows that $\nabla\theta v = \theta\partial v$, where

$$\begin{aligned}
&\partial : \ \Lambda^p \mathfrak{g}^* \otimes \mathbb{V} \to \Lambda^{p+1}\mathfrak{g}^* \otimes \mathbb{V} \qquad \text{is given by} \\
&\qquad v_{\beta\gamma\cdots\delta} \mapsto \rho_{[\alpha} v_{\beta\gamma\cdots\delta]} + (-1)^p (p/2)\Gamma_{[\alpha\beta}{}^\epsilon v_{\gamma\cdots\delta]\epsilon}.
\end{aligned} \tag{26}$$

It also follows that

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathfrak{g}^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2\mathfrak{g}^* \otimes \mathbb{V} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^p\mathfrak{g}^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^{p+1}\mathfrak{g}^* \otimes \mathbb{V} \xrightarrow{\partial} \cdots \tag{27}$$

is a complex of $\mathfrak{g}$-modules. Alternatively, this may be directly verified from (26) using

- $\rho_{[\alpha}\rho_{\beta]} = \frac{1}{2}\Gamma_{\alpha\beta}{}^\gamma \rho_\gamma$ (i.e. that $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ is a representation),
- $\Gamma_{[\alpha\beta}{}^\delta \Gamma_{\gamma]\delta}{}^\epsilon = 0$ (i.e. the Jacobi identity in $\mathfrak{g}$),
- $(Xv)_{\beta\gamma\cdots\delta} = X^\alpha \rho_\alpha v_{\beta\gamma\cdots\delta} + (-1)^p p X^\alpha \Gamma_{\alpha[\beta}{}^\epsilon v_{\gamma\cdots\delta]\epsilon}$ (the action of $\mathfrak{g}$ on $\Lambda^p\mathfrak{g}^* \otimes \mathbb{V}$).

We have shown that there are two ways of defining Lie algebra cohomology as follows.

Theorem 5. *The Lie algebra cohomology $H^r(\mathfrak{g}, \mathbb{V})$ may be defined as either*

- *the cohomology of the Koszul complex* (27)*, or*
- *the cohomology of the complex* (24) *restricted to left-invariant forms.*

Remark 3. Although we use this theorem in the main body of this article, it is easily avoided. In tackling a general parabolic geometry, however, we believe that Theorem 5 will be essential.

Remark 4. As a minor variation on this construction, suppose G is enlarged to Q, a semi-direct product

$$Q = G_0 \ltimes G \qquad \text{i.e.} \qquad \mathrm{Id} \to G \lhd Q \rightleftarrows G_0 \to \mathrm{Id}$$

and suppose that $\mathbb{V}$ extends to a representation of Q. We identify G with the Q-homogeneous space Q/G_0, noting that when the action of Q on $G = Q/G_0$ is restricted to G it coincides with its usual action of G on itself by left translation. The Q-homogeneous bundle $V \equiv Q \times_{G_0} \mathbb{V}$ on Q/G_0 is equipped with a flat connection ∇ by dint of the canonical trivialisation

$$V = Q \times_{G_0} \mathbb{V} \ni (q, v) \mapsto (qG_0, qv) \in Q/G_0 \otimes \mathbb{V} = G \times \mathbb{V},$$

which clearly coincides with ∇ defined by (22). This connection is Q-equivariant. Consequently, not only does the twisted de Rham complex

$$V \xrightarrow{\nabla} \Lambda^1 \otimes V \xrightarrow{\nabla} \Lambda^2 \otimes V \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^p \otimes V \xrightarrow{\nabla} \Lambda^{p+1} \otimes V \xrightarrow{\nabla} \cdots$$

coincide with (24) and thereby compute the Lie algebra cohomology $H^r(\mathfrak{g}, \mathbb{V})$ when restricted to G-invariant forms, but also the complex (27) is automatically one of Q-modules where the Q-action on $\mathbb{V}$ is as supposed and the Q-action on $\mathfrak{g}^*$ is induced by the conjugation action of Q on G (bearing in mind that G is a normal subgroup of Q).

Addendum: A canonical connection on G

Again, although it is unnecessary for the current article and already known to experts, we suspect that the following optional extra will be invaluable in dealing with a general parabolic geometry. Since we already have established suitable notation in the Appendix above, we take the opportunity of presenting it here. Our canonical connection D_a was introduced in [7] as the '(+)-connection' and Lemma 4 is stated without proof as [16, Proposition 2.12].

The trivialisation $T^*G = G \times \mathfrak{g}^*$ provided by the Maurer–Cartan form also equips G with a canonical flat affine connection D_a defined by

$$D_a \omega_b \equiv \theta_b^\beta d_a \omega_\beta,$$

where $\omega_\beta \equiv \phi_\beta^c \omega_c$ and d_a on the right hand side of this equation simply takes the gradient of a function with values in $\mathfrak{g}^*$. If we expand using any torsion-free affine connection ∇_a

$$D_a \omega_b = \theta_b^\beta \nabla_a (\phi_\beta^c \omega_c) = \nabla_a \omega_b + (\theta_b^\beta \nabla_a \phi_\beta^c) \omega_c = \nabla_a \omega_b - (\phi_\beta^c \nabla_a \theta_b^\beta) \omega_c,$$

then we see that, for f a smooth function,

$$D_a D_b f - D_b D_a f = (-\phi_\beta^c \nabla_a \theta_b^\beta + \phi_\beta^c \nabla_b \theta_a^\beta) D_c f = -\phi_\gamma^c (\nabla_a \theta_b^\gamma - \nabla_b \theta_a^\gamma) D_c f$$

and so the canonical connection D_a has torsion

$$T_{ab}{}^c = -(\nabla_a \theta_b^\gamma - \nabla_b \theta_a^\gamma) \phi_\gamma^c.$$

Alternatively, from (20) we see that

$$T_{ab}{}^c \theta_c^\gamma = -(\nabla_a \theta_b^\gamma - \nabla_b \theta_a^\gamma) = \Gamma_{\alpha\beta}{}^\gamma \theta_a^\alpha \theta_b^\beta,$$

which we record as the following lemma.

Lemma 4. *The torsion of D_a coincides with the Lie bracket on $\mathfrak{g}$ under the Maurer–Cartan parallelism.*

Notice that $D_a \theta_b^\beta = 0$. It is another way to characterise D_a and, indeed, is the main point of this construction as follows.

Lemma 5. *Even locally, the kernel of the induced operator*

$$D : \Lambda^p \to \Lambda^1 \otimes \Lambda^p$$

is the left-invariant forms on G.

Proof. Recall that the left-invariant forms are obtained as

$$\Lambda^p \mathfrak{g}^* \ni v_{\alpha\beta\cdots\gamma} \stackrel{\theta}{\longmapsto} v_{ab\cdots c} \equiv \theta_a^\alpha \theta_b^\beta \cdots \theta_c^\gamma v_{\alpha\beta\cdots\gamma} \in \Gamma(G, \Lambda^p)$$

and it clear that such forms are annihilated by D. Conversely, since D is flat and all covariant constant sections are already accounted for, there can be no more, even locally. ∎

Remark 5. Of course, this lemma also holds for $\mathbb{V}$-valued differential forms where the connection is trivially coupled with $\mathbb{V}$ and it is this that we have in mind in constructing the BGG complex in general.

Acknowledgements

Eastwood is supported by the Australian Research Council. Gover is partly supported by the Royal Society of New Zealand (Marsden Grant 10-UOA-113). The authors would like to thank Katharina Neusser for many valuable conversations and the referees for their useful corrections and suggestions.

References

[1] Baston R.J., Almost Hermitian symmetric manifolds. II. Differential invariants, *Duke Math. J.* **63** (1991), 113–138.

[2] Baston R.J., Eastwood M.G., The Penrose transform. Its interaction with representation theory, Oxford University Press, New York, 1989.

[3] Calderbank D.M.J., Diemer T., Differential invariants and curved Bernstein–Gelfand–Gelfand sequences, *J. Reine Angew. Math.* **537** (2001), 67–103, math.DG/0001158.

[4] Calderbank D.M.J., Diemer T., Souček V., Ricci-corrected derivatives and invariant differential operators, *Differential Geom. Appl.* **23** (2005), 149–175, math.DG/0310311.

[5] Čap A., Slovák J., Parabolic geometries. I. Background and general theory, *Mathematical Surveys and Monographs*, Vol. 154, American Mathematical Society, Providence, RI, 2009.

[6] Čap A., Slovák J., Souček V., Bernstein–Gelfand–Gelfand sequences, *Ann. of Math. (2)* **154** (2001), 97–113, math.DG/0001164.

[7] Cartan É., Schouten J.A., On the geometry of the group-manifold of simple and semi-simple groups, *Nederl. Akad. Wetensch. Proc. Ser. A* **29** (1926), 803–815.

[8] Chevalley C.C., Eilenberg S., Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[9] Chow T.Y., You could have invented spectral sequences, *Notices Amer. Math. Soc.* **53** (2006), 15–19.

[10] Eastwood M.G., A duality for homogeneous bundles on twistor space, *J. London Math. Soc. (2)* **31** (1985), 349–356.

[11] Eastwood M.G., Variations on the de Rham complex, *Notices Amer. Math. Soc.* **46** (1999), 1368–1376.

[12] Eastwood M.G., Notes on projective differential geometry, in Symmetries and Overdetermined Systems of Partial Differential Equations, *IMA Vol. Math. Appl.*, Vol. 144, Springer, New York, 2008, 41–60.

[13] Eastwood, M.G., Tod, K.P., Edth – a differential operator on the sphere, *Math. Proc. Cambridge Philos. Soc.* **92** (1982), 317–330.

[14] Fulton W., Harris J., Representation theory, *Graduate Texts in Mathematics*, Vol. 129, Springer-Verlag, New York, 1991.

[15] Gover A.R., Conformally invariant operators of standard type, *Quart. J. Math. Oxford Ser. (2)* **40** (1989), 197–207.

[16] Kobayashi S., Nomizu K., Foundations of differential geometry. II, Wiley Interscience, New York – London – Sydney, 1969.

[17] Kostant B., Lie algebra cohomology and the generalized Borel–Weil theorem, *Ann. of Math. (2)* **74** (1961), 329–387.

[18] Lepowsky J., A generalization of the Bernstein–Gelfand–Gelfand resolution, *J. Algebra* **49** (1977), 496–511.

[19] Olver P., Differential hyperforms I, `http://www.math.umn.edu/~olver/a_/hyper.pdf`.

[20] Penrose R., Rindler W., Spinors and space-time. Vol. 1. Two-spinor calculus and relativistic fields, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 1984.

[21] Rice J.W., Private communication, January 1985.

[22] Schouten J.A., Ricci-calculus. An introduction to tensor analysis and its geometrical applications, Springer-Verlag, Berlin, 1954.

[23] Sternberg S., Lie algebras, `http://www.math.harvard.edu/~shlomo/docs/lie_algebras.pdf`.

[24] Vogan D.A., Representations of real reductive Lie groups, *Progress in Mathematics*, Vol. 15, Birkhäuser, Boston, Mass., 1981.

[25] Woodhouse N.M.J., Geometric quantisation, 2nd ed., Oxford University Press, New York, 1992.