

Dispersionless BKP Hierarchy and Quadrant Löwner Equation^{*}

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Received August 23, 2013, in final form March 10, 2014; Published online March 14, 2014

<http://dx.doi.org/10.3842/SIGMA.2014.023>

Abstract. We show that N -variable reduction of the dispersionless BKP hierarchy is described by a Löwner type equation for the quadrant.

Key words: dBKP hierarchy; quadrant Löwner equation; N -variable reduction

2010 Mathematics Subject Classification: 37K10; 37K20; 30C55

1 Introduction

Since an unexpected connection of the dispersionless KP hierarchy (the Benney equations) to the chordal Löwner equation was found in the seminal paper [9], similar examples have been found by [14, 15, 25, 26] and others. The dispersionless integrable hierarchies are obtained as quasi-classical limits of “dispersionful” integrable hierarchies and thus the Lax operators of the latter are replaced by Lax “functions” (see, for example, [23]).

On the other hand, Löwner type equations are differential equations characterising one-parameter families of conformal mappings between families of domains with growing slits and fixed reference domains (the upper half plane for the chordal Löwner equation and the unit disk or its exterior for the radial Löwner equation). For details of the Löwner equations in the context of complex analysis we refer to [1, 7, 19].

The key point of the “unexpected” relation is that *the Lax functions satisfying the one-variable reduction condition are solutions of the Löwner type equations*. In order to explain more precisely, let us recall the first and most typical example found in [9]: the relation between the dispersionless KP hierarchy and the chordal Löwner equation. We follow formulation in [26]. Suppose that $\mathcal{L}_{\text{KP}}(\tilde{w}; t)$ is a solution of the dispersionless KP hierarchy, which depends on $t = (t_1, t_2, t_3, \dots)$ only through a single variable λ . Namely, there exists a function $f_{\text{KP}}(\tilde{w}; \lambda) = \sum_{n=0}^{\infty} u_n(\lambda) \tilde{w}^{1-n}$ ($u_0 = 1, u_1 = 0$) of two variables (\tilde{w}, λ) and a function $\lambda(t)$ of t such that

$$\mathcal{L}_{\text{KP}}(\tilde{w}; t) = f_{\text{KP}}(\tilde{w}; \lambda(t)) = \sum_{n=0}^{\infty} u_n(\lambda(t)) \tilde{w}^{1-n} = \tilde{w} + u_2(\lambda(t)) \tilde{w}^{-1} + u_3(\lambda(t)) \tilde{w}^{-2} + \dots$$

Then the inverse function $g_{\text{KP}}(\tilde{z}; \lambda)$ of $f_{\text{KP}}(\tilde{w}; \lambda)$ in \tilde{w} satisfies the chordal Löwner equation¹ (5.2) ($\tilde{g} = g_{\text{KP}}, \tilde{u} = u_2$) with respect to λ . There is also a way back and we can construct a solution of the dispersionless KP hierarchy from a solution of the chordal Löwner equation. Similar connections of the dispersionless mKP/Toda hierarchies, the Dym hierarchy, the universal Whitham hierarchy to the various Löwner type equations were found in [14, 15, 24, 25, 26].

^{*}This paper is a contribution to the Special Issue in honor of Anatol Kirillov and Tetsuji Miwa. The full collection is available at <http://www.emis.de/journals/SIGMA/InfiniteAnalysis2013.html>

¹The chordal Löwner equation was first introduced in [12] and rediscovered independently by Gibbons and Tsarev in [9] and by Schramm in [20].

The goal of the present paper is to add another example: we show that, in a similar sense, the one-variable reduction of the dispersionless BKP hierarchy, a variant of the dispersionless KP hierarchy, is characterised by the quadrant Löwner equation (3.2), which is satisfied by a family of conformal mappings from a quadrant slit domain to the quadrant. It is desirable to explain fundamental reason of such relation between dispersionless integrable hierarchies and Löwner type equations in complex analysis.

The other direction, namely, construction of a solution of the dispersionless BKP hierarchy from a solution of the quadrant Löwner type equation, can be generalised to the N -variable case. The celebrated Gibbons–Tsarev system (see, for example, [8, 9, 16, 17]) arises as the compatibility condition of the parameter functions in the quadrant Löwner equation. Our Gibbons–Tsarev system is a particular case in [18] and analysed there in more general context.

Let us comment several differences from previously known results, which are due to our restricting ourselves to the dispersionless BKP hierarchy specifically and not studying general hydrodynamic type integrable systems comprehensively. (Some of the comments also apply to the reduction of similar systems, which we studied earlier in [24, 25, 26].)

- We use the relation of the Grunsky coefficients and the dispersionless tau function (2.7) systematically as in [24, 25, 26]. We believe that this makes proofs transparent.
- The quadrant Löwner equation (3.2) can be obtained if we put $q = g^2$, $B^0 = 2u$ and $p_i = V_i$ in (3.1) of [18]². Instead of studying more general equations in [18], we identify our particular equation (3.2) as the Löwner type equation for the conformal mappings of quadrant domains, which makes it possible to construct an explicit example of solutions by means of complex analysis in Section 6.
- Although Tsarev’s generalised hodograph method is mostly applied to the (1+1)- or (2+1)-dimensional systems (for example, in [16, 17, 28]), there is no need to restrict the number of the independent variables of the dispersionless hierarchy. We show in this paper that the compatibility condition for Tsarev’s generalised hodograph method is satisfied for any independent variables of the dispersionless BKP hierarchy *simultaneously*. See Lemma 4.1 and its proof for details. This also follows from the method by Kodama and Gibbons [11].

This paper is organised as follows. In Section 2 we review the results on the dispersionless BKP hierarchy. Using its dispersionless Hirota equation, we prove that the one-variable reduction of the dispersionless BKP hierarchy is described by the quadrant Löwner equation in Section 3. Conversely we prove in Section 4 that a solution of the dispersionless BKP hierarchy can be constructed from a function depending on N parameters satisfying the quadrant Löwner equations. In Section 5 we explain that the quadrant Löwner equation is obtained by “folding” the chordal Löwner equation by the square root, $w = \sqrt{\tilde{w} - 2u(\lambda)}$, where $2u(\lambda)$ is the suitably chosen centre of the folding. This shows that the quadrant Löwner equation is satisfied by slit mappings between quadrant domains. The last Section 6 is devoted to the simplest non-trivial example.

2 Dispersionless BKP hierarchy

The dispersionless BKP hierarchy (the dBKP hierarchy for short) was first introduced probably by Kupershmidt in [13] as a special case of the Kupershmidt hydrodynamic chain. The formulation of the dBKP hierarchy as a dispersionless limit of the BKP hierarchy by Date, Kashiwara and Miwa [5], which is relevant to us, was due to Takasaki [21, 22] and subsequently studied by Bogdanov and Konopelchenko [3], Chen and Tu [4]. The theory can be developed almost in

²The author is much grateful to one of the referees of the first version of the present paper for pointing this out to him.

the same way as that of the dispersionless KP hierarchy in [23]. We refer further developments from the viewpoint of the Kupersmidt hydrodynamic chain to [2, 13] and [18, especially § 10] (and references therein).

Here we briefly review necessary facts on the dBKP hierarchy, following mainly [21, 22]. Let $\mathcal{L}(w; t)$ be a (formal) Laurent series in w of the form

$$\mathcal{L}(w; t) = w + u_1(t)w^{-1} + u_2(t)w^{-3} + \cdots = \sum_{n=0}^{\infty} u_n(t)w^{1-2n}, \quad u_0 = 1, \quad (2.1)$$

the coefficients of which depend on $t = (t_1, t_3, t_5, \dots)$. We identify another independent variable x with t_1 . The truncation operation $(\cdot)_{\geq 0}$ of the Laurent series is defined by

$$\left(\sum_{n \in \mathbb{Z}} a_n w^n \right)_{\geq 0} = \sum_{n \geq 0} a_n w^n.$$

The Poisson bracket $\{a(w; x), b(w; x)\}$ is defined in the same way as the dispersionless KP hierarchy

$$\{a(w; x), b(w; x)\} = \frac{\partial a}{\partial w} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial w}.$$

The *dispersionless BKP hierarchy* is the system of Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad n \text{ is odd}, \quad (2.2)$$

where $\mathcal{B}_n := (\mathcal{L}^n)_{\geq 0}$. We call \mathcal{L} the *Lax function* of the dBKP hierarchy.

What we need later is the dispersionless Hirota equation of the tau function, which is equivalent to the above Lax representation. In order to define the tau function and also for later use, we introduce several terms in complex analysis. (See [7, 19] for details in the context of complex analysis.)

The *Faber polynomials* $\Phi_n(z)$ and the *Grunsky coefficients* b_{mn} are defined for a function (or a Laurent series³) of the form

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

by the generating functions

$$\begin{aligned} \log \frac{g(z) - w}{z} &= - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Phi_n(w), \\ \log \frac{g(z) - g(w)}{z - w} &= - \sum_{m, n=1}^{\infty} b_{mn} z^{-m} w^{-n}. \end{aligned} \quad (2.3)$$

The Faber polynomial $\Phi_n(w)$ is a monic n -th order polynomial in w , coefficients of which are polynomials in b_k 's. The Grunsky coefficient b_{mn} is a polynomial in b_k 's and symmetric in indices m and n . They are connected by the relation

$$\Phi_n(g(z)) = z^n + n \sum_{m=0}^{\infty} b_{nm} z^{-m}. \quad (2.4)$$

³Throughout this article, we use the word "function" for a formal series.

When $g(z)$ is an odd function, i.e., $b_{\text{even}} = 0$, we have

$$-\sum_{n=1}^{\infty} \frac{(-z)^{-n}}{n} \Phi_n(w) = -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Phi_n(-w)$$

by setting $z \mapsto -z$ or $w \mapsto -w$ in (2.3). Hence the Faber polynomial $\Phi_n(w)$ is an odd (resp. even) polynomial in w for odd (resp. even) n , and consequently $\Phi_n(g(z))$ is an odd (resp. even) function in z . Therefore the relation (2.4) shows that $b_{nm} = 0$ if n and m are of different parity.

We can pick up the Faber polynomials and the Grunsky coefficients with odd indices by the following generating functions

$$\log \frac{g(z) - w}{g(z) + w} = -2 \sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \Phi_n(w), \quad (2.5)$$

$$\log \frac{g(z) - g(w)}{z - w} - \log \frac{g(z) + g(w)}{z + w} = -2 \sum_{1 \leq m, n, \text{ odd}} b_{mn} z^{-m} w^{-n}. \quad (2.6)$$

(The proof is straightforward computation.)

In these terms the dispersionless BKP hierarchy is stated in the following way.

Proposition 2.1. *Let $\mathcal{L}(w; t)$ be a Laurent series of the form (2.1) and $k(z; t) = z + v_1(t)z^{-1} + v_2(t)z^{-3} + \dots$ be its inverse function in the first variable: $\mathcal{L}(k(z; t); t) = z$, $k(\mathcal{L}(w; t); t) = w$.*

The function $\mathcal{L}(w; t)$ is a solution of the dBKP hierarchy (2.2) if and only if there exists a function $F(t)$ such that the Grunsky coefficients $b_{mn}(t)$ of $k(z; t)$ are expressed as

$$b_{mn}(t) = -\frac{2}{mn} \frac{\partial^2}{\partial t_m \partial t_n} F(t), \quad (2.7)$$

for any odd m and n .

The function $F(t)$ is called the *free energy* or the logarithm of the *tau function* $\tau_{\text{dBKP}}(t)$: $F(t) = \log \tau_{\text{dBKP}}(t)$. The proof of the proposition is the same as the corresponding statements for the dispersionless KP hierarchy in § 3.3 of [27].

Since the first Faber polynomial $\Phi_1(w; t)$ of $k(z; t)$ is w , the relation (2.4) means

$$k(z; t) = z + \sum_{1 \leq m, \text{ odd}} b_{1m}(t) z^{-m}. \quad (2.8)$$

Substituting this expression into (2.6), the above condition (2.7) is equivalent to the following equation

$$4D(z_1)D(z_2)F(t) = \log \left(\frac{(z_1 - 2\partial_{t_1} D(z_1)F(t)) - (z_2 - 2\partial_{t_1} D(z_2)F(t))}{z_1 - z_2} \right) - \log \left(\frac{(z_1 - 2\partial_{t_1} D(z_1)F(t)) + (z_2 - 2\partial_{t_1} D(z_2)F(t))}{z_1 + z_2} \right), \quad (2.9)$$

where the operator $D(z)$ is defined by

$$D(z) = \sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n}.$$

Equation (2.9) is the *dispersionless Hirota equation* for the dBKP hierarchy first obtained in [3] (see also [22]).

3 From the dBKP hierarchy to the quadrant Löwner equation

One of the main theorems of the present paper is the following.

Theorem 3.1. *Suppose that $\mathcal{L}(w; t)$ is a solution of the dBKP hierarchy whose dependence on $t = (t_1, t_3, \dots)$ is only through a single variable λ . Namely, there exists a function $f(w; \lambda)$ of w and λ of the form*

$$f(w; \lambda) = \sum_{n=0}^{\infty} u_n(\lambda) w^{1-2n}, \quad u_0 = 1$$

and a function $\lambda(t)$ of t such that

$$\mathcal{L}(w; t) = f(w; \lambda(t)).$$

We assume $du_1/d\lambda \neq 0$ and $\partial\lambda/\partial t_1 \neq 0$. Let a series $g(z; \lambda)$ of the form

$$g(z; \lambda) = \sum_{n=0}^{\infty} v_n(\lambda) z^{1-2n}, \quad v_0 = 1, \quad v_1 = -u_1 \quad (3.1)$$

be the inverse function of $f(w; \lambda)$ in the first variable: $g(f(w; \lambda); \lambda) = w$, $f(g(z; \lambda); \lambda) = z$. Then $g(z; \lambda)$ satisfies the following equation with respect to λ

$$\frac{\partial g}{\partial \lambda} = \frac{g}{V^2 - g^2} \frac{du}{d\lambda}. \quad (3.2)$$

Here $V = V(\lambda)$ is a function of λ , not depending on z , and $u(\lambda) = u_1(\lambda) = -v_1(\lambda)$.

The function $\lambda(t)$ is characterised by the following system

$$\frac{\partial \lambda}{\partial t_n} = \chi_n(\lambda) \frac{\partial \lambda}{\partial t_1} \quad \text{for odd } n. \quad (3.3)$$

The coefficient $\chi_n(\lambda)$ is defined by $\chi_n(\lambda) := \Phi'_n(V(\lambda); \lambda)$, where $\Phi_n(w; \lambda)$ is the n -th Faber polynomial of $g(z; \lambda)$ and $\Phi'_n = \partial\Phi_n/\partial w$.

We call the equation (3.2) the *quadrant Löwner equation* by the reason which we shall explain in Section 5. As in the case of the chordal Löwner equation, we call the function $V(\lambda)$ the *driving function*.

Remark 3.1. As is mentioned in Section 1, the quadrant Löwner equation (3.2) can be obtained if we put $q = g^2$, $B^0 = 2u$ and $p_i = V_i$ in (3.1) of [18]. Correspondingly, the system (3.3) for $n = 3$ is (2.20) of [18] ($\beta = 2$, $t_1 \mapsto x$, $t_3 \mapsto t$). The author thanks the referee for this reference.

Proof. We follow the proof of Proposition 5.1 of [26]. In this situation $g(z; t)$ is $k(z; t)$ in (2.8) and expressed by the free energy as

$$g(z; t) = z - 2\partial_{t_1} D(z)F.$$

Therefore $D(z_1)g(z_2; t) = D(z_2)g(z_1; t)$, from which follows $-D(z)u(\lambda(t)) = \partial_{t_1} g(z; \lambda(t)) = \partial_{\lambda} g(z; \lambda) \partial_{t_1} \lambda(t)$. Thus

$$D(z)\lambda(t) = -\partial_{\lambda} g(z; \lambda) \frac{\partial_{t_1} \lambda(t)}{\partial_{\lambda} u(\lambda(t))}. \quad (3.4)$$

Since $2D(z)\partial_{t_1}F(t) = z - g(z; \lambda(t))$, the above equation and the dispersionless Hirota equation (2.9) differentiated by t_1 imply

$$2\partial_\lambda g(z_1; \lambda)\partial_\lambda g(z_2; \lambda) = \left(\frac{\partial_\lambda g(z_1; \lambda) - \partial_\lambda g(z_2; \lambda)}{g(z_1; \lambda) - g(z_2; \lambda)} - \frac{\partial_\lambda g(z_1; \lambda) + \partial_\lambda g(z_2; \lambda)}{g(z_1; \lambda) + g(z_2; \lambda)} \right) \frac{du}{d\lambda}.$$

This can be rewritten as

$$g(z_1; \lambda)^2 + \frac{g(z_1; \lambda)}{\partial_\lambda g(z_1; \lambda)} \frac{du}{d\lambda} = g(z_2; \lambda)^2 + \frac{g(z_2; \lambda)}{\partial_\lambda g(z_2; \lambda)} \frac{du}{d\lambda},$$

which means that both-hand sides do not depend neither on z_1 nor on z_2 . Defining a function $V(\lambda)$ of λ by

$$V(\lambda)^2 := g(z; \lambda)^2 + \frac{g(z; \lambda)}{\partial_\lambda g(z; \lambda)} \frac{du}{d\lambda},$$

we obtain the equation (3.2). Substituting it into (3.4), we have

$$\sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \frac{\partial \lambda}{\partial t_n} = \frac{g(z; \lambda(t))}{g(z; \lambda(t))^2 - V(\lambda(t))^2} \frac{\partial \lambda}{\partial t_1}. \quad (3.5)$$

Since the derivative of (2.5) with respect to w gives

$$\frac{g(z)}{g(z)^2 - w^2} = \sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \Phi'_n(w), \quad (3.6)$$

the right-hand side of (3.5) is equal to

$$\sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \Phi'_n(V(\lambda(t)); \lambda(t)) \frac{\partial \lambda}{\partial t_1}.$$

Hence equation (3.3) follows from (3.5). ■

4 From the quadrant Löwner equation to the dBKP hierarchy

In this section we prove the converse statement of Theorem 3.1 in a generalised form (N -variable diagonal reduction).

Let $f(w; \boldsymbol{\lambda})$ be a function of the following form

$$f(w; \boldsymbol{\lambda}) = \sum_{n=0}^{\infty} u_n(\boldsymbol{\lambda}) w^{1-2n}, \quad u_0 = 1,$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ is a set of N variables. We denote its inverse function in w by $g(z; \boldsymbol{\lambda})$: $f(g(z; \boldsymbol{\lambda}); \boldsymbol{\lambda}) = z$, $g(f(w; \boldsymbol{\lambda}); \boldsymbol{\lambda}) = w$.

Let us consider a system of differential equations for $g(z; \boldsymbol{\lambda})$ of the type (3.2) in each variable λ_i , namely, the quadrant Löwner equations

$$\frac{\partial g}{\partial \lambda_i}(z; \boldsymbol{\lambda}) = \frac{g(z; \boldsymbol{\lambda})}{V_i^2(\boldsymbol{\lambda}) - g^2(z; \boldsymbol{\lambda})} \frac{\partial u}{\partial \lambda_i}(\boldsymbol{\lambda}), \quad (4.1)$$

with a driving function V_i for $i = 1, \dots, N$, where $u = u_1$. Or equivalently we consider a system of linear partial differential equations for f

$$\frac{\partial f}{\partial \lambda_i}(w; \boldsymbol{\lambda}) = A_i f(w; \boldsymbol{\lambda}), \quad A_i := -\frac{w}{V_i^2(\boldsymbol{\lambda}) - w^2} \frac{\partial u}{\partial \lambda_i}(\boldsymbol{\lambda}) \frac{\partial}{\partial w}. \quad (4.2)$$

When $N \geq 2$, the compatibility conditions $[\partial_{\lambda_i} - A_i, \partial_{\lambda_j} - A_j] = 0$ of this linear system boil down to the following equations for V_i (or V_i^2) and u

$$\frac{\partial V_i^2}{\partial \lambda_j} = \frac{2V_i^2}{V_j^2 - V_i^2} \frac{\partial u}{\partial \lambda_j}, \quad (4.3)$$

$$\frac{\partial^2 u}{\partial \lambda_i \partial \lambda_j} = \frac{2(V_i^2 + V_j^2)}{(V_i^2 - V_j^2)^2} \frac{\partial u}{\partial \lambda_i} \frac{\partial u}{\partial \lambda_j} \quad (4.4)$$

for $i, j = 1, \dots, N, i \neq j$. This is the *Gibbons–Tsarev system* in our case, which is well-known in the literature. (See the remark below.) We assume that these conditions hold and that g and f satisfy (4.1) and (4.2) respectively.

Remark 4.1. This Gibbons–Tsarev system is a specialisation of (3.2) of [18]. For general reference of the Gibbons–Tsarev systems, see [8, 9, 16, 17]. The author thanks the referee for these references.

The converse statement to Theorem 3.1 is the following.

Theorem 4.1. *Suppose each $\lambda_i(t)$ satisfies the following analogue of (3.3) for all odd n*

$$\frac{\partial \lambda_i}{\partial t_n}(t) = \chi_{i,n}(\boldsymbol{\lambda}(t)) \frac{\partial \lambda_i}{\partial t_1}(t), \quad (4.5)$$

where we define the function $\chi_{i,n}(\boldsymbol{\lambda})$ by $\chi_{i,n}(\boldsymbol{\lambda}) := \Phi'_n(V_i(\boldsymbol{\lambda}); \boldsymbol{\lambda})$, using the derivative $\Phi'_n = \partial \Phi_n / \partial w$ of the n -th Faber polynomial $\Phi_n(w; \boldsymbol{\lambda})$ of $g(z; \boldsymbol{\lambda})$.

Then $\mathcal{L}(w; t) := f(w; \boldsymbol{\lambda}(t))$ is a solution of the dBKP hierarchy.

This is the so-called *diagonal N -variable reduction*. We shall discuss the solvability of (4.5) later.

Remark 4.2. We can consider more general N -variable reduction, changing the equation (4.1) to

$$\frac{\partial g}{\partial \lambda_i}(z; \boldsymbol{\lambda}) = R_i(g(z; \boldsymbol{\lambda}); \boldsymbol{\lambda})$$

and correspondingly the equation (4.2) to

$$\frac{\partial f}{\partial \lambda_i}(w; \boldsymbol{\lambda}) = -R_i(w; \boldsymbol{\lambda}) \frac{\partial f}{\partial w}(w; \boldsymbol{\lambda}), \quad (4.6)$$

where the coefficient $R_i(w; \boldsymbol{\lambda})$ is an odd rational function with $2N$ simple poles at $w = \pm V_j(\boldsymbol{\lambda})$ ($j = 1, \dots, N$) which vanishes at ∞ : $R_i(w; \boldsymbol{\lambda}) \rightarrow 0$ ($w \rightarrow \infty$). In other words,

$$R_i(w; \boldsymbol{\lambda}) = \sum_{j=1}^N \frac{\rho_{ij}(\boldsymbol{\lambda}) w}{V_j^2(\boldsymbol{\lambda}) - w^2}.$$

Because of the equation (4.6), $V_i(\boldsymbol{\lambda})$ is a critical point of f under the genericity assumption that f is holomorphic around $w = V_i$. As in [15, § 3.1] (or in [24, § 6]), if we change the coordinates $(\lambda_1, \dots, \lambda_N)$ to the critical values $\zeta_i := f(V_i(\boldsymbol{\lambda}); \boldsymbol{\lambda})$ of f ($i = 1, \dots, N$), then the system (4.6) reduces to the diagonal reduction case (4.2) with respect to $(\zeta_1, \dots, \zeta_N)$. In this context ζ_i 's are the Riemann invariants.

Proof of Theorem 4.1. This is proved in the same way as Proposition 5.5 of [26]. From (2.6) it follows that

$$\begin{aligned} & -2 \sum_{1 \leq m, n, \text{ odd}} \frac{\partial b_{mn}}{\partial t_k} z_1^{-m} z_2^{-n} \\ &= \sum_{i=1}^N \frac{2g(z_1; \boldsymbol{\lambda}(t))g(z_2; \boldsymbol{\lambda}(t))}{(V(\boldsymbol{\lambda}(t))^2 - g(z_1; \boldsymbol{\lambda}(t))^2)(V(\boldsymbol{\lambda}(t))^2 - g(z_2; \boldsymbol{\lambda}(t))^2)} \frac{\partial u}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial t_k} \end{aligned} \quad (4.7)$$

under the assumption (4.1). The right-hand side is rewritten as

$$2 \sum_{i=1}^N \sum_{1 \leq m, n, \text{ odd}} \frac{z_1^{-m} z_2^{-n}}{m n} \chi_{i,m}(\lambda(t)) \chi_{i,n}(\lambda(t)) \chi_{i,k}(\lambda(t)) \frac{\partial u}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial t_1}$$

by virtue of (3.6) and (4.5). Therefore (4.7) implies that $mn \partial b_{mn} / \partial t_k$ is symmetric in (m, n, k) , from which follows the existence of a function $F(t)$ satisfying (2.7). According to Proposition 2.1, $\mathcal{L}(w; t) = f(w; \boldsymbol{\lambda}(t))$ is a solution of the dBKP hierarchy. \blacksquare

The system of the first order partial differential equations (4.5) can be solved by Tsarev's generalised hodograph method [28].

Lemma 4.1. Consider the following system for $R_i = R_i(\boldsymbol{\lambda})$, $i = 1, \dots, N$:

$$\frac{\partial R_i}{\partial \lambda_j} = \Gamma_{ij}(R_j - R_i), \quad i, j = 1, \dots, N, \quad i \neq j, \quad (4.8)$$

where Γ_{ij} is defined by

$$\Gamma_{ij} := \frac{V_i^2 + V_j^2}{(V_i^2 - V_j^2)^2} \frac{\partial u}{\partial \lambda_i}.$$

(When $N = 1$, the condition (4.8) is void.)

(i) The system (4.8) is compatible in the sense of [28, § 3].

(ii) Assume that $R_i(\boldsymbol{\lambda})$ satisfy the system (4.8). If $\lambda(t)$ is defined implicitly by the hodograph relation

$$t_1 + \sum_{1 \leq n, \text{ odd}} \chi_{i,n}(\boldsymbol{\lambda}) t_n = R_i(\boldsymbol{\lambda}), \quad (4.9)$$

then $\boldsymbol{\lambda}(t)$ satisfies (4.5).

This is essentially Theorem 10 of [28], which is well-known to experts, but since the number of the independent variables are infinite in our case, we briefly review the proof and check that it really works. To handle the infinite number of equations simultaneously, we make use of the generating function of the Faber polynomials. We also note that this lemma also follows directly from the generalised hodograph method in the slightly improved version made by Kodama and Gibbons [11] for finitely many independent variables.

Proof. This proof is parallel to the proof of Lemma 4.1 of [24]. First we prove that $\chi_{i,n}$ (n is odd) satisfies the same equation (4.8) as R_i . Since

$$\frac{\partial \chi_{i,n}}{\partial \lambda_j}(\boldsymbol{\lambda}) = \frac{\partial V_i}{\partial \lambda_j}(\boldsymbol{\lambda}) \frac{\partial^2 \Phi_n}{\partial w^2}(V_i(\boldsymbol{\lambda}); \boldsymbol{\lambda}) + \frac{\partial^2 \Phi_n}{\partial w \partial \lambda_j}(V_i(\boldsymbol{\lambda}); \boldsymbol{\lambda}),$$

the generating function for $\partial\chi_{i,n}/\partial\lambda_j$ is obtained by differentiating (3.6). The result is

$$\sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \frac{\partial\chi_{i,n}}{\partial\lambda_j} = \frac{1}{(g^2 - V_i^2)^2} \left(\frac{\partial V_i^2}{\partial\lambda_j} g - (g^2 + V_i^2) \frac{\partial g}{\partial\lambda_j} \right).$$

Substituting the compatibility condition (4.3) and the quadrant Löwner equations (4.1), we obtain

$$\sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} \frac{\partial\chi_{i,n}}{\partial\lambda_j} = \frac{g}{(g^2 - V_j^2)(g^2 - V_i^2)} \frac{V_i^2 + V_j^2}{V_j^2 - V_i^2} \frac{\partial u}{\partial\lambda_j}. \quad (4.10)$$

On the other hand, it follows from (3.6) and the definition of $\chi_{i,n}$ that

$$\sum_{1 \leq n, \text{ odd}} \frac{z^{-n}}{n} (\chi_{j,n} - \chi_{i,n}) = \frac{g(V_j^2 - V_i^2)}{(g^2 - V_j^2)(g^2 - V_i^2)}. \quad (4.11)$$

Comparing (4.10) and (4.11), we have

$$\frac{\partial\chi_{i,n}}{\partial\lambda_j} = \Gamma_{ij}(\chi_{j,n} - \chi_{i,n}), \quad \text{namely,} \quad \Gamma_{ij} = \frac{\partial\lambda_j \chi_{i,n}}{\chi_{j,n} - \chi_{i,n}}, \quad (4.12)$$

for odd n and $i, j = 1, \dots, N$, $i \neq j$. An important point is that this equation holds *for all n simultaneously*. Namely, the coefficient of the equation (4.12) does not depend on n .

(i) Lengthy computation with the help of (4.3) and (4.4) shows that

$$\frac{\partial\Gamma_{ij}}{\partial\lambda_k} = \frac{2(V_i^2(V_j^2 + V_k^2) + 2V_j^2V_k^2)}{(V_j^2 - V_i^2)(V_k^2 - V_i^2)(V_k^2 - V_j^2)^2} \frac{\partial u}{\partial\lambda_j} \frac{\partial u}{\partial\lambda_k}.$$

The right-hand side is symmetric in (j, k) . Hence we have

$$\frac{\partial\Gamma_{ij}}{\partial\lambda_k} = \frac{\partial\Gamma_{ik}}{\partial\lambda_j}.$$

This together with the expression (4.12) is the compatibility condition of (4.8), as shown in § 3 of [28].

(ii) By differentiating the relation (4.9) by t_1 and t_k (k is odd) we obtain

$$\sum_{j=1}^N M_{ij} \frac{\partial\lambda_j}{\partial t_1} = 1, \quad \sum_{j=1}^N M_{ij} \frac{\partial\lambda_j}{\partial t_k} = \chi_{i,k}, \quad (4.13)$$

where

$$M_{ij} := \frac{\partial R_i}{\partial\lambda_j} - \sum_{1 \leq n, \text{ odd}} \frac{\partial\chi_{i,n}}{\partial\lambda_j} t_n.$$

Because of (4.8), (4.12) and the hodograph relation (4.9) the above expression becomes

$$\begin{aligned} M_{ij} &= \Gamma_{ij}(R_j - R_i) - \sum_{1 \leq n, \text{ odd}} \Gamma_{ij}(\chi_{j,n} - \chi_{i,n}) t_n \\ &= \Gamma_{ij} \left((R_j - \sum_{1 \leq n, \text{ odd}} \chi_{j,n} t_n) - (R_i - \sum_{1 \leq n, \text{ odd}} \chi_{i,n} t_n) \right) = 0, \end{aligned}$$

if $i \neq j$. (The fact that the coefficient Γ_{ij} does not depend on n is essential here.) Therefore (4.13) reduces to

$$M_{ii} \frac{\partial\lambda_i}{\partial t_1} = 1, \quad M_{ii} \frac{\partial\lambda_i}{\partial t_k} = \chi_{i,k}.$$

This proves (4.5). ■

5 Quadrant Löwner equation

In this section we show that the quadrant Löwner equation (3.2)

$$\frac{\partial g}{\partial \lambda}(z; \lambda) = \frac{g(z; \lambda)}{V(\lambda)^2 - g(z; \lambda)^2} \frac{du}{d\lambda}(\lambda)$$

is satisfied by slit mappings between quadrant domains.

In fact this equation is connected to the chordal Löwner equation as follows.

Proposition 5.1. (i) *If the function $g(z; \lambda)$ of the form (3.1) satisfies the equation (3.2) ($u = -v_1$), then $\tilde{g}(\tilde{z}; \lambda) := g(\sqrt{\tilde{z}}; \lambda)^2 - 2v_1(\lambda)$ satisfies the hydrodynamic normalisation condition*

$$\tilde{g}(\tilde{z}; \lambda) = \tilde{z} + \tilde{u}(\lambda)\tilde{z}^{-1} + O(\tilde{z}^{-2}) \quad (5.1)$$

and the chordal Löwner equation

$$\frac{\partial \tilde{g}}{\partial \lambda} = \frac{1}{\tilde{g} - U} \frac{d\tilde{u}}{d\lambda}. \quad (5.2)$$

Here the driving function $U = U(\lambda)$ is $U(\lambda) = V(\lambda)^2 - 2v_1(\lambda)$. Moreover, $\tilde{u}(\lambda)$ satisfies

$$\frac{d\tilde{u}}{d\lambda} = -2V^2 \frac{du}{d\lambda}. \quad (5.3)$$

(ii) *Conversely, let $\tilde{g}(\tilde{z}; \lambda) = \tilde{z} + \tilde{u}(\lambda)\tilde{z}^{-1} + \dots$ be a solution of the chordal Löwner equation (5.2) with the driving function $U(\lambda)$. Let $V(\lambda)$ be a solution of the following ordinary differential equation*

$$\frac{d}{d\lambda} V^4 - 2 \frac{dU}{d\lambda} V^2 = 2 \frac{d\tilde{u}}{d\lambda}. \quad (5.4)$$

If we define $u(\lambda)$ by

$$u(\lambda) = \frac{1}{2}(U(\lambda) - V(\lambda)^2), \quad (5.5)$$

then $g(z; \lambda) := \sqrt{\tilde{g}(z^2; \lambda) - 2u(\lambda)}$ is of the form (3.1) and satisfies equation (3.2). (Exactly speaking, we can choose a branch of the square root so that $g(z; \lambda)$ is of the form (3.1) and satisfies (3.2).)

As is well known (cf., for example, [1]), a one-parameter family of conformal mappings from a slit domain to the upper half plane satisfies the chordal Löwner equation⁴. Hence the equation (3.2) is essentially an equation for conformal mappings from a slit domain to the quadrant (Fig. 1). By this reason we call (3.2) the *quadrant Löwner equation*.

Actually, we can “fold” the function $\tilde{g}(z^2; \lambda)$ at any point $c(\lambda)$ on the real axis instead of $2u(\lambda)$ to obtain the conformal mapping to the quadrant: $g_c(z; \lambda) = \sqrt{\tilde{g}(z^2; \lambda) - c(\lambda)}$. As is shown in the proof, the shift by $2u(\lambda)$ is necessary to normalise \tilde{g} by the hydrodynamic normalisation condition.

Proof. Both statements are consequences of straightforward computation.

For example, as for (i), it is easy to check that $\tilde{g}(\tilde{z}; \lambda)$ satisfies the normalisation condition (5.1), since

$$\begin{aligned} \tilde{g}(\tilde{z}; \lambda) &= \left(\sum_{n=0}^{\infty} v_n(\lambda) (\sqrt{\tilde{z}})^{1-2n} \right)^2 - 2v_1(\lambda) = \tilde{z} \left(\sum_{n=0}^{\infty} v_n(\lambda) \tilde{z}^{-n} \right)^2 - 2v_1(\lambda) \\ &= \tilde{z} + (2v_2 + v_1^2) \tilde{z}^{-1} + O(\tilde{z}^{-2}). \end{aligned}$$

⁴Recently a proof without advanced techniques was given in [6].

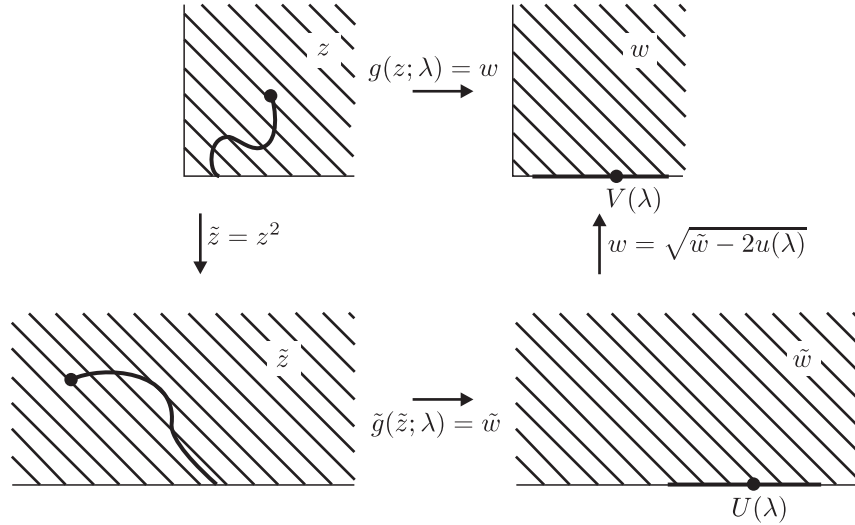


Figure 1. Conformal mapping from a slit domain to the quadrant.

Next, the chordal Löwner equation can be checked as follows (recall that $v_1 = -u$)

$$\frac{\partial \tilde{g}}{\partial \lambda}(\tilde{z}; \lambda) = 2g(\sqrt{\tilde{z}}; \lambda) \frac{\partial g}{\partial \lambda}(\sqrt{\tilde{z}}; \lambda) - 2 \frac{dv_1}{d\lambda} = \left(\frac{2g^2}{V^2 - g^2} + 2 \right) \frac{du}{d\lambda} = \frac{2V^2}{V^2 - g^2} \frac{du}{d\lambda}.$$

Hence summarising, we have

$$\frac{\partial \tilde{g}}{\partial \lambda} = \frac{2V^2}{U - \tilde{g}} \frac{du}{d\lambda}. \quad (5.6)$$

By comparing the coefficients of \tilde{z}^{-1} we have (5.3), which implies (5.2) from (5.6).

The proof of (ii) is similar computation of a Laurent series and derivation. The point is that the conditions (5.4) and (5.5) lead to the equation (5.3). \blacksquare

6 Example

Here, as an application of Theorem 4.1 ($N = 1$) and Proposition 5.1, we construct an example of a solution of the dBKP hierarchy, starting from a solution of the chordal Löwner equation.

The following function $\tilde{g}(\tilde{z}; \lambda)$ is a solution of the chordal Löwner equation with the driving function $U(\lambda) = U$ (a constant)

$$\begin{aligned} \tilde{g}(\tilde{z}; \lambda) &= U + \sqrt{(\tilde{z} - U)^2 + 2\lambda} \\ &= U + \tilde{z} \sqrt{1 - 2U\tilde{z}^{-1} + (U^2 + 2\lambda)\tilde{z}^{-2}} = \tilde{z} + \lambda\tilde{z}^{-1} + \lambda U\tilde{z}^{-2} + \dots \end{aligned} \quad (6.1)$$

(The branch of the square root is chosen so that it has the expansion of the above form.) See, for example, Appendix A.1.1 of [26]. This maps the slit domain $\mathbb{H} \setminus \{U + i\alpha \mid \alpha \in [0, \sqrt{2\lambda}]\}$ to the upper half plane \mathbb{H} . Since the function $\tilde{u}(\lambda)$ is equal to λ , as is seen from the expansion (6.1), the differential equation (5.4) becomes $\frac{d}{d\lambda} V^4 = 2$, which is readily solved:

$$V^2(\lambda) = (2\lambda + c)^{1/2}, \quad (6.2)$$

where c is the integration constant and the branch of the square root can be chosen arbitrarily. Hence $u(\lambda) = (U - V(\lambda)^2)/2 = (U - (2\lambda + c)^{1/2})/2$ and

$$g(z; \lambda) = \sqrt{\tilde{g}(z^2; \lambda) - 2u(\lambda)} = \sqrt{\sqrt{(z^2 - U)^2 + 2\lambda} + (2\lambda + c)^{1/2}}$$

$$\begin{aligned}
&= z \sqrt{\sqrt{(1 - Uz^{-2})^2 + 2\lambda z^{-4}} + (2\lambda + c)^{1/2} z^{-2}} \\
&= z + \frac{-U + (2\lambda + c)^{1/2}}{2} z^{-1} + \left(\frac{\lambda}{4} - \frac{U^2}{8} + \frac{U(2\lambda + c)^{1/2}}{4} - \frac{c}{8} \right) z^{-3} + \dots
\end{aligned}$$

Therefore the Faber polynomials $\Phi_1(w; \lambda)$ and $\Phi_3(w; \lambda)$ are explicitly calculated as

$$\Phi_1(w; \lambda) = w, \quad \Phi_3(w; \lambda) = w^3 - 3 \frac{-U + (2\lambda + c)^{1/2}}{2} w. \quad (6.3)$$

Using (6.2) and (6.3), we have

$$\chi_1(\lambda) = 1, \quad \chi_3(\lambda) = \frac{3}{2}(2\lambda + c)^{1/2} + \frac{3}{2}U.$$

Applying Lemma 4.1 with $R(\lambda) = 0$, $\lambda(t)_{t_5=t_7=\dots=0}$ is determined by the relation

$$t_1 + \frac{3}{2}(U + (2\lambda + c)^{1/2})t_3 = 0,$$

namely,

$$\lambda(t_1, t_3) = \frac{1}{2} \left(\left(\frac{2t_1}{3t_3} + U \right)^2 - c \right).$$

The inverse function $f(w; \lambda)$ of $g(z; \lambda)$ is

$$\begin{aligned}
f(w; \lambda) &= \sqrt{U + \sqrt{(w^2 - (2\lambda + c)^{1/2})^2 - 2\lambda}} \\
&= w + \frac{U - (2\lambda + c)^{1/2}}{2} w^{-1} + \left(-\frac{3\lambda}{4} - \frac{U^2}{8} + \frac{U(2\lambda + c)^{1/2}}{4} - \frac{c}{8} \right) w^{-3} + \dots
\end{aligned}$$

Thus we obtain a solution of the dBKP hierarchy (with respect to t_1 and t_3) by Theorem 3.1:

$$\mathcal{L}(w; t)|_{t_{2n+1}=0(n \geq 2)} = \sqrt{U + \sqrt{\left(w^2 - \frac{2t_1}{3t_3} - U \right)^2 - \left(\frac{2t_1}{3t_3} + U \right)^2} + c}.$$

Remark 6.1. The ordinary differential equation (5.4) for V^2 is a special case of Chini's equation (see [10, C.I.55, p. 303]; $x = \lambda$, $y = V(\lambda)^2$, $n = -1$, $f(x) = d\tilde{u}/d\lambda$, $g(x) = 0$, $h(x) = dU/d\lambda$). It can be solved explicitly only in special cases. The above example is one of them.

Acknowledgements

The author would like to thank Michio Jimbo and Saburo Kakei for interest to this work and for hospitality during his stay in Rikkyo University, where this work was completed. He also express gratitude to Anton Zabrodin for interest and comments. Special thanks of the author are to the referees of the first version of this work, who suggested to study the N -variable reduction, which was absent, and informed many references. This study was carried out within "The National Research University Higher School of Economics" Academic Fund Program in 2013–2014, research grant No. 12-01-0075.

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