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Fusion Procedure for Cyclotomic Hecke Algebras[⋆]

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Abstract. A complete system of primitive pairwise orthogonal idempotents for cyclotomic Hecke algebras is constructed by consecutive evaluations of a rational function in several variables on quantum contents of multi-tableaux. This function is a product of two terms, one of which depends only on the shape of the multi-tableau and is proportional to the inverse of the corresponding Schur element.

Key words: cyclotomic Hecke algebras; fusion formula; idempotents; Young tableaux; Jucys–Murphy elements; Schur element

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1 Introduction

This article is a continuation of the article [14] on the fusion procedure for the complex reflection groups $G(m,1,n)$. The cyclotomic Hecke algebra $H(m,1,n)$, introduced in [2, 3, 4], is a natural flat deformation of the group ring of the complex reflection group $G(m,1,n)$.

In [14], a fusion procedure, in the spirit of [12], for the complex reflection groups $G(m,1,n)$ is suggested: a complete system of primitive pairwise orthogonal idempotents for the groups $G(m,1,n)$ is obtained by consecutive evaluations of a rational function in several variables with values in the group ring $\mathbb{C}G(m,1,n)$. This approach to the fusion procedure relies on the existence of a maximal commutative set of elements of $\mathbb{C}G(m,1,n)$ formed by the Jucys–Murphy elements.

Jucys–Murphy elements for the cyclotomic Hecke algebra $H(m,1,n)$ were introduced in [2] and were used in [13] to develop an inductive approach to the representation theory of the chain of the algebras $H(m,1,n)$. In the generic setting or under certain restrictions on the parameters of the algebra $H(m,1,n)$ (see Section 2 for precise definitions), the Jucys–Murphy elements form a maximal commutative set in the algebra $H(m,1,n)$.

A complete system of primitive pairwise orthogonal idempotents of the algebra $H(m,1,n)$ is indexed by the set of standard m-tableaux of size n. We formulate here the main result of the article. Let $\boldsymbol{\lambda}$ be an m-partition of size n and $\mathcal{T}$ be a standard m-tableau of shape $\boldsymbol{\lambda}$.

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Theorem. *The idempotent $E_{\mathcal{T}}$ of $H(m,1,n)$ corresponding to the standard m-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}} = \mathsf{F}_{\boldsymbol{\lambda}} \Phi(u_1,\dots,u_n)\Big|_{u_1=c_1} \cdots \Big|_{u_{n-1}=c_{n-1}}\Big|_{u_n=c_n} . \tag{1}$$

Here $\Phi(u_1,\dots,u_n)$ is a rational function with values in the algebra $H(m,1,n)$, $\mathsf{F}_{\boldsymbol{\lambda}}$ is an element of the base ring and $c_1,\dots,c_n$ are the quantum contents of the m-nodes of $\mathcal{T}$.

The classical limit of our fusion procedure for algebras $H(m,1,n)$ reproduces the fusion procedure of [14] for the complex reflection groups $G(m,1,n)$. For $\mathbb{C}G(m,1,n)$, the variables of the rational function are split into two parts, one is related to the position of the m-node (its place in the m-tuple) and the other one – to the classical content of the m-node. The position variables can be evaluated simultaneously while the classical content variables have then to be evaluated consequently from 1 to n. For the algebra $H(m,1,n)$, the information about positions and classical contents is fully contained in the quantum contents, and now the function Φ depends on only one set of variables.

Remarkably, the coefficient $\mathsf{F}_{\boldsymbol{\lambda}}$ appearing in (1) depends only on the shape $\boldsymbol{\lambda}$ of the standard m-tableau $\mathcal{T}$ (cf. with the more delicate fusion procedure for the Birman–Murakami–Wenzl algebra [7]). In the classical limit, this coefficient depends only on the usual hook length, see [14]. However, in the deformed situation, the calculation of $\mathsf{F}_{\boldsymbol{\lambda}}$ needs a non-trivial generalization of the hook length. It appears that the coefficient $\mathsf{F}_{\boldsymbol{\lambda}}$ is proportional to the inverse of the *Schur element* (corresponding to the m-partition $\boldsymbol{\lambda}$) associated to a specific symmetrizing form on the algebra $H(m,1,n)$ (see [6, 11] for a calculation of these Schur elements and [5] for an expression in terms of generalized hook lengths); for more precise statements, we refer to [15] where we calculate, using the fusion formula presented here, weights of certain central forms and in particular of these Schur elements.

For $m=1$, the cyclotomic Hecke algebra $H(1,1,n)$ is the Hecke algebra of type A and our fusion procedure reduces to the fusion procedure for the Hecke algebra in [8]. The factors in the rational function are arranged in [8] in such a way that there is a product of "Baxterized" generators on one side and a product of non-Baxterized generators on the other side. For $m>1$ a rearrangement, as for the type A, of the rational function appearing in (1) is no more possible.

The additional, with respect to $H(1,1,n)$, generator of $H(m,1,n)$ satisfies the reflection equation whose "Baxterization" is known [9]. But – and this is maybe surprising – the full Baxterized form is not used in the construction of the rational function in (1). The rational expression involving the additional generator satisfies only a certain limit of the reflection equation with spectral parameters.

The Hecke algebra of type A is the natural quotient of the Birman–Murakami–Wenzl algebra. The fusion procedure, developed in [7], for the Birman–Murakami–Wenzl algebra provides a one-parameter family of fusion procedures for the Hecke algebra of type A. We think that for $m>1$ the fusion procedure (1) can be included into a one-parameter family as well.

2 Definitions

2.1 Cyclotomic Hecke algebra and Baxterized elements

Let $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$. Let $q, v_1, \dots, v_m$ be complex numbers with $q \neq 0$. The cyclotomic Hecke algebra $H(m,1,n+1)$ is the unital associative algebra over $\mathbb{C}$ generated by τ, $\sigma_1,\dots,\sigma_n$ with the defining relations

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i = 1\dots,n-1,\\ &\sigma_i\sigma_j = \sigma_j\sigma_i && \text{for } i,j = 1,\dots,n \text{ such that } |i-j|>1, \end{aligned}$$

$$\begin{aligned}
&\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau, && \\
&\tau\sigma_i = \sigma_i\tau && \text{for } i>1,\\
&\sigma_i^2 = \big(q-q^{-1}\big)\sigma_i + 1 && \text{for } i=1,\dots,n,\\
&(\tau - v_1)\cdots(\tau - v_m) = 0. &&
\end{aligned}$$

We define $H(m,1,0) := \mathbb{C}$. The cyclotomic Hecke algebras $H(m,1,n)$ form a chain (with respect to n) of algebras defined by inclusions $H(m,1,n) \ni \tau,\sigma_1,\dots,\sigma_{n-1} \mapsto \tau,\sigma_1,\dots,\sigma_{n-1} \in H(m,1,n+1)$ for any $n \geq 0$. These inclusions allow to consider (as it will often be done in the article) elements of $H(m,1,n)$ as elements of $H(m,1,n+n')$ for any $n' = 0,1,2,\dots$.

In the sequel we assume the following restrictions on the parameters $q, v_1,\dots,v_m$:

$$1 + q^2 + \dots + q^{2N} \neq 0 \;\; \text{for } N \text{ such that } N \leq n, \tag{2}$$

$$q^{2i}v_j - v_k \neq 0 \;\; \text{for} \;\; i,j,k \;\; \text{such that} \;\; j \neq k \text{ and } -n \leq i \leq n, \tag{3}$$

$$v_j \neq 0 \;\; \text{for} \;\; j = 1,\dots,m. \tag{4}$$

The restrictions (2), (3) are necessary and sufficient for the semi-simplicity of the algebra $H(m,1,n+1)$ [1, main theorem]. The restriction (4) is necessary for the maximality of the commutative set of the Jucys–Murphy elements (as defined in Section 3) [1, Proposition 3.2].

Define the following rational functions in variables a, b with values in $H(m,1,n+1)$:

$$\overline{\sigma}_i(a,b) := \sigma_i + (q-q^{-1})\frac{b}{a-b}, \qquad i = 1,\dots,n. \tag{5}$$

The functions $\overline{\sigma}_i$ are called *Baxterized* elements and the variables a and b are called *spectral parameters*. These Baxterized elements satisfy the Yang–Baxter equation with spectral parameters

$$\overline{\sigma}_i(a,b)\overline{\sigma}_{i+1}(a,c)\overline{\sigma}_i(b,c) = \overline{\sigma}_{i+1}(b,c)\overline{\sigma}_i(a,c)\overline{\sigma}_{i+1}(a,b).$$

The following formula will be used later

$$\overline{\sigma}_i(a,b)\overline{\sigma}_i(b,a) = \frac{(a-q^2b)(a-q^{-2}b)}{(a-b)^2} \qquad \text{for} \quad i = 1,\dots,n. \tag{6}$$

Let $\mathfrak{p}_i$, $i = 1,\dots,m$, be the eigen-idempotents of τ, $\mathfrak{p}_i := \prod_{j:j\neq i} (\tau - v_j)/(v_i - v_j)$, so that $\tau\mathfrak{p}_i = v_i\mathfrak{p}_i$, $\mathfrak{p}_i\mathfrak{p}_j = \delta_{ij}\mathfrak{p}_i$, $\sum_i \mathfrak{p}_i = 1$ and $\tau = \sum_i v_i\mathfrak{p}_i$. Let r be an indeterminate. The resolvent $(r-\tau)^{-1} := \sum_i (r-v_i)^{-1}\mathfrak{p}_i$ of τ is an element of $\mathbb{C}(r) \otimes_{\mathbb{C}} H(m,1,n+1)$. Define a rational function $\overline{\tau}$ with values in $H(m,1,n+1)$:

$$\overline{\tau}(r) := \frac{(r-v_1)(r-v_2)\cdots(r-v_m)}{r-\tau} = \sum_i \left(\prod_{j:j\neq i}(r-v_j)\right)\mathfrak{p}_i \in \mathbb{C}[r] \otimes_{\mathbb{C}} H(m,1,n+1). \tag{7}$$

Remarks. (i) The function $\overline{\tau}(r)$ can be expressed in terms of the complex numbers $a_0, a_1, \dots, a_m$ defined by

$$(X-v_1)(X-v_2)\cdots(X-v_m) = a_0 + a_1X + \dots + a_mX^m,$$

where X is an indeterminate. Let $\mathfrak{a}_i(r)$, $i = 0,\dots,m$, be the polynomials in r given by

$$\mathfrak{a}_i(r) = a_i + ra_{i+1} + \dots + r^{m-i}a_m \qquad \text{for} \quad i = 0,\dots,m. \tag{8}$$

Using that $r\mathfrak{a}_{i+1}(r) = \mathfrak{a}_i(r) - a_i$, for $i = 0, \dots, m-1$, it is straightforward to verify that

$$(r - \tau) \sum_{i=0}^{m-1} \mathfrak{a}_{i+1}(r)\tau^i = \mathfrak{a}_0(r) = (r - v_1)(r - v_2) \cdots (r - v_m). \tag{9}$$

It follows from (9) that

$$\overline{\tau}(r) = \mathfrak{a}_1(r) + \mathfrak{a}_2(r)\tau + \dots + \mathfrak{a}_m(r)\tau^{m-1} = \sum_{i=0}^{m-1} \mathfrak{a}_{i+1}(r)\tau^i, \tag{10}$$

For example, for $m = 1$, we have $\overline{\tau}(r) = 1$; for $m = 2$, we have $\overline{\tau}(r) = \tau + r - v_1 - v_2$; for $m = 3$, we have $\overline{\tau}(r) = \tau^2 + (r - v_1 - v_2 - v_3)\tau + r^2 - r(v_1 + v_2 + v_3) + v_1v_2 + v_1v_3 + v_2v_3$.

(ii) The functions $\overline{\tau}$ and $\overline{\sigma}_1$ satisfy the following equation

$$\overline{\sigma}_1(a, b)\overline{\tau}(a)\sigma_1^{-1}\overline{\tau}(b) = \overline{\tau}(b)\sigma_1^{-1}\overline{\tau}(a)\overline{\sigma}_1(a, b). \tag{11}$$

Indeed, due to (6) and (7), the equality (11) is equivalent to

$$(\tau - b)\sigma_1(\tau - a)\overline{\sigma}_1(b, a) = \overline{\sigma}_1(b, a)(\tau - a)\sigma_1(\tau - b),$$

which is proved by a straightforward calculation. The equation (11) is a certain (we leave the details to the reader) limit of the usual reflection equation with spectral parameters (see, for example, [10]).

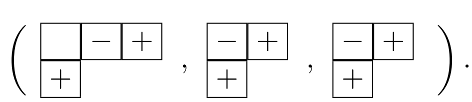
2.2 m-partitions, m-tableaux and generalized hook length

Let $\lambda \vdash n + 1$ be a partition of size $n + 1$, that is, $\lambda = (\lambda_1, \dots, \lambda_l)$, where λ_j, $j = 1, \dots, l$, are positive integers, $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_l$ and $n + 1 = \lambda_1 + \dots + \lambda_l$. We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of rows of nodes containing λ_j nodes in the j-th row, $j = 1, \dots, l$; the rows are numbered from top to bottom. For a node α in line x and column y of a Young diagram, we denote $\alpha = (x, y)$ and call x and y the coordinates of the node.

An m-partition, or a Young m-diagram, of size $n+1$ is an m-tuple of partitions such that the sum of their sizes equals $n + 1$; e.g. the Young 3-diagram (□□, □, □) represents the 3-partition $\big((2), (1), (1)\big)$ of size 4.

We shall understand an m-partition as a set of m-nodes, where an m-node $\boldsymbol{\alpha}$ is a pair $\{\alpha, k\}$ consisting of a node α and an integer $k = 1, \dots, m$, indicating to which diagram in the m-tuple the node belongs. The integer k will be called *position* of the m-node, and we set $\mathrm{pos}(\boldsymbol{\alpha}) := k$.

For an m-partition $\boldsymbol{\lambda}$, an m-node $\boldsymbol{\alpha}$ of $\boldsymbol{\lambda}$ is called *removable* if the set of m-nodes obtained from $\boldsymbol{\lambda}$ by removing $\boldsymbol{\alpha}$ is still an m-partition. An m-node $\boldsymbol{\beta}$ not in $\boldsymbol{\lambda}$ is called *addable* if the set of m-nodes obtained from $\boldsymbol{\lambda}$ by adding $\boldsymbol{\beta}$ is still an m-partition. For an m-partition $\boldsymbol{\lambda}$, we denote by $\mathcal{E}_-(\boldsymbol{\lambda})$ the set of removable m-nodes of $\boldsymbol{\lambda}$ and by $\mathcal{E}_+(\boldsymbol{\lambda})$ the set of addable m-nodes of $\boldsymbol{\lambda}$. For example, the removable/addable m-nodes (marked with $-/+$) for the 3-partition (□□, □, □) are



Let $\boldsymbol{\lambda}$ be an m-diagram of size $n+1$. A standard m-tableau of shape $\boldsymbol{\lambda}$ is obtained by placing the numbers $1, \dots, n + 1$ in the m-nodes of the diagrams of $\boldsymbol{\lambda}$ in such a way that the numbers in the nodes ascend along rows and down columns in every diagram. The *size* of a standard m-tableau is the size of its shape.

Let $q, v_1, \dots, v_m$ be the parameters of the cyclotomic Hecke algebra $H(m,1,n+1)$ and let $\boldsymbol{\alpha} = \{\alpha, k\}$ be an m-node with $\alpha = (x, y)$. We denote by $cc(\boldsymbol{\alpha})$ the classical content of the node α, $cc(\boldsymbol{\alpha}) := y - x$, and by $c(\boldsymbol{\alpha})$ the *quantum content* of the m-node $\boldsymbol{\alpha}$, $c(\boldsymbol{\alpha}) := v_k q^{2cc(\boldsymbol{\alpha})} = v_k q^{2(y-x)}$.

For a standard m-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$ let $\boldsymbol{\alpha}_i$ be the m-node of $\mathcal{T}$ occupied by the number i, $i = 1, \dots, n+1$; we set $c(\mathcal{T}|i) := c(\boldsymbol{\alpha}_i)$, $cc(\mathcal{T}|i) := cc(\boldsymbol{\alpha}_i)$ and $\mathrm{pos}(\mathcal{T}|i) := \mathrm{pos}(\boldsymbol{\alpha}_i)$. For example, for the standard 3-tableau $\mathcal{T} = \left(\boxed{1}\boxed{3}, \boxed{2}, \boxed{4}\right)$ we have

$$\begin{array}{lllll} c(\mathcal{T}|1) = v_1, & c(\mathcal{T}|2) = v_2, & c(\mathcal{T}|3) = v_1 q^2 & \text{and} & c(\mathcal{T}|4) = v_3, \\ cc(\mathcal{T}|1) = 0, & cc(\mathcal{T}|2) = 0, & cc(\mathcal{T}|3) = 1 & \text{and} & cc(\mathcal{T}|4) = 0, \\ \mathrm{pos}(\mathcal{T}|1) = 1, & \mathrm{pos}(\mathcal{T}|2) = 2, & \mathrm{pos}(\mathcal{T}|3) = 1 & \text{and} & \mathrm{pos}(\mathcal{T}|4) = 3, \end{array}$$

Generalized hook length. The hook of a node α of a partition λ is the set of nodes of λ consisting of the node α and the nodes which lie either under α in the same column or to the right of α in the same row; the hook length $h_\lambda(\alpha)$ of α is the cardinality of the hook of α. We extend this definition to m-nodes. For an m-node $\boldsymbol{\alpha} = \{\alpha, k\}$ of an m-partition $\boldsymbol{\lambda}$, the hook length of $\boldsymbol{\alpha}$ in $\boldsymbol{\lambda}$, which we denote by $h_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$, is the hook length of the node α in the k-th partition of $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda}$ be an m-partition. For $j = 1, \dots, m$, let $\mathfrak{l}_{\boldsymbol{\lambda},x,j}$ be the number of nodes in the line x of the j-th diagram of $\boldsymbol{\lambda}$, and $\mathfrak{c}_{\boldsymbol{\lambda},y,j}$ be the number of nodes in the column y of the j-th diagram of $\boldsymbol{\lambda}$. The hook length of an m-node $\boldsymbol{\alpha} = \{(x, y), k\}$ of $\boldsymbol{\lambda}$ can be rewritten as

$$h_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}) = \mathfrak{l}_{\boldsymbol{\lambda},x,k} + \mathfrak{c}_{\boldsymbol{\lambda},y,k} - x - y + 1.$$

Define the generalized hook length of $\boldsymbol{\alpha}$ (see also [5]) by

$$h^{(j)}_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}) := \mathfrak{l}_{\boldsymbol{\lambda},x,j} + \mathfrak{c}_{\boldsymbol{\lambda},y,k} - x - y + 1 \qquad \text{for} \quad j = 1, \dots, m;$$

in particular, $h^{(k)}_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}) = h_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$ is the usual hook length.

For an m-partition $\boldsymbol{\lambda}$, we define

$$\mathsf{F}_{\boldsymbol{\lambda}} = \prod_{\boldsymbol{\alpha} \in \boldsymbol{\lambda}} \left(\frac{q^{cc(\boldsymbol{\alpha})}}{[h_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})]_q} \prod_{\substack{k = 1, \dots, m \\ k \neq \mathrm{pos}(\boldsymbol{\alpha})}} \frac{q^{-cc(\boldsymbol{\alpha})}}{v_{\mathrm{pos}(\boldsymbol{\alpha})} q^{-h^{(k)}_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})} - v_k q^{h^{(k)}_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})}} \right), \tag{12}$$

where $[j]_q := q^{j-1} + q^{j-3} + \dots + q^{-j+1}$ for a non-negative integer j. Under the restrictions (2)–(4), the number $\mathsf{F}_{\boldsymbol{\lambda}}$ is well defined for any m-partition $\boldsymbol{\lambda}$ of size less or equal to $n+1$ since $h_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}) \leq n+1$ and $h^{(k)}_{\boldsymbol{\lambda}}(\boldsymbol{\alpha}) \leq n$ if $k \neq \mathrm{pos}(\boldsymbol{\alpha})$ for any $\boldsymbol{\alpha} \in \boldsymbol{\lambda}$.

3 Idempotents and Jucys–Murphy elements of $H(m, 1, n+1)$

In this section we recall the definition and some properties, from [2], of the Jucys–Murphy elements of the algebra $H(m,1,n+1)$, together with some facts about an explicit realization of the irreducible representations of $H(m,1,n+1)$. We then derive, in the same spirit as in [12], an inductive formula, that we will use in the next section, for the primitive idempotents corresponding to this realization.

The Jucys–Murphy elements J_i, $i = 1, \dots, n+1$, of the algebra $H(m,1,n+1)$ are defined by the following initial condition and recursion

$$J_1 = \tau \qquad \text{and} \qquad J_{i+1} = \sigma_i J_i \sigma_i, \qquad i = 1, \dots, n.$$

We recall that, under the restrictions (2)–(4), the elements J_i, $i = 1, \dots, n+1$, form a maximal commutative set (that is, generate a maximal commutative subalgebra) of $H(m, 1, n+1)$ [2, Proposition 3.17]. Recall also that

$$J_i \sigma_k = \sigma_k J_i \qquad \text{for} \quad k \neq i-1, i.$$

The isomorphism classes of irreducible $\mathbb{C}$-representations of $H(m, 1, n+1)$ are in bijection with the set of m-partitions of size $n+1$. We use the labeling and the explicit realization of the irreducible representations of $H(m, 1, n+1)$ given in [2]. Namely, for any m-partition $\boldsymbol{\lambda}$ of size $n+1$, the irreducible representation $V_{\boldsymbol{\lambda}}$ of $H(m, 1, n+1)$ corresponding to $\boldsymbol{\lambda}$ has a basis $\{v_{\mathcal{T}}\}$ indexed by the set of standard m-tableaux of shape $\boldsymbol{\lambda}$, and is characterized (up to a diagonal change of basis) by the fact that the Jucys–Murphy elements act diagonally by

$$J_i(v_{\mathcal{T}}) = c(\mathcal{T}|i) v_{\mathcal{T}}, \qquad i = 1, \dots, n+1.$$

We will not need the explicit formulas for the action of the generators of $H(m, 1, n+1)$ on basis elements $v_{\mathcal{T}}$.

The restriction of irreducible representations of $H(m, 1, n+1)$ to $H(m, 1, n)$ is determined by inclusion of m-partitions, that is, for $H(m, 1, n)$-modules, we have

$$V_{\boldsymbol{\lambda}} \cong \bigoplus_{\boldsymbol{\mu} \subset \boldsymbol{\lambda},\ \boldsymbol{\mu} \text{ of size } n} V_{\boldsymbol{\mu}}. \tag{13}$$

Moreover, in this decomposition, $V_{\boldsymbol{\mu}}$ is the space spanned by the basis vectors $v_{\mathcal{T}}$, with $\mathcal{T}$ such that the standard m-tableau (of size n) obtained by removing from $\mathcal{T}$ the m-node containing $n+1$ is of shape $\boldsymbol{\mu}$.

For a standard m-tableau $\mathcal{T}$ of size $n+1$, we denote by $E_{\mathcal{T}}$ the primitive idempotent of $H(m, 1, n+1)$ corresponding to $v_{\mathcal{T}}$, uniquely defined by $E_{\mathcal{T}} v_{\mathcal{T}'} = \delta_{\mathcal{T}\mathcal{T}'} v_{\mathcal{T}}$. The results recalled above imply that $\{E_{\mathcal{T}}\}$, where $\mathcal{T}$ runs through the set of standard m-tableaux of size $n+1$, is a complete set of pairwise orthogonal primitive idempotents of $H(m, 1, n+1)$. Moreover, we have by construction

$$J_i E_{\mathcal{T}} = E_{\mathcal{T}} J_i = c(\mathcal{T}|i) E_{\mathcal{T}}, \qquad i = 1, \dots, n+1. \tag{14}$$

Due to the maximality of the commutative set formed by the Jucys–Murphy elements, the idempotent $E_{\mathcal{T}}$ can be expressed in terms of the elements J_i, $i = 1, \dots, n+1$. Let $\boldsymbol{\gamma}$ be the m-node of $\mathcal{T}$ containing the number $n+1$. As the m-tableau $\mathcal{T}$ is standard, the m-node $\boldsymbol{\gamma}$ of $\boldsymbol{\lambda}$ is removable. Let $\mathcal{U}$ be the standard m-tableau obtained from $\mathcal{T}$ by removing the m-node $\boldsymbol{\gamma}$, and let $\boldsymbol{\mu}$ be the shape of $\mathcal{U}$. By (13) and (14), the inductive formula for $E_{\mathcal{T}}$ in terms of the Jucys–Murphy elements reads

$$E_{\mathcal{T}} = E_{\mathcal{U}} \prod_{\substack{\boldsymbol{\beta}:\ \boldsymbol{\beta} \in \mathcal{E}_+(\boldsymbol{\mu}) \\ \boldsymbol{\beta} \neq \boldsymbol{\gamma}}} \frac{J_{n+1} - c(\boldsymbol{\beta})}{c(\boldsymbol{\gamma}) - c(\boldsymbol{\beta})},$$

with the initial condition: $E_{\mathcal{U}_0} = 1$ for the unique m-tableau $\mathcal{U}_0$ of size 0. Here $E_{\mathcal{U}}$ is considered as an element of the algebra $H(m, 1, n+1)$. Note that, due to the restrictions (2)–(4), we have $c(\boldsymbol{\beta}) \neq c(\boldsymbol{\gamma})$ for any $\boldsymbol{\beta} \in \mathcal{E}_+(\boldsymbol{\mu})$ such that $\boldsymbol{\beta} \neq \boldsymbol{\gamma}$.

Let $\{\mathcal{T}_1, \dots, \mathcal{T}_a\}$ be the set of pairwise different standard m-tableaux which can be obtained from $\mathcal{U}$ by adding an m-node with number $n+1$. As a consequence of (13), we have the formula

$$E_{\mathcal{U}} = \sum_{i=1}^{a} E_{\mathcal{T}_i}. \tag{15}$$

The element J_{n+1} satisfies a polynomial equation of finite order so its resolvent is well defined and

$$E_{\mathcal{U}} \frac{u - c(\mathcal{T}|n+1)}{u - J_{n+1}}$$

is a rational function in an indeterminate u with values in $H(m,1,n+1)$. Replacing $E_{\mathcal{U}}$ by the right-hand side of (15) and using (14), we obtain that this function is non-singular at $u = c(\mathcal{T}|n+1)$ and moreover, due to the restrictions (2)–(4),

$$E_{\mathcal{U}} \frac{u - c(\mathcal{T}|n+1)}{u - J_{n+1}}\Big|_{u=c(\mathcal{T}|n+1)} = E_{\mathcal{T}}. \tag{16}$$

4 Fusion formula for the algebra $H(m,1,n+1)$

In this section, we prove, in Theorem 1 below, the fusion formula for the primitive idempotents $E_{\mathcal{T}}$. We use the inductive formula (16) for $E_{\mathcal{T}}$.

Let ϕ_k, for $k = 1, \ldots, n+1$, be the rational functions in variables $u_1, \ldots, u_k$ with values in the algebra $H(m,1,n+1)$ defined by $\phi_1(u_1) := \overline{\tau}(u_1)$ and, for $k = 1, \ldots, n$,

$$\begin{aligned}\phi_{k+1}(u_1, \ldots, u_k, u_{k+1}) &:= \overline{\sigma}_k(u_{k+1}, u_k)\phi_k(u_1, \ldots, u_{k-1}, u_{k+1})\sigma_k^{-1} \\ &= \overline{\sigma}_k(u_{k+1}, u_k)\overline{\sigma}_{k-1}(u_{k+1}, u_{k-1}) \ldots \overline{\sigma}_1(u_{k+1}, u_1)\overline{\tau}(u_{k+1})\sigma_1^{-1} \ldots \sigma_{k-1}^{-1}\sigma_k^{-1}.\end{aligned}$$

Define the following rational function Φ in variables $u_1, \ldots, u_{n+1}$ with values in $H(m,1,n+1)$:

$$\Phi(u_1, \ldots, u_{n+1}) := \phi_{n+1}(u_1, \ldots, u_n, u_{n+1})\phi_n(u_1, \ldots, u_{n-1}, u_n) \cdots \phi_1(u_1).$$

Let $\boldsymbol{\lambda}$ be an m-partition of size $n+1$ and $\mathcal{T}$ a standard m-tableau of shape $\boldsymbol{\lambda}$. For $i = 1, \ldots, n+1$, we set $c_i := c(\mathcal{T}|i)$.

Theorem 1. *The idempotent $E_{\mathcal{T}}$ corresponding to the standard m-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}} = \mathsf{F}_{\boldsymbol{\lambda}} \Phi(u_1, \ldots, u_{n+1})\big|_{u_1=c_1} \cdots \big|_{u_n=c_n}\big|_{u_{n+1}=c_{n+1}},$$

with $\mathsf{F}_{\boldsymbol{\lambda}}$ defined in (12).

We will prove the theorem in this section in several steps.

Until the end of the text, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ denote the m-nodes of $\mathcal{T}$ containing the numbers $n+1$ and n respectively; $\mathcal{U}$ is the standard m-tableau obtained from $\mathcal{T}$ by removing $\boldsymbol{\gamma}$, and $\boldsymbol{\mu}$ is the shape of $\mathcal{U}$; also, $\mathcal{W}$ is the standard m-tableau obtained from $\mathcal{U}$ by removing the m-node $\boldsymbol{\delta}$ and $\boldsymbol{\nu}$ is the shape of $\mathcal{W}$.

For a standard m-tableau $\mathcal{V}$ of size N, we define the following rational function in a variable u with complex values

$$F_{\mathcal{V}}(u) := \frac{u - c(\mathcal{V}|N)}{(u - v_1) \cdots (u - v_m)} \prod_{i=1}^{N-1} \frac{\big(u - c(\mathcal{V}|i)\big)^2}{\big(u - q^2 c(\mathcal{V}|i)\big)\big(u - q^{-2} c(\mathcal{V}|i)\big)}; \tag{17}$$

by convention, $F_{\mathcal{V}}(u) := \frac{u - c(\mathcal{V}|1)}{(u-v_1)\cdots(u-v_m)}$ for $N = 1$.

Proposition 2. *We have*

$$F_{\mathcal{T}}(u)\phi_{n+1}(c_1, \ldots, c_n, u)E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - J_{n+1}} E_{\mathcal{U}}. \tag{18}$$

Proof. We prove (18) by induction on n. As $J_1=\tau$, we have by (7)

$$\frac{u-c_1}{u-J_1}=\frac{u-c_1}{(u-v_1)\cdots(u-v_m)}\overline{\tau}(u),$$

which verifies the basis of induction ($n=0$).

We have: $E_{\mathcal{W}}E_{\mathcal{U}}=E_{\mathcal{U}}$ and $E_{\mathcal{W}}$ commutes with σ_n. Rewrite the left-hand side of (18) as

$$F_{\mathcal{T}}(u)\overline{\sigma}_n(u,c_n)\cdot\phi_n(c_1,\dots,c_{n-1},u)E_{\mathcal{W}}\cdot\sigma_n^{-1}E_{\mathcal{U}}.$$

By the induction hypothesis we have for the left-hand side of (18)

$$F_{\mathcal{T}}(u)\big(F_{\mathcal{U}}(u)\big)^{-1}\overline{\sigma}_n(u,c_n)\frac{u-c_n}{u-J_n}\sigma_n^{-1}E_{\mathcal{U}}.$$

Since J_{n+1} commutes with $E_{\mathcal{U}}$, the equality (18) is equivalent to

$$\begin{aligned}&F_{\mathcal{T}}(u)\big(F_{\mathcal{U}}(u)\big)^{-1}(u-c_n)\sigma_n^{-1}(u-J_{n+1})E_{\mathcal{U}}\\&\qquad=\frac{(u-c_{n+1})(u-c_n)^2}{(u-q^2c_n)(u-q^{-2}c_n)}(u-J_n)\overline{\sigma}_n(c_n,u)E_{\mathcal{U}}\end{aligned}\tag{19}$$

(the inverse of $\overline{\sigma}_n(u,c_n)$ is calculated with the help of (6)). By (17),

$$F_{\mathcal{T}}(u)\big(F_{\mathcal{U}}(u)\big)^{-1}(u-c_n)=(u-c_{n+1})\frac{(u-c_n)^2}{(u-q^2c_n)(u-q^{-2}c_n)}.$$

Therefore, to prove (19), it remains to show that

$$\sigma_n^{-1}(u-J_{n+1})E_{\mathcal{U}}=(u-J_n)\overline{\sigma}_n(c_n,u)E_{\mathcal{U}}.\tag{20}$$

Replacing J_{n+1} by $\sigma_nJ_n\sigma_n$, we write the left-hand side of (20) in the form

$$\big(u\sigma_n^{-1}-J_n\sigma_n\big)E_{\mathcal{U}}.\tag{21}$$

As $J_nE_{\mathcal{U}}=c_nE_{\mathcal{U}}$, the right-hand side of (20) is

$$\left(u\sigma_n-J_n\sigma_n+\big(q-q^{-1}\big)(u-c_n)\frac{u}{c_n-u}\right)E_{\mathcal{U}}$$

and thus coincides with (21). ∎

To prove Theorem 1, we need the following information about the behavior of the rational function $F_{\mathcal{T}}(u)$ at $u=c_{n+1}$.

Proposition 3. *The rational function $F_{\mathcal{T}}(u)$ is non-singular at $u=c_{n+1}$, and moreover*

$$F_{\mathcal{T}}(c_{n+1})=\mathsf{F}_{\boldsymbol{\lambda}}\ \mathsf{F}_{\boldsymbol{\mu}}^{-1},$$

We will prove this proposition with the help of Lemmas 4 and 5 below, which involve the combinatorics of multi-partitions.

Lemma 4. *We have*

$$F_{\mathcal{T}}(u)=(u-c_{n+1})\prod_{\boldsymbol{\beta}\in\mathcal{E}_-(\boldsymbol{\mu})}(u-c(\boldsymbol{\beta}))\prod_{\boldsymbol{\alpha}\in\mathcal{E}_+(\boldsymbol{\mu})}(u-c(\boldsymbol{\alpha}))^{-1}.\tag{22}$$

Proof. The proof is by induction on n. For $n=0$, we have

$$F_{\mathcal{T}}(u)=\frac{u-c_1}{(u-v_1)\cdots(u-v_m)},$$

which is equal to the right-hand side of (22).

Now, for $n>0$, we rewrite (17) for $\mathcal{V}=\mathcal{T}$ as

$$F_{\mathcal{T}}(u)=\frac{u-c_{n+1}}{(u-v_1)\cdots(u-v_m)}\frac{(u-c_n)^2}{(u-q^2c_n)(u-q^{-2}c_n)}\prod_{i=1}^{n-1}\frac{(u-c_i)^2}{(u-q^2c_i)(u-q^{-2}c_i)}.$$

Using the induction hypothesis, we obtain

$$F_{\mathcal{T}}(u)=\frac{(u-c_{n+1})(u-c_n)^2}{(u-q^2c_n)(u-q^{-2}c_n)}\prod_{\boldsymbol{\beta}\in\mathcal{E}_-(\boldsymbol{\nu})}(u-c(\boldsymbol{\beta}))\prod_{\boldsymbol{\alpha}\in\mathcal{E}_+(\boldsymbol{\nu})}(u-c(\boldsymbol{\alpha}))^{-1}. \tag{23}$$

Denote by $\boldsymbol{\delta}_t$ and $\boldsymbol{\delta}_b$ the m-nodes which are, respectively, just above and just below $\boldsymbol{\delta}$, $\boldsymbol{\delta}_l$ and $\boldsymbol{\delta}_r$ the m-nodes which are, respectively, just on the left and just on the right of $\boldsymbol{\delta}$; it might happen that one of the coordinates of $\boldsymbol{\delta}_t$ (or $\boldsymbol{\delta}_l$) is not positive, and in this situation, by definition, $\boldsymbol{\delta}_t\notin\mathcal{E}_-(\boldsymbol{\nu})$ (or $\boldsymbol{\delta}_l\notin\mathcal{E}_-(\boldsymbol{\nu})$). It is straightforward to see that:

- If $\boldsymbol{\delta}_t,\boldsymbol{\delta}_l\notin\mathcal{E}_-(\boldsymbol{\nu})$ then

$$\mathcal{E}_-(\boldsymbol{\mu})=\mathcal{E}_-(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}\}\quad\text{and}\quad\mathcal{E}_+(\boldsymbol{\mu})=(\mathcal{E}_+(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}_b,\boldsymbol{\delta}_r\})\setminus\{\boldsymbol{\delta}\}\ .$$

- If $\boldsymbol{\delta}_t\in\mathcal{E}_-(\boldsymbol{\nu})$ and $\boldsymbol{\delta}_l\notin\mathcal{E}_-(\boldsymbol{\nu})$ then

$$\mathcal{E}_-(\boldsymbol{\mu})=(\mathcal{E}_-(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}\})\setminus\{\boldsymbol{\delta}_t\}\qquad\text{and}\qquad\mathcal{E}_+(\boldsymbol{\mu})=(\mathcal{E}_+(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}_b\})\setminus\{\boldsymbol{\delta}\}.$$

- If $\boldsymbol{\delta}_t\notin\mathcal{E}_-(\boldsymbol{\nu})$ and $\boldsymbol{\delta}_l\in\mathcal{E}_-(\boldsymbol{\nu})$ then

$$\mathcal{E}_-(\boldsymbol{\mu})=(\mathcal{E}_-(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}\})\setminus\{\boldsymbol{\delta}_l\}\qquad\text{and}\qquad\mathcal{E}_+(\boldsymbol{\mu})=(\mathcal{E}_+(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}_r\})\setminus\{\boldsymbol{\delta}\}.$$

- If $\boldsymbol{\delta}_t,\boldsymbol{\delta}_l\in\mathcal{E}_-(\boldsymbol{\nu})$ then

$$\mathcal{E}_-(\boldsymbol{\mu})=(\mathcal{E}_-(\boldsymbol{\nu})\cup\{\boldsymbol{\delta}\})\setminus\{\boldsymbol{\delta}_t,\boldsymbol{\delta}_l\}\qquad\text{and}\qquad\mathcal{E}_+(\boldsymbol{\mu})=\mathcal{E}_+(\boldsymbol{\nu})\setminus\{\boldsymbol{\delta}\}.$$

In each case, using that $c(\boldsymbol{\delta}_t)=c(\boldsymbol{\delta}_r)=q^2c_n$ and $c(\boldsymbol{\delta}_b)=c(\boldsymbol{\delta}_l)=q^{-2}c_n$, it follows that the right-hand side of (23) is equal to

$$(u-c_{n+1})\prod_{\boldsymbol{\beta}\in\mathcal{E}_-(\boldsymbol{\mu})}(u-c(\boldsymbol{\beta}))\prod_{\boldsymbol{\alpha}\in\mathcal{E}_+(\boldsymbol{\mu})}(u-c(\boldsymbol{\alpha}))^{-1},$$

which establishes the formula (22). ■

Lemma 5. *We have*

$$\prod_{\boldsymbol{\beta}\in\mathcal{E}_-(\boldsymbol{\mu})}(c_{n+1}-c(\boldsymbol{\beta}))\prod_{\boldsymbol{\alpha}\in\mathcal{E}_+(\boldsymbol{\mu})\setminus\{\gamma\}}(c_{n+1}-c(\boldsymbol{\alpha}))^{-1}=\mathsf{F}_{\boldsymbol{\lambda}}\mathsf{F}_{\boldsymbol{\mu}}^{-1}.$$

Proof. 1. The definition (12), for a partition λ, reduces to

$$\mathsf{F}_\lambda := \prod_{\alpha\in\lambda} \frac{q^{cc(\alpha)}}{[h_\lambda(\alpha)]_q}.$$

The Lemma 5 for a partition λ is established in [8, Lemma 3.2].

2. Set $k = \mathrm{pos}(\boldsymbol{\gamma})$. Define, for an m-partition $\boldsymbol{\theta}$,

$$\widetilde{\mathsf{F}}_{\boldsymbol{\theta}} := \prod_{\boldsymbol{\alpha}\in\boldsymbol{\theta}} \frac{q^{cc(\boldsymbol{\alpha})}}{[h_{\boldsymbol{\theta}}(\boldsymbol{\alpha})]_q},$$

and, for $j = 1,\dots,m$ such that $j \neq k$,

$$\mathsf{F}^{(j)}_{\boldsymbol{\theta}} := \prod_{\substack{\boldsymbol{\alpha}\in\boldsymbol{\theta}\\ \mathrm{pos}(\boldsymbol{\alpha})=k}} \frac{q^{-cc(\boldsymbol{\alpha})}}{v_k q^{-h^{(j)}_{\boldsymbol{\theta}}(\boldsymbol{\alpha})} - v_j q^{h^{(j)}_{\boldsymbol{\theta}}(\boldsymbol{\alpha})}} \prod_{\substack{\boldsymbol{\alpha}\in\boldsymbol{\theta}\\ \mathrm{pos}(\boldsymbol{\alpha})=j}} \frac{q^{-cc(\boldsymbol{\alpha})}}{v_j q^{-h^{(k)}_{\boldsymbol{\theta}}(\boldsymbol{\alpha})} - v_k q^{h^{(k)}_{\boldsymbol{\theta}}(\boldsymbol{\alpha})}}. \qquad (24)$$

By (12), we have

$$\mathsf{F}_{\boldsymbol{\theta}} = \widetilde{\mathsf{F}}_{\boldsymbol{\theta}} \prod_{\substack{j=1,\dots,m\\ j\neq k}} \mathsf{F}^{(j)}_{\boldsymbol{\theta}}. \qquad (25)$$

Fix $j \in \{1,\dots,m\}$ such that $j \neq k$. We shall show that

$$\prod_{\substack{\boldsymbol{\beta}\in\mathcal{E}_-(\boldsymbol{\mu})\\ \mathrm{pos}(\boldsymbol{\beta})=j}} (c_{n+1} - c(\boldsymbol{\beta})) \prod_{\substack{\boldsymbol{\alpha}\in\mathcal{E}_+(\boldsymbol{\mu})\backslash\{\boldsymbol{\gamma}\}\\ \mathrm{pos}(\boldsymbol{\alpha})=j}} (c_{n+1} - c(\boldsymbol{\alpha}))^{-1} = \mathsf{F}^{(j)}_{\boldsymbol{\lambda}} \big(\mathsf{F}^{(j)}_{\boldsymbol{\mu}}\big)^{-1}. \qquad (26)$$

Let $p_1 < p_2 < \cdots < p_s$ be positive integers such that the j-th partition of $\boldsymbol{\mu}$ is $(\mu_1,\dots,\mu_{p_s})$ with

$$\mu_1 = \cdots = \mu_{p_1} > \mu_{p_1+1} = \cdots = \mu_{p_2} > \cdots > \mu_{p_{s-1}+1} = \cdots = \mu_{p_s} > 0.$$

We set $p_0 := 0$, $p_{s+1} := +\infty$ and $\mu_{p_{s+1}} := 0$. Assume that the m-node $\boldsymbol{\gamma}$ lies in the line x and column y. The left-hand side of (26) is equal to

$$\prod_{b=1}^{s} \big(v_k q^{2(y-x)} - v_j q^{2(\mu_{p_b}-p_b)}\big) \prod_{b=1}^{s+1} \big(v_k q^{2(y-x)} - v_j q^{2(\mu_{p_b}-p_{b-1})}\big)^{-1}. \qquad (27)$$

The factors in the product (24) correspond to m-nodes of an m-partition. The m-nodes lying neither in the column y of the k-th diagrams (of $\boldsymbol{\lambda}$ or $\boldsymbol{\mu}$) nor in the line x of the j-th diagrams do not contribute to the right-hand side of (26). Let $t \in \{0,\dots,s\}$ be such that $p_t < x \leq p_{t+1}$. The contribution from the m-nodes in the column y and lines $1,\dots,p_t$ of the k-th diagrams is

$$\prod_{b=1}^{t} \left(\prod_{a=p_{b-1}+1}^{p_b} \frac{v_k q^{-(\mu_{p_b}-y+x-a)} - v_j q^{(\mu_{p_b}-y+x-a)}}{v_k q^{-(\mu_{p_b}-y+x-a+1)} - v_j q^{(\mu_{p_b}-y+x-a+1)}} \right);$$

the contribution from the m-nodes in the column y and lines $p_t+1,\dots,x$ of the k-th diagrams is

$$\prod_{a=p_t+1}^{x-1} \left(\frac{v_k q^{-(\mu_{p_{t+1}}-y+x-a)} - v_j q^{(\mu_{p_{t+1}}-y+x-a)}}{v_k q^{-(\mu_{p_{t+1}}-y+x-a+1)} - v_j q^{(\mu_{p_{t+1}}-y+x-a+1)}} \right) \frac{q^{-cc(\boldsymbol{\gamma})}}{v_k q^{-(\mu_{p_{t+1}}-y+1)} - v_j q^{(\mu_{p_{t+1}}-y+1)}}.$$

The contribution from the m-nodes lying in the line x of the j-th diagrams is

$$\prod_{b=t+1}^{s} \prod_{a=\mu_{p_{b+1}}+1}^{\mu_{p_b}} \frac{v_j q^{-(y-a+p_b-x)} - v_k q^{(y-a+p_b-x)}}{v_j q^{-(y-a+p_b-x+1)} - v_k q^{(y-a+p_b-x+1)}}.$$

After straightforward simplifications, we obtain for the right-hand side of (26)

$$q^{x-y} \prod_{b=1}^{s} \big(v_k q^{-(\mu_{p_b}-y+x-p_b)} - v_j q^{(\mu_{p_b}-y+x-p_b)}\big) \\ \times \prod_{b=1}^{s+1} \big(v_k q^{-(\mu_{p_b}-y+x-p_{b-1})} - v_j q^{(\mu_{p_b}-y+x-p_{b-1})}\big)^{-1}. \tag{28}$$

The comparison of (27) and (28) concludes the proof of the formula (26).

3. The assertion of the lemma is a consequence of the formulas (25), (26) together with the part **1** of the proof. ■

Proof of the Proposition 3. The formula (22) shows that the rational function $F_{\mathcal{T}}(u)$ is non-singular at $u = c_{n+1}$, and moreover

$$F_{\mathcal{T}}(c_{n+1}) = \prod_{\boldsymbol{\beta} \in \mathcal{E}_-(\boldsymbol{\mu})} (c_{n+1} - c(\boldsymbol{\beta})) \prod_{\boldsymbol{\alpha} \in \mathcal{E}_+(\boldsymbol{\mu}) \setminus \{\boldsymbol{\gamma}\}} (c_{n+1} - c(\boldsymbol{\alpha}))^{-1}.$$

We use the Lemma 5 to conclude the proof of the proposition. ■

Proof of Theorem 1. The theorem follows, by induction on n, from the formula (16) together with Propositions 2 and 3. ■

Example. Consider, for $m = 2$, the standard 2-tableau $\big(\boxed{1}\boxed{3}, \boxed{2}\big)$. The idempotent of the algebra $H(2,1,3)$ corresponding to this standard 2-tableau reads, by the Theorem 1,

$$\frac{\overline{\sigma}_2\big(v_1q^2, v_2\big)\overline{\sigma}_1\big(v_1q^2, v_1\big)\overline{\tau}\big(v_1q^2\big)\sigma_1^{-1}\sigma_2^{-1}\overline{\sigma}_1(v_2, v_1)\overline{\tau}(v_2)\sigma_1^{-1}\overline{\tau}(v_1)}{\big(q+q^{-1}\big)\big(v_1q^{-1} - v_2q\big)(v_1 - v_2)\big(v_2q^{-2} - v_1q^2\big)}.$$

5 Remarks on the classical limit

Recall that the group ring $\mathbb{C}G(m,1,n+1)$ of the complex reflection group $G(m,1,n+1)$ is obtained by taking the classical limit: $q \mapsto \pm 1$ and $v_i \mapsto \xi_i$, $i = 1,\dots,m$, where $\{\xi_1,\dots,\xi_m\}$ is the set of distinct m-th roots of unity. The "classical limit" of the generators $\tau, \sigma_1, \dots, \sigma_n$ of $H(m,1,n+1)$ we denote by $t, s_1, \dots, s_n$.

1. Consider the Baxterized elements (5) with spectral parameters of the form $v_p q^{2a}$ and $v_{p'} q^{2a'}$ with $p, p' \in \{1,\dots,m\}$. One directly finds that

$$\lim_{q\to 1} \lim_{v_i \to \xi_i} \overline{\sigma}_i\big(v_p q^{2a}, v_{p'} q^{2a'}\big) = s_i + \frac{\delta_{p,p'}}{a-a'}. \tag{29}$$

For the Artin generators $\tilde{s}_1, \dots, \tilde{s}_n$ of the symmetric group S_{n+1}, the standard Baxterized elements are given by the rational functions

$$\tilde{s}_i + \frac{1}{a-a'} \qquad \text{for} \quad i = 1,\dots,n.$$

In view of (29), we define generalized Baxterized elements for the group $G(m,1,n+1)$ as the following functions

$$\overline{s}_i(p,p',a,a') := s_i + \frac{\delta_{p,p'}}{a-a'} \qquad \text{for} \quad i=1,\dots,n. \tag{30}$$

These elements satisfy the following Yang–Baxter equation with spectral parameters

$$\overline{s}_i(p,p',a,a')\overline{s}_{i+1}(p,p'',a,a'')\overline{s}_i(p',p'',a',a'') \\ = \overline{s}_{i+1}(p',p'',a',a'')\overline{s}_i(p,p'',a,a'')\overline{s}_{i+1}(p,p',a,a').$$

The Baxterized elements (30) have been used in [14] for a fusion procedure for the complex reflection group $G(m,1,n+1)$.

2. It is immediate that

$$\lim_{v_i\to\xi_i} \mathfrak{a}_0(r) = r^m - 1 \qquad \text{and} \qquad \lim_{v_i\to\xi_i} \mathfrak{a}_i(r) = r^{m-i} \qquad \text{for} \quad i=1,\dots,m,$$

where $\mathfrak{a}_i(r)$, $i=0,\dots,m$, are defined in (8). It follows from (10) that

$$\lim_{v_i\to\xi_i} \overline{\tau}(r) = \sum_{i=0}^{m-1} r^{m-1-i} t^i. \tag{31}$$

The rational function $\overline{t}$ defined by $\overline{t}(r) := \frac{1}{m}\sum\limits_{i=0}^{m-1} r^{m-i}t^i$ with values in $\mathbb{C}G(m,1,n+1)$ was used in [14] for a fusion procedure for the complex reflection group $G(m,1,n+1)$.

3. Define, for an m-partition $\boldsymbol{\lambda}$,

$$f_{\boldsymbol{\lambda}} := \left(\prod_{\boldsymbol{\alpha}\in\boldsymbol{\lambda}} h_{\boldsymbol{\lambda}}(\boldsymbol{\alpha})\right)^{-1}.$$

The classical limit of $\mathsf{F}_{\boldsymbol{\lambda}}$ is proportional to $f_{\boldsymbol{\lambda}}$. More precisely, we have

$$\lim_{q\to 1}\lim_{v_i\to\xi_i} \mathsf{F}_{\boldsymbol{\lambda}} = \mathfrak{x}_{\boldsymbol{\lambda}} f_{\boldsymbol{\lambda}}, \qquad \text{where} \quad \mathfrak{x}_{\boldsymbol{\lambda}} = \frac{1}{m^n}\prod_{\boldsymbol{\alpha}\in\boldsymbol{\lambda}} \xi_{\mathrm{pos}(\boldsymbol{\alpha})}. \tag{32}$$

The formula (32) is obtained directly from (12) since

$$\prod_{\substack{i=1\\ i\neq k}}^{m} (\xi_k - \xi_i) = m/\xi_k \qquad \text{for} \quad k=1,\dots,m.$$

4. Using formulas (29), (31) and (32), it is straightforward to check that the classical limit of the fusion procedure for $H(m,1,n+1)$ given by the Theorem 1 leads to the fusion procedure [14] for the group $G(m,1,n+1)$. Also, for $m=1$, Theorem 1 coincides with the fusion procedure [8] for the Hecke algebra and, in the classical limit, with the fusion procedure [12] for the symmetric group.

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