Direct Connection between the R_{II} Chain and the Nonautonomous Discrete Modified KdV Lattice

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Abstract. The spectral transformation technique for symmetric R_{II} polynomials is developed. Use of this technique reveals that the nonautonomous discrete modified KdV (nd-mKdV) lattice is directly connected with the R_{II} chain. Hankel determinant solutions to the semi-infinite nd-mKdV lattice are also presented.

Key words: orthogonal polynomials; spectral transformation; R_{II} chain; nonautonomous discrete modified KdV lattice

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1   Introduction

In the theory of integrable systems, orthogonal polynomials play an important role. In particular, the spectral transformation technique yields various integrable systems and particular solutions [24, 26, 27]. The spectral transformation for orthogonal polynomials is a mapping from an orthogonal polynomial sequence to another orthogonal polynomial sequence. We can view the three-term recurrence relation and the spectral transformation for orthogonal polynomials as a Lax pair, where the compatibility condition induces an integrable system. Furthermore, the determinant structure of orthogonal polynomials allows us to derive particular solutions to the associated integrable system. During the last fifteen years, many researchers have extended this technique to generalized (bi)orthogonal functions and have exploited novel integrable systems that have rich properties [1, 2, 3, 12, 14, 15, 16, 17, 33, 34, 35, 37]. In this paper, we will extend the spectral transformation technique for symmetric orthogonal polynomials and the associated discrete integrable system to symmetric R_{II} polynomials. Our motivation comes from applications of discrete integrable systems to numerical algorithms. It is well-known that the discrete integrable system associated with orthogonal polynomials is the nonautonomous discrete Toda (nd-Toda) lattice and that its time evolution equation is the same as the recurrence relation of the dqds algorithm [5], a fast and accurate eigenvalue or singular value algorithm. Similarly, the discrete integrable system associated with symmetric orthogonal polynomials is the nonautonomous discrete Lotka–Volterra (nd-LV) lattice, which can compute singular values [9]. By using the spectral transformation technique, we can easily derive a direct connection between the nd-Toda lattice and the nd-LV lattice. This connection was used to develop the mdLVs algorithm, which is an improved version of the singular value algorithm based on the nd-LV lattice [10].

Recently, the authors have been developing a generalized eigenvalue algorithm based on the R_{II} chain [13]. Since the R_{II} chain is associated with R_{II} polynomials, a generalization of orthogonal polynomials from the point of view of Padé approximation or the eigenvalue

problem [8, 18, 28], the proposed algorithm has good properties similar to the dqds algorithm. These studies motivate us to find the discrete integrable system associated with symmetric R_{II} polynomials and its connection with the R_{II} chain. The derived discrete integrable system may become the basis for developing good numerical algorithms for generalized eigenvalue problems or Padé approximations, for example.

This paper is organized as follows. In Section 2, we briefly recall the derivation of the nd-Toda lattice and the nd-LV lattice from the theory of spectral transformations for ordinary and symmetric orthogonal polynomials, respectively. We also review the direct connection (Miura transformation) between the nd-Toda lattice and the nd-LV lattice. In Section 3, we extend the framework presented in Section 2 to R_{II} polynomials. We then demonstrate that the spectral transformations for symmetric R_{II} polynomials give rise to the nonautonomous discrete modified KdV (nd-mKdV) lattice. Particular solutions to the semi-infinite nd-mKdV lattice and the direct connection between the R_{II} chain and the nd-mKdV lattice are also derived. Section 4 is devoted to concluding remarks.

2 Derivation of the nd-Toda lattice and the nd-LV lattice

2.1 Orthogonal polynomials and the nd-Toda lattice

Monic orthogonal polynomials are defined by a three-term recurrence relation in the form

$$\phi_{-1}^{k,t}(x) \coloneqq 0, \qquad \phi_0^{k,t}(x) \coloneqq 1,$$
$$\phi_{n+1}^{k,t}(x) \coloneqq \left(x - a_n^{k,t}\right)\phi_n^{k,t}(x) - b_n^{k,t}\phi_{n-1}^{k,t}(x), \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where $a_n^{k,t} \in \mathbb{R}$, $b_n^{k,t} \in \mathbb{R} \setminus \{0\}$, and $k, t \in \mathbb{Z}$ indicate discrete time. By definition, $\phi_n^{k,t}(x)$ is a monic polynomial of degree n. If some constant $h_0^{k,t} \in \mathbb{R} \setminus \{0\}$ is fixed, then Favard's theorem [4] provides a unique linear functional $\mathcal{L}^{k,t} \colon \mathbb{R}[x] \to \mathbb{R}$ such that the orthogonality relation

$$\mathcal{L}^{k,t}[x^m \phi_n^{k,t}(x)] = h_n^{k,t}\delta_{m,n}, \qquad n = 0, 1, 2, \ldots, \qquad m = 0, 1, \ldots, n, \tag{2}$$

holds, where

$$h_n^{k,t} = h_0^{k,t}b_1^{k,t}b_2^{k,t}\cdots b_n^{k,t}, \qquad n = 1, 2, 3, \ldots,$$

and $\delta_{m,n}$ is the Kronecker delta.

Let us introduce time evolution into the orthogonal polynomials through spectral transformations. First, the spectral transformations for the k-direction are

$$x\phi_n^{k+1,t}(x) = \phi_{n+1}^{k,t}(x) + q_n^{k,t}\phi_n^{k,t}(x), \tag{3a}$$
$$\phi_n^{k,t}(x) = \phi_n^{k+1,t}(x) + e_n^{k,t}\phi_{n-1}^{k+1,t}(x), \tag{3b}$$

where

$$q_n^{k,t} \coloneqq -\frac{\phi_{n+1}^{k,t}(0)}{\phi_n^{k,t}(0)}, \qquad e_n^{k,t} \coloneqq \frac{\mathcal{L}^{k,t}[x^n\phi_n^{k,t}(x)]}{\mathcal{L}^{k+1,t}[x^{n-1}\phi_{n-1}^{k+1,t}(x)]}, \tag{4}$$

$$\mathcal{L}^{k+1,t}[\pi(x)] \coloneqq \mathcal{L}^{k,t}[x\pi(x)] \qquad \text{for all } \pi(x) \in \mathbb{R}[x]. \tag{5}$$

It is readily verified that $\left\{\phi_n^{k+1,t}(x)\right\}_{n=0}^{\infty}$ are monic orthogonal polynomials with respect to the linear functional $\mathcal{L}^{k+1,t}$. Similarly, the spectral transformations for the t-direction are

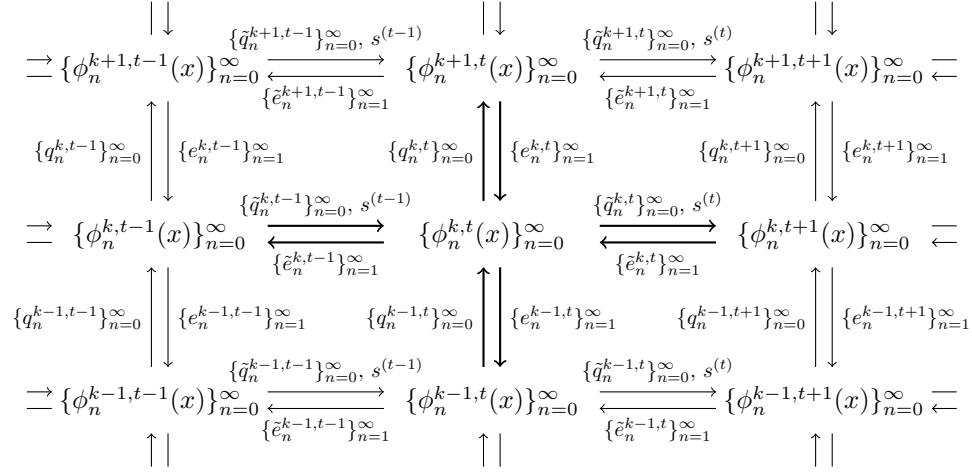$$\left(x + s^{(t)}\right)\phi_n^{k,t+1}(x) = \phi_{n+1}^{k,t}(x) + \tilde{q}_n^{k,t}\phi_n^{k,t}(x), \tag{6a}$$
$$\phi_n^{k,t}(x) = \phi_n^{k,t+1}(x) + \tilde{e}_n^{k,t}\phi_{n-1}^{k,t+1}(x), \tag{6b}$$

where $s^{(t)}$ is a nonzero parameter depending on t and

$$\tilde{q}_n^{k,t} := -\frac{\phi_{n+1}^{k,t}(-s^{(t)})}{\phi_n^{k,t}(-s^{(t)})}, \qquad \tilde{e}_n^{k,t} := \frac{\mathcal{L}^{k,t}[x^n \phi_n^{k,t}(x)]}{\mathcal{L}^{k,t+1}[x^{n-1}\phi_{n-1}^{k,t+1}(x)]}, \tag{7}$$

$$\mathcal{L}^{k,t+1}[\pi(x)] := \mathcal{L}^{k,t}[(x+s^{(t)})\pi(x)] \qquad \text{for all } \pi(x) \in \mathbb{R}[x]. \tag{8}$$

The only difference between the transformations for the k-direction (3) and the t-direction (6) is the parameter $s^{(t)}$. Fig. 1 illustrates the relations among the monic orthogonal polynomials, the spectral transformations and the dependent variables.

Figure 1. Chain of the spectral transformations for monic orthogonal polynomials.

Relations (1), (3) and (6) yield

$$\begin{aligned}
\phi_{n+1}^{k,t}(x) &= (x - a_n^{k,t})\phi_n^{k,t}(x) - b_n^{k,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (q_n^{k,t} + e_n^{k,t}))\phi_n^{k,t}(x) - q_{n-1}^{k,t}e_n^{k,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (q_n^{k-1,t} + e_{n+1}^{k-1,t}))\phi_n^{k,t}(x) - q_n^{k-1,t}e_n^{k-1,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (\tilde{q}_n^{k,t} + \tilde{e}_n^{k,t} - s^{(t)}))\phi_n^{k,t}(x) - \tilde{q}_{n-1}^{k,t}\tilde{e}_n^{k,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (\tilde{q}_n^{k,t-1} + \tilde{e}_{n+1}^{k,t-1} - s^{(t-1)}))\phi_n^{k,t}(x) - \tilde{q}_n^{k,t-1}\tilde{e}_n^{k,t-1}\phi_{n-1}^{k,t}(x).
\end{aligned}$$

Hence, for consistency, the compatibility conditions

$$a_n^{k,t} = q_n^{k,t} + e_n^{k,t} = q_n^{k-1,t} + e_{n+1}^{k-1,t} = \tilde{q}_n^{k,t} + \tilde{e}_n^{k,t} - s^{(t)} = \tilde{q}_n^{k,t-1} + \tilde{e}_{n+1}^{k,t-1} - s^{(t-1)}, \tag{9a}$$

$$b_n^{k,t} = q_{n-1}^{k,t}e_n^{k,t} = q_n^{k-1,t}e_n^{k-1,t} = \tilde{q}_{n-1}^{k,t}\tilde{e}_n^{k,t} = \tilde{q}_n^{k,t-1}\tilde{e}_n^{k,t-1}, \tag{9b}$$

$$e_0^{k,t} = \tilde{e}_0^{k,t} = 0 \qquad \text{for all } k \text{ and } t, \tag{9c}$$

must be satisfied. These are the time evolution equations of the semi-infinite nd-Toda lattice. Equations (9) give the relations among the recurrence coefficients of $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ and the dependent variables around $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ in the diagram (Fig. 1).

Define the moment of the linear functional $\mathcal{L}^{0,t}$ by

$$\mu_m^{(t)} := \mathcal{L}^{0,t}[x^m].$$

Note that (5) gives the relation

$$\mathcal{L}^{k,t}[x^m] = \mathcal{L}^{0,t}[x^{k+m}] = \mu_{k+m}^{(t)}.$$

Further, (8) gives the dispersion relation

$$\mu_m^{(t+1)} = \mu_{m+1}^{(t)} + s^{(t)}\mu_m^{(t)}. \tag{10}$$

We should remark that, if a concrete representation of the initial linear functional $\mathcal{L}^{0,0}$ is given by a weighted integral, then the moment may be represented concretely as

$$\mu_m^{(t)} = \int_\Omega w(x)x^m \prod_{t'=0}^{t-1}\left(x + s^{(t')}\right)\mathrm{d}x,$$

where Ω is some interval on the real line and $w(x)$ is a weight function defined on Ω.

The determinant expression of the monic orthogonal polynomials $\{\phi_n^{k,t}(x)\}_{n=0}^\infty$ is given by

$$\phi_n^{k,t}(x) = \frac{1}{\tau_n^{k,t}}\begin{vmatrix} \mu_k^{(t)} & \mu_{k+1}^{(t)} & \cdots & \mu_{k+n-1}^{(t)} & \mu_{k+n}^{(t)} \\ \mu_{k+1}^{(t)} & \mu_{k+2}^{(t)} & \cdots & \mu_{k+n}^{(t)} & \mu_{k+n+1}^{(t)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{k+n-1}^{(t)} & \mu_{k+n}^{(t)} & \cdots & \mu_{k+2n-2}^{(t)} & \mu_{k+2n-1}^{(t)} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}, \tag{11}$$

where $\tau_n^{k,t}$ is the Hankel determinant of order n:

$$\tau_{-1}^{k,t} := 0, \qquad \tau_0^{k,t} := 1, \qquad \tau_n^{k,t} := \left|\mu_{k+i+j}^{(t)}\right|_{0\le i,j\le n-1}, \qquad n = 1, 2, 3, \ldots.$$

One can readily verify that the right hand side of (11) is a monic polynomial of degree n and satisfies the orthogonality relation (2). This determinant expression (11) and the dispersion relation (10) enable us to give Hankel determinant solutions to the nd-Toda lattice (9); from (4) and (7), we obtain

$$q_n^{k,t} = \frac{\tau_n^{k,t}\tau_{n+1}^{k+1,t}}{\tau_{n+1}^{k,t}\tau_n^{k+1,t}}, \qquad e_n^{k,t} = \frac{\tau_{n+1}^{k,t}\tau_{n-1}^{k+1,t}}{\tau_n^{k,t}\tau_n^{k+1,t}}, \tag{12a}$$

$$\tilde{q}_n^{k,t} = \frac{\tau_n^{k,t}\tau_{n+1}^{k,t+1}}{\tau_{n+1}^{k,t}\tau_n^{k,t+1}}, \qquad \tilde{e}_n^{k,t} = \frac{\tau_{n+1}^{k,t}\tau_{n-1}^{k,t+1}}{\tau_n^{k,t}\tau_n^{k,t+1}}. \tag{12b}$$

2.2   Symmetric orthogonal polynomials and the nd-LV lattice

Next, we consider the polynomial sequence $\{\sigma_n^{k,t}(x)\}_{n=0}^\infty$ defined by

$$\sigma_{2n+i}^{k,t}(x) := x^i\phi_n^{k+i,t}(x^2), \qquad n = 0, 1, 2, \ldots, \qquad i = 0, 1.$$

By definition, $\sigma_n^{k,t}(x)$ is a monic polynomial of degree n and has the symmetry property

$$\sigma_n^{k,t}(-x) = (-1)^n\sigma_n^{k,t}(x).$$

Further, $\{\sigma_n^{k,t}(x)\}_{n=0}^\infty$ are orthogonal with respect to the linear functional $\mathcal{S}^{k,t}$ defined by

$$\mathcal{S}^{k,t}\left[x^{2m}\right] := \mathcal{L}^{k,t}\left[x^m\right], \qquad \mathcal{S}^{k,t}\left[x^{2m+1}\right] := 0, \qquad m = 0, 1, 2, \ldots.$$

$\{\sigma_n^{k,t}(x)\}_{n=0}^\infty$ are called monic symmetric orthogonal polynomials.

From (3), we have the relations

$$x^2\phi_n^{k+1,t}(x^2) = \phi_{n+1}^{k,t}(x^2) + q_n^{k,t}\phi_n^{k,t}(x^2),$$
$$x\phi_n^{k,t}(x^2) = x\phi_n^{k+1,t}(x^2) + e_n^{k,t}x\phi_{n-1}^{k+1,t}(x^2).$$

These relations lead us to the three-term recurrence relation that $\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ satisfy:

$$\sigma_{2n+2}^{k,t}(x) = x\sigma_{2n+1}^{k,t}(x) - q_n^{k,t}\sigma_{2n}^{k,t}(x), \tag{13a}$$
$$\sigma_{2n+1}^{k,t}(x) = x\sigma_{2n}^{k,t}(x) - e_n^{k,t}\sigma_{2n-1}^{k,t}(x). \tag{13b}$$

Spectral transformations for $\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ are also induced from (6):

$$\left(x^2 + s^{(t)}\right)\sigma_{2n}^{k,t+1}(x) = \sigma_{2n+2}^{k,t}(x) + \tilde{q}_n^{k,t}\sigma_{2n}^{k,t}(x), \tag{14a}$$
$$\left(x^2 + s^{(t)}\right)\sigma_{2n+1}^{k,t+1}(x) = \sigma_{2n+3}^{k,t}(x) + \tilde{q}_n^{k+1,t}\sigma_{2n+1}^{k,t}(x), \tag{14b}$$
$$\sigma_{2n}^{k,t}(x) = \sigma_{2n}^{k,t+1}(x) + \tilde{e}_n^{k,t}\sigma_{2n-2}^{k,t+1}(x), \tag{14c}$$
$$\sigma_{2n+1}^{k,t}(x) = \sigma_{2n+1}^{k,t+1}(x) + \tilde{e}_n^{k+1,t}\sigma_{2n-1}^{k,t+1}(x). \tag{14d}$$

Relations (13) and (14) show that there exist variables $v_n^{k,t}$ satisfying the relations

$$\left(x^2 + s^{(t)}\right)\sigma_n^{k,t+1}(x) = x\sigma_{n+1}^{k,t}(x) + \left(s^{(t)} + v_n^{k,t}\right)\sigma_n^{k,t}(x), \tag{15a}$$
$$\left(s^{(t)} + v_n^{k,t}\right)\sigma_n^{k,t}(x) = s^{(t)}\sigma_n^{k,t+1}(x) + v_n^{k,t}x\sigma_{n-1}^{k,t+1}(x). \tag{15b}$$

Relations (15) yield

$$\sigma_{n+1}^{k,t}(x) = x\sigma_n^{k,t}(x) - v_n^{k,t}\left(1 + \left(s^{(t)}\right)^{-1}v_{n-1}^{k,t}\right)\sigma_{n-1}^{k,t}(x)$$
$$= x\sigma_n^{k,t}(x) - v_n^{k,t-1}\left(1 + \left(s^{(t-1)}\right)^{-1}v_{n+1}^{k,t-1}\right)\sigma_{n-1}^{k,t}(x). \tag{16}$$

Hence, the compatibility condition

$$v_n^{k,t}\left(1 + \left(s^{(t)}\right)^{-1}v_{n-1}^{k,t}\right) = v_n^{k,t-1}\left(1 + \left(s^{(t-1)}\right)^{-1}v_{n+1}^{k,t-1}\right), \tag{17a}$$
$$v_0^{k,t} = 0 \qquad \text{for all } k \text{ and } t, \tag{17b}$$

must be satisfied. This is the time evolution equation of the semi-infinite nd-LV lattice.

From relations (13)–(16), we obtain the Miura transformation between the nd-Toda lattice (9) and the nd-LV lattice (17):

$$q_n^{k,t} = v_{2n+1}^{k,t}\left(1 + \left(s^{(t)}\right)^{-1}v_{2n}^{k,t}\right) = v_{2n+1}^{k,t-1}\left(1 + \left(s^{(t-1)}\right)^{-1}v_{2n+2}^{k,t-1}\right),$$
$$e_n^{k,t} = v_{2n}^{k,t}\left(1 + \left(s^{(t)}\right)^{-1}v_{2n-1}^{k,t}\right) = v_{2n}^{k,t-1}\left(1 + \left(s^{(t-1)}\right)^{-1}v_{2n+1}^{k,t-1}\right),$$
$$\tilde{q}_n^{k,t} = s^{(t)}\left(1 + \left(s^{(t)}\right)^{-1}v_{2n+1}^{k,t}\right)\left(1 + \left(s^{(t)}\right)^{-1}v_{2n}^{k,t}\right)$$
$$= s^{(t)}\left(1 + \left(s^{(t)}\right)^{-1}v_{2n+1}^{k-1,t}\right)\left(1 + \left(s^{(t)}\right)^{-1}v_{2n+2}^{k-1,t}\right),$$
$$\tilde{e}_n^{k,t} = \left(s^{(t)}\right)^{-1}v_{2n}^{k,t}v_{2n-1}^{k,t} = \left(s^{(t)}\right)^{-1}v_{2n}^{k-1,t}v_{2n+1}^{k-1,t}.$$

In addition, from relations (15), we obtain

$$1 + \left(s^{(t)}\right)^{-1}v_{2n}^{k,t} = \frac{\sigma_{2n}^{k,t+1}(0)}{\sigma_{2n}^{k,t}(0)} = \frac{\phi_n^{k,t+1}(0)}{\phi_n^{k,t}(0)},$$

$$1 + \left(s^{(t)}\right)^{-1}v_{2n-1}^{k,t} = \left(-s^{(t)}\right)^{-1/2}\frac{\sigma_{2n}^{k,t}\left((-s^{(t)})^{1/2}\right)}{\sigma_{2n-1}^{k,t}\left((-s^{(t)})^{1/2}\right)} = -\left(s^{(t)}\right)^{-1}\frac{\phi_n^{k,t}(-s^{(t)})}{\phi_{n-1}^{k+1,t}(-s^{(t)})}.$$

By using these relations and the solutions to the nd-Toda lattice (12), we obtain Hankel determinant solutions to the nd-LV lattice (17):

$$v_{2n+1}^{k,t} = q_n^{k,t} \frac{\phi_n^{k,t}(0)}{\phi_n^{k,t+1}(0)} = \frac{\tau_n^{k,t+1} \tau_{n+1}^{k+1,t}}{\tau_{n+1}^{k,t} \tau_n^{k+1,t+1}},$$

$$v_{2n}^{k,t} = -s^{(t)} e_n^{k,t} \frac{\phi_{n-1}^{k+1,t}(-s^{(t)})}{\phi_n^{k,t}(-s^{(t)})} = s^{(t)} \frac{\tau_{n+1}^{k,t} \tau_{n-1}^{k+1,t+1}}{\tau_n^{k,t+1} \tau_n^{k+1,t}}.$$

3 Derivation of the R_{II} chain and the nd-mKdV lattice

We will apply the framework constructed in the previous section to monic R_{II} polynomials and derive the nd-mKdV lattice.

3.1 R_{II} polynomials and the R_{II} chain

Monic R_{II} polynomials are defined by the three-term recurrence relation of the form

$$\varphi_{-1}^{k,t}(x) := 0, \qquad \varphi_0^{k,t}(x) := 1,$$

$$\varphi_{n+1}^{k,t}(x) := \left((1 + \beta_n^{k,t})x - \alpha_n^{k,t}\right)\varphi_n^{k,t}(x) - \beta_n^{k,t}(x + \gamma_{k+t+2n-2})(x + \gamma_{k+t+2n-1})\varphi_{n-1}^{k,t}(x),$$

$$n = 0, 1, 2, \ldots,$$

where $\alpha_n^{k,t} \in \mathbb{R}$ and $\beta_n^{k,t}, \gamma_{k+t+n} \in \mathbb{R} \setminus \{0\}$. If nonzero constants $h_0^{k,t}$ and $h_1^{k,t}$ are fixed, then a Favard-type theorem [8] guarantees the existence of a unique linear functional $\mathcal{L}^{k,t}$ such that the orthogonality relation

$$\mathcal{L}^{k,t}\left[\frac{x^m \varphi_n^{k,t}(x)}{\prod_{j=0}^{2n-1}(x + \gamma_{k+t+j})}\right] = h_n^{k,t}\delta_{m,n}, \qquad n = 0, 1, 2, \ldots, \qquad m = 0, 1, \ldots, n,$$

holds, where $h_n^{k,t}$, $n = 2, 3, \ldots$ are nonzero constants. Note that $\mathcal{L}^{k,t}$ is defined on the vector space spanned by $\frac{1}{\prod_{j=0}^{l-1}(x+\gamma_{k+t+j})}$, $l = 0, 1, 2, \ldots$.

As in the case of monic orthogonal polynomials, we introduce the time evolution of the monic R_{II} polynomials by the following spectral transformations:

$$\left(1 + q_n^{k,t}\right)x\varphi_n^{k+1,t}(x) = \varphi_{n+1}^{k,t}(x) + q_n^{k,t}(x + \gamma_{k+t+2n})\varphi_n^{k,t}(x), \tag{18a}$$

$$\left(1 + e_n^{k,t}\right)\varphi_n^{k,t}(x) = \varphi_n^{k+1,t}(x) + e_n^{k,t}(x + \gamma_{k+t+2n-1})\varphi_{n-1}^{k+1,t}(x), \tag{18b}$$

$$\left(1 + \tilde{q}_n^{k,t}\right)\left(x + s^{(t)}\right)\varphi_n^{k,t+1}(x) = \varphi_{n+1}^{k,t}(x) + \tilde{q}_n^{k,t}(x + \gamma_{k+t+2n})\varphi_n^{k,t}(x), \tag{18c}$$

$$\left(1 + \tilde{e}_n^{k,t}\right)\varphi_n^{k,t}(x) = \varphi_n^{k,t+1}(x) + \tilde{e}_n^{k,t}(x + \gamma_{k+t+2n-1})\varphi_{n-1}^{k,t+1}(x). \tag{18d}$$

These spectral transformations were originally introduced by Zhedanov [38]. By choosing the variables $q_n^{k,t}$, $e_n^{k,t}$, $\tilde{q}_n^{k,t}$ and $\tilde{e}_n^{k,t}$ as above, the leading coefficients of both sides of (18) become equal. The time evolution of the linear functional is also given by

$$\mathcal{L}^{k+1,t}[\rho(x)] := \mathcal{L}^{k,t}\left[\frac{x}{x + \gamma_{k+t}}\rho(x)\right], \qquad \mathcal{L}^{k,t+1}[\rho(x)] := \mathcal{L}^{k,t}\left[\frac{x + s^{(t)}}{x + \gamma_{k+t}}\rho(x)\right] \tag{19}$$

for rational functions $\rho(x)$. One can verify that $\left\{\varphi_n^{k+1,t}(x)\right\}_{n=0}^{\infty}$ and $\left\{\varphi_n^{k,t+1}(x)\right\}_{n=0}^{\infty}$ are both also monic R_{II} polynomials. The spectral transformations (18) induce the time evolution equations

of the semi-infinite monic-type R$_{\rm II}$ chain:

$$\alpha_n^{k,t} = \gamma_{k+t+2n} q_n^{k,t} + \gamma_{k+t+2n-1} e_n^{k,t} \frac{1+q_n^{k,t}}{1+q_{n-1}^{k,t}}$$

$$= \gamma_{k+t+2n-1} q_n^{k-1,t} \frac{1+e_{n+1}^{k-1,t}}{1+e_n^{k-1,t}} + \gamma_{k+t+2n} e_{n+1}^{k-1,t}$$

$$= \gamma_{k+t+2n} \tilde{q}_n^{k,t} + \gamma_{k+t+2n-1} \tilde{e}_n^{k,t} \frac{1+\tilde{q}_n^{k,t}}{1+\tilde{q}_{n-1}^{k,t}} - s^{(t)}\left(1+\tilde{q}_n^{k,t}\right)\left(1+\tilde{e}_n^{k,t}\right)$$

$$= \gamma_{k+t+2n-1} \tilde{q}_n^{k,t-1} \frac{1+\tilde{e}_{n+1}^{k,t-1}}{1+\tilde{e}_n^{k,t-1}} + \gamma_{k+t+2n} \tilde{e}_{n+1}^{k,t-1} - s^{(t-1)}\left(1+\tilde{q}_n^{k,t-1}\right)\left(1+\tilde{e}_{n+1}^{k,t-1}\right), \quad (20\text{a})$$

$$\beta_n^{k,t} = q_{n-1}^{k,t} e_n^{k,t} \frac{1+q_n^{k,t}}{1+q_{n-1}^{k,t}} = q_n^{k-1,t} e_n^{k-1,t} \frac{1+e_{n+1}^{k-1,t}}{1+e_n^{k-1,t}}$$

$$= \tilde{q}_{n-1}^{k,t} \tilde{e}_n^{k,t} \frac{1+\tilde{q}_n^{k,t}}{1+\tilde{q}_{n-1}^{k,t}} = \tilde{q}_n^{k,t-1} \tilde{e}_n^{k,t-1} \frac{1+\tilde{e}_{n+1}^{k,t-1}}{1+\tilde{e}_n^{k,t-1}}, \quad (20\text{b})$$

$$e_0^{k,t} = \tilde{e}_0^{k,t} = 0 \qquad \text{for all } k \text{ and } t. \quad (20\text{c})$$

Note that the R$_{\rm II}$ chain was originally introduced by Spiridonov and Zhedanov [28]. The original chain is described by three equations and four types of dependent variables with one constraint. The monic-type version (20) is, however, described by essentially only two equations and two types of dependent variables; since we are now considering two time variables k and t, there are four types of dependent variables.

We define the moment of the linear functional $\mathcal{L}^{0,t}$ by

$$\mu_{m,l}^{(t)} := \mathcal{L}^{0,t}\left[\frac{x^m}{\prod_{j=0}^{l-1}(x+\gamma_{t+j})}\right].$$

Note that the time evolution of the linear functional (19) gives the relation

$$\mathcal{L}^{k,t}\left[\frac{x^m}{\prod_{j=0}^{l-1}(x+\gamma_{k+t+j})}\right] = \mathcal{L}^{0,t}\left[\frac{x^{k+m}}{\prod_{j=0}^{k+l-1}(x+\gamma_{t+j})}\right] = \mu_{k+m,k+l}^{(t)}$$

and the dispersion relations

$$\mu_{m,l}^{(t+1)} = \mu_{m+1,l+1}^{(t)} + s^{(t)} \mu_{m,l+1}^{(t)}, \qquad \mu_{m,l}^{(t)} = \mu_{m+1,l+1}^{(t)} + \gamma_{t+l} \mu_{m,l+1}^{(t)}. \quad (21)$$

Then, the determinant expression of the monic R$_{\rm II}$ polynomials $\{\varphi_n^{k,t}(x)\}_{n=0}^{\infty}$ is given by

$$\varphi_n^{k,t}(x) = \frac{1}{\tau_n^{k,2n,t}}\begin{vmatrix} \mu_{k,k+2n}^{(t)} & \mu_{k+1,k+2n}^{(t)} & \cdots & \mu_{k+n-1,k+2n}^{(t)} & \mu_{k+n,k+2n}^{(t)} \\ \mu_{k+1,k+2n}^{(t)} & \mu_{k+2,k+2n}^{(t)} & \cdots & \mu_{k+n,k+2n}^{(t)} & \mu_{k+n+1,k+2n}^{(t)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{k+n-1,k+2n}^{(t)} & \mu_{k+n,k+2n}^{(t)} & \cdots & \mu_{k+2n-2,k+2n}^{(t)} & \mu_{k+2n-1,k+2n}^{(t)} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad (22)$$

where $\tau_n^{k,l,t}$ is the Hankel determinant of order n:

$$\tau_{-1}^{k,l,t} := 0, \qquad \tau_0^{k,l,t} := 1, \qquad \tau_n^{k,l,t} := \left|\mu_{k+i+j,k+l}^{(t)}\right|_{0\le i,j\le n-1}, \qquad n = 1, 2, 3, \ldots. \quad (23)$$

We should remark that the Casorati determinant representation of the R_{II} polynomials was found by Spiridonov and Zhedanov [31]. The Hankel determinant expression given above reflects the structure of the discrete two-dimensional Toda hierarchy [34]. By using the determinant expression (22), the dispersion relation (21), and a determinant identity called Plücker relation, we can find the Hankel determinant solutions to the semi-infinite monic-type R_{II} chain:

$$q_n^{k,t} = (\gamma_{k+t+2n})^{-1} \frac{\tau_n^{k,2n,t} \tau_{n+1}^{k+1,2n+1,t}}{\tau_{n+1}^{k,2n+2,t} \tau_n^{k+1,2n-1,t}}, \qquad e_n^{k,t} = \gamma_{k+t+2n} \frac{\tau_{n+1}^{k,2n+1,t} \tau_{n-1}^{k+1,2n-2,t}}{\tau_n^{k,2n-1,t} \tau_n^{k+1,2n,t}},$$

$$\tilde{q}_n^{k,t} = (\gamma_{k+t+2n} - s^{(t)})^{-1} \frac{\tau_n^{k,2n,t} \tau_{n+1}^{k,2n+1,t+1}}{\tau_{n+1}^{k,2n+2,t} \tau_n^{k,2n-1,t+1}}, \qquad \tilde{e}_n^{k,t} = (\gamma_{k+t+2n} - s^{(t)}) \frac{\tau_{n+1}^{k,2n+1,t} \tau_{n-1}^{k,2n-2,t+1}}{\tau_n^{k,2n-1,t} \tau_n^{k,2n,t+1}}.$$

3.2   Symmetric R_{II} polynomials and the nd-mKdV lattice

We introduce a symmetric version of the monic R_{II} polynomials, which is an analogue of the monic symmetric orthogonal polynomials.

Let us define a polynomial sequence $\{\varsigma_n^{k,t}(x)\}_{n=0}^{\infty}$ by

$$\varsigma_{2n+i}^{k,t}(x) := x^i \varphi_n^{k+i,t}(x^2), \qquad n = 0, 1, 2, \ldots, \qquad i = 0, 1.$$

The corresponding linear functional $\mathcal{S}^{k,t}$ is given by

$$\mathcal{S}^{k,t}\left[\frac{x^{2m}}{\prod_{j=0}^{l-1}(x^2 + \gamma_{k+t+j})}\right] := \mathcal{L}^{k,t}\left[\frac{x^m}{\prod_{j=0}^{l-1}(x + \gamma_{k+t+j})}\right],$$

$$\mathcal{S}^{k,t}\left[\frac{x^{2m+1}}{\prod_{j=0}^{l-1}(x^2 + \gamma_{k+t+j})}\right] := 0, \qquad l = 0, 1, 2, \ldots, \qquad m = 0, 1, \ldots, l.$$

The spectral transformations (18) yield the three-term recurrence relations

$$\varsigma_{2n+2}^{k,t}(x) = (1 + q_n^{k,t})x\varsigma_{2n+1}^{k,t}(x) - q_n^{k,t}(x^2 + \gamma_{k+t+2n})\varsigma_{2n}^{k,t}(x), \tag{24a}$$

$$\varsigma_{2n+1}^{k,t}(x) = (1 + e_n^{k,t})x\varsigma_{2n}^{k,t}(x) - e_n^{k,t}(x^2 + \gamma_{k+t+2n-1})\varsigma_{2n-1}^{k,t}(x), \tag{24b}$$

and the spectral transformations for the monic symmetric R_{II} polynomials $\{\varsigma_n^{k,t}(x)\}_{n=0}^{\infty}$:

$$(1 + \tilde{q}_n^{k,t})(x^2 + s^{(t)})\varsigma_{2n}^{k,t+1}(x) = \varsigma_{2n+2}^{k,t}(x) + \tilde{q}_n^{k,t}(x^2 + \gamma_{k+t+2n})\varsigma_{2n}^{k,t}(x), \tag{25a}$$

$$(1 + \tilde{q}_n^{k+1,t})(x^2 + s^{(t)})\varsigma_{2n+1}^{k,t+1}(x) = \varsigma_{2n+3}^{k,t}(x) + \tilde{q}_n^{k+1,t}(x^2 + \gamma_{k+t+2n+1})\varsigma_{2n+1}^{k,t}(x), \tag{25b}$$

$$(1 + \tilde{e}_n^{k,t})\varsigma_{2n}^{k,t}(x) = \varsigma_{2n}^{k,t+1}(x) + \tilde{e}_n^{k,t}(x^2 + \gamma_{k+t+2n-1})\varsigma_{2n-2}^{k,t+1}(x), \tag{25c}$$

$$(1 + \tilde{e}_n^{k+1,t})\varsigma_{2n+1}^{k,t}(x) = \varsigma_{2n+1}^{k,t+1}(x) + \tilde{e}_n^{k+1,t}(x^2 + \gamma_{k+t+2n})\varsigma_{2n-1}^{k,t+1}(x). \tag{25d}$$

Relations (24) and (25) show that there exist variables $v_n^{k,t}$ satisfying the relations

$$\begin{aligned}
(\gamma_{k+t+n} &+ v_n^{k,t})(x^2 + s^{(t)})\varsigma_n^{k,t+1}(x) \\
&= (\gamma_{k+t+n} - s^{(t)})x\varsigma_{n+1}^{k,t}(x) + (s^{(t)} + v_n^{k,t})(x^2 + \gamma_{k+t+n})\varsigma_n^{k,t}(x),
\end{aligned} \tag{26a}$$

$$\begin{aligned}
\gamma_{k+t+n}&(s^{(t)} + v_n^{k,t})\varsigma_n^{k,t}(x) \\
&= s^{(t)}(\gamma_{k+t+n} + v_n^{k,t})\varsigma_n^{k,t+1}(x) + (\gamma_{k+t+n} - s^{(t)})v_n^{k,t}x\varsigma_{n-1}^{k,t+1}(x).
\end{aligned} \tag{26b}$$

Relations (26) yield

$$
\begin{aligned}
\varsigma_{n+1}^{k,t}(x) &= \left(1 + (\gamma_{k+t+n-1})^{-1}v_n^{k,t}\frac{1 + \left(s^{(t)}\right)^{-1}v_{n-1}^{k,t}}{1 + (\gamma_{k+t+n-1})^{-1}v_{n-1}^{k,t}}\right)x\varsigma_n^{k,t}(x) \\
&\quad - (\gamma_{k+t+n-1})^{-1}v_n^{k,t}\frac{1 + \left(s^{(t)}\right)^{-1}v_{n-1}^{k,t}}{1 + (\gamma_{k+t+n-1})^{-1}v_{n-1}^{k,t}}\left(x^2 + \gamma_{k+t+n-1}\right)\varsigma_{n-1}^{k,t}(x) \\
&= \left(1 + (\gamma_{k+t+n-1})^{-1}v_n^{k,t-1}\frac{1 + \left(s^{(t-1)}\right)^{-1}v_{n+1}^{k,t-1}}{1 + (\gamma_{k+t+n})^{-1}v_{n+1}^{k,t-1}}\right)x\varsigma_n^{k,t}(x) \\
&\quad - (\gamma_{k+t+n-1})^{-1}v_n^{k,t-1}\frac{1 + \left(s^{(t-1)}\right)^{-1}v_{n+1}^{k,t-1}}{1 + (\gamma_{k+t+n})^{-1}v_{n+1}^{k,t-1}}\left(x^2 + \gamma_{k+t+n-1}\right)\varsigma_{n-1}^{k,t}(x).
\end{aligned}
$$

Hence, the compatibility condition

$$
v_n^{k,t}\frac{1 + \left(s^{(t)}\right)^{-1}v_{n-1}^{k,t}}{1 + (\gamma_{k+t+n-1})^{-1}v_{n-1}^{k,t}} = v_n^{k,t-1}\frac{1 + \left(s^{(t-1)}\right)^{-1}v_{n+1}^{k,t-1}}{1 + (\gamma_{k+t+n})^{-1}v_{n+1}^{k,t-1}}, \tag{27a}
$$

$$
v_0^{k,t} = 0 \qquad \text{for all } k \text{ and } t, \tag{27b}
$$

must be satisfied. This is the time evolution equation of the semi-infinite nd-mKdV lattice, a nonautonomous version of the discrete mKdV lattice [32]. Note that the nd-mKdV lattice (27) reduces to the nd-LV lattice (17) as $\gamma_{k+t+n} \to \infty$.

The Miura transformation between the monic-type R$_{\text{II}}$ chain (20) and the nd-mKdV lattice (27) is obtained as follows:

$$
\begin{aligned}
q_n^{k,t} &= (\gamma_{k+t+2n})^{-1}v_{2n+1}^{k,t}\frac{1 + \left(s^{(t)}\right)^{-1}v_{2n}^{k,t}}{1 + (\gamma_{k+t+2n})^{-1}v_{2n}^{k,t}} = (\gamma_{k+t+2n})^{-1}v_{2n+1}^{k,t-1}\frac{1 + \left(s^{(t-1)}\right)^{-1}v_{2n+2}^{k,t-1}}{1 + (\gamma_{k+t+2n+1})^{-1}v_{2n+2}^{k,t-1}}, \\
e_n^{k,t} &= (\gamma_{k+t+2n-1})^{-1}v_{2n}^{k,t}\frac{1 + \left(s^{(t)}\right)^{-1}v_{2n-1}^{k,t}}{1 + (\gamma_{k+t+2n-1})^{-1}v_{2n-1}^{k,t}} \\
&= (\gamma_{k+t+2n-1})^{-1}v_{2n}^{k,t-1}\frac{1 + \left(s^{(t-1)}\right)^{-1}v_{2n+1}^{k,t-1}}{1 + (\gamma_{k+t+2n})^{-1}v_{2n+1}^{k,t-1}}, \\
\tilde{q}_n^{k,t} &= \left(\gamma_{k+t+2n} - s^{(t)}\right)^{-1}s^{(t)}\left(1 + \left(s^{(t)}\right)^{-1}v_{2n+1}^{k,t}\right)\left(1 + \left(s^{(t)}\right)^{-1}v_{2n}^{k,t}\right) \\
&= \left(\gamma_{k+t+2n} - s^{(t)}\right)^{-1}s^{(t)}\left(1 + \left(s^{(t)}\right)^{-1}v_{2n+1}^{k-1,t}\right)\left(1 + \left(s^{(t)}\right)^{-1}v_{2n+2}^{k-1,t}\right), \\
\tilde{e}_n^{k,t} &= \frac{\left(\gamma_{k+t+2n} - s^{(t)}\right)v_{2n}^{k,t}v_{2n-1}^{k,t}}{s^{(t)}\gamma_{k+t+2n-1}\gamma_{k+t+2n}\left(1 + (\gamma_{k+t+2n})^{-1}v_{2n}^{k,t}\right)\left(1 + (\gamma_{k+t+2n-1})^{-1}v_{2n-1}^{k,t}\right)} \\
&= \frac{\left(\gamma_{k+t+2n} - s^{(t)}\right)v_{2n}^{k-1,t}v_{2n+1}^{k-1,t}}{s^{(t)}\gamma_{k+t+2n-1}\gamma_{k+t+2n}\left(1 + (\gamma_{k+t+2n-1})^{-1}v_{2n}^{k-1,t}\right)\left(1 + (\gamma_{k+t+2n})^{-1}v_{2n+1}^{k-1,t}\right)}.
\end{aligned}
$$

Furthermore, from relations (26), we obtain

$$
\frac{1 + \left(s^{(t)}\right)^{-1}v_{2n}^{k,t}}{1 + (\gamma_{k+t+2n})^{-1}v_{2n}^{k,t}} = \frac{\varphi_n^{k,t+1}(0)}{\varphi_n^{k,t}(0)},
$$

$$
1 + \left(s^{(t)}\right)^{-1}v_{2n-1}^{k,t} = (-s^{(t)})^{-1}\frac{\varphi_n^{k,t}(-s^{(t)})}{\varphi_{n-1}^{k+1,t}(-s^{(t)})},
$$

$$
1 + (\gamma_{k+t+2n-1})^{-1}v_{2n-1}^{k,t} = (-\gamma_{k+t+2n-1})^{-1}\frac{\varphi_n^{k,t}(-\gamma_{k+t+2n-1})}{\varphi_{n-1}^{k+1,t+1}(-\gamma_{k+t+2n-1})}.
$$

Hence, Hankel determinant solutions to the nd-mKdV lattice (27) are given by

$$
v_{2n+1}^{k,t} = \gamma_{k+t+2n} q_n^{k,t} \frac{\varphi_n^{k,t}(0)}{\varphi_n^{k,t+1}(0)} = \frac{\tau_n^{k,2n+1,t+1} \tau_{n+1}^{k+1,2n+1,t}}{\tau_{n+1}^{k,2n+2,t} \tau_n^{k+1,2n-1,t+1}},
$$

$$
v_{2n}^{k,t} = s^{(t)} e_n^{k,t} \frac{\varphi_{n-1}^{k+1,t}(-s^{(t)})}{\varphi_n^{k,t}(-s^{(t)})} \frac{\varphi_n^{k,t}(-\gamma_{k+t+2n-1})}{\varphi_{n-1}^{k+1,t+1}(-\gamma_{k+t+2n-1})} = s^{(t)} \gamma_{k+t+2n} \frac{\tau_{n+1}^{k,2n+1,t} \tau_{n-1}^{k+1,2n-2,t+1}}{\tau_n^{k,2n-1,t+1} \tau_n^{k+1,2n,t}}.
$$

Remark 1. Spiridonov [29] first considered spectral transformations for the (not monic) symmetric R_{II} polynomials. By using spectral transformations, he derived a generalization of the nd-LV lattice (17), which is more complicated than the nd-mKdV lattice (27). We have considered the monic symmetric R_{II} polynomials and their spectral transformations which possess the following symmetry. Consider an independent variable transformation $t' = -k - t - n$ and introduce $\tilde{\varsigma}_n^{k,t'}(x) := \varsigma_n^{k,-k-t'-n+1}(x)$, $\tilde{v}_n^{k,t'} := v_n^{k,-k-t'-n}$, $\tilde{s}_{k+t'+n} := s^{(-k-t'-n)}$ and $\tilde{\gamma}^{(t')} := \gamma_{-t'}$. Then, the spectral transformations for the monic symmetric R_{II} polynomials (26) may be rewritten as

$$
\left(\tilde{s}_{k+t'+n} + \tilde{v}_n^{k,t'}\right)\left(x^2 + \tilde{\gamma}^{(t')}\right)\tilde{\varsigma}_n^{k,t'+1}(x)
$$
$$
= \left(\tilde{s}_{k+t'+n} - \tilde{\gamma}^{(t')}\right)x\tilde{\varsigma}_{n+1}^{k,t'}(x) + \left(\tilde{\gamma}^{(t')} + \tilde{v}_n^{k,t'}\right)\left(x^2 + \tilde{s}_{k+t'+n}\right)\tilde{\varsigma}_n^{k,t'}(x),
$$
$$
\tilde{s}_{k+t'+n}\left(\tilde{\gamma}^{(t')} + \tilde{v}_n^{k,t'}\right)\tilde{\varsigma}_n^{k,t'}(x)
$$
$$
= \tilde{\gamma}^{(t')}\left(\tilde{s}_{k+t'+n} + \tilde{v}_n^{k,t'}\right)\tilde{\varsigma}_n^{k,t'+1}(x) + \left(\tilde{s}_{k+t'+n} - \tilde{\gamma}^{(t')}\right)\tilde{v}_n^{k,t'} x\tilde{\varsigma}_{n-1}^{k,t'+1}(x),
$$

so that the roles of the parameters are replaced. Using the symmetric form of the spectral transformations (26), we can derive the corresponding discrete integrable system in a simpler form.

In another study, Spiridonov et al. [30] derived a discrete integrable system called the FST chain and discussed its connection to the R_{II} chain. The time evolution equation of the FST chain is

$$
\frac{\gamma_{k+t+n} - s^{(t)} + A_n^{k,t} A_{n-1}^{k,t}}{A_n^{k,t}} = \frac{\gamma_{k+t+n-1} - s^{(t-1)} + A_n^{k,t-1} A_{n+1}^{k,t-1}}{A_n^{k,t-1}},
$$
$$
A_{-1}^{k,t} = 0 \qquad \text{for all } k \text{ and } t.
$$

Particular solutions to the FST chain may also be expressed by the Hankel determinant (23):

$$
A_{2n}^{k,t} = \left(\gamma_{k+t+2n} - s^{(t)}\right)\frac{\tau_{n+1}^{k,2n+1,t} \tau_n^{k,2n-1,t+1}}{\tau_n^{k,2n,t} \tau_n^{k,2n,t+1}},
$$
$$
A_{2n+1}^{k,t} = \left(\gamma_{k+t+2n+1} - s^{(t)}\right)\frac{\tau_{n+1}^{k,2n+2,t} \tau_n^{k,2n,t+1}}{\tau_{n+1}^{k,2n+1,t} \tau_{n+1}^{k,2n+1,t+1}}.
$$

Similarly, we have the discrete potential KdV lattice

$$
\left(\epsilon_n^{k,t-1} - \epsilon_n^{k,t}\right)\left(\epsilon_{n+1}^{k,t-1} - \epsilon_{n-1}^{k,t}\right) = \gamma_{k+t+n-1} - s^{(t-1)},
$$
$$
\epsilon_{-1}^{k,t} = 0 \qquad \text{for all } k \text{ and } t,
$$

and its Hankel determinant solutions

$$
\epsilon_{2n}^{k,t} = \frac{\tau_{n+1}^{k,2n,t}}{\tau_n^{k,2n,t}}, \qquad \epsilon_{2n+1}^{k,t} = \frac{\tau_n^{k,2n+1,t}}{\tau_{n+1}^{k,2n+1,t}}.
$$

Therefore, these systems and the nd-mKdV lattice (27) are connected via the bilinear formalism.

4 Concluding remarks

In this paper, we developed the spectral transformation technique for symmetric R$_{II}$ polynomials and derived the nd-mKdV lattice as the compatibility condition. Moreover, we obtained a direct connection between the R$_{II}$ chain and the nd-mKdV lattice. It is easily verified by numerical experiments that the obtained nd-mKdV lattice with a non-periodic finite lattice condition can compute the generalized eigenvalues of the tridiagonal matrix pencil that corresponds to the R$_{II}$ polynomials through the Miura transformation. More practical applications of the nd-mKdV lattice to numerical algorithms are left for future work. In particular, the application to generalized singular value decomposition [36] will be discussed in detail.

In recent studies, various discrete Painlevé equations have been obtained as reductions of discrete integrable systems [6, 7, 20, 21, 22, 23]. On the other hand, it is known that the R$_{II}$ chain and the elliptic Painlevé equation [25] have solutions expressible in terms of the elliptic hypergeometric function $_{10}E_9$ [11, 28]. In addition, it was pointed out that the contiguity relations of the elliptic Painlevé equation are similar to the linear relations of the R$_{II}$ chain [19]. Supported by these evidences, one may believe that a reduction of the R$_{II}$ chain may give rise to the elliptic Painlevé equation. This work linked the nd-mKdV lattice with the R$_{II}$ chain. We are now concerned with its relationship to the discrete Painlevé equations. In particular, we expect that the elliptic Painlevé equation will appear as a reduction of the nd-mKdV lattice.

Acknowledgements

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