Resolutions of Identity for Some Non-Hermitian Hamiltonians. I. Exceptional Point in Continuous Spectrum⋆

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Abstract. Resolutions of identity for certain non-Hermitian Hamiltonians constructed from biorthogonal sets of their eigen- and associated functions are given for the spectral problem defined on entire axis. Non-Hermitian Hamiltonians under consideration possess the continuous spectrum and the following peculiarities are investigated: (1) the case when there is an exceptional point of arbitrary multiplicity situated on a boundary of continuous spectrum; (2) the case when there is an exceptional point situated inside of continuous spectrum. The reductions of the derived resolutions of identity under narrowing of the classes of employed test functions are revealed. It is shown that in the case (1) some of associated functions included into the resolution of identity are normalizable and some of them may be not and in the case (2) the bounded associated function corresponding to the exceptional point does not belong to the physical state space. Spectral properties of a SUSY partner Hamiltonian for the Hamiltonian with an exceptional point are examined.

Key words: non-Hermitian quantum mechanics; supersymmetry; exceptional points; resolution of identity

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1 Introduction

The interest to exceptional points in non-Hermitian quantum dynamical systems has been revoked recently [1, 2, 3, 4, 5] although their very notion exists quite a time [6, 7, 8, 9, 10]. Their appearance can be associated to level coalescence at complex coupling constants for initially Hermitian Hamiltonians [11]. If existing they play an important role in definition of energy spectra and in construction of biorthogonal bases in Riesz spaces [12].

Whereas the appearance of exceptional points in discrete spectrum does not give rise to any principal obstacles for building a resolution of identity, the emergence of exceptional points inside or on the border of continuous spectrum makes the very construction of resolutions of identity rather sophisticated [2]. Their correct description in brief represents the main aim of our paper (entitled as Part I) whereas in the subsequent paper [13] (entitled as Part II) the detailed proofs of the results announced here are presented by one of us (A.V.S.).

Let us start with the notion of exceptional point and further on outline the structure of the present work. The spectrum of a Hermitian Hamiltonian, in general, consists of continuous part

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and discrete points. Meanwhile the spectrum of a non-Hermitian Hamiltonian may contain also a new type of spectral points embedded into a continuous spectrum and/or into a discrete one, namely, exceptional points.

The *exceptional point* of the spectrum of one-dimensional Hamiltonian h defined on entire axis is an eigenvalue λ_0 of this Hamiltonian for which there is a normalizable eigenfunction $\psi_0(x)$ and also a number of associated functions [7] $\psi_j(x)$, $j = 1, \ldots, n-1$:

$$h\psi_0 = \lambda_0\psi_0, \qquad (h - \lambda_0)\psi_j = \psi_{j-1}, \qquad j = 1, \ldots, n-1.$$

For a discrete spectrum the latter ones are typically normalizable [2, 3]. On the other hand some non-normalizable associated functions bounded or even growing at infinity may be involved in building of resolution of identity as well. It will be proven in Part II. The number n is an algebraic multiplicity of λ_0. Thus, in continuous spectrum one can deal with two types of algebraic multiplicities (which are not necessarily equal: their different types for continuous spectrum are discussed in conclusions). If an exceptional point λ_0 belongs to a discrete part of the spectrum then n simultaneously characterizes the order of a pole at $E = \lambda_0$ of the Green function. For an exceptional point λ_0 on the border of continuous spectrum the Green function reveals (see Section 2.3) a branching point with the pole order $2n+1$ in the variable $\sqrt{E - \lambda_0}$, where n is a maximal number of linearly independent eigen- and formal associated functions of h for an eigenvalue λ_0 in the resolution of identity. When an exceptional point lies inside of continuous part of the spectrum the pole order may be larger than n (which has the same meaning as in the previous sentence) that is elucidated in details in conclusions.

In this paper we build resolutions of identity for certain non-Hermitian Hamiltonians constructed from biorthogonal sets of their eigen- and associated functions for the spectral problem defined on entire axis. Non-Hermitian Hamiltonians under consideration are taken with continuous spectrum and the following peculiarities are investigated: in Section 2 the case when there is an exceptional point of arbitrary multiplicity situated on a boundary of continuous spectrum; in Section 3 the case when there is an exceptional point inside of continuous spectrum. In Section 4 in conclusions the different ways to introduce algebraic multiplicities are discussed and the SUSY tools [14, 15, 16, 17, 18, 19, 20, 21, 22] in regulating them are inspected.

More specifically in Sections 2 and 3 the reductions of the resolutions of identity under narrowing of the classes of employed test functions are elaborated. It is shown that the bounded associated function in an exceptional point inside of continuous spectrum does not belong to the physical state space (i.e. does not belong to the complete biorthogonal system built from eigenfunctions of the Hamiltonian and cannot be reproduced with the help of harmonic expansion generated by an appropriate resolution of identity). If an exceptional point lies on a boundary of continuous spectrum then some of associated functions included into the resolution of identity are normalizable and some of them may be not, still being elements of a rigged Hilbert space [23] and its dual one, the Gelfand triple generalized onto biorthogonal resolutions of identity.

2 Resolutions of identity for model Hamiltonians with an exceptional point of arbitrary multiplicity at the bottom of continuous spectrum

2.1 Basic constructions

Let us consider the sequence of Hamiltonians,

$$h_n = -\partial^2 + \frac{n(n+1)}{(x-z)^2}, \qquad x \in \mathbb{R}, \qquad \partial \equiv \frac{d}{dx}, \qquad \operatorname{Im} z \neq 0, \qquad n = 0, 1, 2, \ldots,$$

where h_0 is the Hamiltonian of a free particle and all these Hamiltonians are $\mathcal{PT}$-symmetric [24] for the choice $\operatorname{Re} z = 0$. One can easily check that for the Hamiltonian h_n on the energy level $E = 0$ there is an eigenfunction $\psi_{n0}(x)$ and a chain of formal associated functions $\psi_{nl}(x)$:

$$h_n \psi_{n0} = 0, \qquad h_n \psi_{nl} = \psi_{n,l-1}, \qquad l = 1, 2, 3, \ldots,$$

$$\psi_{n0}(x) = \frac{(-i)^n (2n-1)!!}{\sqrt{2\pi}\,(x-z)^n}, \qquad \psi_{nl}(x) = \frac{(-i)^n (2n-2l-1)!!}{\sqrt{2\pi}(2l)!!(x-z)^{n-2l}}, \qquad l = 0, 1, 2, \ldots,$$

$$0!! = (-1)!! = 1, \qquad (-2m-1)!! = \frac{(-1)^m}{(2m-1)!!}, \qquad m = 1, 2, 3, \ldots. \tag{2.1}$$

Moreover for odd n the functions $\psi_{nl}(x)$; $l = 0, \ldots, [n/2]$ are normalizable (i.e. belong to $L^2(\mathbb{R})$) and when $l > [n/2]$ they are non-normalizable and unboundedly growing for $x \to \pm\infty$. For even n the functions $\psi_{nl}(x)$; $l = 0, \ldots, [n/2] - 1$ are normalizable, the function

$$\psi_{n,n/2}(x) \equiv \frac{(-i)^n (n-1)!!}{\sqrt{2\pi}\,n!!} \tag{2.2}$$

is bounded but non-normalizable and the functions $\psi_{nl}(x)$ for $l > n/2$ are non-normalizable and unboundedly growing for $x \to \pm\infty$.

The Hamiltonians h_n, $n = 0, 1, 2, \ldots$ are intertwined by the operators

$$q_n^{\pm} = \mp\partial - \chi_n(x), \qquad \chi_n(x) = \frac{\psi'_{n0}(x)}{\psi_{n0}(x)} \equiv -\frac{n}{x-z}, \qquad q_n^- \psi_{n0} = 0, \qquad n = 0, 1, 2, \ldots, \tag{2.3}$$

with the help of the chain (ladder) construction [25, 26, 27, 17]

$$h_n q_n^+ = q_n^+ h_{n-1}, \qquad q_n^- h_n = h_{n-1} q_n^-$$

and

$$h_n = q_n^+ q_n^-, \qquad h_{n-1} = q_n^- q_n^+, \qquad n = 0, 1, 2, \ldots, \qquad h_{-1} = -\partial^2 = h_0.$$

One easily check that

$$q_n^- \psi_{nl} = -i\psi_{n-1,l-1}, \qquad n, l = 1, 2, 3, \ldots. \tag{2.4}$$

As well the eigenfunctions $\psi_n(x; k)$ for continuous spectrum of the Hamiltonian h_n can be produced from the eigenfunctions

$$\psi_0(x; k) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \qquad k \in \mathbb{R}$$

for the continuous spectrum of the Hamiltonian h_0 of a free particle with the help of intertwining operators (2.3):

$$\psi_n(x; k) = \frac{1}{\sqrt{2\pi}} \left(\frac{i}{k}\right)^n q_n^+ \cdots q_1^+ e^{ikx},$$

$$h_n \psi_n(x; k) = k^2 \psi_n(x; k), \qquad k \in (-\infty, 0) \cap (0, +\infty), \qquad n = 0, 1, 2, \ldots. \tag{2.5}$$

For the function $\psi_n(x; k)$ one can derive the following explicit representation by induction,

$$\psi_n(x; k) = \frac{1}{\sqrt{2\pi}} e^{ikx} \sum_{m=0}^{n} \frac{(n+m)!}{2^m m!(n-m)!} \frac{i^m}{k^m (x-z)^m}, \tag{2.6}$$

where from it follows that the cofactor $(i/k)^n$ in (2.5) provides the asymptotic form for

$$\psi_n(x;k) = \frac{1}{\sqrt{2\pi}} e^{ikx} \left[1 + O\left(\frac{1}{x}\right) \right], \qquad x \to \pm\infty. \tag{2.7}$$

As well, using (2.3) and (2.5), we can get the following representations for $\psi_n(x;k)$:

$$\psi_n(x;k) = \frac{1}{\sqrt{2\pi}} e^{ikz} \left(\frac{d}{dt} - \frac{n}{t} \right) \left(\frac{d}{dt} - \frac{n-1}{t} \right) \cdots \left(\frac{d}{dt} - \frac{1}{t} \right) e^t \Big|_{t=ik(x-z)}$$

$$= \frac{e^{ikz}}{\sqrt{2\pi} i^n (x-z)^n} \left(\frac{\partial}{\partial k} - \frac{n}{k} \right) \left(\frac{\partial}{\partial k} - \frac{n-1}{k} \right) \cdots \left(\frac{\partial}{\partial k} - \frac{1}{k} \right) e^{ik(x-z)}. \tag{2.8}$$

At last, it follows from (2.5) and (2.8) that

$$\psi_n(x;k) = \frac{i}{k} q_n^+ \psi_{n-1}(x;k)$$

$$= \frac{e^{ikz}}{i(x-z)} \left(\frac{\partial}{\partial k} - \frac{n}{k} \right) \left[e^{-ikz} \psi_{n-1}(x;z) \right], \qquad n = 1, 2, 3, \ldots. \tag{2.9}$$

Let us find now the connection between $\psi_n(x;k)$ and $\psi_{nl}(x)$, $l = 0, 1, 2, \ldots$. For this purpose we calculate the following derivative, using Leibnitz formula and the relation 4.2.7.14 from [28],

$$\lim_{k \to 0} \frac{\partial^m}{\partial k^m} \left[e^{-ikz} k^n \psi_n(x;k) \right] = \frac{\partial^m}{\partial k^m} \left[\frac{1}{\sqrt{2\pi}} e^{ik(x-z)} \sum_{s=0}^{n} \frac{(n+s)!}{2^s s! (n-s)!} \frac{i^s k^{n-s}}{(x-z)^s} \right] \Big|_{k=0}$$

$$= \frac{i^{n+m} n!}{\sqrt{2\pi}\, 2^n (x-z)^{n-m}} \sum_{j=0}^{\min\{m,n\}} (-2)^j C_m^j C_{2n-j}^n = \frac{i^{n+m}}{\sqrt{2\pi}\, (x-z)^{n-m}} \prod_{s=1}^{n} (2s - m - 1)$$

$$= \begin{cases} (-1)^n m! \psi_{n,m/2}(x), & m \text{ is even}, \\ 0, & m \text{ is odd and } \leqslant 2n - 1, \\ \dfrac{i^{m-n}(m-1)!!}{\sqrt{2\pi}(m-2n-1)!!} \dfrac{1}{(x-z)^{n-m}}, & m \text{ is odd and } > 2n - 1, \end{cases} \tag{2.10}$$

where $C_n^m \equiv n!/[m!(n-m)!]$ is a binomial coefficient. Thus,

$$\psi_{nl}(x) = \frac{(-1)^n}{(2l)!} \lim_{k \to 0} \frac{\partial^{2l}}{\partial k^{2l}} \left[e^{-ikz} k^n \psi_n(x;k) \right], \qquad l = 0, 1, 2, \ldots. \tag{2.11}$$

Since the continuous spectrum of the Hamiltonian h_n, $n = 1, 2, 3, \ldots$ coincide with $[0, +\infty)$ (see (2.5)), the eigenvalue $E = 0$ of this Hamiltonian for the chain of functions $\psi_{nl}(x)$, $l = 0, 1, 2, \ldots$ is situated at the bottom of continuous spectrum. It will be shown in Section 2.3 which of these functions are included in the resolution of identity constructed from eigen- and associated functions of h_n.

2.2   Biorthogonality relations

The biorthogonality relations between functions $\psi_{nl}(x)$, $l = 0, \ldots, n-1$ follow from (2.1):

$$\int_{-\infty}^{+\infty} \psi_{nl}(x) \psi_{nl'}(x) \, dx = 0, \qquad l + l' \leqslant n - 1. \tag{2.12}$$

The biorthogonality relations between $\psi_{n0}(x)$ and $\psi_n(x;k)$ can be derived with the help of (2.3), (2.7) and (2.9):

$$\int_{-\infty}^{+\infty} \psi_{n0}(x) \left[k^n \psi_n(x;k) \right] dx = i \int_{-\infty}^{+\infty} \psi_{n0}(x) \left(-\partial + \frac{n}{x-z} \right) \left[k^{n-1} \psi_{n-1}(x;k) \right] dx$$

$$= -i\psi_{n0}(x)\big[k^{n-1}\psi_{n-1}(x;k)\big]\Big|_{-\infty}^{+\infty}$$

$$+ i\int_{-\infty}^{+\infty}\big[q_n^-\psi_{n0}(x)\big]\big[k^{n-1}\psi_{n-1}(x;k)\big]\,dx = 0, \qquad n = 1, 2, 3, \ldots. \tag{2.13}$$

The biorthogonality relations between normalizable associated functions $\psi_{nl}(x)$ and $\psi_n(x;k)$ can be derived in the same way with the help of (2.4) by induction,

$$\int_{-\infty}^{+\infty}\psi_{nl}(x)[k^n\psi_n(x;k)]\,dx = i\int_{-\infty}^{+\infty}\psi_{nl}(x)\left(-\partial + \frac{n}{x-z}\right)\big[k^{n-1}\psi_{n-1}(x;k)\big]\,dx$$

$$= -i\psi_{nl}(x)[k^{n-1}\psi_{n-1}(x;k)]\Big|_{-\infty}^{+\infty} + i\int_{-\infty}^{+\infty}[q_n^-\psi_{nl}(x)]\big[k^{n-1}\psi_{n-1}(x;k)\big]\,dx$$

$$= \int_{-\infty}^{+\infty}\psi_{n-1,l-1}(x)\big[k^{n-1}\psi_{n-1}(x;k)\big]\,dx = \cdots$$

$$= \int_{-\infty}^{+\infty}\psi_{n-l,0}(x)\big[k^{n-l}\psi_{n-l}(x;k)\big]\,dx = 0, \qquad l = 1, \ldots, \left[\frac{n-1}{2}\right]. \tag{2.14}$$

The biorthogonality relations between non-normalizable formal associated functions $\psi_{nl}(x)$, $l = [(n+1)/2], \ldots, n-1$ and $\psi_n(x;k)$,

$$\int_{-\infty}^{+\infty}\psi_{nl}(x)\big[k^n\psi_n(x;k)\big]\,dx = 0, \qquad l = \left[\frac{n+1}{2}\right], \ldots, n-1, \tag{2.15}$$

can be derived with the help of (2.1), (2.6), the Jordan lemma and the relation 4.2.7.17 from [28] as follows,

$$\int_{-\infty}^{+\infty}\psi_{nl}(x)[k^n\psi_n(x;k)]\,dx$$

$$= (-i)^n\frac{(2n-2l-1)!!}{2\pi(2l)!!}\sum_{m=0}^{2l-n}\frac{i^m(n+m)!}{2^m m!(n-m)!}k^{n-m}\int_{-\infty}^{+\infty}(x-z)^{2l-n-m}e^{ikx}\,dx$$

$$+ (-i)^n\frac{(2n-2l-1)!!}{2\pi(2l)!!}\sum_{m=2l-n+1}^{n}\frac{i^m(n+m)!}{2^m m!(n-m)!}k^{n-m}\int_{-\infty}^{+\infty}\frac{e^{ikx}\,dx}{(x-z)^{n-2l+m}}$$

$$= (-i)^n\frac{(2n-2l-1)!!}{(2l)!!}\sum_{m=0}^{2l-n}\frac{i^m(n+m)!}{2^m m!(n-m)!}k^{n-m}\left(-i\frac{d}{dk}-z\right)^{2l-n-m}\delta(k)$$

$$+ \operatorname{sign}(\operatorname{Im}z)\theta(\operatorname{sign}(\operatorname{Im}z)\,k)(-i)^n\frac{(2n-2l-1)!!}{(2l)!!}e^{ikz}$$

$$\times \sum_{m=2l-n+1}^{n}\frac{i^{n-2l+2m}(n+m)!}{2^m m!(n-m)!(n-2l+m-1)!}k^{2n-2l-1}$$

$$= (-i)^n\frac{(2n-2l-1)!!}{(2l)!!}\sum_{m=0}^{2l-n}\frac{i^m(n+m)!}{2^m m!(n-m)!}k^{n-m}\left(-i\frac{d}{dk}-z\right)^{(n-m)-2(n-l)}\delta(k)$$

$$+ \operatorname{sign}(\operatorname{Im}z)\theta(\operatorname{sign}(\operatorname{Im}z)\,k)(-1)^l 2^n\frac{(2n-2l-1)!!(2l+1)!!}{(2n)!!}k^{2n-2l-1}e^{ikz}$$

$$\times \sum_{m=2l-n+1}^{n}\frac{(-1)^m}{2^m}C_n^m C_{m+n}^{2l+1} = 0,$$

where

$$\theta(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

The formal associated functions $\psi_{nl}(x)$, $l = n, n+1, n+2, \ldots$ are not contained in the resolutions of identity (see Section 2.3), but it is interesting that one can write the biorthogonality relations for these functions with $\psi_n(x; k)$ as well,

$$
\int_{-\infty}^{+\infty} \psi_{nl}(x) \left[e^{-ikz} k^n \psi_n(x; k) \right] dx
$$

$$
= \frac{(-1)^l i^n}{2\pi (2l)!!(2l-2n-1)!!} e^{-ikz} \sum_{m=0}^{n} \frac{i^m (n+m)!}{2^m m!(n-m)!} k^{n-m} \int_{-\infty}^{+\infty} (x-z)^{2l-n-m} e^{ikx} dx
$$

$$
= \frac{(-1)^l i^n}{(2l)!!(2l-2n-1)!!} e^{-ikz} \sum_{m=0}^{n} \frac{i^m (n+m)!}{2^m m!(n-m)!} k^{n-m} \left(-i\frac{d}{dk} - z \right)^{2l-n-m} \delta(k)
$$

$$
= \frac{(-1)^n}{(2l)!!(2l-2n-1)!!} e^{-ikz} \sum_{m=0}^{n} \frac{(-1)^m (n+m)!}{2^m m!(n-m)!} k^{n-m} \left(\frac{d}{dk} - iz \right)^{2l-n-m} \left[e^{ikz} \delta(k) \right]
$$

$$
= \frac{(-1)^n}{(2l)!!(2l-2n-1)!!} \sum_{m=0}^{n} \frac{(-1)^m (n+m)!}{2^m m!(n-m)!} k^{n-m} \delta^{(2l-n-m)}(k)
$$

$$
= \frac{1}{(2l)!!(2l-2n-1)!!(2l-2n)!} \delta^{(2l-2n)}(k) \sum_{m=0}^{n} \frac{(n+m)!(2l-n-m)!}{2^m m!(n-m)!}
$$

$$
= \frac{1}{(2l-2n)!} \delta^{(2l-2n)}(k), \qquad l = n, n+1, n+2, \ldots, \tag{2.16}
$$

where we have used (2.1), (2.6) and the relation

$$
\sum_{m=0}^{n} \frac{(n+m)!(s-m)!}{2^m m!(n-m)!} = \sum_{m=0}^{n} \frac{(n-m+2m)(n-1+m)!(s-m)!}{2^m m!(n-m)!}
$$

$$
= \sum_{m=0}^{n-1} \frac{(n-1+m)!(s-m)!}{2^m m!(n-1-m)!} + \sum_{m=1}^{n} \frac{(n-1+m)!(s-m)!}{2^{m-1}(m-1)!(n-m)!}
$$

$$
= \sum_{m=0}^{n-1} \frac{(n-1+m)!(s-m)!}{2^m m!(n-1-m)!} + \sum_{m=0}^{n-1} \frac{(n+m)!(s-1-m)!}{2^m m!(n-1-m)!}
$$

$$
= (s+n) \sum_{m=0}^{n-1} \frac{(n-1+m)!(s-1-m)!}{2^m m!(n-1-m)!} = \cdots = (s+n) \cdots (s-n+2)(s-n)!
$$

$$
= (s-n-1)!!(s+n)!!, \qquad s \geqslant n.
$$

At last, the biorthogonality relations between eigenfunctions for continuous spectrum of the Hamiltonian h_n are proved in [13] and take the following form:

$$
\int_{-\infty}^{+\infty} [k^n \psi_n(x; k)][(k')^n \psi_n(x; -k')] dx = (k')^{2n} \delta(k - k'). \tag{2.17}
$$

Let us notice that (2.13)–(2.15) contain (2.12) for $l = 0, \ldots, n-1$, $l' = 0$ and (2.17) contains (2.12) for $l = l' = 0$ due to (2.11). The relations (2.16) can be derived with the help of differentiation from (2.17) also in view of (2.11).

2.3   Resolutions of identity

The initial resolution of identity constructed from $\psi_n(x; k)$ holds [13],

$$
\delta(x - x') = \int_{\mathcal{L}} \psi_n(x; k) \psi_n(x'; -k) \, dk, \tag{2.18}
$$

where $\mathcal{L}$ is an integration path in complex k plane, obtained from the real axis by its deformation near the point $k = 0$ upwards or downwards (the direction of this deformation is of no difference since the residue of the integrand for the point $k = 0$ is equal zero in view of (2.6) and (2.10)) and the direction of $\mathcal{L}$ is specified from $-\infty$ to $+\infty$. This resolution of identity is valid for test functions belonging to $CL_\gamma \equiv C_{\mathbb{R}}^\infty \cap L_2(\mathbb{R}; (1 + |x|)^\gamma)$, $\gamma > -1$ as well as for some bounded and even slowly increasing test functions (more details are presented in [13]) and, in particular, for eigenfunctions $\psi_n(x; k)$ and for the associated function (2.2).

One can rearrange the resolution of identity (2.18) for any $\varepsilon > 0$ to the forms

$$
\begin{aligned}
\delta(x - x') &= \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_n(x; k)\psi_n(x'; -k)\, dk \\
&+ \sum_{l=0}^{n-1} \left(\frac{x' - z}{x - z} \right)^l \frac{\psi_{n-l-1}(x; k)\psi_{n-l}(x'; -k)}{i(x - z)} \Big|_{-\varepsilon}^{\varepsilon} + \left(\frac{x' - z}{x - z} \right)^n \frac{\sin \varepsilon(x - x')}{\pi(x - x')} \\
&\equiv \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_n(x; k)\psi_n(x'; -k)\, dk + \frac{\sin \varepsilon(x - x')}{\pi(x - x')} \\
&- \frac{\cos \varepsilon(x - x')}{2\pi\varepsilon(x - z)(x' - z)} \sum_{l=0}^{n-1} \frac{(-1/4)^l}{\varepsilon^{2l}(x' - z)^{2l}} \sum_{m=0}^{\min\{2l, n-1\}} C_{2l+1,m,n} \frac{(n+2l+1-m)!}{(n-1-m)!} \left(\frac{x' - z}{x - z} \right)^m \\
&+ \frac{\sin \varepsilon(x - x')}{\pi(x - z)} \sum_{l=1}^{n-1} \frac{(-1/4)^l}{\varepsilon^{2l}(x' - z)^{2l}} \sum_{m=0}^{\min\{2l-1, n-1\}} C_{2l,m,n} \frac{(n+2l-m)!}{(n-1-m)!} \left(\frac{x' - z}{x - z} \right)^m, \quad (2.19)
\end{aligned}
$$

$$
C_{lmn} = \frac{1}{l} \sum_{j=0}^{m} (-1)^j C_l^j C_{n-m-1+2j}^{l-1},
$$

$$
n = 1, 2, 3, \ldots, \qquad l = 1, \ldots, 2n - 1, \qquad m = 0, \ldots, \min\{l - 1, n - 1\}
$$

(cf. with (68) in [2]) and, consequently, to the form

$$
\begin{aligned}
\delta(x - x') &= \lim_{\varepsilon \downarrow 0}{}' \Bigg\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_n(x; k)\psi_n(x'; -k)\, dk + \frac{\sin \varepsilon(x - x')}{\pi(x - x')} \\
&- \frac{\cos \varepsilon(x - x')}{2\pi\varepsilon(x - z)(x' - z)} \sum_{l=0}^{n-1} \frac{(-1/4)^l}{\varepsilon^{2l}(x' - z)^{2l}} \sum_{m=0}^{\min\{2l, n-1\}} C_{2l+1,m,n} \frac{(n+2l+1-m)!}{(n-1-m)!} \left(\frac{x' - z}{x - z} \right)^m \\
&+ \frac{\sin \varepsilon(x - x')}{\pi(x - z)} \sum_{l=1}^{n-1} \frac{(-1/4)^l}{\varepsilon^{2l}(x' - z)^{2l}} \sum_{m=0}^{\min\{2l-1, n-1\}} C_{2l,m,n} \frac{(n+2l-m)!}{(n-1-m)!} \left(\frac{x' - z}{x - z} \right)^m \Bigg\}, \quad (2.20)
\end{aligned}
$$

where the prime $'$ at the limit symbol emphasizes that this limit is regarded as a limit in the space of distributions (see details in [13]).

From (2.19) and (2.20) one can try to derive various reduced resolutions of identity similar to the resolutions (69) and (70) of [2], which correspond to the partial case $n = 1$, or to the resolutions (15) and (16) of [29]. In particular, by virtue of Lemma 3.7 from [13], for test functions from CL_γ, $\gamma > -1$ the resolution (2.20) can be reduced to the form

$$
\begin{aligned}
\delta(x - x') &= \lim_{\varepsilon \downarrow 0}{}' \Bigg\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_n(x; k)\psi_n(x'; -k)\, dk \\
&- \frac{\cos \varepsilon(x - x')}{2\pi\varepsilon(x - z)(x' - z)} \sum_{l=0}^{n-1} \frac{(-1/4)^l}{\varepsilon^{2l}(x' - z)^{2l}} \sum_{m=0}^{\min\{2l, n-1\}} C_{2l+1,m,n} \frac{(n+2l+1-m)!}{(n-1-m)!} \left(\frac{x' - z}{x - z} \right)^m
\end{aligned}
$$

$$+ \frac{\sin \varepsilon(x - x')}{\pi(x - z)} \sum_{l=1}^{n-1} \frac{(-1/4)^l}{\varepsilon^{2l}(x' - z)^{2l}} \sum_{m=0}^{\min\{2l-1,n-1\}} C_{2l,m,n} \frac{(n + 2l - m)!}{(n - 1 - m)!} \left(\frac{x' - z}{x - z}\right)^m \Bigg\}. \quad (2.21)$$

More reduced resolutions of identity for the partial case $n = 2$ are presented in the forthcoming Section 2.4.

There is another way to transform the resolution (2.18) as well. This way was used for obtaining of the resolution of identity (26) from [29]. The integral from the right-hand part of (2.18) is understood [13] as follows:

$$\int_{\mathcal{L}} \psi_n(x; k)\psi_n(x'; -k)\, dk = \lim_{A \to +\infty}{}' \int_{\mathcal{L}(A)} \psi_n(x; k)\psi_n(x'; -k)\, dk, \quad (2.22)$$

where $\mathcal{L}(A)$ is a path in complex k plane, made of the segment $[-A, A]$ by its deformation near the point $k = 0$ upwards or downwards (the direction of this deformation is of no difference as well as in the case with $\mathcal{L}$) and the direction of $\mathcal{L}(A)$ is specified from $-A$ to A. When using the facts that the integral in the right-hand part of (2.22) is a standard integral (not a distribution) and that $k^{2n}\psi_n(x; k)\psi_n(x'; -k)$ is an entire function of k (see (2.6)) as well as employing the Leibniz formula and the formulae (2.10) and (2.11), we can transform (2.22) as follows,

$$\int_{\mathcal{L}} \psi_n(x; k)\psi_n(x'; -k)\, dk = \lim_{A \to +\infty}{}' \lim_{\varepsilon \downarrow 0} \left\{ \left(\int_{-A}^{-\varepsilon} + \int_{\varepsilon}^{A}\right) \psi_n(x; k)\psi_n(x'; -k)\, dk \right.$$

$$+ \sum_{j=0}^{2n-1} \frac{1}{j!} \frac{\partial^j}{\partial k^j}\left[k^{2n}\psi_n(x; k)\psi_n(x'; -k)\right]\Big|_{k=0} \int_{\mathcal{L}(\varepsilon)} \frac{dk}{k^{2n-j}} \Bigg\}$$

$$= \lim_{A \to +\infty}{}' \lim_{\varepsilon \downarrow 0} \left\{ \left(\int_{-A}^{-\varepsilon} + \int_{\varepsilon}^{A}\right) \psi_n(x; k)\psi_n(x'; -k)\, dk \right.$$

$$\left. - 2(-1)^n \sum_{l=0}^{n-1} \frac{1}{(2n - 2l - 1)\varepsilon^{2n-2l-1}} \sum_{m=0}^{l} \psi_{nm}(x)\psi_{n,l-m}(x') \right\}$$

$$= \lim_{\varepsilon \downarrow 0}{}^* \left\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty}\right) \psi_n(x; k)\psi_n(x'; -k)\, dk \right.$$

$$\left. - 2(-1)^n \sum_{l=0}^{n-1} \frac{1}{(2n - 2l - 1)\varepsilon^{2n-2l-1}} \sum_{m=0}^{l} \psi_{nm}(x)\psi_{n,l-m}(x') \right\}, \quad (2.23)$$

where the latter equality is considered as a definition and the limit $\lim_{\varepsilon \downarrow 0}$ is regarded as point-wise one (not as a limit in a function space).

Let us show that the terms outside the integral in the last line of (2.23) can be rearranged as follows,

$$-2(-1)^n \sum_{l=0}^{n-1} \frac{1}{(2n - 2l - 1)\varepsilon^{2n-2l-1}} \sum_{m=0}^{l} \psi_{nm}(x)\psi_{n,l-m}(x')$$

$$= \sum_{l=0}^{n-1} \psi_{nl}(x; \varepsilon)\psi_{n,n-1-l}(x'; \varepsilon), \quad (2.24)$$

where $\psi_{nl}(x; \varepsilon)$, $l = 0, \ldots, n - 1$ is the chain of eigenfunction and associated functions (formal for $l = [(n + 1)/2], \ldots, n - 1$) of the Hamiltonian h_n for the eigenvalue $E = 0$ of the form

$$\psi_{nl}(x; \varepsilon) = \sum_{j=0}^{l} \alpha_j(\varepsilon)\psi_{n,l-j}(x), \qquad l = 0, \ldots, n - 1, \quad (2.25)$$

$$h_n \psi_{n0}(x;\varepsilon) = 0, \qquad h_n \psi_{nl}(x;\varepsilon) = \psi_{n,l-1}(x;\varepsilon), \qquad l = 1,\ldots,n-1$$

with $\alpha_j(\varepsilon)$, $j = 0,\ldots,n-1$ being unknown coefficients which will be found below. Using (2.24) and (2.25), it is easy to check that (2.24) is valid iff the coefficients $\alpha_j(x)$ satisfy the following system,

$$\sum_{j=0}^{l} \alpha_j(\varepsilon)\alpha_{l-j}(\varepsilon) = -\frac{2(-1)^n}{(2l+1)\varepsilon^{2l+1}}, \qquad l = 0,\ldots,n-1. \tag{2.26}$$

After the redefinition

$$\alpha_j(\varepsilon) = \frac{\sqrt{2}\, i^{n+1}}{\varepsilon^{2j}\sqrt{\varepsilon}}\beta_j, \qquad j = 0,\ldots,n-1,$$

where β_j, $j = 0,\ldots,n-1$ are new unknown coefficients, the system (2.26) takes the form

$$\sum_{j=0}^{l} \beta_j \beta_{l-j} = \frac{1}{2l+1}, \qquad l = 0,\ldots,n-1.$$

The general solution of the latter system can be found in the recurrent form,

$$\beta_0 = \pm 1, \qquad \beta_1 = \frac{1}{6\beta_0}, \qquad \beta_l = \frac{1}{2\beta_0}\left(\frac{1}{2l+1} - \sum_{j=1}^{l-1}\beta_j\beta_{l-j}\right), \qquad l = 2,\ldots,n-1.$$

The first terms of the sequence β_j, $j = 0,\ldots,n-1$ in the case $\beta_0 = 1$ are the following ones,

$$\beta_0 = 1, \qquad \beta_1 = \frac{1}{6}, \qquad \beta_2 = \frac{31}{360}, \qquad \beta_3 = \frac{863}{15120}, \qquad \beta_4 = \frac{76813}{1814400}, \qquad \ldots.$$

Thus, we can choose the functions $\psi_{nl}(x;\varepsilon)$ in the form,

$$\psi_{nl}(x;\varepsilon) = i^{n+1}\sqrt{\frac{2}{\varepsilon}}\sum_{j=0}^{l}\frac{\beta_j}{\varepsilon^{2j}}\psi_{n,l-j}(x), \qquad l = 0,\ldots,n-1 \tag{2.27}$$

and the resolution of identity holds,

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}^*\left\{\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty}\right)\psi_n(x;k)\psi_n(x';-k)\,dk\right.$$

$$\left. + \sum_{l=0}^{n-1}\psi_{nl}(x;\varepsilon)\psi_{n,n-1-l}(x';\varepsilon)\right\}, \tag{2.28}$$

(cf. with (69) from [2] for the case $n = 1$). Moreover, this resolution is equivalent to (2.18), i.e. it is valid for all test functions for which (2.18) is valid (cf. with the analogous results in Section 3 and in [29]).

The resolution of identity (2.28) contains all n functions from the chain $\psi_{nl}(x;\varepsilon)$, $l = 0,\ldots,n-1$ and in this resolution for the eigenvalue $E = 0$ there are no other eigen- or associated functions of the Hamiltonian h_n. The order of the pole $k = 0$ for the Green function

$$G_n(x,x';E) = \left[\frac{\pi i}{k}\psi_n(x_>;k)\psi_n(x_<;-k)\right]\Big|_{k=\sqrt{E}},$$

$$x_> = \max\{x,x'\}, \qquad x_< = \{x,x'\}, \qquad \operatorname{Im}\sqrt{E} \geqslant 0, \qquad (h_n - E)G_n = \delta(x-x') \tag{2.29}$$

considered as a function of $k = \sqrt{E}$ is equal to $2n + 1$ in view of (2.6), and the exceptional point $E = 0$ coincides with the branch point of this Green function as a function of E. One can consider this pole of the order $2n + 1$ as a result of confluence of the pole of the order n for the Green function as a function of $E = k^2$ and of the factor $k \equiv \sqrt{E}$ from the denominator of the Green function (see (2.29)). Thus, the number n of linearly independent eigen- and (formal) associated functions of the Hamiltonian h_n incorporated in the resolution of identity (2.28) for the eigenvalue $E = 0$ (exceptional point) is equal to the order of the "pole" $E = 0$ for the Green function $G_n(x, x'; E)$ in the sense elucidated above or, more rigorously, the order of the pole of $G_n(x, x'; E)$ as a function of $k = \sqrt{E}$ is $2n + 1$ expressed in terms of the number n.

Let us notice that in view of (2.27) the functions $\psi_{nl}(x; \varepsilon)$, $l = 0, \ldots, n - 1$ satisfy the same biorthogonality relations from Section 2.2 as the functions $\psi_{nl}(x)$, $l = 0, \ldots, n - 1$.

2.4   Example: case $n = 2$

For the Hamiltonian

$$h_2 = -\partial^2 + \frac{6}{(x - z)^2}, \qquad x \in \mathbb{R}, \qquad \mathrm{Im}\, z \neq 0,$$

there are continuous spectrum eigenfunctions

$$\psi_2(x; k) = \frac{1}{\sqrt{2\pi}} \left[1 - \frac{3}{ik(x - z)} - \frac{3}{k^2(x - z)^2} \right] e^{ikx},$$

$$h_2 \psi_2(x; k) = k^2 \psi_2(x; k), \qquad k \in (-\infty, 0) \cup (0, +\infty) \tag{2.30}$$

(see (2.6)) and also the normalizable eigenfunction $\psi_{20}(x)$ and the bounded associated function $\psi_{21}(x)$ on the level $E = 0$,

$$\psi_{20}(x) = -\frac{3}{\sqrt{2\pi}\,(x - z)^2}, \qquad \psi_{21}(x) = -\frac{1}{2\sqrt{2\pi}}, \qquad h_2 \psi_{20} = 0, \qquad h_2 \psi_{21} = \psi_{20},$$

(see (2.1) and (2.2)). In this case at the point $k = 0$ there is a fifth order pole in the Green function $G_2(x, x'; E)$ considered as a function of $k = \sqrt{E}$ (see (2.29) and (2.30)) and thereby the exceptional point $E = 0$ of the spectrum of the Hamiltonian h_2 coincides with the branch point for the Green function as a function of E.

In the case under consideration the biorthogonality relations (2.12)–(2.15) and (2.17) take the form,

$$\int_{-\infty}^{+\infty} \psi_{20}^2(x)\, dx = 0, \qquad \int_{-\infty}^{+\infty} \psi_{20}(x) \left[k^2 \psi_2(x; k) \right] dx = 0, \tag{2.31}$$

$$\int_{-\infty}^{+\infty} \psi_{20}(x) \psi_{21}(x)\, dx = 0, \qquad \int_{-\infty}^{+\infty} \psi_{21}(x) \left[k^2 \psi_2(x; k) \right] dx = 0, \tag{2.32}$$

$$\int_{-\infty}^{+\infty} \left[k^2 \psi_2(x; k) \right] \left[(k')^2 \psi_2(x; -k') \right] dx = (k')^4 \delta(k - k'), \tag{2.33}$$

where (2.31) are included in (2.33) due to the equality

$$\psi_{20}(x) = \lim_{k \to 0} \left[k^2 \psi_2(x; k) \right] \tag{2.34}$$

(see (2.11)) and (2.32) can be derived from (2.33) in view of (2.34) and of the equality

$$\psi_{21}(x) = \frac{1}{2} \lim_{k \to 0} \frac{\partial^2}{\partial k^2} \left[e^{-ikz} k^2 \psi_2(x; k) \right]$$

(see (2.11) as well).

It is straightforward to check that the resolution of identity (2.19) (which is valid for test functions from CL_γ, $\gamma > -1$ as well as for some bounded and even slowly increasing test functions, see Section 2.3), can be rewritten in the form

$$\delta(x - x') = \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_2(x; k)\psi_2(x'; -k)\, dk$$

$$+ \left[\psi_{20}(x; \varepsilon)\psi_{21}(x'; \varepsilon) + \psi_{21}(x; \varepsilon)\psi_{20}(x'; \varepsilon) \right] + \frac{\sin \varepsilon(x - x')}{\pi(x - x')} + \frac{6 \sin^2 \frac{\varepsilon}{2}(x - x')}{\pi\varepsilon(x - z)(x' - z)}$$

$$+ \frac{12(x - x') \sin^2 \frac{\varepsilon}{4}(x - x') \sin \frac{\varepsilon}{2}(x - x')}{\pi\varepsilon^2(x - z)^2(x' - z)^2} + \frac{3[\varepsilon(x - x') - 2 \sin \frac{\varepsilon}{2}(x - x')]^2}{2\pi\varepsilon^3(x - z)^2(x' - z)^2}, \tag{2.35}$$

where the eigenfunction $\psi_{20}(x; \varepsilon)$ and the associated function $\psi_{21}(x; \varepsilon)$ of the Hamiltonian h_2 read

$$\psi_{20}(x; \varepsilon) = -i\sqrt{\frac{2}{\varepsilon}} \psi_{20}(x) \equiv \frac{3i}{\sqrt{\pi\varepsilon}\,(x - z)^2},$$

$$\psi_{21}(x; \varepsilon) = -i\sqrt{\frac{2}{\varepsilon}} \left[\psi_{21}(x) + \frac{1}{6\varepsilon^2}\psi_{20}(x) \right] \equiv \frac{i}{2\sqrt{\pi\varepsilon}} \left[1 + \frac{1}{\varepsilon^2(x - z)^2} \right],$$

$$h_2\psi_{20}(x; \varepsilon) = 0, \qquad h_2\psi_{21}(x; \varepsilon) = \psi_{20}(x; \varepsilon).$$

The eigenfunction $\psi_{20}(x; \varepsilon)$ and the associated function $\psi_{21}(x; \varepsilon)$ obviously satisfy the biorthogonality relations similar to (2.31) and (2.32).

It is shown in [13] that the resolution of identity (2.35) can be reduced: a) for test functions from CL_γ, $\gamma > -1$ to the form

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}' \left\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_2(x; k)\psi_2(x'; -k)\, dk \right.$$

$$+ \left[\psi_{20}(x; \varepsilon)\psi_{21}(x'; \varepsilon) + \psi_{21}(x; \varepsilon)\psi_{20}(x'; \varepsilon) \right] + \frac{6 \sin^2 \frac{\varepsilon}{2}(x - x')}{\pi\varepsilon(x - z)(x' - z)}$$

$$\left. + \frac{12(x - x') \sin^2 \frac{\varepsilon}{4}(x - x') \sin \frac{\varepsilon}{2}(x - x')}{\pi\varepsilon^2(x - z)^2(x' - z)^2} + \frac{3[\varepsilon(x - x') - 2 \sin \frac{\varepsilon}{2}(x - x')]^2}{2\pi\varepsilon^3(x - z)^2(x' - z)^2} \right\} \tag{2.36}$$

identical to (2.21), b) for test functions from CL_γ, $\gamma > 1$ to the form

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}' \left\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_2(x; k)\psi_2(x'; -k)\, dk \right.$$

$$+ \left[\psi_{20}(x; \varepsilon)\psi_{21}(x'; \varepsilon) + \psi_{21}(x; \varepsilon)\psi_{20}(x'; \varepsilon) \right]$$

$$\left. + \frac{12(x - x') \sin^2 \frac{\varepsilon}{4}(x - x') \sin \frac{\varepsilon}{2}(x - x')}{\pi\varepsilon^2(x - z)^2(x' - z)^2} + \frac{3[\varepsilon(x - x') - 2 \sin \frac{\varepsilon}{2}(x - x')]^2}{2\pi\varepsilon^3(x - z)^2(x' - z)^2} \right\} \tag{2.37}$$

and c) for test functions from CL_γ, $\gamma > 3$ to the form

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}' \left\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_2(x; k)\psi_2(x'; -k)\, dk \right.$$

$$\left. + \psi_{20}(x; \varepsilon)\psi_{21}(x'; \varepsilon) + \psi_{21}(x; \varepsilon)\psi_{20}(x'; \varepsilon) \right\}. \tag{2.38}$$

The latter of these resolutions of identity seems to have a more natural form than the previous ones, but the right-hand part of the latter resolution cannot reproduce the *normalizable* eigenfunction

$$\psi_{20}(x) \notin CL_\gamma \equiv C_{\mathbb{R}}^\infty \cap L^2(\mathbb{R}; (1 + |x|)^\gamma), \qquad \gamma > 3$$

because of the biorthogonality relations. With the help of the Jordan lemma one can check that

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \frac{12(x - x') \sin^2 \frac{\varepsilon}{4}(x - x') \sin \frac{\varepsilon}{2}(x - x')}{\pi \varepsilon^2 (x - z)^2 (x' - z)^2} \psi_{20}(x)\, dx$$

$$= \lim_{\varepsilon \downarrow 0} \left\{ \left[-\frac{3}{4} e^{\pm i\varepsilon(z - x')/2} \mp \frac{i}{8} \varepsilon(z - x') e^{\pm i\varepsilon(z - x')/2} + \frac{3}{2} e^{\pm i\varepsilon(z - x')} \right. \right.$$

$$\left. \left. \pm \frac{i}{2} \varepsilon(z - x') e^{\pm i\varepsilon(z - x')} \right] \psi_{20}(x') \right\} = \frac{3}{4} \psi_{20}(x') \tag{2.39}$$

and

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \frac{3[\varepsilon(x - x') - 2 \sin \frac{\varepsilon}{2}(x - x')]^2}{2\pi \varepsilon^3 (x - z)^2 (x' - z)^2} \psi_{20}(x)\, dx \tag{2.40}$$

$$= \lim_{\varepsilon \downarrow 0} \left\{ \left[\frac{3}{4} e^{\pm i\varepsilon(z - x')/2} \pm \frac{i}{8} \varepsilon(z - x') e^{\pm i\varepsilon(z - x')/2} - \frac{1}{2} e^{\pm i\varepsilon(z - x')} \right] \psi_{20}(x') \right\} = \frac{1}{4} \psi_{20}(x'),$$

where the upper (lower) signs correspond to the case $\text{Im}\, z > 0$ ($\text{Im}\, z < 0$). Hence, just two last terms of the resolution of identity (2.37) and the corresponding terms in the resolutions of identity (2.35) and (2.36) give a chance to reproduce $\psi_{20}(x)$ by these resolutions (cf. with the analogous results in Section 6.1 of [2], in Section 2 of [29] and in Section 3 of the present paper). It is interesting that contributions of these terms in the resolution of identity are (see Remark 3.4 in [13]) singular discontinuous functionals whose supports consist of the only element which is the infinity (cf. with the analogous comments in Section 2 of [29] and in Section 3 of the present paper).

In the case under consideration the resolution of identity (2.28) takes the following form,

$$\delta(x - x') = \lim_{\varepsilon \downarrow 0}{}^* \left\{ \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \psi_2(x; k) \psi_2(x'; -k)\, dk \right.$$

$$\left. + \psi_{20}(x; \varepsilon) \psi_{21}(x'; \varepsilon) + \psi_{21}(x; \varepsilon) \psi_{20}(x'; \varepsilon) \right\},$$

(cf. with (2.38)) and it is valid for test functions from CL_γ, $\gamma > -1$ as well as for some bounded and even slowly increasing test functions (see Section 2.3).

3 Resolutions of identity for the model Hamiltonian with exceptional point inside of the continuous spectrum

For the Hamiltonian

$$h = -\partial^2 + 16\alpha^2 \frac{\alpha(x - z) \sin 2\alpha x + 2 \cos^2 \alpha x}{[\sin 2\alpha x + 2\alpha(x - z)]^2},$$

$$x \in \mathbb{R}, \qquad \partial \equiv \frac{d}{dx}, \qquad \alpha > 0, \qquad \text{Im}\, z \neq 0$$

there are [2] continuous spectrum eigenfunctions

$$\psi(x;k) = \frac{1}{\sqrt{2\pi}} \left[1 + \frac{ik}{k^2 - \alpha^2} \frac{W'(x)}{W(x)} - \frac{1}{2(k^2 - \alpha^2)} \frac{W''(x)}{W(x)}\right] e^{ikx},$$
$$W(x) = \sin 2\alpha x + 2\alpha(x - z),$$
$$h\psi(x;k) = k^2\psi(x;k), \qquad k \in (-\infty, -\alpha) \cup (-\alpha, \alpha) \cup (\alpha, +\infty). \tag{3.1}$$

As well for the level $E = \alpha^2$ there is the normalizable eigenfunction $\psi_0(x)$ and the bounded associated function[1] $\psi_1(x)$,

$$\psi_0(x) = \frac{(2\alpha)^{3/2} \cos \alpha x}{\sin 2\alpha x + 2\alpha(x - z)}, \qquad \psi_1(x) = \frac{2\alpha(x - z)\sin \alpha x + \cos \alpha x}{\sqrt{2\alpha}\,[\sin 2\alpha x + 2\alpha(x - z)]},$$
$$\psi_1(x) = \frac{i}{2\sqrt{2\alpha}} \left[e^{-i\alpha x} - e^{i\alpha x}\right] + O\left(\frac{1}{x}\right), \qquad x \to \pm\infty, \tag{3.2}$$

such that

$$h\psi_0 = \alpha^2\psi_0, \qquad (h - \alpha^2)\psi_1 = \psi_0.$$

The exceptional point $E = \alpha^2$ is a pole for the Green function

$$G(x, x'; E) = \left[\frac{\pi i}{k} \psi_n(x_>; k)\psi_n(x_<; -k)\right]\Bigg|_{k=\sqrt{E}},$$
$$x_> = \max\{x, x'\}, \qquad x_< = \{x, x'\}, \qquad \mathrm{Im}\,\sqrt{E} \geqslant 0, \qquad (h - E)\,G = \delta(x - x').$$

This pole is a pole of second order, it is replicated on both sides of the cut $E > 0$ and there are no other poles for $G(x, x'; E)$.

One can show [2] that the eigenfunctions and the associated function of h obey the biorthogonality relations,

$$\int_{-\infty}^{+\infty} \psi_0^2(x)\,dx = 0, \qquad \int_{-\infty}^{+\infty} \psi_0(x)\left[(k^2 - \alpha^2)\psi(x;k)\right]dx = 0, \tag{3.3}$$

$$\int_{-\infty}^{+\infty} \psi_0(x)\psi_1(x)\,dx = 0, \qquad \int_{-\infty}^{+\infty} \psi_1(x)\left[(k^2 - \alpha^2)\psi(x;k)\right]dx = 0, \tag{3.4}$$

$$\int_{-\infty}^{+\infty} \left[(k^2 - \alpha^2)\psi(x;k)\right]\left[((k')^2 - \alpha^2)\psi(x;-k')\right]dx = \left((k')^2 - \alpha^2\right)^2 \delta(k - k'), \tag{3.5}$$

where (3.3) are included in (3.5) due to the equality

$$\psi_0(x) = \mp i\sqrt{\frac{\pi}{\alpha}} \lim_{k \to \pm\alpha} \left[(k^2 - \alpha^2)\psi(x;k)\right] \tag{3.6}$$

and (3.4) follow from (3.5) in view of (3.6) and of the equality

$$\psi_1(x) = \pm i\sqrt{\frac{\pi}{\alpha}} \lim_{k \to \mp\alpha} \left\{\frac{1}{2k}\frac{\partial}{\partial k}\left[(k^2 - \alpha^2)\right]\psi(x;k)\right\} - \frac{1 \mp 2i\alpha z}{4\alpha^2}\psi_0(x).$$

The resolution of identity constructed from $\psi(x;k)$ holds [13],

$$\delta(x - x') = \int_{\mathcal{L}} \psi(x;k)\psi(x';-k)\,dk, \tag{3.7}$$

---

[1]There is a misprint in the normalization of $\psi_0(x)$ and $\psi_1(x)$ in [2].

where $\mathcal{L}$ is an integration path in complex k plane, obtained from the real axis by its simultaneous deformation near the points $k = -\alpha$ and $k = \alpha$ upwards or downwards (the direction of this deformation is of no difference since for the points $k = -\alpha$ and $k = \alpha$ the sum of residues of the integrand is equal to zero). The direction of $\mathcal{L}$ is specified from $-\infty$ to $+\infty$. This resolution of identity is valid for test functions belonging to $CL_\gamma \equiv C_{\mathbb{R}}^\infty \cap L_2(\mathbb{R}; (1 + |x|)^\gamma)$, $\gamma > -1$ as well as for some bounded and even slowly increasing test functions and, in particular, for eigenfunctions $\psi(x; k)$ and for the associated function $\psi_1(x)$.

One can rearrange [13] the resolution of identity (3.7) for any $\varepsilon \in (0, \alpha)$ to the form

$$\delta(x - x') = \left(\int_{-\infty}^{-\alpha-\varepsilon} + \int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} + \int_{\alpha+\varepsilon}^{+\infty} \right) \psi(x; k)\psi(x'; -k)\, dk$$
$$+ \frac{2}{\pi} \cos\alpha(x - x') \frac{\sin\varepsilon(x - x')}{x - x'} - \frac{1}{\pi\alpha}\psi_0(x)\psi_0(x') \left[\frac{1}{\varepsilon} \left[1 - 2\sin^2\frac{\varepsilon}{2}(x - x') \right] \right.$$
$$\left. - \frac{\varepsilon}{4\alpha^2 - \varepsilon^2} \cos 2\alpha(x - x') \cos\varepsilon(x - x') - \frac{2\alpha}{4\alpha^2 - \varepsilon^2} \sin 2\alpha(x - x')\sin\varepsilon(x - x') \right]$$
$$- \frac{1}{\pi} [\psi_0(x)\psi_1(x') + \psi_1(x)\psi_0(x')] \int_{2\alpha-\varepsilon}^{2\alpha+\varepsilon} \cos t(x - x') \frac{dt}{t} \tag{3.8}$$

and, consequently, to the form

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}' \left\{ \left(\int_{-\infty}^{-\alpha-\varepsilon} + \int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} + \int_{\alpha+\varepsilon}^{+\infty} \right) \psi(x; k)\psi(x'; -k)\, dk \right.$$
$$+ \frac{2}{\pi} \cos\alpha(x - x') \frac{\sin\varepsilon(x - x')}{x - x'} - \frac{1}{\pi\alpha}\psi_0(x)\psi_0(x') \left[\frac{1}{\varepsilon} \left[1 - 2\sin^2\frac{\varepsilon}{2}(x - x') \right] \right.$$
$$\left. - \frac{\varepsilon}{4\alpha^2 - \varepsilon^2} \cos 2\alpha(x - x') \cos\varepsilon(x - x') - \frac{2\alpha}{4\alpha^2 - \varepsilon^2} \sin 2\alpha(x - x')\sin\varepsilon(x - x') \right]$$
$$\left. - \frac{1}{\pi} [\psi_0(x)\psi_1(x') + \psi_1(x)\psi_0(x')] \int_{2\alpha-\varepsilon}^{2\alpha+\varepsilon} \cos t(x - x') \frac{dt}{t} \right\}, \tag{3.9}$$

where the prime $'$ at the limit symbol emphasizes that this limit is regarded as a limit in the space of distributions.

One can reduce [13] the resolutions of identity (3.8) and (3.9) for test functions from CL_γ, $\gamma > -1$ to the form

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}' \left\{ \left(\int_{-\infty}^{-\alpha-\varepsilon} + \int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} + \int_{\alpha+\varepsilon}^{+\infty} \right) \psi(x; k)\psi(x'; -k)\, dk \right.$$
$$\left. - \frac{1}{\pi\varepsilon\alpha} \left[1 - 2\sin^2\frac{\varepsilon}{2}(x - x') \right] \psi_0(x)\psi_0(x') \right\} \tag{3.10}$$

and for test functions from CL_γ, $\gamma > 1$ to a more simple form

$$\delta(x - x') = \lim_{\varepsilon\downarrow 0}{}' \left\{ \left(\int_{-\infty}^{-\alpha-\varepsilon} + \int_{-\alpha+\varepsilon}^{\alpha-\varepsilon} + \int_{\alpha+\varepsilon}^{+\infty} \right) \psi(x; k)\psi(x'; -k)\, dk - \frac{1}{\pi\varepsilon\alpha}\psi_0(x)\psi_0(x') \right\}. \tag{3.11}$$

The latter of these resolutions seems to have a more natural form than the previous ones, but it cannot reproduce the *normalizable* eigenfunction

$$\psi_0(x) \notin CL_\gamma \equiv C_{\mathbb{R}}^\infty \cap L^2(\mathbb{R}; (1 + |x|)^\gamma), \qquad \gamma > 1$$

because of the biorthogonality relations. With the help of (3.2), Lemma 4.8 from [13] and the Jordan lemma one can check that

$$
\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \left[\frac{2}{\pi \varepsilon \alpha} \sin^2 \frac{\varepsilon}{2} (x - x') \psi_0(x) \psi_0(x') \right] \psi_0(x) \, dx
$$

$$
= \lim_{\varepsilon \downarrow 0} \left\{ \psi_0(x') \int_{-\infty}^{+\infty} \left[\frac{2}{\pi \varepsilon \alpha} \sin^2 \frac{\varepsilon}{2} (x - x') \right] \right.
$$

$$
\left. \times \left[2\alpha \frac{\cos^2 \alpha x}{(x - z)^2} - \frac{\sin 2\alpha x [\sin 2\alpha x + 4\alpha(x - z)]}{4\alpha^2 (x - z)^2} \psi_0^2(x) \right] dx \right\}
$$

$$
= \lim_{\varepsilon \downarrow 0} \left\{ \psi_0(x') \int_{-\infty}^{+\infty} \left[\frac{2}{\pi \varepsilon \alpha} \sin^2 \frac{\varepsilon}{2} (x - x') \right] \left[2\alpha \frac{\cos^2 \alpha x}{(x - z)^2} \right] dx \right\} \qquad (3.12)
$$

$$
= \lim_{\varepsilon \downarrow 0} \left\{ \left[e^{\pm i \varepsilon (z - x')} - 4 \frac{\alpha}{\varepsilon} \sin^2 \frac{\varepsilon}{2} (z - x') \pm i e^{\pm 2 i \alpha z} \sin \varepsilon (z - x') \right] \psi_0(x') \right\} = \psi_0(x'),
$$

where the upper (lower) signs correspond to the case $\mathrm{Im}\, z > 0$ ($\mathrm{Im}\, z < 0$). Hence, just the term

$$
\frac{2}{\pi \varepsilon \alpha} \sin^2 \frac{\varepsilon}{2} (x - x') \psi_0(x) \psi_0(x') \qquad (3.13)
$$

in the resolutions of identity (3.8)–(3.10) gives an opportunity to reproduce $\psi_0(x)$ by these resolutions (cf. with the analogous results in Section 6.1 of [2], in Section 2 of [29] and in Section 2.4 of the present paper). It is interesting that the contribution of the term (3.13) in the resolutions of identity (3.8)–(3.10) is a singular discontinuous functional (see Remark 4.2 in [13]) which support consists of the only element – the infinity (cf. with the analogous comments in Section 2 of [29] and in Section 2.4 of the present paper).

There is another way to transform the resolution (3.7) as well. This way was used for obtaining of the resolutions of identity (26) in [29] and (2.28) in Section 2.3. The integral in the right-hand part of (3.7) is understood as follows,

$$
\int_{\mathcal{L}} \psi(x; k) \psi(x'; -k) \, dk = \lim_{A \to +\infty}' \int_{\mathcal{L}(A)} \psi(x; k) \psi(x'; -k) \, dk, \qquad (3.14)
$$

where $\mathcal{L}(A)$ is an integration path in complex k plane, obtained from the segment $[-A, A]$ by its simultaneous deformation near the points $k = -\alpha$ and $k = \alpha$ upwards or downwards (the direction of this deformation is of no difference as well as in the case with $\mathcal{L}$) and the direction of $\mathcal{L}(A)$ is specified from $-A$ to A. Using

(1) the fact that the integral in the right-hand part of (3.14) is a standard integral (not a distribution);

(2) the fact that $(k \mp \alpha)^2 \psi(x; k) \psi(x'; -k)$ is a holomorphic function of k in a neighborhood of $k = \pm \alpha$ (see (3.1));

(3) the Leibniz formula and the formulae (3.2);

(4) the notation $\mathcal{L}(k_0; \varepsilon)$ with fixed $k_0 \in \mathbb{R}$ and $\varepsilon > 0$ for the path in complex k plane defined by

$$
k = k_0 + \varepsilon[\cos(\pi - \vartheta) \pm i \sin(\pi - \vartheta)], \qquad 0 \leqslant \vartheta \leqslant \pi,
$$

where the upper (lower) sign corresponds to the case of upper (lower) deformations in $\mathcal{L}$ and the direction of $\mathcal{L}(k_0; \varepsilon)$ is specified from $\vartheta = 0$ to $\vartheta = \pi$;

we can transform (3.14) as follows,

$$
\begin{aligned}
\int_{\mathcal{L}} \psi(x;k)\psi(x';-k)\,dk &= \lim_{A\to+\infty}{}' \lim_{\varepsilon\downarrow 0}\left\{\left(\int_{-A}^{-\alpha-\varepsilon}+\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon}+\int_{\alpha+\varepsilon}^{A}\right)\psi(x;k)\psi(x';-k)\,dk\right.\\
&\quad +\sum_{j=0}^{1}\frac{1}{j!}\frac{\partial^j}{\partial k^j}\Big[(k+\alpha)^2\psi(x;k)\psi_n(x';-k)\Big]\Big|_{k=-\alpha}\int_{\mathcal{L}(-\alpha;\varepsilon)}\frac{dk}{(k+\alpha)^{2-j}}\\
&\quad \left.+\sum_{j=0}^{1}\frac{1}{j!}\frac{\partial^j}{\partial k^j}\Big[(k-\alpha)^2\psi(x;k)\psi_n(x';-k)\Big]\Big|_{k=\alpha}\int_{\mathcal{L}(\alpha;\varepsilon)}\frac{dk}{(k-\alpha)^{2-j}}\right\}\\
&= \lim_{A\to+\infty}{}' \lim_{\varepsilon\downarrow 0}\left\{\left(\int_{-A}^{-\alpha-\varepsilon}+\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon}+\int_{\alpha+\varepsilon}^{A}\right)\psi(x;k)\psi(x';-k)\,dk\right.\\
&\quad +\frac{1}{4\pi\alpha}\psi_0(x)\psi_0(x')\left[\int_{\mathcal{L}(-\alpha;\varepsilon)}\frac{dk}{(k+\alpha)^2}+\int_{\mathcal{L}(\alpha;\varepsilon)}\frac{dk}{(k-\alpha)^2}\right]\\
&\quad \left.+\frac{1}{2\pi}[\psi_0(x)\psi_1(x')+\psi_1(x)\psi_0(x')]\left[\int_{\mathcal{L}(\alpha;\varepsilon)}\frac{dk}{k-\alpha}-\int_{\mathcal{L}(-\alpha;\varepsilon)}\frac{dk}{k+\alpha}\right]\right\}\\
&= \lim_{A\to+\infty}{}' \lim_{\varepsilon\downarrow 0}\left\{\left(\int_{-A}^{-\alpha-\varepsilon}+\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon}+\int_{\alpha+\varepsilon}^{A}\right)\psi(x;k)\psi(x';-k)\,dk-\frac{1}{\pi\varepsilon\alpha}\psi_0(x)\psi_0(x')\right\}\\
&= \lim_{\varepsilon\downarrow 0}{}^*\left\{\left(\int_{-\infty}^{-\alpha-\varepsilon}+\int_{-\alpha+\varepsilon}^{\alpha-\varepsilon}+\int_{\alpha+\varepsilon}^{+\infty}\right)\psi(x;k)\psi(x';-k)\,dk-\frac{1}{\pi\varepsilon\alpha}\psi_0(x)\psi_0(x')\right\}, \quad (3.15)
\end{aligned}
$$

where the latter equality is considered as a definition for $\lim^*$ and the limit $\lim_{\varepsilon\downarrow 0}$ is regarded as a pointwise one (not as a limit in a function space). The resolution of identity (3.15) is equivalent to (3.7), i.e. it is valid for all test functions for which (3.7) is valid (cf. with (3.11) and with the similar results in Section 2 of [29] and in Section 2.3 of the present paper).

Let us note that the associated function $\psi_1(x)$ does not appear in the derived resolutions of identity and is not expandable with the help of the resolution (3.11). Thereby, this associated function does not belong to the physical state space (rigged Hilbert space).

Let us notice also that the number (equal to 1) of linearly independent eigen- and (formal) associated functions of the Hamiltonian h for the eigenvalue $E=\alpha^2$ included into the resolutions of identity (3.11) and (3.15) is less than the order (equal to 2) of the pole $E=\alpha^2$ of the Green function $G(x,x';E)$.

4 Conclusions: indexes of exceptional points and SUSY

We remark that, in general, one can introduce, at least, three different number indexes of exceptional point $E=\lambda_0$ of a Hamiltonian h:

(1) $n_1(\lambda_0)$ to be a maximal number of linearly independent normalizable eigenfunctions and associated functions of h for the eigenvalue $E=\lambda_0$;

(2) $n_2(\lambda_0)$ to be a maximal number of linearly independent eigenfunctions and formal associated functions of h for the eigenvalue $E=\lambda_0$ appeared in the resolution of identity constructed from biorthogonal set of eigenfunctions and associated functions of h;

(3) $n_3(\lambda_0)$ to be an order of the pole in the point $E=\lambda_0$ for the Green function for h as a function of E (in the case with the Hamiltonian $h=h_n$ (see Section 2), where the exceptional point $E=\lambda_0=0$ coincides with the branch point of the Green function; it is natural to assume (see Section 2.3) that $n_3(0)=n$).

It can be proven (by methods of [7], see as well the example in Section 5.1 of [2]) that for an exceptional point outside of continuous spectrum all these indexes are identical,

$$n_1(\lambda_0) = n_2(\lambda_0) = n_3(\lambda_0)$$

and represent the algebraic multiplicity of the eigenvalue λ_0. In the cases where an exceptional point is situated on the border of continuous spectrum or inside of it these indexes may be different. For example, in the case of an exceptional point $E = 0$ at the bottom of continuous spectrum of the Hamiltonian h_n in Section 2,

$$n_1(0) = \left[\frac{n+1}{2}\right], \qquad n_2(0) = n, \qquad n_3(0) = n \qquad \Rightarrow \qquad n_1(0) \leqslant n_2(0) = n_3(0)$$

and in the case of an exceptional point $E = \alpha^2$ inside of continuous spectrum of the Hamiltonian h in Section 3,

$$n_1(\alpha^2) = 1, \qquad n_2(\alpha^2) = 1, \qquad n_3(\alpha^2) = 2 \qquad \Rightarrow \qquad n_1(\alpha^2) = n_2(\alpha^2) < n_3(\alpha^2).$$

Thus, one can consider these indexes as different generalizations of the notion of algebraic multiplicity.

One can use SUSY technique [14, 15, 16, 17, 18, 19, 20] in order to regulate the algebraic multiplicity (in any sense mentioned above) of an exceptional point in the spectrum of a SUSY partner Hamiltonian with respect to the order of this exceptional point in the spectrum of a given Hamiltonian (originally proposed in [21] and elaborated in details in [3, 4]):

(1) in order to increase the multiplicity of an exceptional point λ_0 in the spectrum of a SUSY partner Hamiltonian with respect to its order in the spectrum of a given Hamiltonian, one must take a formal eigenfunction (and a chain of formal associated functions) for the spectral value λ_0 of the latter Hamiltonian as transformation function(s) which tends (tend) to infinity for $x \to \pm\infty$;

(2) in order to decrease the multiplicity of an exceptional point λ_0 in the spectrum of a SUSY partner Hamiltonian with respect to its multiplicity in the spectrum of a given Hamiltonian, one must take a normalizable eigenfunction (and a chain of normalizable associated functions) of the latter Hamiltonian for eigenvalue λ_0 as transformation function(s).

These statements can be clarified by the following simple example. For the eigenvalue $E = \lambda_0 = 0$ of the Hamiltonian h_n, $n = 1, 2, 3, \ldots$ in Section 2 there is a chain (2.1) of the eigenfunction $\psi_{n0}(x)$ and associated functions $\psi_{nl}(x)$, $l = 1, \ldots, [(n-1)/2]$. As well one can check that for the spectral value $E = 0$ of the Hamiltonian h_n, $n = 0, 1, 2, \ldots$ there is a chain of the formal eigenfunction $\varphi_{n0}(x)$ and formal associated functions $\varphi_{nl}(x)$,

$$\varphi_{n0}(x) = (x - z)^{n+1}, \qquad \varphi_{nl}(x) = \frac{(-1)^l(2n+1)!!}{(2l)!!(2n+2l+1)!!}(x - z)^{n+2l+1},$$

$$h_n\varphi_{n0} = 0, \qquad h_n\varphi_{nl} = \varphi_{n,l-1}, \qquad l = 1, 2, 3, \ldots,$$

which tend to infinity for $x \to \pm\infty$. Thus, it can be easily found that a) if to use $\varphi_{nl}(x)$, $l = 0, \ldots, m$ as transformation functions for h_n, then the resulting Hamiltonian is h_{n+m+1} with the exceptional point $E = 0$ of larger algebraic multiplicity (except for the case with the indexes $n_1(0)$, $m = 0$ and odd n, where the indexes $n_1(0)$ for h_n and $h_{n+m+1} \equiv h_{n+1}$ are equal), and b) if to use $\psi_{nl}(x)$, $l = 0, \ldots, m \leqslant [(n-1)/2]$ as transformation functions for h_n, then the resulting Hamiltonian is h_{n-m-1} with the exceptional point $E = 0$ of smaller algebraic multiplicity (except for the case with indexes $n_1(0)$, $m = 0$ and even n, where the indexes $n_1(0)$ for h_n and $h_{n-m-1} \equiv h_{n-1}$ are equal).

Acknowledgments

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