On the N-Solitons Solutions in the Novikov–Veselov Equation

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Abstract. We construct the N-solitons solution in the Novikov–Veselov equation from the extended Moutard transformation and the Pfaffian structure. Also, the corresponding wave functions are obtained explicitly. As a result, the property characterizing the N-solitons wave function is proved using the Pfaffian expansion. This property corresponding to the discrete scattering data for N-solitons solution is obtained in [arXiv:0912.2155] from the $\bar\partial$-dressing method.

Key words: Novikov–Veselov equation; N-solitons solutions; Pfaffian expansion; wave functions

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1 Introduction

The Novikov–Veselov equation [3, 9, 34, 41] is defined by

$$\begin{aligned} &U_t = \partial_z^3 U + \bar\partial_z U + 3\partial_z(VU) + 3\bar\partial_z(V^* U), \\ &\bar\partial_z V = \partial_z U, \qquad \partial_z V^* = \bar\partial_z U. \end{aligned} \tag{1}$$

When $z = \bar z = x$, we get the famous KdV equation ($U = \bar U = V = \bar V$)

$$U_t = 2U_{xxx} + 12UU_x.$$

The equation (1) can be represented as the form of Manakov's triad [24]

$$H_t = [A, H] + BH,$$

where H is the two-dimension Schrödinger operator

$$H = \partial_z \bar\partial_z + U$$

and

$$A = \partial_z^3 + V\partial_z + \bar\partial_z^3 + \bar V \bar\partial_z, \qquad B = V_z + \bar V_{\bar z}.$$

It is equivalent to the linear representation

$$H\psi = 0, \qquad \partial_t \psi = A\psi. \tag{2}$$

We see that the Novikov–Veselov equation (1) preserves a class of the purely potential self-adjoint operators H. Here the pure potential means H has no external electric and magnetic fields. The periodic inverse spectral problem for the two-dimensional Schrödinger operator H

was investigated in terms of the Riemann surfaces with some group of involutions and the corresponding Prym Θ-functions [5, 10, 22, 27, 28, 33, 37]. On the other hand, it is known that the Novikov–Veselov hierarchy is a special reduction of the two-component BKP hierarchy [23, 36, 40] (and references therein). In [23], the authors showed that the Drinfeld–Sokolov hierarchy of D-type is a reduction of the two-component BKP hierarchy using two different types of pseudo-differential operators, which is different from Shiota's point of view [37]. Also, in [26], it is shown that the Tzitzeica equation is a stationary symmetry of the Novikov–Veselov equation. Finally, it is worthwhile to notice that the Novikov–Veselov equation (1) is a special reduction of the Davey–Stewartson equation [20, 21].

Let $H\psi = H\omega = 0$. Then via the Moutard transformation [1, 29, 30, 31]

$$\begin{aligned}
&U(z,\bar z) \longrightarrow \hat U(z,\bar z) = U(z,\bar z) + 2\partial\bar\partial \ln\omega,\\
&\psi \longrightarrow \theta = \frac{1}{\omega}\int (\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar\partial\omega - \omega\bar\partial\psi)d\bar z,
\end{aligned} \tag{3}$$

one can construct a new Schrödinger operator $\hat H = \partial_z\bar\partial_z + \hat U$ and $\hat H\theta = 0$. We remark that the Moutard transformation (3) is utilized to construct the N-solitons solutions of the Tzitzeica equation [15].

The extended Moutard transformation was established such that $\hat U(t,z,\bar z)$ and $\hat V(t,z,\bar z)$ defined by [13, 25]

$$\hat U(t,z,\bar z) = U(t,z,\bar z) + 2\partial\bar\partial \ln W(\psi,\omega), \qquad \hat V(t,z,\bar z) = V(t,z,\bar z) + 2\partial\partial \ln W(\psi,\omega),$$

where the skew product (alternating bilinear form) W is defined by

$$\begin{aligned}
W(\psi,\omega) = \int &(\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar\partial\omega - \omega\bar\partial\psi)d\bar z + \big[\psi\partial^3\omega - \omega\partial^3\psi + \omega\bar\partial^3 - \psi\bar\partial^3\omega\\
&+ 2\big(\partial^2\psi\partial\omega - \partial\psi\partial^2\omega\big) - 2\big(\bar\partial^2\psi\bar\partial\omega - \bar\partial\psi\bar\partial^2\omega\big) + 3V(\psi\partial\omega - \omega\partial\psi)\\
&- 3\bar V(\psi\bar\partial\omega - \omega\bar\partial\psi)\big]dt,
\end{aligned} \tag{4}$$

will also satisfy the Novikov–Veselov equation.

In [2, 6, 7, 8], the rational solutions and line solitons of the Novikov–Veselov equation (1) are constructed by the $\bar\partial$-dressing method. To get these kinds of solutions, the scattering datum have to be delta-type and the reality of U also puts some extra constraints on them. In [39], the singular rational solutions are obtained using the extended Moutard transformation (4); however, the non-singular rational solutions are constructed in [4].

Next, we construct Pfaffian-type solutions. Given any N wave functions $\psi_1, \psi_2, \psi_3, \dots, \psi_N$ (or their linear combinations) of (2) for fixed potential $U(z,\bar z,t)$, the N-step extended Moutard transformation can be obtained in the Pfaffian [1, 31] (also see [12, 35])

$$P(\psi_1,\psi_2,\psi_3,\dots,\psi_N) = \begin{cases}\mathrm{Pf}(\psi_1,\psi_2,\psi_3,\dots,\psi_N) & \text{if } N \text{ even},\\ \widetilde{\mathrm{Pf}}(\psi_1,\psi_2,\psi_3,\dots,\psi_N) & \text{if } N \text{ odd},\end{cases}$$

$$\mathrm{Pf}(\psi_1,\psi_2,\psi_3,\dots,\psi_N) = \sum_\sigma \epsilon(\sigma) W_{\sigma_1\sigma_2}W_{\sigma_3\sigma_4}\cdots W_{\sigma_{N-1}\sigma_N}, \tag{5}$$

$$\widetilde{\mathrm{Pf}}(\psi_1,\psi_2,\psi_3,\dots,\psi_N) = \sum_\sigma \epsilon(\sigma) W_{\sigma_1\sigma_2}W_{\sigma_3\sigma_4}\cdots W_{\sigma_{N-2}\sigma_{N-1}}\psi_{\sigma_N}, \tag{6}$$

where $W_{\sigma_i\sigma_j} = W(\psi_{\sigma(i)},\psi_{\sigma(j)})$ is defined by the skew product (4). The summations σ in (5) and (6) run from over the permutations of $\{1,2,3,\dots,N\}$ such that $\sigma_1 < \sigma_2$, $\sigma_3 < \sigma_4$, $\sigma_5 < \sigma_6$, … and $\sigma_1 < \sigma_3 < \sigma_5 < \sigma_7 < \cdots$, with $\epsilon(\sigma) = 1$ for the even permutations and $\epsilon(\sigma) = -1$ for the odd permutations. Then the solution U and V can be expressed as [1]

$$U = U_0 + 2\partial\bar\partial[\ln P(\psi_1,\psi_2,\psi_3,\dots,\psi_N)], \qquad V = V_0 + 2\partial\partial[\ln P(\psi_1,\psi_2,\psi_3,\dots,\psi_N)],$$

and the corresponding wave function is

$$\varphi = \frac{P(\psi_1, \psi_2, \psi_3, \dots, \psi_N, \vartheta)}{P(\psi_1, \psi_2, \psi_3, \dots, \psi_N)}, \tag{7}$$

where ϑ is an arbitrary wave function different from $\psi_1, \psi_2, \psi_3, \dots, \psi_N$.

The paper is organized as follows. In Section 2, we obtain the N-solitons solutions using the extended Moutard transformation and the Pfaffian expansion. Several examples are given. In Section 3, the N-solitonic wave function is derived using (7) and the Pfaffian expansion. Section 4 is used to prove a special property to characterize the N-solitons wave function. Section 5 is devoted to the concluding remarks.

2 N-solitons solutions

In this section, one uses successive iterations of the extended Moutard transformation (4) to construct N-solitons solutions.

To obtain the N-solitons solutions, we assume that $V = 0$ in (1) and recall that $\partial\bar{\partial} = \frac{1}{4}\triangle$. One considers $U = -\epsilon \neq 0$, i.e.,

$$\partial\bar{\partial}\varphi = \epsilon\varphi, \qquad \varphi_t = \varphi_{zzz} + \varphi_{\bar{z}\bar{z}\bar{z}}, \tag{8}$$

where ϵ is non-zero real constant. The general solution of (8) can be expressed as

$$\varphi(z, \bar{z}, t) = \int_\Gamma e^{(i\lambda)z + (i\lambda)^3 t + \frac{\epsilon}{i\lambda}\bar{z} + \frac{\epsilon^3}{(i\lambda)^3}t} \nu(\lambda) d\lambda, \tag{9}$$

where $\nu(\lambda)$ is an arbitrary distribution and Γ is an arbitrary path of integration such that the r.h.s. of (9) is well defined.

Next, using (5) and (9), one can construct the N-solitons solutions. Let's take $\nu_m(\lambda) = \delta(\lambda - p_m)$ and $\nu_n(\lambda) = a_n\delta(\lambda - q_n)$, where p_m, a_n, q_n are complex numbers. Then one defines

$$\phi_m = \frac{\varphi(p_m)}{\sqrt{3}} = \frac{1}{\sqrt{3}} e^{F(p_m)}, \qquad \psi_n = a_n \frac{\varphi(q_n)}{\sqrt{3}} = \frac{a_n}{\sqrt{3}} e^{F(q_n)},$$

where

$$F(\lambda) = (i\lambda)z + (i\lambda)^3 t + \frac{\epsilon}{i\lambda}\bar{z} + \frac{\epsilon^3}{(i\lambda)^3}t.$$

Then a direct calculation of the extended Moutard transformation (4) can yield

$$W(\phi_m, \psi_n) = i a_n \frac{q_n - p_m}{q_n + p_m} e^{F(p_m)+F(q_n)}, \qquad W(\phi_m, \phi_n) = i \frac{p_n - p_m}{p_n + p_m} e^{F(p_m)+F(p_n)},$$
$$W(\psi_m, \psi_n) = i a_m a_n \frac{q_n - q_m}{q_n + q_m} e^{F(q_m)+F(q_n)}. \tag{10}$$

The N-solitons solutions are defined by

$$U(z, \bar{z}, t) = -\epsilon + 2\partial\bar{\partial} \ln \tau_N(z, \bar{z}, t), \qquad V(z, \bar{z}, t) = 2\partial\partial \ln \tau_N(z, \bar{z}, t),$$

and then

$$U(z, \bar{z}, t) \to -\epsilon \qquad \text{as} \quad z\bar{z} \to \infty,$$

where t is fixed. The τ-functions are defined as follows. For simplicity, let's denote

$$W(p_m, q_n) = W(\phi_m, \psi_n), \qquad W(p_m, p_n) = W(\phi_m, \phi_n), \qquad W(q_m, q_n) = W(\psi_m, \psi_n), \tag{11}$$

and notice that $F(-\lambda) = -F(\lambda)$. The τ_N is defined as

$$\tau_N(z, \bar{z}, t) = \mathrm{Pf}(-p_1, q_1, -p_2, q_2, -p_3, q_3, \ldots, -p_N, q_N), \tag{12}$$

where

$$\begin{aligned} &(-p_m, -p_n) = W(-p_m, -p_n), \qquad (-p_m, q_n) = W(-p_m, q_n) + \delta_{mn}, \\ &(q_m, q_n) = W(q_m, q_n). \end{aligned} \tag{13}$$

To get the expansion of (12), we use the following useful formula [14, 38]

$$\mathrm{Pf}(\mathbf{A} + \mathbf{B}) = \sum_{r=0}^{s} \sum_{\alpha \in I_{2r}^m} (-1)^{|\alpha| - r}\, \mathrm{Pf}(\mathbf{A}_\alpha)\, \mathrm{Pf}(\mathbf{B}_{\alpha^c}), \tag{14}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $m \times m$ matrices and $s = [m/2]$ is the integer part of $m/2$; moreover, we denote by α^c the complementary set of α in the subset $\{1, 2, 3, \ldots, m\}$ which is arranged in increasing order, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{2r}$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2r})$. For the case (12), one has

$$\mathbf{A}_N(z, \bar{z}, t) = \begin{bmatrix} 0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) & \cdots & (-p_1, q_N) \\ (q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) & \cdots & (q_1, q_N) \\ (-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) & \cdots & (-p_2, q_N) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ (q_N, -p_1) & (q_N, q_1) & (q_N, -p_2) & (q_N, q_2) & \cdots & 0 \end{bmatrix}$$

and

$$\mathbf{B}_N = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

where $\mathbf{A}_N$ and $\mathbf{B}_N$ are $2N \times 2N$ matrices. Hence by (14) one can have the expansion of (12), i.e.,

$$\tau_N = 1 + \sum_{\ell=1}^{N} f_\ell + \sum_{m=2} \left(\sum_{1 \le \ell_1 < \ell_2 < \cdots < \ell_m \le N} f_{\ell_1} f_{\ell_2} \cdots f_{\ell_m} \prod_{1 \le j < k \le m} \mathbb{P}_{\ell_j \ell_k} \right), \tag{15}$$

where

$$f_\ell = i a_\ell \frac{p_\ell + q_\ell}{q_\ell - p_\ell} e^{F(q_\ell) - F(p_\ell)}, \qquad \mathbb{P}_{\ell_j \ell_k} = \frac{(p_{\ell_j} - p_{\ell_k})(q_{\ell_j} - q_{\ell_k})(p_{\ell_j} + q_{\ell_k})(q_{\ell_j} + p_{\ell_k})}{(p_{\ell_j} + p_{\ell_k})(q_{\ell_j} + q_{\ell_k})(p_{\ell_j} - q_{\ell_k})(q_{\ell_j} - p_{\ell_k})}.$$

Here we have utilized the formula that if $\mathbf{C}$ is a $2N \times 2N$ matrix with (i, j)-th entry $\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}$, then one has the Schur identity [32, 36]

$$\mathrm{Pf}(\mathbf{C}) = \prod_{1 \le i < j \le 2N} \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right). \tag{16}$$

Next, we illustrate the formula (15) (or 12) with several examples.

(1) One-soliton solution:

$$\mathbf{A}_1(z,\bar{z},t) = \begin{bmatrix} 0 & (-p_1,q_1) \\ (q_1,-p_1) & 0 \end{bmatrix}, \qquad \mathbf{B}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\tau_1 = \mathrm{Pf}(\mathbf{A}_1 + \mathbf{B}_1) = 1 + ia_1 \frac{q_1+p_1}{q_1-p_1} e^{F(q_1)-F(p_1)}.$$

(2) Two-solitons solution:

$$\mathbf{A}_2(z,\bar{z},t) = \begin{bmatrix} 0 & (-p_1,q_1) & (-p_1,-p_2) & (-p_1,q_2) \\ (q_1,-p_1) & 0 & (q_1,-p_2) & (q_1,q_2) \\ (-p_2,-p_1) & (-p_2,q_1) & 0 & (-p_2,q_2) \\ (q_2,-p_1) & (q_2,q_1) & (q_2,-p_2) & 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \tau_2 = \mathrm{Pf}(\mathbf{A}_2 + \mathbf{B}_2) &= 1 + ia_1 \frac{p_1+q_1}{q_1-p_1} e^{F(q_1)-F(p_1)} + ia_2 \frac{p_2+q_2}{q_2-p_2} e^{F(q_2)-F(p_2)} \\ &+ ia_1 ia_2 \frac{p_1+q_1}{q_1-p_1} \frac{p_2+q_2}{q_2-p_2} \frac{p_2-p_1}{p_2+p_1} \frac{q_2-q_1}{q_2+q_1} \frac{p_1+q_2}{q_2-p_1} \frac{p_2+q_1}{p_2-q_1} e^{F(q_1)-F(p_1)+F(q_2)-F(p_2)} \end{aligned}$$

or

$$\tau_2 = 1 + f_1 + f_2 + \mathbb{P}_{12} f_1 f_2, \tag{17}$$

where

$$f_1 = ia_1 \frac{p_1+q_1}{q_1-p_1} e^{F(q_1)-F(p_1)}, \qquad f_2 = ia_2 \frac{p_2+q_2}{q_2-p_2} e^{F(q_2)-F(p_2)},$$

$$\mathbb{P}_{12} = \frac{p_1-p_2}{p_1+p_2} \frac{q_1-q_2}{q_1+q_2} \frac{p_1+q_2}{p_1-q_2} \frac{q_1+p_2}{q_1-p_2}.$$

The τ_2 soliton (17) is also found in [2] using the $\overline{\partial}$-dressing method.

(3) Three-solitons solutin:

$$\mathbf{A}_3(z,\bar{z},t) = \begin{bmatrix} 0 & (-p_1,q_1) & (-p_1,-p_2) & (-p_1,q_2) & (-p_1,-p_3) & (-p_1,q_3) \\ (q_1,-p_1) & 0 & (q_1,-p_2) & (q_1,q_2) & (q_1,-p_3) & (q_1,q_3) \\ (-p_2,-p_1) & (-p_2,q_1) & 0 & (-p_2,q_2) & (-p_2,-p_3) & (-p_2,q_3) \\ (q_2,-p_1) & (q_2,q_1) & (q_2,-p_2) & 0 & (q_2,-p_3) & (q_2,q_3) \\ (-p_3,-p_1) & (-p_3,q_1) & (-p_3,-p_2) & (-p_3,q_2) & 0 & (-p_3,q_3) \\ (q_3,-p_1) & (q_3,q_1) & (q_3,-p_2) & (q_3,q_2) & (q_3,-p_3) & 0 \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned}\tau_3 &= \mathrm{Pf}(\mathbf{A}_3+\mathbf{B}_3)\\ &= 1+f_1+f_2+f_3+\mathbb{P}_{12}f_1f_2+\mathbb{P}_{13}f_1f_3+\mathbb{P}_{23}f_2f_3+\mathbb{P}_{12}\mathbb{P}_{13}\mathbb{P}_{23}f_1f_2f_3,\end{aligned} \tag{18}$$

where

$$\begin{aligned}&f_1 = ia_1\frac{p_1+q_1}{q_1-p_1}e^{F(q_1)-F(p_1)}, \qquad f_2 = ia_2\frac{p_2+q_2}{q_2-p_2}e^{F(q_2)-F(p_2)},\\ &f_3 = ia_3\frac{p_3+q_3}{q_3-p_3}e^{F(q_3)-F(p_3)}, \qquad \mathbb{P}_{12} = \frac{p_1-p_2}{p_1+p_2}\frac{q_1-q_2}{q_1+q_2}\frac{p_1+q_2}{p_1-q_2}\frac{q_1+p_2}{q_1-p_2},\\ &\mathbb{P}_{13} = \frac{p_1+q_3}{p_1-q_3}\frac{q_1+p_3}{q_1-p_3}\frac{p_1-p_3}{p_1+p_3}\frac{q_1-q_3}{q_1+q_3}, \qquad \mathbb{P}_{23} = \frac{p_2+q_3}{p_2-q_3}\frac{q_2+p_3}{q_2-p_3}\frac{p_2-p_3}{p_2+p_3}\frac{q_2-q_3}{q_2+q_3}.\end{aligned}$$

3 The wave functions

In this section, one uses (7) to construct the corresponding wave function of the τ function (15).

From (7), one knows that the corresponding wave function of the N-solitons (15) can be written as

$$\varphi_N = \frac{P\left(\frac{\varphi(-p_1)}{\sqrt{3}},\frac{\varphi(q_1)}{\sqrt{3}},\frac{\varphi(-p_2)}{\sqrt{3}},\frac{\varphi(q_2)}{\sqrt{3}},\ldots,\frac{\varphi(-p_N)}{\sqrt{3}},\frac{\varphi(q_N)}{\sqrt{3}},\frac{\varphi(\lambda)}{\sqrt{3}}\right)}{\tau_N}.$$

Using the notations in (11), (12) and (13), we can express φ_N as

$$\varphi_N = \frac{P(-p_1,q_1,-p_2,q_2,\ldots,-p_N,q_N,\lambda)}{\tau_N}. \tag{19}$$

But we notice that

$$(-p_m,\lambda)^\sharp = W(-p_m,\lambda), \qquad (q_m,\lambda)^\sharp = W(q_m,\lambda), \tag{20}$$

where $(\cdot)^\sharp$ means there is no δ_{mn} here when compared with (13). Now, let's compute $P(-p_1,q_1,-p_2,q_2,\ldots,-p_N,q_N,\lambda)$ using (14). In this case,

$$P(-p_1,q_1,-p_2,q_2,\ldots,-p_N,q_N,\lambda) = \mathrm{Pf}(M_N+Q_N),$$

where

$$M_N = \begin{bmatrix} & & & & (-p_1,\lambda) & \frac{\varphi(-p_1)}{\sqrt{3}}\\ & & & & (q_1,\lambda) & \frac{\varphi(q_1)}{\sqrt{3}}\\ & A_N(z,\bar{z},t) & & & \vdots & \vdots\\ & & & & (q_N,\lambda) & \frac{\varphi(q_N)}{\sqrt{3}}\\ (\lambda,-p_1) & (\lambda,q_1) & \cdots & (\lambda,q_N) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}}\\ -\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & \cdots & -\frac{\varphi(q_N)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0 \end{bmatrix}$$

and

$$Q_N = \begin{bmatrix} & & & & 0 & 0\\ & & & & 0 & 0\\ & B_N & & & \vdots & \vdots\\ & & & & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Using (16), a simple calculation yields

$$\begin{aligned}
&\mathrm{Pf}(M_N+Q_N)\\
&\quad=\phi\left[1+\sum_{\ell=1}^{N}h_\ell(\lambda)+\sum_{m=2}^{N}\left(\sum_{1\leqq\ell_1<\ell_2<\ell_3<\cdots<\ell_m\leqq N}h_{\ell_1}h_{\ell_2}\cdots h_{\ell_m}\prod_{1\leqq j<k\leqq m}\mathbb{P}_{\ell_j\ell_k}\right)\right]\\
&\quad=\phi\hat{\chi}_N(\lambda),
\end{aligned}\tag{21}$$

where

$$\phi=\frac{\varphi(\lambda)}{\sqrt{3}},\qquad h_\ell(\lambda)=ia_\ell\frac{p_\ell+q_\ell}{q_\ell-p_\ell}\frac{p_\ell+\lambda}{p_\ell-\lambda}\frac{q_\ell-\lambda}{q_\ell+\lambda}e^{F(q_\ell)-F(p_\ell)}=f_\ell\frac{p_\ell+\lambda}{p_\ell-\lambda}\frac{q_\ell-\lambda}{q_\ell+\lambda},$$

$$\hat{\chi}_N(\lambda)=1+\sum_{\ell=1}^{N}h_\ell(\lambda)+\sum_{m=2}^{N}\left(\sum_{1\leqq\ell_1<\ell_2<\ell_3<\cdots<\ell_m\leqq N}h_{\ell_1}h_{\ell_2}\cdots h_{\ell_m}\prod_{1\leqq j<k\leqq m}\mathbb{P}_{\ell_j\ell_k}\right),$$

and $\mathbb{P}_{\ell_j\ell_k}$ is defined in (15).

We give several examples here.

(1) The one-soliton wave function:

$$M_1=\begin{bmatrix}
\multicolumn{2}{c}{\multirow{2}{*}{$A_1(z,\bar{z},t)$}} & (-p_1,\lambda) & \frac{\varphi(-p_1)}{\sqrt{3}}\\
 & & (q_1,\lambda) & \frac{\varphi(q_1)}{\sqrt{3}}\\
(\lambda,-p_1) & (\lambda,q_1) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}}\\
-\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0
\end{bmatrix},\qquad
Q_1=\begin{bmatrix}
\multicolumn{2}{c}{\multirow{2}{*}{B_1}} & 0 & 0\\
 & & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{bmatrix}.$$

Then

$$\varphi_1=\frac{P(-p_1,q_1,\lambda)}{\tau_1}=\frac{\mathrm{Pf}(M_1+Q_1)}{\tau_1}=\frac{\phi}{\tau_1}\left(1+ia_1\frac{p_1+q_1}{q_1-p_1}\frac{p_1+\lambda}{p_1-\lambda}\frac{q_1-\lambda}{q_1+\lambda}e^{F(q_1)-F(p_1)}\right).$$

We remark that

$$\begin{aligned}
\varphi_1&=\frac{\phi}{1+f_1}\left[1+ia_1\frac{p_1+\lambda}{p_1-\lambda}\frac{q_1-\lambda}{q_1+\lambda}f_1\right]=\frac{\phi}{1+f_1}\left[1+ia_1\left(\frac{2p_1}{p_1-\lambda}-1\right)\left(\frac{2q_1}{q_1+\lambda}-1\right)f_1\right]\\
&=\phi\left[\frac{(1+f_1)+2ia_1\left(\frac{p_1}{p_1-\lambda}-\frac{q_1}{q_1+\lambda}\right)e^{F(q_1)-F(p_1)}}{1+f_1}\right]\\
&=\phi\left[1-2ia_1\left(\frac{p_1}{\lambda-p_1}+\frac{q_1}{\lambda+q_1}\right)\frac{e^{F(q_1)-F(p_1)}}{\tau_1}\right].
\end{aligned}\tag{22}$$

This is the one-soliton wave function in [2, p. 9].

(2) The two-soliton wave function:

$$M_2=\begin{bmatrix}
\multicolumn{4}{c}{\multirow{4}{*}{$A_2(z,\bar{z},t)$}} & (-p_1,\lambda) & \frac{\varphi(-p_1)}{\sqrt{3}}\\
 & & & & (q_1,\lambda) & \frac{\varphi(q_1)}{\sqrt{3}}\\
 & & & & (-p_2,\lambda) & \frac{\varphi(-p_2)}{\sqrt{3}}\\
 & & & & (q_2,\lambda) & \frac{\varphi(q_2)}{\sqrt{3}}\\
(\lambda,-p_1) & (\lambda,q_1) & (\lambda,-p_2) & (\lambda,q_2) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}}\\
-\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & -\frac{\varphi(-p_2)}{\sqrt{3}} & -\frac{\varphi(q_2)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0
\end{bmatrix},$$

$$Q_2 = \begin{bmatrix} & & & & 0 & 0 \\ & B_2 & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then using (21), one has

$$\begin{aligned}\varphi_2 &= \frac{P(-p_1, q_1, -p_2, q_2, \lambda)}{\tau_2} = \frac{\mathrm{Pf}(M_2 + Q_2)}{\tau_2} = \frac{\phi}{\tau_2}[1 + h_1(\lambda) + h_2(\lambda) + \mathbb{P}_{12} h_1(\lambda) h_2(\lambda)] \\ &= \frac{\phi}{\tau_2}\bigg(1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1}\frac{p_1 + \lambda}{p_1 - \lambda}\frac{q_1 - \lambda}{q_1 + \lambda} e^{F(q_1) - F(p_1)} + ia_2 \frac{p_2 + q_2}{q_2 - p_2}\frac{p_2 + \lambda}{p_2 - \lambda}\frac{q_2 - \lambda}{q_2 + \lambda} e^{F(q_2) - F(p_2)} \\ &\quad + ia_1 ia_2 \frac{p_1 + q_1}{q_1 - p_1}\frac{p_1 + \lambda}{p_1 - \lambda}\frac{q_1 - \lambda}{q_1 + \lambda}\frac{p_2 + q_2}{q_2 - p_2}\frac{p_2 + \lambda}{p_2 - \lambda}\frac{q_2 - \lambda}{q_2 + \lambda} e^{F(q_1) + F(q_2) - F(p_1) - F(p_2)}\bigg).\end{aligned}$$

(3) Three-solitons wave function:

$$M_3 = \begin{bmatrix} & & & & & & (-p_1, \lambda) & \frac{\varphi(-p_1)}{\sqrt{3}} \\ & & & & & & (q_1, \lambda) & \frac{\varphi(q_1)}{\sqrt{3}} \\ & & & A_3(z, \bar{z}, t) & & & (-p_2, \lambda) & \frac{\varphi(-p_2)}{\sqrt{3}} \\ & & & & & & (q_2, \lambda) & \frac{\varphi(q_2)}{\sqrt{3}} \\ & & & & & & (-p_3, \lambda) & \frac{\varphi(-p_3)}{\sqrt{3}} \\ & & & & & & (q_3, \lambda) & \frac{\varphi(q_3)}{\sqrt{3}} \\ (\lambda, -p_1) & (\lambda, q_1) & (\lambda, -p_2) & (\lambda, q_2) & (\lambda, -p_3) & (\lambda, q_3) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}} \\ -\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & -\frac{\varphi(-p_2)}{\sqrt{3}} & -\frac{\varphi(q_2)}{\sqrt{3}} & -\frac{\varphi(-p_3)}{\sqrt{3}} & -\frac{\varphi(q_3)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & B_3 & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (21), we get

$$\begin{aligned}\varphi_3 &= \frac{P(-p_1, q_1, -p_2, q_2, -p_3, q_3, \lambda)}{\tau_3} = \frac{\mathrm{Pf}(M_3 + Q_3)}{\tau_3} \\ &= \frac{\phi}{\tau_3}[1 + h_1(\lambda) + h_2(\lambda) + h_3(\lambda) + \mathbb{P}_{12} h_1(\lambda) h_2(\lambda) + \mathbb{P}_{13} h_1(\lambda) h_3(\lambda) + \mathbb{P}_{23} h_2(\lambda) h_3(\lambda) \\ &\quad + \mathbb{P}_{12}\mathbb{P}_{13}\mathbb{P}_{23} h_1(\lambda) h_2(\lambda) h_3(\lambda)],\end{aligned}$$

where

$$h_1 = ia_1 \frac{p_1 + \lambda}{p_1 - \lambda}\frac{q_1 - \lambda}{q_1 + \lambda}\frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}, \qquad h_2 = ia_2 \frac{p_2 + \lambda}{p_2 - \lambda}\frac{q_2 - \lambda}{q_2 + \lambda}\frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)},$$

$$h_3 = ia_3 \frac{p_3 + \lambda}{p_3 - \lambda}\frac{q_3 - \lambda}{q_3 + \lambda}\frac{p_3 + q_3}{q_3 - p_3} e^{F(q_3) - F(p_3)}$$

and $\mathbb{P}_{12}$, $\mathbb{P}_{13}$ and $\mathbb{P}_{23}$ are defined in (18).

4 A property of N-solitons wave function

In this section, we will express the wave function (19) as another form to generalize the equation (22) to N-solitons case.

Firstly, according to the Pfaffian expansion in [11], it is not difficult to see that

$$\begin{aligned}&\widetilde{\mathrm{Pf}}(b_1,b_2,b_3,b_4,\dots,b_{2n-1},b_{2n},b_{2n+1})\\&\quad=\sum_{m=1}^{2n+1}(-1)^{j+m}(b_j,b_m)\widetilde{\mathrm{Pf}}(b_1,b_2,\dots,\hat{b}_j,\dots,\hat{b}_m,\dots,b_{2n},b_{2n+1}),\\&\quad\text{for}\quad j=1,2,\dots,2n+1,\end{aligned}\tag{23}$$

where $\hat{b}_j$ and $\hat{b}_m$ mean these two terms are omitted.

Secondly, noticing (10) and letting $\lambda=-p_\alpha$ or $\lambda=q_\alpha$, $\alpha=1,2,\dots,n$, we have

$$\begin{aligned}&\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_\alpha,q_\alpha,-p_{\alpha+1},q_{\alpha+1},\dots,-p_N,q_N,-p_\alpha)\\&\quad=\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_\alpha,-p_{\alpha+1},q_{\alpha+1},\dots,-p_N,q_N)=\phi(-p_\alpha)\hat{\chi}_N(-p_\alpha),\\&\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,q_{\alpha-1},-p_\alpha,q_\alpha,-p_{\alpha+1},q_{\alpha+1},\dots,-p_N,q_N,q_\alpha)\\&\quad=\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,q_{\alpha-1},q_\alpha,-p_{\alpha+1},q_{\alpha+1},\dots,-p_N,q_N)=a_\alpha\phi(q_\alpha)\hat{\chi}_N(q_\alpha).\end{aligned}\tag{24}$$

They can be seen as follows. By (23), one has

$$\begin{aligned}&\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_\alpha,q_\alpha,-p_{\alpha+1},q_{\alpha+1},\dots,-p_N,q_N,-p_\alpha)\\&=-(q_\alpha,-p_1)\widetilde{\mathrm{Pf}}(q_1,-p_2,q_2,\dots,-p_{\alpha-1},q_{\alpha-1},-p_\alpha,-p_{\alpha+1},\dots,-p_N,q_N,-p_\alpha)\\&\quad+(q_\alpha,q_1)\widetilde{\mathrm{Pf}}(-p_1,-p_2,q_2,\dots,-p_{\alpha-1},q_{\alpha-1},-p_\alpha,-p_{\alpha+1},\dots,-p_N,q_N,-p_\alpha)-\cdots\\&\quad-(q_\alpha,-p_\alpha)\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_{\alpha-1},q_{\alpha-1},-p_{\alpha+1},\dots,-p_N,q_N,-p_\alpha)+\cdots\\&\quad+(q_\alpha,-p_\alpha)^\sharp\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_{\alpha-1},q_{\alpha-1},-p_\alpha,-p_{\alpha+1},\dots,-p_N,q_N)\\&=[-(q_\alpha,-p_\alpha)+(q_\alpha,-p_\alpha)^\sharp]\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_{\alpha-1},q_{\alpha-1},-p_\alpha,-p_{\alpha+1},\dots,-p_N,q_N)\\&=\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_{\alpha-1},q_{\alpha-1},-p_\alpha,-p_{\alpha+1},q_{\alpha+1},\dots,-p_N,q_N),\end{aligned}$$

where $(q_\alpha,-p_\alpha)^\sharp$ is defined in (20) and we know that

$$\widetilde{\mathrm{Pf}}(\dots,-p_\alpha,\dots,-p_\alpha)=0.$$

The second equation of (24) can be proved similarly.

Finally, from (23), one yields

$$\begin{aligned}&\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\dots,-p_N,q_N,\lambda)=(\lambda,-p_1)\widetilde{\mathrm{Pf}}(q_1,-p_2,q_2,\dots,-p_N,q_N)\\&\quad-(\lambda,q_1)\widetilde{\mathrm{Pf}}(-p_1,-p_2,q_2,\dots,-p_N,q_N)+(\lambda,-p_2)\widetilde{\mathrm{Pf}}(q_1,-p_1,q_2,\dots,-p_N,q_N)\\&\quad-(\lambda,q_2)\widetilde{\mathrm{Pf}}(q_1,-p_1,-p_2,\dots,-p_N,q_N)+\cdots\\&\quad+(\lambda,-p_N)\widetilde{\mathrm{Pf}}(-p_1,-p_2,q_2,\dots,-p_{N-1},q_{N-1},q_N)\\&\quad-(\lambda,q_N)\widetilde{\mathrm{Pf}}(-p_1,-p_2,q_2,\dots,-p_{N-1},q_{N-1},-p_N)+\phi(\lambda)\tau_N.\end{aligned}$$

Also, the wave function (21) can be written as

$$\varphi_N(\lambda)=\phi(\lambda)\chi_N(\lambda),$$

where $\chi_N(\lambda)=\frac{\hat{\chi}_N(\lambda)}{\tau_N}$. Therefore, using (24) and letting $\Delta F_n=F(q_n)-F(p_n)$, we get

$$\begin{aligned}
\chi_N(\lambda) &= 1-\frac{p_1+\lambda}{\lambda-p_1}ia_1e^{\Delta F_1}\chi_N(q_1)-\frac{q_1-\lambda}{q_1+\lambda}ia_1e^{\Delta F_1}\chi_N(-p_1)\\
&\quad -\frac{p_2+\lambda}{\lambda-p_2}ia_2e^{\Delta F_2}\chi_N(q_2)-\frac{q_2-\lambda}{q_2+\lambda}ia_2e^{\Delta F_2}\chi_N(-p_2)-\cdots\\
&\quad -\frac{p_N+\lambda}{\lambda-p_N}ia_Ne^{\Delta F_N}\chi_N(q_N)-\frac{q_N-\lambda}{q_N+\lambda}ia_Ne^{\Delta F_N}\chi_N(-p_N)\\
&= 1-\frac{2p_1}{\lambda-p_1}ia_1e^{\Delta F_1}\chi_N(q_1)-\frac{2q_1}{q_1+\lambda}ia_1e^{\Delta F_1}\chi_N(-p_1)\\
&\quad -\frac{2p_2}{\lambda-p_2}ia_2e^{\Delta F_2}\chi_N(q_2)-\frac{2q_2}{q_2+\lambda}ia_2e^{\Delta F_2}\chi_N(-p_2)-\cdots\\
&\quad -\frac{2p_N}{\lambda-p_N}ia_Ne^{\Delta F_N}\chi_N(q_N)-\frac{2q_N}{q_N+\lambda}ia_Ne^{\Delta F_N}\chi_N(-p_N)\\
&\quad +\big[-ia_1e^{\Delta F_1}\chi_N(q_1)+ia_1e^{\Delta F_1}\chi_N(-p_1)-ia_2e^{\Delta F_2}\chi_N(q_2)\\
&\quad +ia_2e^{\Delta F_2}\chi_N(-p_2)-\cdots-ia_Ne^{\Delta F_N}\chi_N(q_N)+ia_Ne^{\Delta F_N}\chi_N(-p_N)\big]. \qquad (25)
\end{aligned}$$

Since $\widetilde{\mathrm{Pf}}(-p_1,q_1,-p_2,q_2,\ldots,-p_N,q_N,0)=\frac{\tau_N}{\sqrt{3}}$, we have $\hat{\chi}_N(0)=\tau_N$. Then the last term in $[\cdots]$ of (25) is zero (or $\lim\limits_{\lambda\to\infty}\chi_N(\lambda)=1$). Hence one has

$$\begin{aligned}
\chi_N(\lambda) &= 1-\frac{2p_1}{\lambda-p_1}ia_1e^{\Delta F_1}\chi_N(q_1)-\frac{2q_1}{q_1+\lambda}ia_1e^{\Delta F_1}\chi_N(-p_1)\\
&\quad -\frac{2p_2}{\lambda-p_2}ia_2e^{\Delta F_2}\chi_N(q_2)-\frac{2q_2}{q_2+\lambda}ia_2e^{\Delta F_2}\chi_N(-p_2)-\cdots\\
&\quad -\frac{2p_N}{\lambda-p_N}ia_Ne^{\Delta F_N}\chi_N(q_N)-\frac{2q_N}{q_N+\lambda}ia_Ne^{\Delta F_N}\chi_N(-p_N).
\end{aligned}$$

This formula is also obtained by the d-bar dressing method when the d-bar data is the degenerate delta kernel [2, p. 6].

When n=1, we have (22). For n=2, from (21), one knows that

$$\begin{aligned}
\chi_2(-p_1) &= \left[1+ia_2\frac{p_2+q_2}{q_2-p_2}\frac{p_2-p_1}{p_2+p_1}\frac{q_2+p_1}{q_2-p_1}e^{\triangle F_2}\right]/\tau_2,\\
\chi_2(q_1) &= \left[1+ia_2\frac{p_2+q_2}{q_2-p_2}\frac{p_2+q_1}{p_2-q_1}\frac{q_2-q_1}{q_2+q_1}e^{\triangle F_2}\right]/\tau_2,\\
\chi_2(-p_2) &= \left[1+ia_1\frac{p_1+q_1}{q_1-p_1}\frac{p_1-p_2}{p_1+p_2}\frac{q_1+p_2}{q_1-p_2}e^{\triangle F_1}\right]/\tau_2,\\
\chi_2(q_2) &= \left[1+ia_1\frac{p_1+q_1}{q_1-p_1}\frac{p_1+q_2}{p_1-q_2}\frac{q_1-q_2}{q_1+q_2}e^{\triangle F_1}\right]/\tau_2.
\end{aligned}$$

Then

$$\begin{aligned}
\chi_2(\lambda) &= 1-\frac{2p_1}{\lambda-p_1}ia_1e^{\Delta F_1}\chi_2(q_1)-\frac{2q_1}{q_1+\lambda}ia_1e^{\Delta F_1}\chi_2(-p_1)\\
&\quad -\frac{2p_2}{\lambda-p_2}ia_2e^{\Delta F_2}\chi_2(q_2)-\frac{2q_2}{q_2+\lambda}ia_2e^{\Delta F_2}\chi_2(-p_2).
\end{aligned}$$

We remark that this formula also appears in [2, p. 10], the parameters being different. For n=3, by (21), we obtain

$$\chi_3(-p_1)=\Bigg[1+ia_2\frac{p_2+q_2}{q_2-p_2}\frac{p_2-p_1}{p_2+p_1}\frac{q_2+p_1}{q_2-p_1}e^{\triangle F_2}+ia_3\frac{p_3+q_3}{q_3-p_3}\frac{p_3-p_1}{p_3+p_1}\frac{q_3+p_1}{q_3-p_1}e^{\triangle F_3}$$

$$+ ia_2 ia_3 \frac{p_2+q_2}{q_2-p_2}\frac{p_2-p_1}{p_2+p_1}\frac{q_2+p_1}{q_2-p_1}\frac{p_3+q_3}{q_3-p_3}\frac{p_3-p_1}{p_3+p_1}\frac{q_3+p_1}{q_3-p_1}\mathbb{P}_{23}e^{\triangle F_2+\triangle F_3}\Bigg]/\tau_3,$$

$$\chi_3(q_1) = \Bigg[1 + ia_2 \frac{p_2+q_2}{q_2-p_2}\frac{p_2+q_1}{p_2-q_1}\frac{q_2-q_1}{q_2+q_1}e^{\triangle F_2} + ia_3\frac{p_3+q_3}{q_3-p_3}\frac{p_2+q_1}{p_2-q_1}\frac{q_3-q_1}{q_3+q_1}e^{\triangle F_3}$$
$$+ ia_2 ia_3 \frac{p_2+q_2}{q_2-p_2}\frac{p_2+q_1}{p_2-q_1}\frac{q_2-q_1}{q_2+q_1}\frac{p_3+q_3}{q_3-p_3}\frac{p_2+q_1}{p_2-q_1}\frac{q_3-q_1}{q_3+q_1}\mathbb{P}_{23}e^{\triangle F_2+\triangle F_3}\Bigg]/\tau_3,$$

$$\chi_3(-p_2) = \Bigg[1 + ia_1 \frac{p_1+q_1}{q_1-p_1}\frac{p_1-p_2}{p_1+p_2}\frac{q_1+p_2}{q_1-p_2}e^{\triangle F_1} + ia_3\frac{p_3+q_3}{p_3-q_3}\frac{p_3-p_2}{p_3+p_2}\frac{q_3+p_2}{q_3-p_2}e^{\triangle F_3}$$
$$+ ia_1 ia_3 \frac{p_1+q_1}{q_1-p_1}\frac{p_1-p_2}{p_1+p_2}\frac{q_1+p_2}{q_1-p_2}\frac{p_3+q_3}{p_3-q_3}\frac{p_3-p_2}{p_3+p_2}\frac{q_3+p_2}{q_3-p_2}\mathbb{P}_{13}e^{\triangle F_1+\triangle F_3}\Bigg]/\tau_3,$$

$$\chi_3(q_2) = \Bigg[1 + ia_1 \frac{p_1+q_1}{q_1-p_1}\frac{p_1+q_2}{p_1-q_2}\frac{q_1-q_2}{q_1+q_2}e^{\triangle F_1} + ia_3\frac{p_3+q_3}{q_3-p_3}\frac{p_3+q_2}{p_3-q_2}\frac{q_3-q_2}{q_3+q_2}e^{\triangle F_3}$$
$$+ ia_1 ia_3 \frac{p_1+q_1}{q_1-p_1}\frac{p_1+q_2}{p_1-q_2}\frac{q_1-q_2}{q_1+q_2}\frac{p_3+q_3}{q_3-p_3}\frac{p_3+q_2}{p_3-q_2}\frac{q_3-q_2}{q_3+q_2}\mathbb{P}_{13}e^{\triangle F_1+\triangle F_3}\Bigg]/\tau_3,$$

$$\chi_3(-p_3) = \Bigg[1 + ia_1 \frac{p_1+q_1}{p_1-q_1}\frac{p_1-p_3}{p_1+p_3}\frac{q_1+p_3}{q_1-p_3}e^{\triangle F_1} + ia_2\frac{p_2+q_2}{q_2-p_2}\frac{p_2-p_3}{p_2+p_3}\frac{q_2+p_3}{q_2-p_3}e^{\triangle F_2}$$
$$+ ia_1 ia_2 \frac{p_1+q_1}{p_1-q_1}\frac{p_1-p_3}{p_1+p_3}\frac{q_1+p_3}{q_1-p_3}\frac{p_2+q_2}{q_2-p_2}\frac{p_2-p_3}{p_2+p_3}\frac{q_2+p_3}{q_2-p_3}\mathbb{P}_{12}e^{\triangle F_1+\triangle F_2}\Bigg]/\tau_3,$$

$$\chi_3(q_3) = \Bigg[1 + ia_1 \frac{p_1+q_1}{q_1-p_1}\frac{p_1+q_3}{p_1-q_3}\frac{q_1-q_3}{q_1+q_3}e^{\triangle F_1} + ia_2\frac{p_2+q_2}{q_2-p_2}\frac{p_2+q_3}{p_2-q_3}\frac{q_2-q_3}{q_2+q_3}e^{\triangle F_2}$$
$$+ ia_1 ia_2 \frac{p_1+q_1}{q_1-p_1}\frac{p_1+q_3}{p_1-q_3}\frac{q_1-q_3}{q_1+q_3}\frac{p_2+q_2}{q_2-p_2}\frac{p_2+q_3}{p_2-q_3}\frac{q_2-q_3}{q_2+q_3}\mathbb{P}_{12}e^{\triangle F_1+\triangle F_2}\Bigg]/\tau_3,$$

where $\mathbb{P}_{12}$, $\mathbb{P}_{13}$ and $\mathbb{P}_{23}$ are defined in (18). Then

$$\chi_3(\lambda) = 1 - \frac{2p_1}{\lambda-p_1}ia_1e^{\Delta F_1}\chi_3(q_1) - \frac{2q_1}{q_1+\lambda}ia_1e^{\Delta F_1}\chi_3(-p_1) - \frac{2p_2}{\lambda-p_2}ia_2e^{\Delta F_2}\chi_3(q_2)$$
$$- \frac{2q_2}{q_2+\lambda}ia_2e^{\Delta F_2}\chi_3(-p_2) - \frac{2p_3}{\lambda-p_3}ia_3e^{\Delta F_3}\chi_3(q_3) - \frac{2q_3}{q_3+\lambda}ia_3e^{\Delta F_3}\chi_3(-p_3).$$

5 Concluding remarks

In this paper, we have used the extended Moutard transformation to construct the N-solitons solutions. The basic idea comes from the successive iterations of solitons solutions, as remains to be the simple method to obtain the N-solitons solutions. Also, the corresponding wave functions are constructed by the Pfaffian expansion of the sum of two anti-symmetric matrices (14) when compared with the $\overline{\partial}$-dressing method [2].

To obtain real N-solitons solutions of the Novikov–Veselov equation (1), one has to put extra relations between $-p_i$ and q_i [2]. It could be interesting to investigate these real solutions for the Schrödinger operator (self-adjoint). On the other hand, the resonance of N-solitons solutions of DKP or KP theory has been studied in [16, 17, 18, 19]. And then the resonance of N-solitons solutions of Pfaffian type (15) deserves to be investigated. These issues will be published elsewhere.

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References

[1] Athorne C., Nimmo J.J.C., On the Moutard transformation for integrable partial differential equations, *Inverse Problems* **7** (1991), 809–826.

[2] Basalaev M.Yu., Dubrovsky V.G., Topovsky A.V., New exact solutions with constant asymptotic values at infinity of the NVN integrable nonlinear evolution equation via $\overline{\partial}$-dressing method, arXiv:0912.2155.

[3] Bogdanov L.V., Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg–de Vries equation, *Theoret. Math. Phys.* **70** (1987), 219–223.

[4] Chang J.H., The Gould–Hopper polynomials in the Novikov–Veselov equation, *J. Math. Phys.* **52** (2011), 092703, 15 pages, arXiv:1011.1614.

[5] Dubrovin B.A., Krichever I.M., Novikov S.P., The Schrödinger equation in a periodic field and Riemann surfaces, *Sov. Math. Dokl.* **17** (1976), 947–952.

[6] Dubrovsky V.G., Formusatik I.B., New lumps of Veselov–Novikov integrable nonlinear equation and new exact rational potentials of two-dimensional stationary Schrödinger equation via $\overline{\partial}$-dressing method, *Phys. Lett. A* **313** (2003), 68–76.

[7] Dubrovsky V.G., Formusatik I.B., The construction of exact rational solutions with constant asymptotic values at infinity of two-dimensional NVN integrable nonlinear evolution equations via the $\overline{\partial}$-dressing method, *J. Phys. A: Math.Gen.* **34** (2001), 1837–1851.

[8] Grinevich P.G., Rational solitons of the Veselov–Novikov equations are reflectionless two-dimensional potentials at fixed energy, *Theoret. Math. Phys.* **69** (1986), 1170–1172.

[9] Grinevich P.G., Manakov S.V., Inverse scattering problem for the two-dimensional Schrödinger operator, the $\overline{\partial}$-method and nonlinear equations, *Funct. Anal. Appl.* **20** (1986), 94–103.

[10] Grinevich P.G., Mironov A.E., Novikov S.P., New reductions and nonlinear systems for 2D Schrödinger operators, arXiv:1001.4300.

[11] Hirota R., The direct method in soliton theory, *Cambridge Tracts in Mathematics*, Vol. 155, Cambridge University Press, Cambridge, 2004.

[12] Hu H.C., Lou S.Y., Construction of the Darboux transformaiton and solutions to the modified Nizhnik–Novikov–Veselov equation, *Chinese Phys. Lett.* **21** (2004), 2073–2076.

[13] Hu H.C., Lou S.Y., Liu Q.P., Darboux transformation and variable separation approach: the Nizhnik–Novikov–Veselov equation, *Chinese Phys. Lett.* **20** (2003), 1413–1415, nlin.SI/0210012.

[14] Ishikawa M., Wakayama M., Applications of minor-summation formula. II. Pfaffians and Schur polynomials, *J. Combin. Theory Ser. A* **88** (1999), 136–157.

[15] Kaptsov O.V., Shan'ko Yu.V., Trilinear representation and the Moutard transformation for the Tzitzéica equation, solv-int/9704014.

[16] Kodama Y., KP solitons in shallow water, *J. Phys. A: Math. Gen.* **43** (2010), 434004, 54 pages, arXiv:1004.4607.

[17] Kodama Y., Maruno K., N-soliton solutions to the DKP equation and Weyl group actions, *J. Phys. A: Math. Gen.* **39** (2006), 4063–4086, nlin.SI/0602031.

[18] Kodama Y., Williams L.K., KP solitons and total positivity for the Grassmannian, arXiv:1106.0023.

[19] Kodama Y., Williams L.K., KP solitons, total positivity, and cluster algebras, *Proc. Natl. Acad. Sci. USA* **108** (2011), 8984–8989, arXiv:1105.4170.

[20] Konopelchenko B.G., Introduction to multidimensional integrable equations. The inverse spectral transform in $2+1$ dimensions, Plenum Press, New York, 1992.

[21] Konopelchenko B.G., Landolfi G., Induced surfaces and their integrable dynamics. II. Generalized Weierstrass representations in 4D spaces and deformations via DS hierarchy, *Stud. Appl. Math.* **104** (2000), 129–169.

[22] Krichever I.M., A characterization of Prym varieties, *Int. Math. Res. Not.* **2006** (2006), Art. ID 81476, 36 pages, math.AG/0506238.

[23] Liu S.Q., Wu C.Z., Zhang Y., On the Drinfeld–Sokolov hierarchies of D type, *Int. Math. Res. Not.* **2011** (2011), 1952–1996, arXiv:0912.5273.

[24] Manakov S.V., The method of the inverse scattering problem, and two-dimensional evolution equations, *Russian Math. Surveys* **31** (1976), no. 5, 245–246.

[25] Matveev V.B., Salle M.A., Darboux transformations and solitons, *Springer Series in Nonlinear Dynamics*, Springer-Verlag, Berlin, 1991.

[26] Mironov A.E., A relationship between symmetries of the Tzitzéica equation and the Veselov–Novikov hierarchy, *Math. Notes* **82** (2007), 569–572.

[27] Mironov A.E., Finite-gap minimal Lagrangian surfaces in $\mathbb{C}P^2$, in Riemann Surfaces, Harmonic Maps and Visualization, *OCAMI Stud.*, Vol. 3, Osaka Munic. Univ. Press, Osaka, 2010, 185–196, arXiv:1005.3402.

[28] Mironov A.E., The Veselov–Novikov hierarchy of equations, and integrable deformations of minimal Lagrangian tori in $\mathbb{C}P^2$, *Sib. Electron. Math. Rep.* **1** (2004), 38–46, math.DG/0607700.

[29] Moutard M., Note sur les équations différentielles linéaires du second ordre, *C.R. Acad. Sci. Paris* **80** (1875), 729–733.

[30] Moutard M., Sur la construction des équations de la forme $\frac{1}{z}\frac{\partial^2 z}{\partial x \partial y} = \lambda(xy)$, qui admettent une intégrale générale explicite, *J. de. l'Éc. Polyt.* **28** (1878), 1–12.

[31] Nimmo J.J.C., Darboux transformations in $(2+1)$-dimensions, in Applications of Analytic and Geometric Methods to Nonlinear Differential Equations (Exeter, 1992), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, Vol. 413, Kluwer Acad. Publ., Dordrecht, 1993, 183–192.

[32] Nimmo J.J.C., Hall–Littlewood symmetric functions and the BKP equation, *J. Phys. A: Math. Gen.* **23** (1990), 751–760.

[33] Novikov S.P., Two-dimensional Schrödinger operators in periodic fields, *J. Sov. Math.* **28** (1985), 1–20.

[34] Novikov S.P., Veselov A.P., Two-dimensional Schrödinger operator: inverse scattering transform and evolutional equations, *Phys. D* **18** (1986), 267–273.

[35] Ohta Y., Pfaffian solutions for the Veselov–Novikov equation, *J. Phys. Soc. Japan* **61** (1992), 3928–3933.

[36] Orlov A.Yu., Shiota T., Takasaki K., Pfaffian structures and certain solutions to BKP hierarchies. I. Sums over partitions, arXiv:1201.4518.

[37] Shiota T., Prym varieties and soliton equations, in Infinite-Dimensional Lie Algebras and Groups (Luminy-Marseille, 1988), *Adv. Ser. Math. Phys.*, Vol. 7, World Sci. Publ., Teaneck, NJ, 1989, 407–448.

[38] Stembridge J.R., Nonintersecting paths, Pfaffians, and plane partitions, *Adv. Math.* **83** (1990), 96–131.

[39] Taimanov I.A., Tsarev S.P., Two-dimensional rational solitons and their blowup via the Moutard transformation, *Theoret. Math. Phys.* **157** (2008), 1525–1541, arXiv:0801.3225.

[40] Takasaki K., Dispersionless Hirota equations of two-component BKP hierarchy, *SIGMA* **2** (2006), 057, 22 pages, nlin.SI/0604003.

[41] Veselov A.P., Novikov S.P., Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations, *Sov. Math. Dokl.* **30** (1984), 588–591.