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Special Solutions and Linear Monodromy for the Two-Dimensional Degenerate Garnier System G(1112)

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Abstract. We have classified special solutions around the origin for the two-dimensional degenerate Garnier system G(1112) with generic values of complex parameters, whose linear monodromy can be calculated explicitly.

Key words: two-dimensional degenerate Garnier system; monodromy data

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1 Introduction

We have studied special solutions with generic values of complex parameters for the fourth, fifth, sixth and third Painlevé equations, for which the monodromy data of the associated linear equation (we call linear monodromy) can be calculated explicitly [9, 11, 10, 12]. These papers are based on A.V. Kitaev's idea who calculated first the linear monodromy with generic value of complex parameter explicitly by taking examples of the first and second Painlevé equations [14]. We remark that P. Appell [1] also studied the symmetric solutions to the first and second Painlevé equations, but he did not study linear monodromy problems.

The Garnier system was derived by R. Garnier (1912) as the extension of the sixth Painlevé equation [4]. The original Garnier system has n variables and is expressed in the nonlinear partial differential equations system, whose dimension of the solution space is $2n$. There are few research for the special solutions to the Garnier system compared with Painlevé equations. We will study the Garnier transcendents by applying first the same method to the two-dimensional Garnier system, which we have used for the Painlevé equations above. Some new discovery is expected by viewing Painlevé equations from the Garnier system.

Two-dimensional Garnier system has the following degeneration diagram similar to the Painlevé equations [13]:

$$
\begin{array}{ccccccccc}
G(11111) & \to & G(1112) & \to & G(122) & \to & G(23) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G(113) & \to & G(14) & \to & G(5) & \to & G(9/2)
\end{array}
$$

(The degeneration from G(113) to G(23) also exists.) Numbers in brackets represents a partition of 5. The number 1 represents the regular singular point and the number $r+1$ represents an irregular singular point of Poincaré rank r. The two-dimensional Garnier system G(11111) which is the extension of the sixth Painlevé equation P_{VI} degenerates step by step to the two-dimensional degenerate Garnier system G(9/2) which is the extension of the first Painlevé equation P_{I}.

The purpose of this paper is to obtain the special solutions to the system G(1112), for which the linear monodromy $\{M_0, M_1 = S_1^{(1)} S_2^{(1)} e^{2\pi i T_1}, M_{t_2}, M_\infty\}$ can be calculated explicitly.

The two-dimensional degenerate Garnier system G(1112) $\{K_1, K_2, \lambda_1, \lambda_2, \mu_1, \mu_2, t_1, t_2\}$ is derived as the extension of the fifth Painlevé equation by the isomonodromic deformation of the second kind, non-Fuchsian ordinary differential equation, which has three regular singularities and one irregular singularity of Poincaré rank 1 on the Riemann sphere [13, 16]:

$$\begin{aligned}\frac{d^2\psi}{dx^2} &+ \left[\frac{1-\alpha_0}{x} + \frac{\eta t_1}{(x-1)^2} + \frac{2-\alpha_1}{x-1} + \frac{1-\alpha_2}{x-t_2} - \frac{1}{x-\lambda_1} - \frac{1}{x-\lambda_2}\right]\frac{d\psi}{dx} \\ &+ \left[\frac{\nu(\nu+\alpha_\infty)}{x(x-1)} - \frac{t_1 K_1}{x(x-1)^2} - \frac{t_2(t_2-1)K_2}{x(x-1)(x-t_2)}\right. \\ &\left. + \frac{\lambda_1(\lambda_1-1)\mu_1}{x(x-1)(x-\lambda_1)} + \frac{\lambda_2(\lambda_2-1)\mu_2}{x(x-1)(x-\lambda_2)}\right]\psi = 0, \end{aligned} \tag{1.1}$$

where K_1 and K_2 are Hamiltonians, λ_1, λ_2, μ_1 and μ_2 are the Garnier functions, t_1 and t_2 are deformation parameters and α_j $(j = 0, 1, 2, \infty)$, $\nu \in \mathbb{C}$ and $\eta \in \mathbb{C}^\times$ are complex parameters. The Riemann scheme of (1.1) is

$$P\begin{pmatrix} x=0 & \overbrace{x=1} & & x=t_2 & x=\lambda_1 & x=\lambda_2 & x=\infty & \\ 0 & 0 & 0 & 0 & 0 & 0 & \nu & \\ \alpha_0 & \eta t_1 & \alpha_1 & \alpha_2 & 2 & 2 & \nu+\alpha_\infty & ;x \end{pmatrix},$$

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_\infty = 1 - 2\nu.$$

This is also derived by the confluence of two regular singularities $x = t_1$ and $x = 1$ in the two-dimensional Garnier system G(11111).

G(1112) has movable algebraic branch points and Hamiltonian structure expressed in rational function. We have the two-dimensional degenerate Garnier system $\mathcal{H}_2(1112)\{H_1, H_2, q_1, q_2, p_1, p_2, s_1, s_2\}$ by the canonical transformations:

$$s_1 = \frac{1}{t_1}, \qquad s_2 = \frac{t_2}{t_2-1}, \qquad -t_1(t_2-1)q_1 = (\lambda_1-1)(\lambda_2-1),$$

$$(t_2-1)^2 q_2 = (\lambda_1-t_2)(\lambda_2-t_2), \qquad \mu_i = \frac{q_1 p_1}{\lambda_i - 1} + \frac{q_2 p_2}{\lambda_i - t_2}, \qquad i = 1, 2,$$

$$H_1 = -t_1^2\left(K_1 + \sum_{j=1}^{2} p_j \frac{\partial q_j}{\partial t_1}\right), \qquad H_2 = -(t_2-1)^2\left(K_2 + \sum_{j=1}^{2} p_j \frac{\partial q_j}{\partial t_2}\right),$$

$$\sum_{j=1}^{2}(dp_j \wedge dq_j - dH_j \wedge ds_j) = \sum_{j=1}^{2}(d\mu_j \wedge d\lambda_j - dK_j \wedge dt_j).$$

$\mathcal{H}_2$ has the Painlevé property and the polynomial Hamiltonian structure [7, 13, 16]. We obtain the special solutions in the Hamiltonian system $\mathcal{H}_2$ and then inversely transform them to the solutions in the Hamiltonian system G(1112), which are substituted into the linear equation (1.1). We obtain eight meromorphic solutions with generic values of complex parameters around the origin $(t_1, t_2) = (0, 0)$, which we name the solutions (1), (2), …, (8).

The calculation of the linear monodromy consists of three steps. The first step is taking the limit $(t_1, t_2) \to (0, 0)$ after substituting the solution into the linear equation (1.1). We call this step "the first limit", in which the linear monodromy matrices M_0 and M_{t_2} are calculated as the confluent linear monodromy $M_{t_2}M_0$. In the second step, we separate this confluent linear monodromy $M_{t_2}M_0$. After transforming the linear equation (1.1) by putting $x = t_2\xi$ and substituting the solution into the linear equation (1.1), we take the limit $(t_1, t_2) \to (0, 0)$. We call this step “the second limit”. In the third step, we transform the linear equation (1.1) by putting $x-1=\eta t_1/z$ which keeps the irregularity at $x=1$ so that we can calculate the Stokes matrices $\{S_1^{(1)}, S_2^{(1)}\}$. We call this step “the third limit”.

Each of the obtained eight meromorphic solutions with generic values of complex parameters around the origin $(t_1,t_2)=(0,0)$ has the remarkable characteristics, respectively. The four solutions make the two monodromy matrices commutable and the Stokes matrices around $x=1$ unity and the other four solutions make the three monodromy matrices commutable, which are summarized in Theorem 4.

In Appendix A, we show the fundamental solutions and the associated monodromy matrices of Gauss hypergeometric equation and Kummer's equation. In Appendix B, we show the Briot–Bouquet's theorem for a system of partial differential equations in two variables and short comment on it, how it proves convergence of the eight solutions.

2 The two-dimensional degenerate Garnier system $\mathcal{H}_2(1112)$

In this section, we write down the polynomial Hamiltonians H_1, H_2 and the Hamiltonian system $\mathcal{H}_2(1112)$.

- Hamiltonians H_1 and H_2:

$$\begin{aligned}
s_1^2H_1 &= q_1^2(q_1-s_1)p_1^2+2q_1^2q_2p_1p_2+q_1q_2(q_2-s_2)p_2^2\\
&\quad-\Big[(\alpha_0+\alpha_2-1)q_1^2+\alpha_1q_1(q_1-s_1)+\eta(q_1-s_1)+\eta s_1q_2\Big]p_1\\
&\quad-\Big[(\alpha_0+\alpha_1-1)q_1q_2+\alpha_2q_1(q_2-s_2)-\eta(s_2-1)q_2\Big]p_2+\nu(\nu+\alpha_\infty)q_1,\\
s_2(s_2-1)H_2 &= q_1^2q_2p_1^2+2q_1q_2(q_2-s_2)p_1p_2\\
&\quad+\Big[q_2(q_2-1)(q_2-s_2)+\frac{s_2(s_2-1)}{s_1}q_1q_2\Big]p_2^2\\
&\quad-\Big[(\alpha_0+\alpha_1-1)q_1q_2+\alpha_2q_1(q_2-s_2)-\eta(s_2-1)q_2\Big]p_1\\
&\quad-\Big[(\alpha_0-1)q_2(q_2-1)+\alpha_1q_2(q_2-s_2)+\alpha_2(q_2-1)(q_2-s_2)\\
&\quad+\frac{s_2(s_2-1)}{s_1}(\alpha_2q_1+\eta q_2)\Big]p_2+\nu(\nu+\alpha_\infty)q_2.
\end{aligned}$$

- Hamiltonian system $\mathcal{H}_2(1112)$:

$$\begin{aligned}
&-t_1\frac{\partial q_1}{\partial t_1}=s_1\frac{\partial H_1}{\partial p_1}, \qquad -t_1\frac{\partial q_2}{\partial t_1}=s_1\frac{\partial H_1}{\partial p_2}, \qquad t_1\frac{\partial p_1}{\partial t_1}=s_1\frac{\partial H_1}{\partial q_1}, \qquad t_1\frac{\partial p_2}{\partial t_1}=s_1\frac{\partial H_1}{\partial p_1},\\
&\frac{\partial q_1}{\partial s_2}=\frac{\partial H_2}{\partial p_1}, \qquad \frac{\partial q_2}{\partial s_2}=\frac{\partial H_2}{\partial p_2}, \qquad -\frac{\partial p_1}{\partial s_2}=\frac{\partial H_2}{\partial q_1}, \qquad -\frac{\partial p_2}{\partial s_2}=\frac{\partial H_2}{\partial q_2}.
\end{aligned}$$

Remark 1. We use t_1 $(=1/s_1)$ instead of s_1 to apply the Briot–Bouquet's theorem [3] at the origin.

3 Meromorphic solutions around the origin $(t_1=1/s_1, s_2)=(0,0)$

In this section, we give the calculated meromorphic solutions around $(t_1=1/s_1, s_2)=(0,0)$, which are satisfied with the Hamiltonian system $\mathcal{H}_2(1112)$.

When q_i and p_i $(i=1,2)$ are meromorphic, they have at most a simple pole around $(t_1 = 1/s_1, s_2) = (0,0)$. Let $R = \mathbb{C}\{\{t_1, s_2\}\}$ be the ring of convergent power series of t_1 $(=1/s_1)$ and s_2 around the origin and for $u_1, u_2, \ldots, u_n \in R$, let $\langle u_1, u_2, \ldots, u_n\rangle$ be the ideal of R generated by $u_1, u_2, \ldots, u_n$.

Theorem 1. *For generic values of the complex parameters $\{\alpha_0, \alpha_1, \alpha_2, \alpha_\infty, \nu, \eta\}$, the Hamiltonian system $\mathcal{H}_2(1112)$ has the following eight meromorphic solutions around $(t_1 = 1/s_1, s_2) = (0,0)$:*

$$(1)\quad q_1 = \frac{\eta}{\alpha_\infty} + \langle t_1, s_2\rangle, \qquad q_2 = \frac{\alpha_\infty + \alpha_1}{\alpha_\infty} + \langle t_1, s_2\rangle,$$
$$p_1 = \frac{(\alpha_\infty + \alpha_1)(\nu + \alpha_2)}{\alpha_\infty(1 - \alpha_\infty - \alpha_1)} t_1 s_2 + \langle t_1^3, t_1^2 s_2, s_2^2\rangle, \qquad p_2 = \frac{-\nu\alpha_\infty}{\alpha_\infty + \alpha_1} + \langle t_1, s_2\rangle,$$
$$(2)\quad q_1 = \frac{-\eta}{\alpha_\infty} + \langle t_1, s_2\rangle, \qquad q_2 = \frac{\alpha_\infty - \alpha_1}{\alpha_\infty} + \langle t_1, s_2\rangle,$$
$$p_1 = \frac{(\alpha_\infty - \alpha_1)(\nu + \alpha_2 + \alpha_\infty)}{\alpha_\infty(1 + \alpha_\infty - \alpha_1)} t_1 s_2 + \langle t_1^3, t_1^2 s_2, s_2^2\rangle, \qquad p_2 = \frac{-(\alpha_\infty + \nu)}{\alpha_\infty - \alpha_1}\alpha_\infty + \langle t_1, s_2\rangle,$$
$$(3)\quad q_1 = \frac{-\eta}{\alpha_1} + \langle t_1, s_2\rangle, \qquad q_2 = s_2\left(\frac{\alpha_2}{\alpha_0 + \alpha_2} + \langle t_1, s_2\rangle\right),$$
$$p_1 = \frac{\nu(\nu + \alpha_\infty)}{1 - \alpha_1} t_1 + \langle s_2, t_1^2\rangle, \qquad p_2 = \frac{1}{s_2}\left(\frac{\nu(\nu + \alpha_\infty)}{1 - \alpha_0 - \alpha_2} s_2 + \langle t_1^2, t_1 s_2, s_2^2\rangle\right),$$
$$(4)\quad q_1 = \frac{-\eta}{\alpha_1} + \langle t_1, s_2\rangle, \qquad q_2 = s_2\left(\frac{\alpha_2}{\alpha_2 - \alpha_0} + \langle t_1, s_2\rangle\right),$$
$$p_1 = \frac{(\nu + \alpha_2)(\nu + \alpha_2 + \alpha_\infty)}{1 - \alpha_1} t_1 + \langle s_2, t_1^2\rangle, \qquad p_2 = \frac{1}{s_2}\left(\alpha_0 - \alpha_2 + \langle t_1, s_2\rangle\right),$$
$$(5)\quad q_1 = \frac{1}{t_1}\left(\frac{\alpha_\infty + \alpha_0 + \alpha_2}{\alpha_\infty} + \langle t_1, s_2\rangle\right), \qquad q_2 = \frac{-\alpha_2}{\alpha_\infty} s_2 + \langle t_1^2, t_1 s_2, s_2^2\rangle,$$
$$p_1 = t_1\left(\frac{-\nu\alpha_\infty}{\alpha_\infty + \alpha_0 + \alpha_2} + \langle t_1, s_2\rangle\right), \qquad p_2 = \frac{\eta\nu\alpha_\infty t_1}{(\alpha_1 + 2\nu)(\alpha_\infty + \alpha_0 + \alpha_2)} + \langle s_2, t_1^2\rangle,$$
$$(6)\quad q_1 = \frac{1}{t_1}\left(\frac{\alpha_\infty - \alpha_0 - \alpha_2}{\alpha_\infty} + \langle t_1, s_2\rangle\right), \qquad q_2 = \frac{\alpha_2}{\alpha_\infty} s_2 + \langle t_1^2, t_1 s_2, s_2^2\rangle,$$
$$p_1 = t_1\left(\frac{-(\nu + \alpha_\infty)\alpha_\infty}{\alpha_\infty - \alpha_0 - \alpha_2} + \langle t_1, s_2\rangle\right),$$
$$p_2 = \frac{\eta\alpha_\infty(\nu + \alpha_\infty) t_1}{(\alpha_\infty - \alpha_0 - \alpha_2)(1 - \alpha_\infty + \alpha_0 + \alpha_2)} + \langle s_2, t_1^2\rangle,$$
$$(7)\quad q_1 = \frac{1}{t_1}\left(\frac{\alpha_\infty + \alpha_0 - \alpha_2}{\alpha_\infty} + \langle t_1, s_2\rangle\right), \qquad q_2 = s_2\left(\frac{\alpha_2}{\alpha_\infty} + \langle t_1, s_2\rangle\right),$$
$$p_1 = t_1\left(\frac{-(\nu + \alpha_2)\alpha_\infty}{\alpha_\infty + \alpha_0 - \alpha_2} + \langle t_1, s_2\rangle\right), \qquad p_2 = \frac{1}{s_2}\left(\alpha_\infty + \langle t_1, s_2\rangle\right),$$
$$(8)\quad q_1 = \frac{1}{t_1}\left(\frac{\alpha_\infty - \alpha_0 + \alpha_2}{\alpha_\infty} + \langle t_1, s_2\rangle\right), \qquad q_2 = s_2\left(\frac{-\alpha_2}{\alpha_\infty} + \langle t_1, s_2\rangle\right),$$
$$p_1 = t_1\left(\frac{-(\nu + \alpha_2 + \alpha_\infty)\alpha_\infty}{\alpha_\infty - \alpha_0 + \alpha_2} + \langle t_1, s_2\rangle\right), \qquad p_2 = \frac{1}{s_2}\left(-\alpha_\infty + \langle t_1, s_2\rangle\right).$$

Remark 2.

1. Higher order expansions of these solutions are uniquely determined recursively by the Hamiltonian system and do not contain any other parameter than the complex parameters $\{\alpha_0, \alpha_1, \alpha_2, \alpha_\infty, \nu, \eta\}$.

2. These solutions are convergent by Briot–Bouquet's theorem (see Appendix B).
3. The values of complex parameters are generic and should be excluded the values with which the denominator of the coefficients become zero in the solutions above, that is, $\{\alpha_1, \alpha_\infty, \alpha_1 \pm \alpha_\infty, \alpha_0 \pm \alpha_2, \alpha_0 \pm \alpha_2 \pm \alpha_\infty\} \notin \mathbb{Z}$.

4 The linear monodromy

In this section, we calculate the linear monodromy for the solutions (1) and (5).

4.1 For the solution (1)

4.1.1 The first limit

After substituting the solution (1) into the linear equation (1.1), we take the limit $(t_1, t_2) \to (0, 0)$. Hereafter we call this as the first limit. Then the linear equation (1.1) becomes

$$\frac{d^2\psi_1}{dx^2} + \left(\frac{2-\alpha_0-\alpha_2}{x} + \frac{1-\alpha_1}{x-1} - \frac{1}{x-b_0}\right)\frac{d\psi_1}{dx} + \left[\frac{\nu(\nu+\alpha_\infty)}{x(x-1)} - k_2\left(\frac{1}{x(x-1)} - \frac{1}{x^2}\right) + \frac{m_2}{x(x-1)(x-b_0)}\right]\psi_1 = 0, \tag{4.1}$$

where $b_0 = \frac{\alpha_\infty+\alpha_1}{\alpha_\infty}$, $k_2 = \nu(1-\alpha_0-\alpha_2-\nu)$ and $m_2 = \frac{-\nu\alpha_1}{\alpha_\infty}$. This is a Heun's type equation with the Riemann scheme

$$P\begin{pmatrix} x = 0\cdot t_2 & x=1 & x=b_0 & x=\infty & \\ -\nu & 0 & 0 & \nu & ;x \\ -1+\alpha_0+\alpha_2+\nu & \alpha_1 & 2 & \nu+\alpha_\infty & \end{pmatrix}.$$

The general solution of (4.1) is

$$\psi_1 = c_1 x^{-\nu} + c_2 x^{-1+\alpha_0+\alpha_2+\nu}(x-1)^{\alpha_1}, \qquad c_1, c_2 \in \mathbb{C}.$$

By taking the first limit, two regular singular points become confluent as a regular singular point [2, 8]. The linear monodromy around $x = 0\cdot t_2$ is obtained as a confluent one $\widetilde{M}_{t_2}\widetilde{M}_0$.

The linear monodromy $\{\widetilde{M}_{t_2}\widetilde{M}_0, \widetilde{M}_1, \widetilde{M}_\infty\}$ of (4.1) is

$$\widetilde{M}_{t_2}\widetilde{M}_0 = \begin{pmatrix} e^{-2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\alpha_0+\alpha_2+\nu)} \end{pmatrix}, \qquad \widetilde{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix},$$

$$\widetilde{M}_\infty = \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix}, \qquad \widetilde{M}_\infty\widetilde{M}_1\widetilde{M}_{t_2}\widetilde{M}_0 = I_2.$$

We should separate the confluent linear monodromy $\widetilde{M}_{t_2}\widetilde{M}_0$.

4.1.2 The second limit

In this section, we separate the confluent linear monodromy $\widetilde{M}_{t_2}\widetilde{M}_0$. After transforming the linear equation (1.1) with $x = t_2\xi$ and substituting the solution (1) into (1.1), we take the limit $(t_1, t_2) \to (0, 0)$. Hereafter we call this as the second limit. By taking the second limit, $x = 0$ and $x = t_2$ are separated and $x = 1$ and $x = \infty$ become confluent (see Remark 4). Then $\psi_2(\xi) = \psi(t_2\xi, t_1, t_2)$ satisfies the following Gauss hypergeometric equation after taking the limit $(t_1, t_2) \longrightarrow (0, 0)$,

$$\frac{d^2\psi_2}{d\xi^2} + \left(\frac{1-\alpha_0}{\xi} + \frac{1-\alpha_2}{\xi-1}\right)\frac{d\psi_2}{d\xi} + \frac{k_2}{\xi(\xi-1)}\psi_2 = 0 \tag{4.2}$$

with the Riemann scheme

$$P\begin{pmatrix} (x=0) & (x=t_2) & (x=1\cdot\infty) & \\ \xi=0 & \xi=1 & \xi=\infty & \\ 0 & 0 & \nu & ;\xi \\ \alpha_0 & \alpha_2 & 1-\alpha_0-\alpha_2-\nu & \end{pmatrix}.$$

The linear monodromy $\{M_0, M_{t_2}, M_\infty M_1\}$ of (4.2) is

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0}\end{pmatrix}, \qquad M_{t_2} = C_{01}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2}\end{pmatrix} C_{01},$$

$$M_\infty M_1 = C_{0\infty}^{-1}\begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\alpha_1+\nu+\alpha_\infty)}\end{pmatrix} C_{0\infty}, \qquad M_\infty M_1 M_{t_2} M_0 = I_2,$$

where C_{01} and $C_{0\infty}$ are the connection matrices of the Gauss hypergeometric function (see Appendix A, Lemma 1). The linear monodromy $\{M_0, M_{t_2}, M_\infty M_1\}$ of (4.2) is equivalent to $\{\widetilde{M}_{t_2}\widetilde{M}_0, \widetilde{M}_1, \widetilde{M}_\infty\}$ of (4.1). We have

$$M_1 = P^{-1}\widetilde{M}_1 P, \qquad M_\infty = P^{-1}\widetilde{M}_\infty P, \qquad M_\infty M_1 = P^{-1}\widetilde{M}_\infty\widetilde{M}_1 P$$

for a matrix $P \in \mathrm{GL}(2,\mathbb{C})$. Therefore, $PC_{0\infty}^{-1} \in \mathrm{GL}(2,\mathbb{C})$ is a diagonal matrix. We have

$$M_1 = C_{0\infty}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1}\end{pmatrix} C_{0\infty}, \qquad M_\infty = C_{0\infty}^{-1}\begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)}\end{pmatrix} C_{0\infty}.$$

Remark 3. The formal solution of the linear equation (1.1) around $x = 1$ has the form

$$\Psi^{(1)} = \left(I_2 + \sum_{k=1}^{\infty} \widehat{\Psi}_k^{(1)} (x-1)^k \right) (x-1)^{T_1} e^{\frac{T}{x-1}},$$

where $T = \mathrm{diag}(0, \eta t_1)$, $T_1 = \mathrm{diag}(0, \alpha_1)$, $\widehat{\Psi}_k^{(1)} = \widehat{\Psi}_k^{(1)}(\lambda_1, \lambda_2, \mu_1, \mu_2, t_1, t_2, \alpha_j, \eta)$.

The series $\widehat{\Psi}^{(1)} = I_2 + \sum\limits_{k=1}^{\infty} \widehat{\Psi}_k^{(1)}(x-1)^k$ may be a divergent series since $x = 1$ is an irregular singularity of the Poincaré rank one.

The Stokes regions $\widetilde{\mathcal{S}}_j$ around $x = 1$ are given by

$$\widetilde{\mathcal{S}}_j = \{x \in \mathbb{C} \mid -\varepsilon + (j-1)\pi < \arg(x-1) < j\pi + \varepsilon,\ |x-1| < r\},$$

where ε and r are sufficiently small. There exist holomorphic functions $\widetilde{\Psi}_j(x)$ of (1.1) on $\widetilde{\mathcal{S}}_j$ such that

$$\widetilde{\Psi}_j(x) \sim \widehat{\Psi}^{(1)} \quad \text{for} \quad x \to 1 \quad \text{and} \quad \Psi_j(x) = \widetilde{\Psi}_j(x)(x-1)^{T_1} e^{\frac{T}{x-1}}$$

is a solution of (1.1) on $\widetilde{\mathcal{S}}_j$. The Stokes matrix S_j is defined by

$$\Psi_{j+1} = \Psi_j S_j.$$

We notice that $\Psi_3 = \Psi_1(xe^{-2\pi i})e^{2\pi i T_1}$.

First by taking a limit $t_1 \to 0$ after substituting the solution (1) into the linear equation (1.1), $x = 1$ of (1.1) becomes a regular singular point. Since the coefficients $\widehat{\Psi}_k^{(1)}$ are finite in the limit $t_1 \to 0$, the formal solution $\Psi^{(1)}$ exists even in the limit $t_1 \to 0$. Therefore,

$$\Psi^{(1)}|_{t_1=0} = \left(I_2 + \sum_{k=1}^{\infty} \big[\widehat{\Psi}_k^{(1)}\big]_{t_1=0} (x-1)^k \right) (x-1)^{T_1}$$

has a regular singularity at $x = 1$. Thus the Stokes matrix S_j become I_2 for $j = 1, 2$. Then taking a limit $t_2 \to 0$ makes two regular singular points $x = 1$ and $x = \infty$ confluent [2, 8].

4.1.3 The third limit

In this section, we have Stokes matrices around the irregular singular point $x = 1$ by the transformation of the linear equation (1.1), which keeps the irregularity at $x = 1$. Put

$$x - 1 = \frac{\eta t_1}{z},$$

then $\psi_3(z) = \psi(\frac{\eta t_1}{z} + 1, t_1, t_2)$ satisfies the following degenerate Kummer's equation after taking the limit $(t_1, t_2) \to (0, 0)$,

$$\frac{d^2\psi_3}{dz^2} + \left(\frac{1+\alpha_1}{z} - \frac{1}{z-\alpha_1} - 1\right)\frac{d\psi_3}{dz} = 0.$$

We have the general solution

$$\psi_3 = c_3 e^z z^{-\alpha_1} + c_4, \qquad c_3, c_4 \in \mathbb{C}.$$

This means the formal solution around the irregular singular point $x = 1$ $(z = \infty)$ becomes convergent and Stokes matrices around $x = 1$ become I_2.

Therefore, we have the linear monodromy for the solution (1) explicitly.

Theorem 2. *For the solution* (1)*, the linear monodromy of* (1.1) *is*

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix}, \qquad M_{t_2} = C_{01}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix} C_{01},$$

$$M_1 = S_1^{(1)} S_2^{(1)} C_{0\infty}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix} C_{0\infty}, \qquad S_1^{(1)} = S_2^{(1)} = I_2,$$

$$M_\infty = C_{0\infty}^{-1} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C_{0\infty},$$

$$C_{01} = \begin{pmatrix} \frac{\Gamma(1-\alpha_0)\Gamma(\alpha_2)}{\Gamma(1-\alpha_0-\nu)\Gamma(\alpha_2+\nu)} & \frac{\Gamma(1+\alpha_0)\Gamma(\alpha_2)}{\Gamma(1-\nu)\Gamma(\alpha_0+\alpha_2+\nu)} \\ \frac{\Gamma(1-\alpha_0)\Gamma(-\alpha_2)}{\Gamma(\nu)\Gamma(1-\alpha_0-\alpha_2-\nu)} & \frac{\Gamma(1+\alpha_0)\Gamma(-\alpha_2)}{\Gamma(\alpha_0+\nu)\Gamma(1-\alpha_2-\nu)} \end{pmatrix},$$

$$C_{0\infty} = \begin{pmatrix} \frac{e^{\pi i\nu}\Gamma(1-\alpha_0)\Gamma(1-\alpha_0-\alpha_2-2\nu)}{\Gamma(1-\alpha_0-\alpha_2-\nu)\Gamma(1-\alpha_0-\nu)} & \frac{e^{\pi i(\alpha_0+\nu)}\Gamma(1+\alpha_0)\Gamma(1-\alpha_0-\alpha_2-2\nu)}{\Gamma(1-\nu)\Gamma(1-\alpha_2-\nu)} \\ \frac{e^{\pi i(1-\alpha_0-\alpha_2-\nu)}\Gamma(1-\alpha_0)\Gamma(\alpha_0+\alpha_2+2\nu-1)}{\Gamma(\nu)\Gamma(\alpha_2+\nu)} & \frac{e^{\pi i(1-\alpha_2-\nu)}\Gamma(1+\alpha_0)\Gamma(\alpha_0+\alpha_2+2\nu-1)}{\Gamma(\alpha_0+\alpha_2+\nu)\Gamma(\alpha_0+\nu)} \end{pmatrix},$$

$$M_\infty M_1 M_{t_2} M_0 = I_2,$$

for which $[M_1, M_\infty] = 0$ *holds.*

4.2 For the solutions (2), (3) and (4)

Also for the solutions (2), (3) and (4), we have the similar results, which are summarized as follows.

- For the solution (2):

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix}, \qquad M_{t_2} = C_{01}^{(2)-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix} C_{01}^{(2)},$$

$$M_1 = S_1^{(1)} S_2^{(1)} C_{0\infty}^{(2)-1} \begin{pmatrix} e^{2\pi i\alpha_1} & 0 \\ 0 & 1 \end{pmatrix} C_{0\infty}^{(2)}, \qquad S_1^{(1)} = S_2^{(1)} = I_2,$$

$$M_\infty = C_{0\infty}^{(2)-1} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C_{0\infty}^{(2)}, \qquad C_{01}^{(2)} = C_{01}(\nu+\alpha_1, \nu+\alpha_\infty, 1-\alpha_0),$$

$$C_{0\infty}^{(2)} = C_{0\infty}(\nu+\alpha_1, \nu+\alpha_\infty, 1-\alpha_0), \qquad M_\infty M_1 M_{t_2} M_0 = I_2,$$

for which $[M_1, M_\infty] = 0$ holds.

- For the solution (3):

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix}, \qquad M_{t_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix},$$
$$M_1 = S_1^{(1)} S_2^{(1)} C_{01}^{(3)-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix} C_{01}^{(3)}, \qquad S_1^{(1)} = S_2^{(1)} = I_2,$$
$$M_\infty = C_{0\infty}^{(3)-1} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C_{0\infty}^{(3)}, \qquad C_{01}^{(3)} = C_{01}(\nu, \nu+\alpha_\infty, 1-\alpha_0-\alpha_2),$$
$$C_{0\infty}^{(3)} = C_{0\infty}(\nu, \nu+\alpha_\infty, 1-\alpha_0-\alpha_2), \qquad M_\infty M_1 M_{t_2} M_0 = I_2,$$

for which $[M_0, M_{t_2}] = 0$ holds.

- For the solution (4):

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix}, \qquad M_{t_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix},$$
$$M_1 = S_1^{(1)} S_2^{(1)} C_{01}^{(4)-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix} C_{01}^{(4)}, \qquad S_1^{(1)} = S_2^{(1)} = I_2,$$
$$M_\infty = C_{0\infty}^{(4)-1} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C_{0\infty}^{(4)}, \qquad C_{01}^{(4)} = C_{01}(\nu, \nu+\alpha_\infty, 1-\alpha_2),$$
$$C_{0\infty}^{(4)} = C_{0\infty}(\nu, \nu+\alpha_\infty, 1-\alpha_2), \qquad M_\infty M_1 M_{t_2} M_0 = I_2,$$

for which $[M_0, M_{t_2}] = 0$ holds.

4.3 For the solution (5)

In this section, we calculate the linear monodromy for the solution (5) by the similar way to Subsection 4.1.

4.3.1 The first limit

After substituting the solution (5) into the linear equation (1.1), we take the limit $(t_1, t_2) \to (0, 0)$. At first we take a limit $t_1 \to 0$ keeping t_2 as a non-zero constant, then we take $t_2 \to 0$. Then the linear equation (1.1) becomes

$$\frac{d^2\psi_1}{dx^2} + \left(\frac{1-\alpha_0-\alpha_2}{x} + \frac{2-\alpha_1}{x-1} - \frac{1}{x-b_0} \right) \frac{d\psi_1}{dx} + \left[\frac{\nu(\nu+\alpha_\infty)}{x(x-1)} - \frac{k_1}{x(x-1)^2} + \frac{m_1}{x(x-1)(x-b_0)} \right] \psi_1 = 0, \tag{4.3}$$

where $b_0 = \frac{\alpha_0+\alpha_2}{-\alpha_\infty}$, $k_1 = \nu(\nu+\alpha_1-1)$ and $m_1 = \frac{\nu(\alpha_0+\alpha_2)}{\alpha_\infty}$. This is a Heun's type equation with the Riemann scheme

$$P \begin{pmatrix} x = 0 \cdot t_2 & x = 1 & x = b_0 & x = \infty & \\ 0 & -\nu & 0 & \nu & \\ \alpha_0+\alpha_2 & \nu+\alpha_1-1 & 2 & \nu+\alpha_\infty & ;x \end{pmatrix}.$$

The general solution of (4.3) is

$$\psi_1 = c_5(x-1)^{-\nu} + c_6 x^{\alpha_0+\alpha_2}(x-1)^{\nu+\alpha_1-1}, \qquad c_5, c_6 \in \mathbb{C}.$$

The linear monodromy $\{\widetilde{M}_{t_2}\widetilde{M}_0, \widetilde{M}_1, \widetilde{M}_\infty\}$ of (4.3) is

$$\widetilde{M}_{t_2}\widetilde{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(\alpha_0+\alpha_2)} \end{pmatrix}, \qquad \widetilde{M}_1 = \begin{pmatrix} e^{-2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_1)} \end{pmatrix},$$
$$\widetilde{M}_\infty = \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix}, \qquad \widetilde{M}_\infty\widetilde{M}_1\widetilde{M}_{t_2}\widetilde{M}_0 = I_2.$$

We should separate the confluent linear monodromy $\widetilde{M}_{t_2}\widetilde{M}_0$.

4.3.2 The second limit

In this section, we separate the confluent linear monodromy $\widetilde{M}_{t_2}\widetilde{M}_0$. After transforming the linear equation (1.1) with $x = t_2\xi$ and substituting the solution (5) into (1.1), we take the limit $(t_1, t_2) \to (0, 0)$ as the same as the first limit.

By taking the second limit, $x = 0$ and $x = t_2$ are separated and $x = 1$ and $x = \infty$ become confluent. Then $\psi_2(\xi) = \psi(t_2\xi, t_1, t_2)$ satisfies the following degenerate Heun's equation after taking the limit $(t_1, t_2) \to (0, 0)$,

$$\frac{d^2\psi_2}{d\xi^2} + \left(\frac{1-\alpha_0}{\xi} + \frac{1-\alpha_2}{\xi-1} - \frac{1}{\xi-\xi_{\lambda_2}}\right)\frac{d\psi_2}{d\xi} = 0, \tag{4.4}$$

where $\xi_{\lambda_2} = \frac{\alpha_0}{\alpha_0+\alpha_2}$. We have the general solution

$$\psi_2 = c_7 + c_8\xi^{\alpha_0}(\xi-1)^{\alpha_2}, \qquad c_7, c_8 \in \mathbb{C}.$$

The linear monodromy $\{M_0, M_{t_2}, M_\infty M_1\}$ of (4.4) is

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix}, \qquad M_{t_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix}, \qquad M_\infty M_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i(\alpha_0+\alpha_2)} \end{pmatrix},$$
$$M_\infty M_1 M_{t_2} M_0 = I_2.$$

4.3.3 The third limit

In this section, we calculate the Stokes matrices around the irregular singular point $x = 1$ by the transformation of the linear equation (1.1), which keeps the irregularity at $x = 1$. Put

$$x - 1 = \frac{\eta t_1}{z}, \qquad \psi = \left(\eta^{-1}z\right)^{\nu}\psi_3(z, t_1, t_2),$$

then $\psi_3(z, t_1, t_2) = (\eta^{-1}z)^{-\nu}\psi(\frac{\eta t_1}{z} + 1, t_1, t_2)$ satisfies the following Kummer's confluent hypergeometric equation after taking the limit $(t_1, t_2) \longrightarrow (0, 0)$,

$$\frac{d^2\psi_3}{dz^2} + \left(\frac{2\nu+\alpha_1}{z} - 1\right)\frac{d\psi_3}{dz} - \frac{\nu}{z}\psi_3 = 0.$$

A system of the fundamental solutions is

$$\left({}_1F_1(\nu, 2\nu+\alpha_1; z),\ z^{1-2\nu-\alpha_1}{}_1F_1(1-\nu-\alpha_1, 2-2\nu-\alpha_1; z)\right).$$

We have the Stokes matrices $S_1^{(1)}$ and $S_2^{(1)}$ around the irregular singular point $x = 1$ $(z = \infty)$ (see Appendix A, Lemma 2).

Summarizing the calculations above, we have the following theorem:

Theorem 3. *For the solution* (5), *the linear monodromy of* (1.1) *is*

$$M_0 = C^{(5)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix} C^{(5)-1}, \qquad M_{t_2} = C^{(5)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix} C^{(5)-1},$$

$$M_1 = S_1^{(1)} S_2^{(1)} e^{2\pi i T_1}, \qquad e^{2\pi i T_1} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix}, \qquad S_1^{(1)} = \begin{pmatrix} 1 & 0 \\ \frac{-2\pi i e^{\pi i\alpha_1}}{\Gamma(\nu)\Gamma(1-\nu-\alpha_1)} & 1 \end{pmatrix},$$

$$S_2^{(1)} = \begin{pmatrix} 1 & \frac{-2\pi i e^{-2\pi i\alpha_1}}{\Gamma(1-\nu)\Gamma(\nu+\alpha_1)} \\ 0 & 1 \end{pmatrix}, \qquad M_\infty = C^{(5)} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C^{(5)-1},$$

$$C^{(5)} = C(\nu, 2\nu+\alpha_1) = \begin{pmatrix} \frac{\Gamma(2\nu+\alpha_1)}{\Gamma(\nu+\alpha_1)} e^{\pi i\nu} & \frac{\Gamma(2-2\nu-\alpha_1)}{\Gamma(1-\nu)} e^{\pi i(1-\nu-\alpha_1)} \\ \frac{\Gamma(2\nu+\alpha_1)}{\Gamma(\nu)} & \frac{\Gamma(2-2\nu-\alpha_1)}{\Gamma(1-\nu-\alpha_1)} \end{pmatrix},$$

$$M_\infty M_{t_2} M_0 S_1^{(1)} S_2^{(1)} e^{2\pi i T_1} = I_2,$$

for which $[M_0, M_{t_2}] = 0$, $[M_0, M_\infty] = 0$ *and* $[M_{t_2}, M_\infty] = 0$ *hold and* $C^{(5)}$ *is the connection matrix of Kummer's confluent hypergeometric function* (*see Appendix* A, *Lemma* 2).

Remark 4. If we take a limit $(t_1, t_2) \to (0, 0)$ along a curve $s_2 \sim A t_1^2 (A \in \mathbb{C}^\times)$, the limit of the last term in (1.1)

$$\frac{\lambda_2(\lambda_2 - 1)\mu_2}{x(x-1)(x-\lambda_2)}$$

is not zero. In our calculation, we take a special path from (t_1, t_2) to $(0, 0)$, such that the numerator of the above term tends to zero. Therefore we obtain different limit equations when we choose different paths for the first and the second limit equations. It may be a contradiction. But the whole of linear monodromy is the same even though some limit equations are different, since we have the Riemann–Hilbert correspondence. In our case, the third limit is the same for any path from (t_1, t_2) to $(0, 0)$, which is the main part of the linear monodromy for the solution (5).

4.4 For the solutions (6), (7) and (8)

We can determine the linear monodromy for the other solutions (6), (7) and (8), which are summarized as follows.

- For the solution (6):

$$M_0 = C^{(6)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix} C^{(6)-1}, \qquad M_{t_2} = C^{(6)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix} C^{(6)-1},$$

$$M_\infty = C^{(6)} \begin{pmatrix} e^{2\pi i(\nu+\alpha_\infty)} & 0 \\ 0 & e^{2\pi i\nu} \end{pmatrix} C^{(6)-1}, \qquad M_1 = S_1^{(1)} S_2^{(1)} e^{2\pi i T_1},$$

$$e^{2\pi i T_1} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix}, \qquad C^{(6)} = C(\nu+\alpha_\infty, 2\nu+2\alpha_\infty+\alpha_1),$$

$$S_1^{(1)} = S_1^{(\infty)}(\nu+\alpha_\infty, 2\nu+2\alpha_\infty+\alpha_1), \qquad S_2^{(1)} = S_2^{(\infty)}(\nu+\alpha_\infty, 2\nu+2\alpha_\infty+\alpha_1),$$

$$M_\infty M_{t_2} M_0 S_1^{(1)} S_2^{(1)} e^{2\pi i T_1} = I_2,$$

for which $[M_0, M_{t_2}] = 0$, $[M_0, M_\infty] = 0$ and $[M_{t_2}, M_\infty] = 0$ hold.

- For the solution (7):

$$M_0 = C^{(7)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_0} \end{pmatrix} C^{(7)-1}, \qquad M_{t_2} = C^{(7)} \begin{pmatrix} e^{2\pi i\alpha_2} & 0 \\ 0 & 1 \end{pmatrix} C^{(7)-1},$$

$$M_\infty = C^{(7)} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C^{(7)-1}, \qquad M_1 = S_1^{(1)} S_2^{(1)} e^{2\pi i T_1},$$
$$e^{2\pi i T_1} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix}, \qquad C^{(7)} = C(\nu+\alpha_2, 2\nu+2\alpha_2+\alpha_1),$$
$$S_1^{(1)} = S_1^\infty(\nu+\alpha_2, 2\nu+2\alpha_2+\alpha_1), \qquad S_2^{(1)} = S_2^\infty(\nu+\alpha_2, 2\nu+2\alpha_2+\alpha_1),$$
$$M_\infty M_{t_2} M_0 S_1^{(1)} S_2^{(1)} e^{2\pi i T_1} = I_2,$$

for which $[M_0, M_{t_2}] = 0$, $[M_0, M_\infty] = 0$ and $[M_{t_2}, M_\infty] = 0$ hold.

- For the solution (8):

$$M_0 = C^{(8)} \begin{pmatrix} e^{2\pi i\alpha_0} & 0 \\ 0 & 1 \end{pmatrix} = C^{(8)-1}, \qquad M_{t_2} = C^{(8)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_2} \end{pmatrix} C^{(8)-1},$$
$$M_\infty = C^{(8)} \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{2\pi i(\nu+\alpha_\infty)} \end{pmatrix} C^{(8)-1}, \qquad M_1 = S_1^{(1)} S_2^{(1)} e^{2\pi i T_1},$$
$$e^{2\pi i T_1} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\alpha_1} \end{pmatrix}, \qquad C^{(8)} = C(\nu+\alpha_2+\alpha_\infty, 2\nu+2\alpha_2+2\alpha_\infty+\alpha_1),$$
$$S_1^{(1)} = S_1^\infty(\nu+\alpha_2+\alpha_\infty, 2\nu+2\alpha_2+2\alpha_\infty+\alpha_1),$$
$$S_2^{(1)} = S_2^\infty(\nu+\alpha_2+\alpha_\infty, 2\nu+2\alpha_2+2\alpha_\infty+\alpha_1),$$
$$M_\infty M_{t_2} M_0 S_1^{(1)} S_2^{(1)} e^{2\pi i T_1} = I_2,$$

for which $[M_0, M_\infty] = 0$, $[M_0, M_{t_2}] = 0$ and $[M_{t_2}, M_\infty] = 0$ hold.

Summarizing the all calculations above, we have the following theorem:

Theorem 4. *The eight meromorphic solutions around the origin of the two-dimensional degenerate Garnier system* $\mathrm{G}_2(1112)$ *have the following characteristics:*

- *For the solution* (1) *and* (2), $[M_1, M_\infty] = 0$ *and* $S_1^{(1)} = S_2^{(1)} = I_2$ *hold.*
- *For the solution* (3) *and* (4), $[M_0, M_{t_2}] = 0$ *and* $S_1^{(1)} = S_2^{(1)} = I_2$ *hold.*
- *For the solution* (5), (6), (7) *and* (8), $[M_0, M_\infty] = 0$, $[M_0, M_{t_2}] = 0$ *and* $[M_{t_2}, M_\infty] = 0$ *hold.*

Remark 5. For the linear monodromy data $\{M_0, M_\infty, S_1^{(1)}, S_2^{(1)}, e^{2\pi i T_1}\}$ of the fifth Painlevé equation, there are three meromorphic solutions around the origin: two solutions such that $[M_0, M_\infty] = 0$ and one solution such that $S_1^{(1)} = S_2^{(1)} = I_2$ [11].

A Gauss hypergeometric equation and Kummer's equation

In this appendix, we show the fundamental solutions and the associated monodromy matrices of Gauss hypergeometric equation [15] and Kummer's equation [6].

A.1 Gauss hypergeometric equation

The Gauss hypergeometric equation is

$$x(1-x)\frac{d^2\psi}{dx^2} + (\gamma - (\alpha+\beta+1)x)\frac{d\psi}{dx} - \alpha\beta\psi = 0. \tag{A.1}$$

The Riemann scheme of (A.1) is

$$P\begin{pmatrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & \alpha & \\ 1-\gamma & \gamma-\alpha-\beta & \beta & ;x \end{pmatrix},$$

We list fundamental systems of solutions for (A.1) around $x=0,1,\infty$.

- Around $x=0$:

$$\psi^{(0)}=\left({}_2F_1(\alpha,\beta,\gamma;x)\quad x^{1-\gamma}{}_2F_1(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;x)\right).$$

- Around $x=1$:

$$\psi^{(1)}=\left(\psi_1^{(1)}\quad\psi_2^{(1)}\right),$$
$$\psi_1^{(1)}={}_2F_1(\alpha,\beta,\alpha+\beta-\gamma+1;1-x),$$
$$\psi_2^{(1)}=(1-x)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma+1-\alpha-\beta;1-x).$$

- Around $x=\infty$:

$$\psi^{(\infty)}=\left(\psi_1^{(\infty)}\quad\psi_2^{(\infty)}\right),$$
$$\psi_1^{(\infty)}=x^{-\alpha}{}_2F_1\left(\alpha,\alpha-\gamma+1,\alpha+1-\beta;x^{-1}\right),$$
$$\psi_2^{(\infty)}=x^{-\beta}{}_2F_1\left(\beta,\beta-\gamma+1,\beta+1-\alpha;x^{-1}\right).$$

The associated monodromy matrices M_j $(j=0,1,\infty)$ are as follows

$$M_0=\begin{pmatrix}1 & 0\\ 0 & e^{-2\pi i\gamma}\end{pmatrix},\qquad M_1=C_{01}(\alpha,\beta,\gamma)^{-1}\begin{pmatrix}1 & 0\\ 0 & e^{2\pi i(\gamma-\alpha-\beta)}\end{pmatrix}C_{01}(\alpha,\beta,\gamma),$$
$$M_\infty=C_{0\infty}(\alpha,\beta,\gamma)^{-1}\begin{pmatrix}e^{2\pi i\alpha} & 0\\ 0 & e^{2\pi i\beta}\end{pmatrix}C_{0\infty}(\alpha,\beta,\gamma),\qquad M_\infty M_1M_0=I_2,$$

where $C_{01}(\alpha,\beta,\gamma)$ and $C_{0\infty}(\alpha,\beta,\gamma)$ are connection matrices which are shown in the following lemma.

Lemma 1. *The Gauss hypergeometric function which is the solution of* (A.1) *has the following connection matrices between fundamental solutions around two singularities:*

$$\psi^{(i)}=\psi^{(j)}C_{ij}(\alpha,\beta,\gamma),\qquad i,j\in\{0,1,\infty\},$$

where $\psi^{(\nu)}$ $(\nu\in\{0,1,\infty\})$ is the fundamental solution around the singularity ν and $C_{ij}(\alpha,\beta,\gamma)$ are the connection matrices which are shown as follows

$$C_{01}(\alpha,\beta,\gamma)=\begin{pmatrix}\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\\ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} & \frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)}\end{pmatrix},$$
$$C_{0\infty}(\alpha,\beta,\gamma)=\begin{pmatrix}\frac{e^{\alpha\pi i}\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} & \frac{e^{(\alpha-\gamma+1)\pi i}\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(1-\gamma+\beta)}\\ \frac{e^{\beta\pi i}\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} & \frac{e^{(\beta-\gamma+1)\pi i}\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(1-\gamma+\alpha)}\end{pmatrix},$$
$$C_{\infty1}(\alpha,\beta,\gamma)=\begin{pmatrix}\frac{\Gamma(1+\alpha-\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(1-\beta)} & \frac{\Gamma(1+\beta-\alpha)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)}\\ \frac{e^{(\gamma-\alpha-\beta)\pi i}\Gamma(1+\alpha-\beta)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(\alpha)} & \frac{e^{(\gamma-\alpha-\beta)\pi i}\Gamma(1+\beta-\alpha)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1+\beta-\gamma)\Gamma(\beta)}\end{pmatrix}.$$

A.2 Kummer's equation

The Kummer's equation is

$$\frac{d^2\phi}{dx^2} + \left(\frac{\gamma}{x} - 1\right)\frac{d\phi}{dx} - \frac{\alpha}{x}\phi = 0. \tag{A.2}$$

The Riemann scheme of (A.2) is

$$P\begin{pmatrix} x=0 & \overbrace{x=\infty \quad\quad}^{} & & \\ 0 & 0 \quad \alpha & & \\ 1-\gamma & 1 \quad \gamma-\alpha & & ;x \end{pmatrix}.$$

Fundamental systems of solutions for (A.2) is given by

$$\phi^{(0)} = \begin{pmatrix} {}_1F_1(\alpha,\gamma;x) & x^{1-\gamma}{}_1F_1(\alpha+1-\gamma, 2-\gamma; x)\end{pmatrix}.$$

Asymtotic solutions around $x=\infty$ is given by

$$\phi_1^{(\infty)}(e^{-\pi i}x) \sim x^{-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha)_k(\alpha+1-\gamma)_k}{k!(e^{-\pi i}x)^k},$$
$$\phi_2^{(\infty)}(x) \sim e^x x^{\alpha-\gamma}\sum_{k=0}^{\infty}\frac{(\gamma-\alpha)_k(1-\alpha)_k}{k!x^k}.$$

The associated monodromy matrices are as follows

$$M_0 = C(\alpha,\gamma)\begin{pmatrix}1 & 0\\ 0 & e^{-2\pi i\gamma}\end{pmatrix}C(\alpha,\gamma)^{-1}, \qquad M_\infty = S_1^{(\infty)}(\alpha,\gamma)S_2^{(\infty)}(\alpha,\gamma)e^{2\pi i T_\infty},$$
$$e^{2\pi i T_\infty} = \begin{pmatrix}e^{2\pi i\alpha} & 0\\ 0 & e^{2\pi i(\gamma-\alpha)}\end{pmatrix}, \qquad M_0 M_\infty = I_2,$$

where $C(\alpha,\gamma)$, $S_1^{(\infty)}(\alpha,\gamma)$ and $S_2^{(\infty)}(\alpha,\gamma)$ are connection matrix and Stokes matrices respectively which are shown in the following lemma.

Lemma 2. *The Kummer's confluent hypergeometric function which is the solution of* (A.2) *has the following connection matrix between fundamental solutions around $x=0$ and $x=\infty$ and Stokes matrices around $x=\infty$:*

- *Connection matrix:*

$$\begin{pmatrix}{}_1F_1(\alpha,\gamma;x) & x^{1-\gamma}{}_1F_1(\alpha+1-\gamma,2-\gamma;x)\end{pmatrix} = \begin{pmatrix}\phi_1^{(\infty)}(e^{-\pi i}x) & \phi_2^{(\infty)}(x)\end{pmatrix}C(\alpha,\gamma),$$

where $\phi_i^{(\infty)}$ $(i\in\{1,2\})$ is the fundamental solutions around the singularity $x=\infty$ and $C(\alpha,\gamma)$ is the connection matrix which is shown as follows

$$C(\alpha,\gamma) = \begin{pmatrix}\frac{\Gamma(\gamma)e^{\alpha\pi i}}{\Gamma(\gamma-\alpha)} & \frac{\Gamma(2-\gamma)e^{\pi i(1+\alpha-\gamma)}}{\Gamma(1-\alpha)}\\ \frac{\Gamma(\gamma)}{\Gamma(\alpha)} & \frac{\Gamma(2-\gamma)}{\Gamma(1+\alpha-\gamma)}\end{pmatrix}.$$

- *Stokes matrices:*

$$S_1^{(\infty)}(\alpha,\gamma) = \begin{pmatrix}1 & 0\\ \frac{-2\pi i e^{\pi i(\gamma-2\alpha)}}{\Gamma(\alpha)\Gamma(1+\alpha-\gamma)} & 1\end{pmatrix}, \qquad S_2^{(\infty)}(\alpha,\gamma) = \begin{pmatrix}1 & \frac{-2\pi i e^{\pi i(4\alpha-2\gamma)}}{\Gamma(1-\alpha)\Gamma(\gamma-\alpha)}\\ 0 & 1\end{pmatrix}.$$

B Briot–Bouquet's theorem for a system of partial differential equations in two variables

Briot and Bouquet [3] showed that existence of a holomorphic solution for a special type of nonlinear ordinary differential equations. In this section we explain the Briot–Bouquet's type theorem for a system of partial differential equations in two variables following [5].

B.1 Briot–Bouquet's theorem

Briot and Bouquet studied a nonlinear ordinary differential equation

$$x\frac{dz}{dx} = h(z, x), \qquad z = (z_1, \dots, z_n) \tag{B.1}$$

for $h(0, 0) = 0$. They have shown that if the eigenvalues of the Jacobi matrix $(\frac{\partial h}{\partial z}(0, 0))$ are not positive integers, then (B.1) has a convergent holomorphic solution.

R. Gerard and Y. Sibuya [5] studied the Briot–Bouquet's type theorem for a system of partial differential equations in two variables. They have shown that a formal solution will be convergent:

Lemma 3. *Assuming that h_1 and h_2 are holomorphic functions of z, x_1 and x_2 and $z(0, 0) = 0$, $h_1(0, 0, 0) = h_2(0, 0, 0) = 0$. If the simultaneous equations*

$$x_1\frac{\partial z}{\partial x_1} = h_1(z, x_1, x_2), \qquad x_2\frac{\partial z}{\partial x_2} = h_2(z, x_1, x_2)$$

have the formal solutions around $(x_1, x_2) = (0, 0)$ expressed in power series of x_1 and x_2, they are convergent.

B.2 Convergence of the solutions

Solutions (1) and (2) are convergent by Lemma 3. For the solutions with a pole, for example, solutions (7) and (8), we put

$$q_1 = \frac{Q_1}{t_1}, \qquad p_1 = t_1 P_1, \qquad q_2 = s_2 Q_2, \qquad p_2 = \frac{P_2}{s_2},$$

where Q_1, P_1, Q_2 and P_2 are holomorphic functions of t_1 and s_2 near $(t_1, s_2) = (0, 0)$. Substituting these into the Hamiltonian system $\mathcal{H}_2$, it becomes all Briot–Bouquet's type differential equations with respect to Q_1, P_1, Q_2 and P_2.

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