Hypergeometric Solutions of the $A_4^{(1)}$-Surface q-Painlevé IV Equation

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Abstract. We consider a q-Painlevé IV equation which is the $A_4^{(1)}$-surface type in the Sakai's classification. We find three distinct types of classical solutions with determinantal structures whose elements are basic hypergeometric functions. Two of them are expressed by $_2\varphi_1$ basic hypergeometric series and the other is given by $_2\psi_2$ bilateral basic hypergeometric series.

Key words: q-Painlevé equation; basic hypergeometric function; affine Weyl group; τ-function; projective reduction; orthogonal polynomial

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1   Introduction

The focus of this paper is on the following single second-order ordinary difference equation:

$$(X_{n+1}X_n - 1)(X_{n-1}X_n - 1)$$
$$= q^{(-N+2n-m-1)/2}a_0a_1^{3/2}a_2^2 \frac{\left(X_n + q^{(N-m)/2}a_1^{1/2}\right)\left(X_n + q^{(-N+m)/2}a_1^{-1/2}\right)}{X_n + q^{(-N+n-m)/2}a_1^{1/2}a_2}, \tag{1.1}$$

where $n \in \mathbb{Z}$ is the independent variable, $X_n = X_n(m,N)$ is the dependent variable and $m, N \in \mathbb{Z}$ and a_0, a_1, a_2, q $(|q| < 1) \in \mathbb{C}^*$ are parameters. Equation (1.1) is known as a q-discrete analog of the Painlevé IV equation (q-P$_{\text{IV}}$) [35].

In 2001, Sakai introduced a geometric approach to the theory of the Painlevé and discrete Painlevé equations (Painlevé systems) and showed the classifications of Painlevé systems by the rational surface [37]. The rational surface can be identified with the space of initial condition, and the group of Cremona isometries associated with the surface generate the affine Weyl group. He also showed that the translation part of the affine Weyl group gives rise to various discrete Painlevé equations. Then, such discrete Painlevé equations are said to have the affine Weyl group symmetries.

In 2004, q-P$_{\text{IV}}$ (1.1) was generalized to the following simultaneous first-order ordinary difference equations by the singularity confinement criterion [38]:

$$(y_{k+1}x_k - 1)(y_kx_k - 1)$$
$$= q^{(-N+4k-m+2l-2)/2}a_0a_1^{3/2}a_2^2a_3 \frac{\left(x_k + q^{(N-m)/2}a_1^{1/2}\right)\left(x_k + q^{(-N+m)/2}a_1^{-1/2}\right)}{x_k + q^{(-N+2k-m+2l)/2}a_1^{1/2}a_2}, \tag{1.2a}$$

$$(y_kx_k - 1)(y_kx_{k-1} - 1)$$
$$= q^{(-N+4k-m+2l-4)/2}a_0a_1^{3/2}a_2^2a_3 \frac{\left(y_k + q^{(N-m)/2}a_1^{1/2}\right)\left(y_k + q^{(-N+m)/2}a_1^{-1/2}\right)}{y_k + q^{(-N+2k-m-2)/2}a_1^{1/2}a_2a_3}, \tag{1.2b}$$

where $k \in \mathbb{Z}$ is the independent variable, $x_k = x_k(l, m, N)$ and $y_k = y_k(l, m, N)$ are dependent variables and $l, m, N \in \mathbb{Z}$ and a_0, a_1, a_2, a_3, q $(|q| < 1) \in \mathbb{C}^*$ are parameters. System (1.2) is known as a q-discrete analog of the Painlevé V equation (q-$\mathrm{P_V}$). It is also known that q-$\mathrm{P_V}$ (1.2) is the $A_4^{(1)}$-surface type in the Sakai's classification and has the affine Weyl group symmetry of type $A_4^{(1)}$.

Conversely, q-$\mathrm{P_{IV}}$ (1.1) can be recovered from q-$\mathrm{P_V}$ (1.2) by putting

$$a_3 = q^{1/2}, \qquad l = 0,$$

and replacing the independent variable and the dependent variables by

$$2k = n, \qquad x_k = X_n, \qquad y_k = X_{n-1}.$$

This procedure is referred to as "*symmetrization*" of q-$\mathrm{P_V}$ (1.2), which comes from the terminology of the Quispel–Roberts–Thompson (QRT) mapping [33, 34]. After this terminology, q-$\mathrm{P_V}$ (1.2) is sometimes called the "*asymmetric discrete Painlevé equation*", and q-$\mathrm{P_{IV}}$ (1.1) is called the "*symmetric discrete Painlevé equation*". It appears as though the symmetrization is a simple specialization on the level of the equation, but the following problems were known:

(i) According to Sakai's theory [37], the discrete Painlevé equations arise as the birational mappings corresponding to the translations of the affine Weyl groups. The asymmetric discrete Painlevé equations are characterized in this manner, however, it was not known how to characterize the symmetric discrete Painlevé equations as the action of affine Weyl groups;

(ii) Painlevé systems admit the particular solutions expressible in terms of the hypergeometric type functions (*hypergeometric solutions*) when some of the parameters take special values (see, for example, [13, 14] and references therein). However, the hypergeometric solutions to the symmetric discrete Painlevé equation cannot be obtained by the naïve specialization of those to the corresponding asymmetric equation.

In [17], the mechanism of the symmetrization was investigated and the nontrivial inconsistency among the hypergeometric solutions were explained in detail by taking an example of q-Painlevé equation with the affine Weyl group symmetry of type $(A_2 + A_1)^{(1)}$. The key to characterize the symmetric discrete Painlevé equation as the action of affine Weyl group is taking the half-step translation instead of a translation as a time evolution. In general, various discrete dynamical systems of Painlevé type can be obtained from elements of infinite order that are not necessarily translations in the affine Weyl group by taking the projection on appropriate subspaces of the parameter spaces. Such a procedure is called a "*projective reduction*".

It is well known that the τ-functions play a crucial role in the theory of integrable systems [25], and it is also possible to introduce them in the theory of Painlevé systems [6, 7, 8, 27, 29, 30, 31, 32]. A representation of the affine Weyl groups can be lifted on the level of the τ-functions [12, 15, 39], which gives rise to various bilinear equations of Hirota type satisfied the τ-functions. Usually, the hypergeometric solutions are derived by reducing the bilinear equations to the Plücker relations by using the contiguity relations satisfied by the entries of determinants [2, 3, 9, 10, 11, 18, 19, 20, 21, 28, 36]. This method is elementary, but it encounters technical difficulties for Painlevé systems with large symmetries. In order to overcome this difficulty, Masuda has proposed a method of constructing hypergeometric solutions under a certain boundary condition on the lattice where the τ-functions live (*hypergeometric τ-functions*), so that they are consistent with the action of the affine Weyl groups [23, 24, 26]. Although this requires somewhat complex calculations, the merit is that it is systematic and that it can be applied to the systems with large symmetries.

In [16], the list of the simplest hypergeometric solutions to the symmetric q-Painlevé equations are shown. In general, hypergeometric solutions of Painlevé systems can be expressed by determinants whose entries are given by hypergeometric type functions. Therefore, it is natural to be curious about the determinant formulae of them. The purpose of this paper is to obtain the determinant formulae of the hypergeometric solutions to the q-P$_{\mathrm{IV}}$ via the construction of the hypergeometric τ-functions and the theory of orthogonal polynomials.

This paper is organized as follows: in Section 2, we first introduce a representation of the affine Weyl group of type $A_4^{(1)}$. Next, we show how q-P$_{\mathrm{V}}$ (1.2) and q-P$_{\mathrm{IV}}$ (1.1) can be derived from the representation. In Section 3, we construct the hypergeometric τ-functions for the q-P$_{\mathrm{IV}}$ and obtain the hypergeometric solutions of the q-P$_{\mathrm{IV}}$ which are expressed by basic hypergeometric series (see Theorems 1 and 2). In Section 4, we obtain the hypergeometric solutions of the q-P$_{\mathrm{IV}}$ which are expressed by bilateral basic hypergeometric series via the theory of orthogonal polynomials (see Theorem 3). Some concluding remarks are given in Section 5.

We use the following conventions of q-analysis with $|q|, |p| < 1$ throughout this paper [1, 22]:

- q-shifted factorials:

$$(a;q)_\infty = \prod_{i=1}^\infty \left(1 - aq^{i-1}\right), \qquad (a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty},$$

  where $\lambda \in \mathbb{C}$;

- Jacobi theta function:

$$\Theta(a;q) = (a;q)_\infty \left(qa^{-1};q\right)_\infty;$$

- Elliptic gamma function:

$$\Gamma(a;p,q) = \frac{\left(pqa^{-1};p,q\right)_\infty}{(a;p,q)_\infty},$$

  where

$$(a;p,q)_k = \prod_{i,j=0}^{k-1} \left(1 - p^i q^j a\right);$$

- Basic hypergeometric series:

$$_s\varphi_r \left(\begin{matrix} a_1, \ldots, a_s \\ b_1, \ldots, b_r \end{matrix}; q, z\right) = \sum_{n=0}^\infty \frac{(a_1, \ldots, a_s;q)_n}{(b_1, \ldots, b_r;q)_n (q;q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+r-s} z^n,$$

  where

$$(a_1, \ldots, a_s;q)_n = \prod_{j=1}^s (a_j;q)_n;$$

- Bilateral basic hypergeometric series:

$$_s\psi_r \left(\begin{matrix} a_1, \ldots, a_s \\ b_1, \ldots, b_r \end{matrix}; q, z\right) = \sum_{n=-\infty}^\infty \frac{(a_1, \ldots, a_s;q)_n}{(b_1, \ldots, b_r;q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{r-s} z^n;$$

- Bilateral q-integral:

$$\int_{-\infty}^{\infty} f(t)\mathrm{d}_q t = (1-q) \sum_{n=-\infty}^{\infty} \big(f(q^n) + f(-q^n)\big)q^n.$$

We note that the following formulae hold:

$$\frac{(a;q)_{\lambda+1}}{(a;q)_\lambda} = 1 - aq^\lambda, \qquad \frac{\Theta(qa;q)}{\Theta(a;q)} = -a^{-1}, \qquad \frac{\Gamma(qa;q,q)}{\Gamma(a;q,q)} = \Theta(a;q).$$

2   Affine Weyl group of type $A_4^{(1)}$

2.1   Birational representation of the affine Weyl group of type $A_4^{(1)}$

In this section, we formulate the family of Bäcklund transformations of q-$\mathrm{P_V}$ (1.2) as a birational representation of the affine Weyl group of type $A_4^{(1)}$.

Let s_i $(i = 0, 1, 2, 3, 4)$, σ and ι be transformations of the parameters a_k $(k = 0, 1, 2, 3, 4)$ and the variables f_j $(j = 0, 1, 2, 3, 4)$. The action of the transformations on the parameters is given by

$$s_i(a_j) = a_j a_i^{-a_{ij}}, \qquad \sigma(a_i) = a_{i+1},$$
$$\iota: \; (a_0, a_1, a_2, a_3, a_4) \mapsto \big(a_0^{-1}, a_4^{-1}, a_3^{-1}, a_2^{-1}, a_1^{-1}\big),$$

where $i, j \in \mathbb{Z}/5\mathbb{Z}$ and the symmetric 5×5 matrix

$$A = (a_{ij})_{i,j=0}^4 = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

is the Cartan matrix of type $A_4^{(1)}$. Moreover, the action on the variables is given by

$$s_i(f_{i+2}) = \frac{a_{i+3}a_{i+4}(a_i a_{i+1} + a_{i+3}f_i)}{a_{i+1}^2 f_{i+3}}, \qquad s_i(f_{i+4}) = \frac{a_{i+4}(a_{i+2} + a_{i+4}a_i f_{i+1})}{a_i a_{i+1} a_{i+2}^2 f_{i+3}},$$
$$s_i(f_j) = f_j, \quad j \neq i+2, i+4, \qquad \sigma(f_i) = f_{i+1},$$
$$\iota: \; (f_0, f_1, f_2, f_3, f_4) \mapsto (f_1, f_0, f_4, f_3, f_2),$$

where $i \in \mathbb{Z}/5\mathbb{Z}$. Note that the variables satisfy the following conditions:

$$a_{i+3}^2 a_{i+4} f_i = a_{i+1}(a_i a_{i+1} f_{i+2} f_{i+3} - a_{i+3}a_{i+4}),$$

where $i \in \mathbb{Z}/5\mathbb{Z}$. The conditions above look like five, but they are essentially three. Therefore, variables f_i are essentially two.

Proposition 1 ([2, 37, 39]). *The group of birational transformations $\widetilde{W}\big(A_4^{(1)}\big) = \langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle$ gives a representation of the (extended) affine Weyl group of type $A_4^{(1)}$. Namely, the transformations satisfy the fundamental relations*

$$s_i^2 = 1, \qquad (s_i s_{i\pm1})^3 = 1, \qquad (s_i s_j)^2 = 1, \quad j \neq i \pm 1, \qquad \sigma^5 = 1, \qquad \sigma s_i = s_{i+1}\sigma,$$
$$\iota^2 = 1, \qquad \iota s_0 = s_0 \iota, \qquad \iota s_1 = s_4 \iota, \qquad \iota s_2 = s_3 \iota,$$

where $i, j \in \mathbb{Z}/5\mathbb{Z}$.

In general, for a function $F = F(a_i, f_j)$, we let an element $w \in \widetilde{W}(A_4^{(1)})$ act as $w.F(a_i, f_j) = F(w.a_i, w.f_j)$, that is, w acts on the arguments from the left. Note that $q = a_0 a_1 a_2 a_3 a_4$ is invariant under the action of $\langle s_0, s_1, s_2, s_3, s_4, \sigma \rangle$. We define the translations T_i $(i = 0, 1, 2, 3, 4)$ by

$$
\begin{aligned}
T_0 &= \sigma s_4 s_3 s_2 s_1, & T_1 &= \sigma s_0 s_4 s_3 s_2, & T_2 &= \sigma s_1 s_0 s_4 s_3, \\
T_3 &= \sigma s_2 s_1 s_0 s_4, & T_4 &= \sigma s_3 s_2 s_1 s_0,
\end{aligned}
\tag{2.1}
$$

whose action on the parameters is given by

$$
\begin{aligned}
T_0 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (qa_0, q^{-1}a_1, a_2, a_3, a_4), \\
T_1 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, qa_1, q^{-1}a_2, a_3, a_4), \\
T_2 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1, qa_2, q^{-1}a_3, a_4), \\
T_3 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1, a_2, qa_3, q^{-1}a_4), \\
T_4 &: (a_0, a_1, a_2, a_3, a_4) \mapsto (q^{-1}a_0, a_1, a_2, a_3, qa_4).
\end{aligned}
$$

Note that T_i $(i = 0, 1, 2, 3, 4)$ commute with each other and $T_0 T_1 T_2 T_3 T_4 = 1$.

2.2 Derivations of the q-Painlevé equations

In this section, we derive the q-Painlevé equations from $\widetilde{W}(A_4^{(1)})$. The action of $T_{23} = T_2 T_3$ on f-variables can be expressed as

$$
(T_{23}(y)x - 1)(yx - 1) = q^{-1} a_0 a_1^{3/2} a_2^2 a_3 \frac{\left(x + a_1^{1/2}\right)\left(x + a_1^{-1/2}\right)}{x + a_1^{1/2} a_2},
\tag{2.2}
$$

$$
(yx - 1)\left(y T_{23}^{-1}(x) - 1\right) = q^{-2} a_0 a_1^{3/2} a_2^2 a_3 \frac{\left(y + a_1^{1/2}\right)\left(y + a_1^{-1/2}\right)}{y + q^{-1} a_1^{1/2} a_2 a_3},
\tag{2.3}
$$

where

$$
x = a_0 a_1^{1/2} a_3^{-1} f_2, \qquad y = a_1^{-1/2} a_2^{-1} a_4^{-1} s_4(f_1).
$$

Applying $T_{23}^k T_2^l T_0^m T_1^N$ on equations (2.2) and (2.3) and putting

$$
x_k(l, m, N) = T_{23}^k T_2^l T_0^m T_1^N(x), \qquad y_k(l, m, N) = T_{23}^k T_2^l T_0^m T_1^N(y),
$$

we obtain q-P$_\mathrm{V}$ (1.2). Then, we can regard T_{23} and T_i $(i = 0, 1, 2, 3, 4)$ as the time evolution and the Bäcklund transformations of q-P$_\mathrm{V}$ (1.2), respectively. We note that considering the action of T_0:

$$
T_0(g)g = \frac{\left(f + a_0^{-3/4} a_1^{1/4} a_3^{-1/4}\right)\left(f + a_0^{-3/4} a_1^{1/4} a_3^{-1/4} a_4^{-1}\right)}{1 + a_0^{-1/4} a_1^{-1/4} a_3^{1/4} f},
$$

$$
T_0^{-1}(f)f = \frac{\left(g + a_0^{-1/4} a_1^{3/4} a_3^{1/4}\right)\left(g + a_0^{-1/4} a_1^{3/4} a_2 a_3^{1/4}\right)}{1 + a_0^{1/4} a_1^{1/4} a_3^{-1/4} g},
$$

where

$$
f = a_0^{-3/4} a_1^{-3/4} a_3^{3/4} f_0, \qquad g = a_0^{3/4} a_1^{3/4} a_3^{-3/4} f_2,
$$

we obtain another q-discrete analog of Painlevé V equation [37]:

$$
g_{n+1} g_n = \frac{\left(f_n + q^{-n + \frac{k+l-m}{4}} a_0^{-\frac{3}{4}} a_1^{\frac{1}{4}} a_3^{-\frac{1}{4}}\right)\left(f_n + q^{-n + \frac{k+l+3m}{4}} a_0^{-\frac{3}{4}} a_1^{\frac{1}{4}} a_3^{-\frac{1}{4}} a_4^{-1}\right)}{1 + q^{\frac{-k-l+m}{4}} a_0^{-\frac{1}{4}} a_1^{-\frac{1}{4}} a_3^{\frac{1}{4}} f_n},
\tag{2.4a}
$$

$$f_{n+1}f_n = \frac{\left(g_{n+1}+q^{-n-1+\frac{3k-l+m}{4}}a_0^{-\frac{1}{4}}a_1^{\frac{3}{4}}a_3^{\frac{1}{4}}\right)\left(g_{n+1}+q^{-n-1+\frac{-k+3l+m}{4}}a_0^{-\frac{1}{4}}a_1^{\frac{3}{4}}a_2a_3^{\frac{1}{4}}\right)}{1+q^{\frac{k+l-m}{4}}a_0^{\frac{1}{4}}a_1^{\frac{1}{4}}a_3^{-\frac{1}{4}}g_{n+1}}, \qquad (2.4b)$$

where

$$f_n = f_n(k,l,m) = T_0^n T_1^k T_2^l T_3^m(f), \qquad g_n = g_n(k,l,m) = T_0^n T_1^k T_2^l T_3^m(g).$$

Thus, q-P_V (1.2) and equation (2.4) are the Bäcklund transformations each other.

In order to derive q-P_{IV} (1.1), we factorize T_{23} as follows

$$T_{23} = R_{23}^2,$$

where R_{23} is given by

$$R_{23} = \sigma s_1 s_0 s_4. \qquad (2.5)$$

The action of R_{23} on the parameters is given by

$$R_{23}: \ (a_0,a_1,a_2,a_3,a_4) \mapsto \left(a_0,a_1,a_2a_3,qa_3^{-1},q^{-1}a_3a_4\right).$$

Let us consider the projection of the action of R_{23} on the line

$$a_3 = q^{1/2}. \qquad (2.6)$$

Then, the action on the parameters becomes translational motion:

$$R_{23}: \ (a_0,a_1,a_2,a_4) \mapsto \left(a_0,a_1,q^{1/2}a_2,q^{-1/2}a_4\right).$$

Since the action of R_{23} on the variable f_2 is given by

$$R_{23}(f_2) = \frac{q}{a_0^2 a_1 f_2}\left(1 + \frac{a_0\left(a_0 f_2 + q^{1/2}\right)\left(a_0 a_1 f_2 + q^{1/2}\right)}{q^{1/2}\left(q^{1/2}a_2 + a_0 f_2\right)f_4}\right),$$
$$R_{23}^{-1}(f_2) = \frac{q}{a_0^2 a_1 f_2}\left(1 + \frac{a_1 a_2^2}{q^{1/2}}f_4\right),$$

we obtain

$$\left(R_{23}(X)X - 1\right)\left(R_{23}^{-1}(X)X - 1\right) = q^{-1/2}a_0 a_1^{3/2}a_2^2 \frac{\left(X + a_1^{1/2}\right)\left(X + a_1^{-1/2}\right)}{X + a_1^{1/2}a_2}, \qquad (2.7)$$

where

$$X = q^{-1/2}a_0 a_1^{1/2} f_2. \qquad (2.8)$$

Applying $R_{23}^n T_0^m T_1^N$ on equation (2.7) and putting

$$X_n(m,N) = R_{23}^n T_0^m T_1^N(X),$$

we obtain q-P_{IV} (1.1). Note that R_{23} commute with T_i ($i = 0,1,4$) and $T_0 T_1 R_{23}^2 T_4 = 1$. Then, R_{23} and T_i ($i = 0,1,4$) are regarded as the time evolution and the Bäcklund transformations of q-P_{IV} (1.1), respectively.

3   Hypergeometric solutions of the q-P_{IV} (I)

In this section, we obtain the hypergeometric solutions of q-P_{IV} (1.1) by constructing the hypergeometric τ-functions for the q-P_{IV}.

3.1 τ-functions

In this section, we define the τ-functions.

We introduce the new variables τ_i $(i = 1, 2, \ldots, 7)$ with

$$f_2 = \frac{\tau_4 \tau_5}{\tau_6 \tau_7}, \qquad f_4 = \frac{\tau_1 \tau_2}{\tau_3 \tau_7}, \tag{3.1}$$

and lift the representation of $\widetilde{W}(A_4^{(1)})$ on their level:

$$s_0(\tau_1) = \frac{a_4(a_0\tau_3\tau_4\tau_5 + a_2a_3\tau_1\tau_2\tau_6 + a_0a_3\tau_3\tau_6\tau_7)}{a_0^2 a_1 a_2 \tau_4 \tau_7}, \qquad s_0(\tau_i) = \tau_i, \quad i = 2, 3, 5, 6,$$

$$s_0(\tau_4) = \frac{a_0 a_4 (a_0\tau_3\tau_4\tau_5 + a_2a_3\tau_1\tau_2\tau_6 + a_3\tau_3\tau_6\tau_7)}{a_1 a_2 \tau_1 \tau_7},$$

$$s_0(\tau_7) = \frac{a_4 \left(a_0^2 \tau_3\tau_4\tau_5 + a_3 a_0 \tau_3\tau_6\tau_7 + a_2 a_3 \tau_1\tau_2\tau_6 \right)}{a_0 a_1 a_2 \tau_1 \tau_4}, \qquad s_1(\tau_1) = \tau_2, \qquad s_1(\tau_2) = \tau_1,$$

$$s_1(\tau_i) = \tau_i, \quad i = 3, \ldots, 7, \qquad s_2(\tau_1) = \frac{a_0 a_1 (a_0\tau_4\tau_5 + a_2 a_3 \tau_6\tau_7)}{a_3^2 \tau_3},$$

$$s_2(\tau_3) = \frac{a_0 a_1 (a_0\tau_4\tau_5 + a_3 \tau_6\tau_7)}{a_2 a_3^2 \tau_1}, \qquad s_2(\tau_i) = \tau_i, \quad i = 2, 4, 5, 6, 7,$$

$$s_3(\tau_4) = \frac{a_2 (a_2 a_3 \tau_1\tau_2 + a_0 \tau_3\tau_7)}{a_0^2 a_3 a_4 \tau_6}, \qquad s_3(\tau_6) = \frac{a_2 a_3 (a_2 \tau_1\tau_2 + a_0 \tau_3\tau_7)}{a_0^2 a_4 \tau_4},$$

$$s_3(\tau_i) = \tau_i, \quad i = 1, 2, 3, 5, 7, \qquad s_4(\tau_4) = s_4(\tau_5), \qquad s_4(\tau_5) = s_4(\tau_4),$$

$$s_4(\tau_i) = \tau_i, \quad i = 1, 2, 3, 6, 7, \qquad \iota: (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7) = (\tau_4, \tau_5, \tau_6, \tau_1, \tau_2, \tau_3, \tau_7),$$

$$\sigma(\tau_1) = \frac{a_0 a_1 (a_0\tau_4\tau_5 + a_3 \tau_6\tau_7)}{a_2 a_3^2 \tau_1}, \qquad \sigma(\tau_2) = \tau_3, \qquad \sigma(\tau_3) = \tau_6,$$

$$\sigma(\tau_4) = \frac{a_4 \left(a_0^2 \tau_3\tau_4\tau_5 + a_3 a_0 \tau_3\tau_6\tau_7 + a_2 a_3 \tau_1\tau_2\tau_6 \right)}{a_0 a_1 a_2 \tau_1 \tau_4}, \qquad \sigma(\tau_5) = \tau_7,$$

$$\sigma(\tau_6) = \tau_5, \qquad \sigma(\tau_7) = \tau_2.$$

Then, we get the following proposition:

Proposition 2 ([39]). *The transformations: s_0, s_1, s_2, s_3, s_4, σ, ι, on the variables τ_i ($i = 1, 2, \ldots, 7$) also realize the (extended) affine Weyl group of type $A_4^{(1)}$.*

Let us define the τ-functions $\tau_N^{n,m}$ $(n, m, N \in \mathbb{Z})$ by

$$\tau_N^{n,m} = R_{23}^n T_0^m T_1^N(\tau_4).$$

By definition, every τ-function can be determined by a rational function in 7 initial variables τ_i $(i = 1, 2, \ldots, 7)$. We note that the 7 initial variables are expressed by the τ-functions as the following (see Fig. 1):

$$\tau_1 = \tau_1^{1,0}, \qquad \tau_2 = \tau_0^{1,1}, \qquad \tau_3 = \tau_1^{1,1}, \qquad \tau_4 = \tau_0^{0,0},$$

$$\tau_5 = \tau_1^{3,1}, \qquad \tau_6 = \tau_1^{2,1}, \qquad \tau_7 = \tau_0^{1,0}. \tag{3.2}$$

3.2 Hypergeometric τ-functions for the q-P_{IV}

The aim of this section is to construct the hypergeometric τ-functions for the q-P_{IV}. We define the hypergeometric τ-functions for the q-P_{IV} by $\tau_N^{n,m}$ consistent with the action of $\langle R_{23}, T_0 \rangle$. Here, we mean $\tau = \tau(\alpha)$ consistent with an action of transformation r as
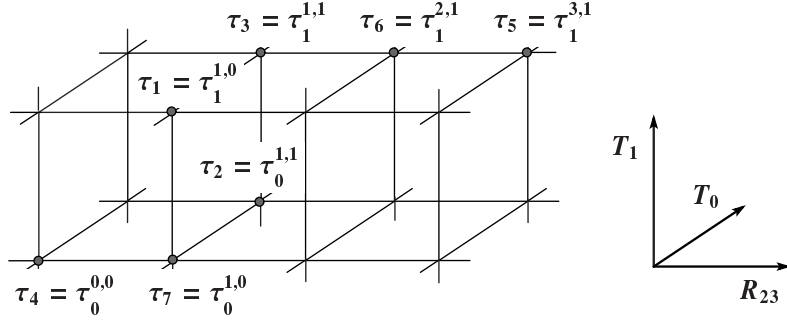
$$r.\tau = \tau(r.\alpha).$$

Figure 1. Configuration of the τ-functions on the 3D-lattice.

Hereinafter, we impose the condition (2.6), and then regard τ-functions $\tau_N^{n,m}$ as the functions in a_0 and a_2 consistent with the action of $\langle R_{23}, T_0 \rangle$, i.e.,

$$\tau_N^{n,m} = \tau_N^{0,0}(q^m a_0, q^{n/2} a_2).$$

By definition, every τ-function $\tau_N^{n,m}$ is determined by a rational function in $\tau_0^{n,m}$ and $\tau_1^{n,m}$ (or $\tau_1, \ldots, \tau_7$). Thus, our purpose is determining $\tau_0^{n,m}$ and $\tau_1^{n,m}$ consistent with the action of $\langle R_{23}, T_0 \rangle$ and constructing the closed-form expressions of $\tau_N^{n,m}$ ($N \geq 2$) under the condition

$$a_0 a_1 = q, \tag{3.3}$$

and the boundary condition

$$\tau_N^{n,m} = 0, \qquad N < 0. \tag{3.4}$$

Henceforth, we construct the hypergeometric τ-functions for the q-$\mathrm{P_{IV}}$ in the following four steps.

Step 1. Conditions of $\tau_0^{n,m}$

In the first step, we obtain the condition of $\tau_0^{n,m}$, which follows from the boundary condition (3.4).

Lemma 1. *The following bilinear equations hold:*

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n-1,m-1} - q^{(-n-4m+4N+7)/2}a_0^{-2}a_2^{-1}\tau_N^{n,m-1}\tau_N^{n-1,m}$$
$$- q^{(-n-2m+4N+4)/2}a_0^{-1}a_2^{-1}\tau_N^{n,m}\tau_N^{n-1,m-1} = 0, \tag{3.5}$$

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n-1,m} + q^{2N-n+1}a_2^{-2}\big(q^{(-n+2N+1)/2}a_2^{-1} - 1\big)\tau_N^{n-1,m}\tau_N^{n,m}$$
$$- q^{(-3n+6N+3)/2}a_2^{-3}\tau_N^{n-2,m}\tau_N^{n+1,m} = 0, \tag{3.6}$$

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n,m} + q^{(-2n+6N+1)/2}a_2^{-2}\big(1 - q^{N-m+1}a_0^{-1}\big)\big(\tau_N^{n,m}\big)^2$$
$$- q^{4N-4m+4}a_0^{-4}\tau_N^{n,m-1}\tau_N^{n,m+1} = 0. \tag{3.7}$$

Proof. Application of T_1 on τ_3 yields the following bilinear equations:

$$T_1(\tau_3)\tau_4 - q^2 a_0^{-1}a_1 a_2^{-1}\tau_1 R_{23}^{-1}(\tau_3) - q^{3/2}a_1 a_2^{-1}\tau_3 R_{23}^{-1}(\tau_1) = 0, \tag{3.8}$$

$$T_1(\tau_3)\tau_2 + q^{3/2}a_0 a_1 a_2^{-2}(1 - a_1)(\tau_3)^2 - a_1^4 \tau_1 T_0(\tau_3) = 0. \tag{3.9}$$

Applying $R_{23}^{n-1}T_0^{m-1}T_1^{N-1}$ on equations (3.8) and (3.9) and substituting condition (3.3) in them, we obtain equations (3.5) and (3.7), respectively. Similarly, application of T_1 on τ_6 yields

$$T_1(\tau_6)\tau_2 + q a_2^{-2}\big(q^{1/2}a_2^{-1} - 1\big)\tau_3\tau_6 - q^{3/2}a_2^{-3}R_{23}^{-1}(\tau_3)\tau_5 = 0. \tag{3.10}$$

Then, applying $R_{23}^{n-2}T_0^{m-1}T_1^{N-1}$ on equation (3.10) and substituting condition (3.3) in it, we obtain equation (3.6). Although we do not write the action of R_{23} and T_0 on the variables τ_i here, it will be described in the next step. ∎

Putting $N = 0$ in equations (3.5)–(3.7) and using the boundary condition (3.4), we get the following conditions:

$$\frac{\tau_0^{n+1,m}\tau_0^{n,m+1}}{\tau_0^{n,m}\tau_0^{n+1,m+1}} = -q^{(2m-1)/2}a_0, \tag{3.11}$$

$$\frac{\tau_0^{n,m}\tau_0^{n+3,m}}{\tau_0^{n+1,m}\tau_0^{n+2,m}} = 1 - q^{(n+1)/2}a_2, \tag{3.12}$$

$$\frac{\tau_0^{n,m}\tau_0^{n,m+2}}{\left(\tau_0^{n,m+1}\right)^2} = q^{(-2n+8m+1)/2}a_0^4 a_2^{-2}\left(1 - q^{-m}a_0^{-1}\right). \tag{3.13}$$

Step 2. Conditions of $\tau_1^{n,m}$

In the second step, we shall get the conditions of $\tau_1^{n,m}$ from the consistency with the action of $\langle R_{23}, T_0\rangle$. By definitions (2.1) and (2.5) and Proposition 2, the action of T_0 and R_{23} are given by the follows:

$$T_0(\tau_1) = \tau_3, \qquad T_0(\tau_7) = \tau_2, \qquad R_{23}(\tau_3) = \tau_6, \qquad R_{23}(\tau_4) = \tau_7, \qquad R_{23}(\tau_6) = \tau_5,$$

$$T_0(\tau_2) = \frac{q^{3/2}a_0 a_1^{-1}a_2^{-1}\tau_2 T_0(\tau_6) + qa_0^2 a_2^{-1}T_0(\tau_4)T_0(\tau_5)}{\tau_6}, \tag{3.14}$$

$$T_0(\tau_3) = \frac{a_0^2 a_1 T_0(\tau_4)T_0(\tau_5) + q^{-1/2}a_0 a_1 a_2 T_0(\tau_6)\tau_2}{R_{23}(\tau_2)}, \tag{3.15}$$

$$T_0(\tau_4) = \frac{a_2 \tau_2 \tau_6 + qa_0^{-1}a_1^{-2}\tau_3 R_{23}(\tau_2)}{\tau_5}, \tag{3.16}$$

$$T_0(\tau_5) = \frac{qa_0^{-2}a_1^{-2}a_2^{-1}\tau_2 \tau_6 + q^{3/2}a_0^{-2}a_1^{-3}a_2^{-1}\tau_3 R_{23}(\tau_2)}{\tau_4}, \tag{3.17}$$

$$T_0(\tau_6) = \frac{qa_1^{-2}a_2^{-1}\tau_2 \tau_6 + q^{3/2}a_0^{-1}a_1^{-3}a_2^{-1}\tau_3 R_{23}(\tau_2)}{\tau_7}, \tag{3.18}$$

$$T_0^{-1}(\tau_1) = \frac{q^{-2}a_0^2 a_1 a_2^{-1}T_0^{-1}(\tau_4)T_0^{-1}(\tau_5) + q^{-1/2}a_0 a_1 a_2^{-1}T_0^{-1}(\tau_6)T_0^{-1}(\tau_7)}{R_{23}(\tau_7)}, \tag{3.19}$$

$$T_0^{-1}(\tau_4) = \frac{qa_0^{-2}a_1^{-2}a_2^{-1}\tau_7 T_0^{-1}(\tau_6) + q^{1/2}a_0^{-2}a_1^{-3}a_2^{-1}\tau_1 R_{23}(\tau_7)}{\tau_5}, \tag{3.20}$$

$$T_0^{-1}(\tau_5) = \frac{a_2 \tau_7 T_0^{-1}(\tau_6) + a_0^{-1}a_1^{-2}\tau_1 R_{23}(\tau_7)}{\tau_4}, \tag{3.21}$$

$$T_0^{-1}(\tau_6) = \frac{q^{-1}a_0^2 a_2^{-1}\tau_4 \tau_5 + q^{-1/2}a_0 a_1^{-1}a_2^{-1}\tau_6 \tau_7}{\tau_2}, \tag{3.22}$$

$$T_0^{-1}(\tau_7) = \frac{q^{-1}a_1^{-2}a_2^{-1}\tau_7 T_0^{-1}(\tau_6) + q^{-1/2}a_0^{-1}a_1^{-3}a_2^{-1}\tau_1 R_{23}(\tau_7)}{\tau_6}, \tag{3.23}$$

$$R_{23}(\tau_1) = \frac{q^{-1}a_0^2 a_2^{-1}\tau_4 \tau_5 + q^{-1/2}a_0 a_1^{-1}a_2^{-1}\tau_6 \tau_7}{\tau_2}, \tag{3.24}$$

$$R_{23}(\tau_2) = \frac{q^{-1}a_0^2 a_1 a_2^{-1}\tau_4 \tau_5 + q^{-1/2}a_0 a_1 a_2^{-1}\tau_6 \tau_7}{\tau_1}, \tag{3.25}$$

$$R_{23}(\tau_5) = \frac{q^{3/2}a_0^{-2}a_2^2 R_{23}(\tau_1)R_{23}(\tau_2) + qa_0^{-2}a_1^{-1}\tau_6 R_{23}(\tau_7)}{\tau_4}, \tag{3.26}$$

$$R_{23}(\tau_7) = \frac{q^{-1}a_0^2a_1\tau_4\tau_5 + q^{-1/2}a_0a_1a_2\tau_6\tau_7}{\tau_3},\tag{3.27}$$

$$R_{23}^{-1}(\tau_1) = \frac{q^{-1/2}a_0^2a_1a_2^{-1}R_{23}^{-1}(\tau_4)\tau_6 + a_0a_1a_2^{-1}\tau_3\tau_4}{\tau_2},\tag{3.28}$$

$$R_{23}^{-1}(\tau_2) = \frac{q^{-1/2}a_0^2a_2^{-1}R_{23}^{-1}(\tau_4)\tau_6 + a_0a_1^{-1}a_2^{-1}\tau_3\tau_4}{\tau_1},\tag{3.29}$$

$$R_{23}^{-1}(\tau_3) = \frac{q^{-1}a_0^2a_1R_{23}^{-1}(\tau_4)\tau_6 + q^{-1}a_0a_1a_2\tau_3\tau_4}{\tau_7},\tag{3.30}$$

$$R_{23}^{-1}(\tau_4) = \frac{q^{1/2}a_0^{-2}a_2^2\tau_1\tau_2 + qa_0^{-2}a_1^{-1}\tau_3\tau_7}{\tau_5}.\tag{3.31}$$

Using notation (3.2) and condition (3.3), we can rewrite equations (3.14)–(3.31) as

$$a_2\tau_1^{2,1}\tau_0^{1,2} = q^{1/2}a_0^2\tau_0^{1,1}\tau_1^{2,2} + qa_0^2\tau_0^{0,1}\tau_1^{3,2},\tag{3.32}$$

$$\tau_0^{2,1}\tau_1^{1,2} = qa_0\tau_0^{0,1}\tau_1^{3,2} + q^{1/2}a_2\tau_1^{2,2}\tau_0^{1,1},\tag{3.33}$$

$$q\tau_1^{3,1}\tau_0^{0,1} = qa_2\tau_0^{1,1}\tau_1^{2,1} + a_0\tau_1^{1,1}\tau_0^{2,1},\tag{3.34}$$

$$q^{3/2}a_2\tau_0^{0,0}\tau_1^{3,2} = q^{1/2}\tau_0^{1,1}\tau_1^{2,1} + a_0\tau_1^{1,1}\tau_0^{2,1},\tag{3.35}$$

$$q^{3/2}a_2\tau_0^{1,0}\tau_1^{2,2} = q^{1/2}a_0^2\tau_0^{1,1}\tau_1^{2,1} + a_0^2\tau_1^{1,1}\tau_0^{2,1},\tag{3.36}$$

$$a_2\tau_0^{2,0}\tau_1^{1,-1} = q^{-1}a_0\tau_0^{0,-1}\tau_1^{3,0} + q^{1/2}\tau_0^{2,0}\tau_0^{1,-1},\tag{3.37}$$

$$q^{5/2}a_2\tau_1^{3,1}\tau_0^{0,-1} = q^{3/2}\tau_0^{1,0}\tau_1^{2,0} + a_0\tau_1^{1,0}\tau_0^{2,0},\tag{3.38}$$

$$q^2\tau_0^{0,0}\tau_1^{3,0} = q^2a_2\tau_0^{1,0}\tau_1^{2,0} + a_0\tau_1^{1,0}\tau_0^{2,0},\tag{3.39}$$

$$q^{3/2}a_2\tau_0^{1,1}\tau_1^{2,0} = q^{1/2}a_0^2\tau_0^{0,0}\tau_1^{3,1} + a_0^2\tau_1^{2,1}\tau_0^{1,0},\tag{3.40}$$

$$q^{7/2}a_2\tau_1^{2,1}\tau_0^{1,-1} = q^{1/2}a_0^2\tau_0^{1,0}\tau_1^{2,0} + a_0^2\tau_1^{1,0}\tau_0^{2,0},\tag{3.41}$$

$$q^{3/2}a_2\tau_0^{1,1}\tau_1^{2,0} = q^{1/2}a_0^2\tau_0^{0,0}\tau_1^{3,1} + a_0^2\tau_1^{2,1}\tau_0^{1,0},\tag{3.42}$$

$$a_2\tau_1^{1,0}\tau_0^{2,1} = a_0\tau_0^{0,0}\tau_1^{3,1} + q^{1/2}\tau_1^{2,1}\tau_0^{1,0},\tag{3.43}$$

$$a_0^2\tau_0^{0,0}\tau_1^{4,1} = q^{3/2}a_2^2\tau_1^{2,0}\tau_0^{2,1} + a_0\tau_1^{2,1}\tau_0^{2,0},\tag{3.44}$$

$$\tau_1^{1,1}\tau_0^{2,0} = a_0\tau_0^{0,0}\tau_1^{3,1} + q^{1/2}a_2\tau_1^{2,1}\tau_0^{1,0},\tag{3.45}$$

$$a_2\tau_0^{1,1}\tau_1^{0,0} = q^{1/2}a_0\tau_0^{-1,0}\tau_1^{2,1} + q\tau_1^{1,1}\tau_0^{0,0},\tag{3.46}$$

$$qa_2\tau_1^{1,0}\tau_0^{0,1} = q^{1/2}a_0^2\tau_0^{-1,0}\tau_1^{2,1} + a_0^2\tau_1^{1,1}\tau_0^{0,0},\tag{3.47}$$

$$\tau_0^{1,0}\tau_1^{0,1} = a_0\tau_0^{-1,0}\tau_1^{2,1} + a_2\tau_1^{1,1}\tau_0^{0,0},\tag{3.48}$$

$$a_0^2\tau_1^{3,1}\tau_0^{-1,0} = q^{1/2}a_2^2\tau_1^{1,0}\tau_0^{1,1} + a_0\tau_1^{1,1}\tau_0^{1,0},\tag{3.49}$$

respectively. By setting

$$\tau_1^{n,m} = \left(q^{(n-1)/2}a_2; q^{1/2}\right)_\infty \tau_0^{n,m}F_{n,m},\tag{3.50}$$

and using conditions (3.11)–(3.13), equations (3.32)–(3.49) can be reduced to the following contiguity relations:

$$F_{n+2,m} - q^{(n-1)/2}a_2F_{n+1,m} - q^{m-2}a_0\left(1 - q^{(n-1)/2}a_2\right)F_{n,m} = 0,\tag{3.51}$$

$$F_{n+1,m+1} - q^{m-1}a_0F_{n,m+1} - q^{(n-2)/2}a_2F_{n,m} = 0,\tag{3.52}$$

$$q^{1/2}F_{n+2,m+1} - q^{1/2}F_{n+1,m+1} + q^{(n-1)/2}a_2\left(1 - q^{(n-1)/2}a_2\right)F_{n,m} = 0,\tag{3.53}$$

$$\left(1 - q^{m-1}a_0\right)F_{n+1,m+1} - q^{(n-1)/2}a_2F_{n+1,m} - q^{n/2-1}a_2\left(1 - q^{(n-1)/2}a_2\right)F_{n,m} = 0,\tag{3.54}$$

$$q^{3/2}\left(1 - q^{m-1}a_0\right)F_{n+2,m+1} - q^{(n+2)/2}a_2 F_{n+1,m}$$
$$- q^{(2m+n-1)/2}a_0 a_2\left(1 - q^{(n-1)/2}a_2\right)F_{n,m} = 0, \tag{3.55}$$
$$qF_{n+2,m+1} - q^m a_0 F_{n,m+1} - q^{(2n-1)/2}a_2^2 F_{n,m} = 0. \tag{3.56}$$

The correspondence between equations (3.32)–(3.49) and equations (3.51)–(3.56) is established as follows:

(3.33), (3.34), (3.39), (3.45), (3.48) $\Rightarrow$ (3.51),
(3.32), (3.40), (3.42), (3.47) $\Rightarrow$ (3.52),
(3.37), (3.43), (3.46) $\Rightarrow$ (3.53),
(3.36), (3.41) $\Rightarrow$ (3.54),
(3.35), (3.38) $\Rightarrow$ (3.55),
(3.44), (3.49) $\Rightarrow$ (3.56).

Step 3. Determination of $\tau_0^{n,m}$ and $\tau_1^{n,m}$

In this step, we determine $\tau_0^{n,m}$ and $\tau_1^{n,m}$, i.e., we solve equations (3.11)–(3.13) and equations (3.51)–(3.56). It is easily verified that the function

$$\tau_0^{n,m} = \left(q^m a_0; q, q\right)_\infty \left(q^{(n+1)/2}a_2; q^{1/2}, q\right)_\infty \Gamma\left(q^{(2n+2m-3)/4}a_0^{1/2}a_2; q^{1/2}, q^{1/2}\right)$$
$$\times \frac{\Gamma\left(q^{(n-m+1)/4}a_0^{-1/4}a_2^{1/2}; q^{1/4}, q^{1/4}\right)\Gamma\left(q^{(n-m)/4}a_0^{-1/4}a_2^{1/2}; q^{1/4}, q^{1/4}\right)}{\Gamma\left(-q^{3m-1}a_0^3; q^3, q^3\right)\Gamma\left(-q^{2n}a_2^4; q^2, q^2\right)}, \tag{3.57}$$

is the solution of equations (3.11)–(3.13). Therefore, the aim of this step is to solve the equations (3.51)–(3.56). Since equations (3.51)–(3.56) are overdetermined system, let us first consider the essential conditions of $F_{n,m}$.

Lemma 2. *Equations* (3.51) *and* (3.52) *are essential conditions for* $F_{n,m}$.

Proof. Eliminating $F_{n,m+1}$ from equations (3.51)$_{m\to m+1}$ and (3.52), we obtain equation (3.53). In a similar manner, equations (3.54)–(3.56) can be proven by the following procedures: eliminating $F_{n+2,m+1}$ from equations (3.52)$_{n\to n+1}$ and (3.53), we obtain equation (3.54); eliminating $F_{n+1,m+1}$ from equations (3.52)$_{n\to n+1}$ and (3.53), we obtain equation (3.55); eliminating $F_{n+1,m+1}$ from equations (3.52) and (3.53), we obtain equation (3.56). These calculations mean that if $F_{n,m}$ satisfies conditions (3.51) and (3.52), then it also satisfies conditions (3.53)–(3.56). Therefore we have completed the proof. ■

Next, we solve equations (3.51) and (3.52).

Lemma 3. *The general solution of contiguity relations* (3.51) *and* (3.52) *is given by*

$$F_{n,m} = A_{n,m} \frac{\Theta\left(q^{n/2}a_2; q^{1/2}\right)\Theta\left(q^{(2m-1)/2}a_0; q\right)\left(q^{(m-1)/2}a_0^{1/2}; q^{1/2}\right)_\infty}{\Theta\left(q^{(n+m-2)/2}a_0^{1/2}a_2; q^{1/2}\right)}$$
$$\times {}_2\varphi_1\left(\begin{matrix} 0, q^{(-m+2)/2}a_0^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{(n-1)/2}a_2\right)$$
$$+ B_{n,m} \frac{\Theta\left(q^{n/2}a_2; q^{1/2}\right)\Theta\left(q^{(2m-1)/2}a_0; q\right)\left(-q^{(m-1)/2}a_0^{1/2}; q^{1/2}\right)_\infty}{\Theta\left(-q^{(n+m-2)/2}a_0^{1/2}a_2; q^{1/2}\right)}$$
$$\times {}_2\varphi_1\left(\begin{matrix} 0, -q^{(-m+2)/2}a_0^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{(n-1)/2}a_2\right),$$

where $A_{n,m}$ and $B_{n,m}$ are periodic functions of period one for n and m, i.e.,

$$A_{n,m} = A_{n+1,m} = A_{n,m+1}, \qquad B_{n,m} = B_{n+1,m} = B_{n,m+1}.$$

Proof. For convenience, we put

$$t = q^{n/2} a_2, \qquad \alpha = q^m a_0, \qquad F_{n,m} = F(t, \alpha).$$

Then, equations (3.51) and (3.52) can be rewritten as

$$F(qt, \alpha) - q^{-1/2} t F(q^{1/2}t, \alpha) - q^{-2}\alpha(1 - q^{-1/2}t)F(t, \alpha) = 0, \tag{3.58}$$
$$F(q^{1/2}t, q\alpha) - q^{-1}\alpha F(t, q\alpha) - q^{-1}t F(t, \alpha) = 0, \tag{3.59}$$

respectively. Substituting

$$F(t, \alpha) = D(t, \alpha) \sum_{k=0}^{\infty} C_k(\alpha) t^k,$$

in equation (3.58), we obtain

$$D(qt, \alpha) = q^{-2}\alpha D(t, \alpha), \tag{3.60}$$
$$C_k(\alpha) = \frac{\left(q^2 D(q^{1/2}t, \alpha) D(t, \alpha)^{-1}\alpha^{-1}; q^{1/2}\right)_k}{q^{k/2}\left(-q^{1/2}, q^{1/2}; q^{1/2}\right)_k} C_0(\alpha).$$

Therefore, we obtain the solution of equation (3.58):

$$F(t, \alpha) = D_1(t, \alpha) \, {}_2\varphi_1\left(\begin{matrix} 0, q\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{-1/2}t\right)$$
$$+ D_2(t, \alpha) \, {}_2\varphi_1\left(\begin{matrix} 0, -q\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{-1/2}t\right), \tag{3.61}$$

where $D_1(t, \alpha)$ and $D_2(t, \alpha)$ are the solutions of (3.60) which satisfy

$$D_1(q^{1/2}t, \alpha) = q^{-1}\alpha^{1/2} D_1(t, \alpha), \tag{3.62}$$
$$D_2(q^{1/2}t, \alpha) = -q^{-1}\alpha^{1/2} D_2(t, \alpha), \tag{3.63}$$

respectively. Substituting (3.61) in equation (3.59), we can obtain the following equations:

$$q^{-1/2}\alpha^{1/2} \, {}_2\varphi_1\left(\begin{matrix} 0, q^{1/2}\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, t\right) - q^{-1}\alpha \, {}_2\varphi_1\left(\begin{matrix} 0, q^{1/2}\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{-1/2}t\right)$$
$$- q^{-1}t\frac{D_1(t, \alpha)}{D_1(t, q\alpha)} \, {}_2\varphi_1\left(\begin{matrix} 0, q\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{-1/2}t\right) = 0, \tag{3.64}$$
$$q^{-1/2}\alpha^{1/2} \, {}_2\varphi_1\left(\begin{matrix} 0, -q^{1/2}\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, t\right) + q^{-1}\alpha \, {}_2\varphi_1\left(\begin{matrix} 0, -q^{1/2}\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{-1/2}t\right)$$
$$+ q^{-1}t\frac{D_2(t, \alpha)}{D_2(t, q\alpha)} \, {}_2\varphi_1\left(\begin{matrix} 0, -q\alpha^{-1/2} \\ -q^{1/2} \end{matrix}; q^{1/2}, q^{-1/2}t\right) = 0. \tag{3.65}$$

By the definition of basic hypergeometric series ${}_2\varphi_1$, it is easily verified that

$${}_2\varphi_1\left(\begin{matrix} 0, a \\ c \end{matrix}; q^{1/2}, z\right) - a^{-1} \, {}_2\varphi_1\left(\begin{matrix} 0, a \\ c \end{matrix}; q^{1/2}, q^{-1/2}z\right)$$

$$- \left(1 - a^{-1}\right) {}_2\varphi_1 \left(\begin{matrix} 0, q^{1/2}a \\ c \end{matrix}; q^{1/2}, q^{-1/2}z \right) = 0. \tag{3.66}$$

Therefore, we obtain the following conditions from equations (3.64) and (3.65) by using equation (3.66):

$$D_1(t, q\alpha) = -\frac{t}{\alpha(1 - q^{1/2}\alpha^{-1/2})} D_1(t, \alpha), \tag{3.67}$$

$$D_2(t, q\alpha) = -\frac{t}{\alpha(1 + q^{1/2}\alpha^{-1/2})} D_2(t, \alpha). \tag{3.68}$$

By setting

$$D_1(t, \alpha) = \frac{\Theta(t; q^{1/2}) \Theta(q^{-1/2}\alpha; q) (q^{-1/2}\alpha^{1/2}; q^{1/2})_\infty}{\Theta(q^{-1}\alpha^{1/2}t; q^{1/2})} A(t, \alpha),$$

$$D_2(t, \alpha) = \frac{\Theta(t; q^{1/2}) \Theta(q^{-1/2}\alpha; q) (-q^{-1/2}\alpha^{1/2}; q^{1/2})_\infty}{\Theta(-q^{-1}\alpha^{1/2}t; q^{1/2})} B(t, \alpha),$$

equations (3.62), (3.63), (3.67) and (3.68) can be rewritten as

$$A\left(q^{1/2}t, \alpha\right) = A(t, \alpha), \qquad B\left(q^{1/2}t, \alpha\right) = B(t, \alpha),$$
$$A(t, q\alpha) = A(t, \alpha), \qquad B(t, q\alpha) = B(t, \alpha),$$

respectively. This completes the proof. ∎

It was shown that hypergeometric solutions of a symmetric discrete Painlevé equation, which can be obtained by projective reduction, have two expressions and there are the following differences between the two expressions (see [17, Section 2.3]):

(i) the bases of hypergeometric series appearing in the solutions are different;

(ii) the periodicities of periodic functions appearing in the solutions are different.

The differences between these two expressions can be explained by the factorization of the linear difference operators associated with the three-term relation of the hypergeometric functions (see [17, Section 3.2]). Namely, we can see these differences by comparing Lemmas 3 and 6. To get another expression, we first reselect essential conditions of $F_{n,m}$.

Lemma 4. *Equations* (3.52) *and*

$$q^{m-1}a_0\left(1 - q^m a_0\right) F_{n,m+2} - q^{(n-5)/2}a_2\left(\left(1 + q^{1/2}\right)q^m a_0 - q^{n/2}a_2\right) F_{n,m+1}$$
$$- q^{(2n-5)/2}a_2^2 F_{n,m} = 0, \tag{3.69}$$

are essential conditions of $F_{n,m}$.

Proof. Eliminating $F_{n,m}$ from equations (3.52) and (3.69), we obtain

$$q^{m-2}a_0\left(1 - q^{m-1}a_0\right) F_{n,m+1} - q^{(n-3)/2}a_2 F_{n+1,m}$$
$$- q^{(n-5)/2}a_2\left(q^{(2m-1)/2}a_0 - q^{n/2}a_2\right) F_{n,m} = 0. \tag{3.70}$$

Similarly, eliminating $F_{n,m+1}$ from equations (3.52) and (3.70), we obtain

$$\left(1 - q^{m-1}a_0\right) F_{n+1,m+1} - q^{(n-1)/2}a_2 F_{n+1,m} - q^{(n-2)/2}a_2\left(1 - q^{(n-1)/2}a_2\right) F_{n,m} = 0. \tag{3.71}$$

Finally, eliminating $F_{n+1,m+1}$ from equations $(3.70)_{n\to n+1}$ and (3.71), we obtain equation (3.51). This result together with Lemma 2 complete the proof. ∎

By setting

$$F_{n,m} = \frac{\Theta(q^m a_0; q)\Theta(q^{n/2} a_2; q^{1/2})}{\Theta(-q^{n/2} a_2; q^{1/2})} G_{n-3,m-1}, \tag{3.72}$$

equations (3.52) and (3.69) can be rewritten as

$$q^{-m+1} a_0^{-1} G_{n-2,m} + G_{n-3,m} - q^{n/2} a_2 G_{n-3,m-1} = 0, \tag{3.73}$$

$$G_{n,m-2} + \left(q^{-m+1} a_0^{-1} - (1+q^{1/2}) q^{(-n-3)/2} a_2^{-1}\right) G_{n,m-1}$$
$$- q^{(-2n-5)/2} a_2^{-2} \left(q^{-m+1} a_0^{-1} - 1\right) G_{n,m} = 0, \tag{3.74}$$

respectively. Before solving equations (3.73) and (3.74), we prepare the following lemma:

Lemma 5. *The following recurrence relations hold:*

$$_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, z\right) - {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, qz\right) = \frac{(1-a)(1-b)z}{1-c} \, _2\varphi_1\left(\begin{matrix} qa, qb \\ qc \end{matrix}; q, z\right), \tag{3.75}$$

$$\left(q^{-1}c - 1\right) {}_2\varphi_1\left(\begin{matrix} a,b \\ q^{-1}c \end{matrix}; q, z\right) + {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, z\right) - q^{-1}c \, _2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, qz\right) = 0. \tag{3.76}$$

Proof. Substituting

$$_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, z\right) = 1 + \sum_{n=0}^{\infty} \frac{(qa, qb; q)_n}{(qc, q; q)_n} \frac{(1-a)(1-b)}{(1-c)(1-q^{n+1})} z^{n+1},$$

in the left-hand side of equation (3.75), we obtain the right-hand side. Equation (3.76) can be easily verified as the following:

$$_2\varphi_1\left(\begin{matrix} a,b \\ q^{-1}c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a,b; q)_n}{(c, q; q)_n} \frac{1 - q^{n-1}c}{1 - q^{-1}c} z^n$$

$$= \frac{1}{1 - q^{-1}c} \, _2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, z\right) - \frac{q^{-1}c}{1 - q^{-1}c} \, _2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix}; q, qz\right).$$

Therefore we have completed the proof. ∎

Using Lemma 5, we obtain the following lemma:

Lemma 6. *The general solution of equations* (3.73) *and* (3.74) *is given by*

$$G_{n,m} = \Lambda_{n,m} \frac{\Theta(q^{(n-2m+2)/2} a_0^{-1} a_2; q)}{\Theta(q^{-m} a_0^{-1}; q)\Theta(q^{n/2} a_2; q)(q^{(n+3)/2} a_2; q)_\infty (q^{-1/2}; q)_\infty}$$

$$\times {}_2\varphi_1\left(\begin{matrix} 0, q^{(n+3)/2} a_2 \\ q^{3/2} \end{matrix}; q, q^{-m+1} a_0^{-1}\right)$$

$$+ \Lambda_{n+1,m} \frac{q^{1/2}\Theta(q^{(n-2m+3)/2} a_0^{-1} a_2; q)}{\Theta(q^{-m} a_0^{-1}; q)\Theta(q^{(n+1)/2} a_2; q)(q^{(n+2)/2} a_2; q)_\infty (q^{1/2}; q)_\infty}$$

$$\times {}_2\varphi_1\left(\begin{matrix} 0, q^{(n+2)/2} a_2 \\ q^{1/2} \end{matrix}; q, q^{-m+1} a_0^{-1}\right),$$

where $\Lambda_{n,m}$ *is a periodic function of period two for* n *and period one for* m, *i.e.,*

$$\Lambda_{n+2,m} = \Lambda_{n,m+1} = \Lambda_{n,m}.$$

Proof. For convenience, we put

$$t = q^{-m+1}a_0^{-1}, \qquad \alpha = q^{n/2}a_2, \qquad G_{n,m} = G(t,\alpha).$$

Then, equations (3.73) and (3.74) can be rewritten as

$$tG(t, q^{-1}\alpha) + G(t, q^{-3/2}\alpha) - \alpha G(qt, q^{-3/2}\alpha) = 0, \tag{3.77}$$
$$G(q^2t, \alpha) + (t - (1 + q^{1/2})q^{-3/2}\alpha^{-1})G(qt, \alpha) - q^{-5/2}\alpha^{-2}(t-1)G(t, \alpha) = 0, \tag{3.78}$$

respectively. Substituting

$$G(t, \alpha) = D(t, \alpha)\sum_{k=0}^{\infty} C_k(\alpha)t^k,$$

in equation (3.78), we obtain

$$G(t, \alpha) = D_1(t, \alpha)\,{}_2\varphi_1\left(\begin{matrix}0, q^{3/2}\alpha\\q^{3/2}\end{matrix}; q, t\right) + D_2(t, \alpha)\,{}_2\varphi_1\left(\begin{matrix}0, q\alpha\\q^{1/2}\end{matrix}; q, t\right), \tag{3.79}$$

where $D_1(t, \alpha)$ and $D_2(t, \alpha)$ satisfy

$$D_1(qt, \alpha) = q^{-1}\alpha^{-1}D_1(t, \alpha), \tag{3.80}$$
$$D_2(qt, \alpha) = q^{-3/2}\alpha^{-1}D_2(t, \alpha), \tag{3.81}$$

respectively. Substituting (3.79) in equation (3.77), we can obtain the following equations:

$$_2\varphi_1\left(\begin{matrix}0, q^{-1/2}\alpha\\q^{1/2}\end{matrix}; q, t\right) - {}_2\varphi_1\left(\begin{matrix}0, q^{-1/2}\alpha\\q^{1/2}\end{matrix}; q, qt\right) = -t\frac{D_1(t, q^{-1}\alpha)}{D_2(t, q^{-3/2}\alpha)}\,{}_2\varphi_1\left(\begin{matrix}0, q^{1/2}\alpha\\q^{3/2}\end{matrix}; q, t\right), \tag{3.82}$$

$$t\frac{D_2(t, q^{-1}\alpha)}{D_1(t, q^{-3/2}\alpha)}\,{}_2\varphi_1\left(\begin{matrix}0, \alpha\\q^{1/2}\end{matrix}; q, t\right) + {}_2\varphi_1\left(\begin{matrix}0, \alpha\\q^{3/2}\end{matrix}; q, t\right) - q^{1/2}\,{}_2\varphi_1\left(\begin{matrix}0, \alpha\\q^{3/2}\end{matrix}; q, qt\right) = 0. \tag{3.83}$$

Therefore, we obtain

$$D_1(t, q^{1/2}\alpha) = -\frac{1 - q\alpha}{1 - q^{1/2}}D_2(t, \alpha), \tag{3.84}$$

$$D_2(t, q^{1/2}\alpha) = -\frac{1 - q^{1/2}}{t}D_1(t, \alpha), \tag{3.85}$$

from equations (3.82) and (3.83) by using equations (3.75) and (3.76), respectively. By setting

$$D_1(t, \alpha) = \frac{\Theta(\alpha t; q)}{(q^{-1/2}; q)_\infty (q^{3/2}\alpha; q)_\infty \Theta(q^{-1}t; q)\Theta(\alpha; q)}\Lambda(t, \alpha),$$

$$D_2(t, \alpha) = \frac{q^{1/2}\Theta(q^{1/2}\alpha t; q)}{(q^{1/2}; q)_\infty (q\alpha; q)_\infty \Theta(q^{-1}t; q)\Theta(q^{1/2}\alpha; q)}\Lambda(t, q^{1/2}\alpha),$$

equations (3.80), (3.81), (3.84) and (3.85) can be reduced to

$$\Lambda(t, q\alpha) = \Lambda(qt, \alpha) = \Lambda(t, \alpha).$$

This completes the proof. ∎

Step 4. Constructing the closed-form expressions of $\tau_N^{n,m}$ $(N \geq 2)$

In this final step, constructing the closed-form expressions of $\tau_N^{n,m}$ $(N \geq 2)$, we obtain the hypergeometric τ-functions for the q-$\mathrm{P_{IV}}$.

Let

$$\tau_N^{n,m} = (-1)^{N(N-1)/2} q^{3N(N-1)(N-n+1)/4} a_2^{-3N(N-1)/2} \left(\prod_{k=1}^{N} \left(q^{(n-2k+1)/2} a_2; q^{1/2} \right)_\infty \right)$$

$$\times \left(q^m a_0; q, q \right)_\infty \left(q^{(n+1)/2} a_2; q^{1/2}, q \right)_\infty \Gamma\left(q^{(2n+2m-3)/4} a_0^{1/2} a_2; q^{1/2}, q^{1/2} \right)$$

$$\times \frac{\Gamma\left(q^{(n-m+1)/4} a_0^{-1/4} a_2^{1/2}; q^{1/4}, q^{1/4} \right) \Gamma\left(q^{(n-m)/4} a_0^{-1/4} a_2^{1/2}; q^{1/4}, q^{1/4} \right)}{\Gamma\left(-q^{3m-1} a_0^3; q^3, q^3 \right) \Gamma\left(-q^{2n} a_2^4; q^2, q^2 \right)} \phi_N^{n,m}.$$

From (3.4), (3.50) and (3.57), we get

$$\phi_N^{n,m} = 0, \qquad N < 0, \qquad \phi_0^{n,m} = 1, \qquad \phi_1^{n,m} = F_{n,m}.$$

Moreover, it is easily verified that equation (3.6) can be rewritten as

$$\phi_{N+1}^{n,m} \phi_{N-1}^{n-1,m} - \phi_N^{n-1,m} \phi_N^{n,m} + \phi_N^{n-2,m} \phi_N^{n+1,m} = 0. \tag{3.86}$$

In general, equation (3.86) admits a solution expressed in terms of Jacobi–Trudi type determinant

$$\phi_N^{n,m} = \det(c_{n-2i+j+1,m})_{i,j=1,\dots,N}, \qquad N > 0,$$

under the boundary conditions

$$\phi_N^{n,m} = 0, \qquad N < 0, \qquad \phi_0^{n,m} = 1, \qquad \phi_1^{n,m} = c_{n,m},$$

where $c_{n,m}$ is an arbitrary function. Therefore, we obtain the following lemma:

Lemma 7. *Under the assumptions*

$$a_0 a_1 = q, \qquad \tau_N^{n,m} = 0, \qquad N < 0,$$

the hypergeometric τ-functions for the q-$\mathrm{P_{IV}}$ are given as

$$\tau_N^{n,m} = (-1)^{N(N-1)/2} q^{3N(N-1)(N-n+1)/4} a_2^{-3N(N-1)/2} \left(\prod_{k=1}^{N} \left(q^{(n-2k+1)/2} a_2; q^{1/2} \right)_\infty \right)$$

$$\times \left(q^m a_0; q, q \right)_\infty \left(q^{(n+1)/2} a_2; q^{1/2}, q \right)_\infty \Gamma\left(q^{(2n+2m-3)/4} a_0^{1/2} a_2; q^{1/2}, q^{1/2} \right)$$

$$\times \frac{\Gamma\left(q^{(n-m+1)/4} a_0^{-1/4} a_2^{1/2}; q^{1/4}, q^{1/4} \right) \Gamma\left(q^{(n-m)/4} a_0^{-1/4} a_2^{1/2}; q^{1/4}, q^{1/4} \right)}{\Gamma\left(-q^{3m-1} a_0^3; q^3, q^3 \right) \Gamma\left(-q^{2n} a_2^4; q^2, q^2 \right)} \phi_N^{n,m},$$

where

$$\phi_N^{n,m} = \begin{vmatrix} F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m} \\ F_{n-2,m} & F_{n-1,m} & \cdots & F_{n+N-3,m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-2N+2,m} & F_{n-2N+3,m} & \cdots & F_{n-N+1,m} \end{vmatrix}, \qquad \phi_0^{n,m} = 1, \quad \phi_{-N}^{n,m} = 0, \quad N > 0.$$

Here, $F_{n,m}$ is given in Lemma 3.

We also show another expression of the hypergeometric τ-functions for the q-P_{IV}. From relation (3.72), $\phi_N^{n,m}$ can be rewritten as

$$\phi_N^{n,m} = \Theta\big(q^m a_0; q\big)^N \left(\prod_{k=0}^{N-1} \frac{\Theta\big(q^{(n+k)/2} a_2; q^{1/2}\big)}{\Theta\big(-q^{(n+k)/2} a_2; q^{1/2}\big)}\right) \psi_N^{n-3,m-1},$$

where

$$\psi_N^{n,m} = \begin{vmatrix} G_{n,m} & G_{n+1,m} & \cdots & G_{n+N-1,m} \\ G_{n-2,m} & G_{n-1,m} & \cdots & G_{n+N-3,m} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n-2N+2,m} & G_{n-2N+3,m} & \cdots & G_{n-N+1,m} \end{vmatrix}, \quad \psi_0^{n,m} = 1, \quad \psi_{-N}^{n,m} = 0, \quad N > 0.$$

This gives the following lemma:

Lemma 8. *Under the assumptions*

$$a_0 a_1 = q, \qquad \tau_N^{n,m} = 0, \qquad N < 0,$$

the hypergeometric τ-functions for the q-P_{IV} are given as

$$\tau_N^{n,m} = (-1)^{N(N-1)/2} q^{3N(N-1)(N-n+1)/4} a_2^{-3N(N-1)/2} \left(\prod_{k=1}^{N} \big(q^{(n-2k+1)/2} a_2; q^{1/2}\big)_\infty\right)$$

$$\times \big(q^m a_0; q, q\big)_\infty \big(q^{(n+1)/2} a_2; q^{1/2}, q\big)_\infty \Gamma\big(q^{(2n+2m-3)/4} a_0^{1/2} a_2; q^{1/2}, q^{1/2}\big)$$

$$\times \frac{\Gamma\big(q^{(n-m+1)/4} a_0^{-1/4} a_2^{1/2}; q^{1/4}, q^{1/4}\big) \Gamma\big(q^{(n-m)/4} a_0^{-1/4} a_2^{1/2}; q^{1/4}, q^{1/4}\big)}{\Gamma\big(-q^{3m-1} a_0^3; q^3, q^3\big) \Gamma\big(-q^{2n} a_2^4; q^2, q^2\big)}$$

$$\times \Theta\big(q^m a_0; q\big)^N \left(\prod_{k=0}^{N-1} \frac{\Theta\big(q^{(n+k)/2} a_2; q^{1/2}\big)}{\Theta\big(-q^{(n+k)/2} a_2; q^{1/2}\big)}\right) \psi_N^{n-3,m-1},$$

where

$$\psi_N^{n,m} = \begin{vmatrix} G_{n,m} & G_{n+1,m} & \cdots & G_{n+N-1,m} \\ G_{n-2,m} & G_{n-1,m} & \cdots & G_{n+N-3,m} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n-2N+2,m} & G_{n-2N+3,m} & \cdots & G_{n-N+1,m} \end{vmatrix}, \quad \psi_0^{n,m} = 1, \quad \psi_{-N}^{n,m} = 0, \quad N > 0.$$

Here, $G_{n,m}$ is given in Lemma 6.

3.3 Hypergeometric solutions of the q-P_{IV}

In this section, we show the hypergeometric solutions of q-P_{IV} (1.1).

From relations (2.8) and (3.1), the variable for q-P_{IV} (1.1) is expressed by the τ-functions as the following:

$$X_n(m, N) = q^{(m+N-1)/2} a_0 a_1^{1/2} \frac{\tau_N^{n,m} \tau_{N+1}^{n+3,m+1}}{\tau_{N+1}^{n+2,m+1} \tau_N^{n+1,m}}.$$

Therefore, from Lemmas 7 and 8, we obtain the following theorems:

Theorem 1. *The hypergeometric solutions of q-P_{IV} (1.1) with*

$$a_0 a_1 = q, \qquad N \geq 0, \tag{3.87}$$

is given by

$$X_n(m, N) = -q^{(-2N-m+1)/2} a_0^{-1/2} \frac{\phi_N^{n,m} \phi_{N+1}^{n+3,m+1}}{\phi_{N+1}^{n+2,m+1} \phi_N^{n+1,m}},$$

where

$$\phi_N^{n,m} = \begin{vmatrix} F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m} \\ F_{n-2,m} & F_{n-1,m} & \cdots & F_{n+N-3,m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-2N+2,m} & F_{n-2N+3,m} & \cdots & F_{n-N+1,m} \end{vmatrix}, \qquad \phi_0^{n,m} = 1.$$

Here, $F_{n,m}$ is given in Lemma 3.

Theorem 2. *The hypergeometric solutions of q-P_{IV} (1.1) with the condition (3.87) is given by*

$$X_n(m, N) = q^{(-2N-m+1)/2} a_0^{-1/2} \frac{\psi_N^{n-3,m-1} \psi_{N+1}^{n,m}}{\psi_{N+1}^{n-1,m} \psi_N^{n-2,m-1}},$$

where

$$\psi_N^{n,m} = \begin{vmatrix} G_{n,m} & G_{n+1,m} & \cdots & G_{n+N-1,m} \\ G_{n-2,m} & G_{n-1,m} & \cdots & G_{n+N-3,m} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n-2N+2,m} & G_{n-2N+3,m} & \cdots & G_{n-N+1,m} \end{vmatrix}, \qquad \psi_0^{n,m} = 1.$$

Here, $G_{n,m}$ is given in Lemma 6.

4 Hypergeometric solutions of the q-P_{IV} (II)

In this section, we show that q-P_{IV} (1.1) also has the hypergeometric solutions expressed by bilateral basic hypergeometric series.

First, we recall the definitions of orthogonal polynomials.

Definition 1. A polynomial sequence $(P_n(t))_{n=0}^{\infty}$ which satisfies the following conditions is called an orthogonal polynomial sequence over the field $\mathcal{K}$, and each term $P_n(t)$ is called an orthogonal polynomial over the field $\mathcal{K}$.

(i) $\deg(P_n(t)) = n$;

(ii) there exists a linear functional $\mathcal{L} : \mathcal{K}(t) \to \mathcal{K}$ which holds the orthogonal condition:

$$\mathcal{L}[t^k P_n(t)] = h_n \delta_{n,k}, \qquad n \geq k,$$

where $\delta_{n,k}$ is Kronecker's symbol. Here, h_n and $\mu_n = \mathcal{L}[t^n]$ are called a normalization factor and a moment, respectively.

Definition 2. An orthogonal polynomial sequence whose leading coefficient is 1 is called a monic orthogonal polynomial sequence (MOPS). Let $(P_n(t))_{n=0}^\infty$ be a MOPS. Then, polynomial $P_n(t)$ and its normalization factor h_n are expressed by the moment μ_n as the following:

$$P_n = \frac{1}{\Phi_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ 1 & t & \cdots & t^{n-1} & t^n \end{vmatrix}, \qquad P_0 = 1, \qquad h_n = \frac{\Phi_{n+1}}{\Phi_n}, \qquad h_0 = \mu_0, \quad (4.1)$$

where Φ_n is the Hankel determinant given by

$$\Phi_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}. \tag{4.2}$$

We assume that $(P_n)_{n=0}^\infty = (P_n(t))_{n=0}^\infty$ and $(\hat{P}_n)_{n=0}^\infty = (\hat{P}_n(t))_{n=0}^\infty$ are MOPSs which satisfy the following orthogonal conditions:

$$\mathcal{L}\big[t^k P_n(t)\big] = h_n \delta_{n,k}, \quad n \ge k, \qquad \hat{\mathcal{L}}\big[t^k \hat{P}_n(t)\big] = \hat{h}_n \delta_{n,k}, \quad n \ge k,$$

respectively. In addition, we put the case that P_n and $\hat{P}_n$ are related by the Christoffel transformation (or Geronimus transformation), that is, the linear functionals satisfy the following relation for an arbitrary function $f(t)$:

$$\mathcal{L}[f(t)] = \hat{\mathcal{L}}\left[\frac{f(t)}{t - c_0} + \delta(t - c_0)\right],$$

where $\delta(x)$ is the Dirac delta function and $c_0 \in \mathbb{C}$ is an additional parameter. For these MOPSs, the following relations hold [4, 40, 41]:

$$(t - c_0)\hat{P}_n = P_{n+1} + \frac{\hat{h}_n}{h_n} P_n, \tag{4.3}$$

$$P_n = \hat{P}_n + \frac{h_n}{\hat{h}_{n-1}} \hat{P}_{n-1}. \tag{4.4}$$

Eliminating P_n from equation (4.3) by using equation (4.4), we obtain the following three-term recurrence relation:

$$t\hat{P}_n = \hat{P}_{n+1} + \left(\frac{h_{n+1}}{\hat{h}_n} + \frac{\hat{h}_n}{h_n} + c_0\right)\hat{P}_n + \frac{\hat{h}_n}{\hat{h}_{n-1}}\hat{P}_{n-1}. \tag{4.5}$$

Let

$$\hat{P}_n(t) = \frac{\tilde{h}_n(c_1 t; p)}{c_1^n}, \qquad c_1 > 0,$$

where $\tilde{h}_n(t; q)$ is the discrete q-Hermite II polynomial:

$$\tilde{h}_n(t; q) = t^n {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix}; q^2, -\frac{q^2}{t^2}\right).$$

Then, the linear functional, the normalization factor and the three-term recurrence relation for $\hat{P}_n$ are given by

$$\hat{\mathcal{L}}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{\left(-c_1^2 t^2; p^2\right)_{\infty}} \mathrm{d}_p t,$$

$$\hat{h}_n = \frac{2}{p^{n^2} c_1^{2n}} \frac{(p;p)_n \left(p^2; p^2\right)_{\infty} \Theta\left(-pc_1^2; p^2\right)}{\left(p^3; p^2\right)_{\infty} \Theta\left(-c_1^2; p^2\right)}, \tag{4.6}$$

$$t\hat{P}_n = \hat{P}_{n+1} + p^{-2n+1}\left(1 - p^n\right) c_1^{-2} \hat{P}_{n-1}, \tag{4.7}$$

respectively. We note that these properties of q-Hermite II polynomials are given in [22]. We here impose the condition $c_0 \neq p^a$ for all $a \in \mathbb{Z}$ since the linear functional for P_n is given by

$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} \frac{f(t)}{(t - c_0)\left(-c_1^2 t^2; p^2\right)_{\infty}} \mathrm{d}_p t.$$

In addition, the moment μ_n can be obtained by

$$\begin{aligned}
\mu_n &= -\frac{1-p}{c_0^2} \sum_{k=-\infty}^{\infty} \frac{\left(1 - (-1)^n\right) p^k + \left(1 + (-1)^n\right) c_0}{\left(1 - p^{2k} c_0^{-2}\right)\left(-c_1^2 p^{2k}; p^2\right)_{\infty}} p^{k(n+1)} \\
&= \frac{2(1-p)}{\left(1 - c_0^2\right)\left(-c_1^2; p^2\right)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{\left(-c_1^2, c_0^{-2}; p^2\right)_k}{\left(p^2 c_0^{-2}; p^2\right)_k} \left(\frac{1 - (-1)^n}{2} p^k + \frac{1 + (-1)^n}{2} c_0\right) p^{k(n+1)} \\
&= \begin{cases}
\dfrac{2(1-p)}{\left(1 - c_0^2\right)\left(-c_1^2; p^2\right)_{\infty}} \, {}_2\psi_2\left(\begin{matrix} -c_1^2, c_0^{-2} \\ 0, p^2 c_0^{-2} \end{matrix}; p^2, p^{2k+1}\right), & n = 2k - 1, \\[3mm]
\dfrac{2c_0(1-p)}{\left(1 - c_0^2\right)\left(-c_1^2; p^2\right)_{\infty}} \, {}_2\psi_2\left(\begin{matrix} -c_1^2, c_0^{-2} \\ 0, p^2 c_0^{-2} \end{matrix}; p^2, p^{2k+1}\right), & n = 2k.
\end{cases}
\end{aligned} \tag{4.8}$$

Comparing the coefficients of equations (4.5) and (4.7), we obtain the following equations:

$$\frac{h_{n+1}}{\hat{h}_n} + \frac{\hat{h}_n}{h_n} + c_0 = 0, \tag{4.9}$$

$$\frac{\hat{h}_n}{\hat{h}_{n-1}} = p^{-2n+1}\left(1 - p^n\right) c_1^{-2}. \tag{4.10}$$

From equations (4.9) and (4.10), we obtain

$$\frac{\hat{h}_n}{h_{n+1}} = -\frac{h_n}{p^{-2n+1}(1 - p^n)c_1^{-2}\hat{h}_{n-1} + c_0 h_n}. \tag{4.11}$$

By setting

$$X_n = \mathrm{i}\frac{1 - p^{n+1}}{p^n c_1} \frac{\hat{h}_n}{h_{n+1}}, \tag{4.12}$$

equation (4.11) can be rewritten as the following discrete Riccati equation:

$$X_n = \frac{1 - p^{n+1}}{X_{n-1} + \mathrm{i}p^n c_1 c_0}. \tag{4.13}$$

Since in the case of

$$a_0^{1/2} a_1^{1/2} = q^{1/2}, \qquad a_2 = 1, \qquad N = -1,$$

q-P$_{\mathrm{IV}}$ (1.1) admits the reduction to

$$X_n = \frac{1 - q^{(n+1)/2}}{X_{n-1} + q^{(n-m+1)/2}a_0^{-1/2}},$$

which is equivalent to equation (4.13) with the following correspondence:

$$q^{m/2}a_0^{1/2} = -\mathrm{i}q^{1/2}c_0^{-1}c_1^{-1}, \qquad q^{1/2} = p,$$

(4.1), (4.2), (4.6), (4.8) and (4.12) give the hypergeometric solutions of q-P$_{\mathrm{IV}}$ (1.1). Therefore, we finally obtain the following theorem:

Theorem 3. *In the case of*

$$a_0^{1/2}a_1^{1/2} = q^{1/2}, \qquad a_2 = 1, \qquad N = -1, \qquad n \geq 0,$$

q-P$_{\mathrm{IV}}$ (1.1) *with*

$$q^{m/2}a_0^{1/2} = -\mathrm{i}q^{1/2}c_0^{-1}c_1^{-1},$$

admits the following hypergeometric solution:

$$X_n = \frac{2\mathrm{i}\left(1 - q^{(n+1)/2}\right)\left(q^{1/2}; q^{1/2}\right)_n (q; q)_\infty \Theta\left(-q^{1/2}c_1^2; q\right)}{q^{n(n+1)/2}c_1^{2n+1}\left(q^{3/2}; q\right)_\infty \Theta\left(-c_1^2; q\right)} \frac{\Phi_{n+1}}{\Phi_{n+2}},$$

where

$$\Phi_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix},$$

$$\mu_n = \begin{cases} \dfrac{2\left(1 - q^{1/2}\right)}{\left(1 - c_0^2\right)\left(-c_1^2; q\right)_\infty} {}_2\psi_2\left(\begin{matrix} -c_1^2, c_0^{-2} \\ 0, qc_0^{-2} \end{matrix}; q, q^{(2k+1)/2}\right), & n = 2k - 1, \\[4mm] \dfrac{2c_0\left(1 - q^{1/2}\right)}{\left(1 - c_0^2\right)\left(-c_1^2; q\right)_\infty} {}_2\psi_2\left(\begin{matrix} -c_1^2, c_0^{-2} \\ 0, qc_0^{-2} \end{matrix}; q, q^{(2k+1)/2}\right), & n = 2k. \end{cases}$$

Here, $c_0 \neq q^{a/2}$ for all $a \in \mathbb{Z}$.

5 Concluding remarks

In this paper, we have constructed the hypergeometric solutions of q-P$_{\mathrm{IV}}$ (1.1) via the construction of the hypergeometric τ-functions and the theory of orthogonal polynomials. We showed that the hypergeometric solutions of the q-P$_{\mathrm{IV}}$ can be expressed by the three expressions. We note that the hypergeometric solutions of Painlevé systems expressed by the determinants whose sizes do not depend on the independent variable are called the lattice type solutions, while those expressed by the determinants whose sizes depend on the independent variable are called molecule type solutions. Thus, the hypergeometric solutions given in Theorems 1 and 2 are lattice type solutions whereas those given in Theorem 3 are molecule type solutions.

Before closing, we mention the bilateral type hypergeometric solutions here. It is well known that the coalescence cascade of hypergeometric functions, from the Gauss's hypergeometric function to the Airy function, corresponds to the diagram of degeneration of the Painlevé equations,

from the Painlevé VI equation to the Painlevé II equation, in the sense of the hypergeometric solutions [5]:

$$
\begin{array}{ccccc}
\mathrm{P_{VI}} & \to & \mathrm{P_V} & \to & \mathrm{P_{III}} \\
\text{Gauss} & & \text{Kummer} & & \text{Bessel} \\
& & \downarrow & & \downarrow \\
& & \mathrm{P_{IV}} & \to & \mathrm{P_{II}} \\
& & \text{Weber} & & \text{Airy}
\end{array}
$$

Similarly, the relations between basic hypergeometric series and q-Painlevé equations are also investigated [14, 16]. However, the hypergeometric solutions described by bilateral basic hypergeometric series have not been considered. It might be an interesting future problem to make a list of the bilateral basic hypergeometric series that appear as the solutions of the q-Painlevé equations.

Acknowledgments

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