Symmetry, Integrability and Geometry: Methods and Applications SIGMA **12** (2016), 103, 15 pages

Strictly Positive Definite Kernels on a Product of Spheres II

Jean C. GUELLA, Valdir A. MENEGATTO and Ana P. PERON

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, 13560-970, São Carlos - SP, Brazil

E-mail: *jeanguella@gmail.com, menegatt@icmc.usp.br, apperon@icmc.usp.br*

Received June 01, 2016, in final form October 24, 2016; Published online October 28, 2016
http://dx.doi.org/10.3842/SIGMA.2016.103

Abstract. We present, among other things, a necessary and sufficient condition for the strict positive definiteness of an isotropic and positive definite kernel on the cartesian product of a circle and a higher dimensional sphere. The result complements similar results previously obtained for strict positive definiteness on a product of circles [*Positivity*, to appear, arXiv:1505.01169] and on a product of high dimensional spheres [*J. Math. Anal. Appl.* **435** (2016), 286–301, arXiv:1505.03695].

Key words: positive definite kernels; strictly positive definiteness; isotropy; covariance functions; sphere; circle

2010 Mathematics Subject Classification: 33C50; 33C55; 42A16; 42A82; 42C10; 43A35

1 Introduction

The theory of positive definite and strictly positive definite kernels on manifolds and groups can not be separated from the seminal paper of I.J. Schoenberg [17] published in the first half of the past century. The major contribution in that paper refers to kernels of the form $(x,y) \in S^m \times S^m \to f(x\cdot y)$, in which $\cdot$ is the usual inner product of $\mathbb{R}^{m+1}$ and the "generating function" f is real and continuous in $[-1,1]$. The kernel is positive definite if, and only if, the generating function f has a series representation in the form

$$f(t) = \sum_{k=0}^{\infty} a_k^m P_k^m(t), \qquad t \in [-1,1],$$

where all the coefficients a_k^m are nonnegative, P_k^m denotes the usual Gegenbauer polynomial of degree k attached to the rational number $(m-1)/2$ and $\sum_k a_k^m P_k^m(1) < \infty$. The normalization for the Gegenbauer polynomials used here is

$$P_n^m(1) = \binom{n+m-2}{n}, \qquad n = 0,1,\dots.$$

Recall that the positive definiteness of a general real kernel $(x,y) \in X^2 \to K(x,y)$ on a nonempty set X, demands that $K(x,y) = K(y,x)$, $x,y \in X$, and the inequality

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu K(x_\mu, x_\nu) \geq 0,$$

whenever n is a positive integer, $x_1, x_2, \dots, x_n$ are n distinct points on X and $c_1, c_2, \dots, c_n$ are real scalars. For complex kernels we do not need the symmetry assumption and we have to use complex scalars instead. Isotropy on S^m refers to the fact that the variables x and y of S^m are tied to each other via the inner product of $\mathbb{R}^{m+1}$.

Some fifty years later, the very same positive definite functions were found useful for solving scattered data interpolation problems on spheres. But that demanded *strictly* positive definite functions, and thus a characterization of these functions was needed at the start. We recall that the strict positive definiteness of a general positive definite kernel as in the above definition requires that the previous inequalities be strict whenever at least one c_μ is nonzero. In other words, the *interpolation matrices* $[K(x_\mu, x_\nu)]_{\mu,\nu=1}^n$ of K at each set $\{x_1, x_2, \ldots, x_n\}$ need to be positive definite. The strict positive definiteness on spheres was an issue for some time until Schoenberg's result was complemented by a result of Debao Chen et al. in 2003 [6] and by Menegatto et al. [14]. A real, continuous, isotropic and positive definite kernel $(x, y) \in S^m \times S^m \to f(x \cdot y)$ is strictly positive definite if, and only if, the following additional condition holds for the coefficients in Schoenberg's series representation for the generating function f:

– $(m = 1)$ the set $\{k \in \mathbb{Z}\colon a_{|k|}^1 > 0\}$ intersects each full arithmetic progression in $\mathbb{Z}$;
– $(m \geq 2)$ the set $\{k\colon a_k^m > 0\}$ contains infinitely many even and infinitely many odd integers.

As a matter of fact, positive definite and strictly positive definite functions and kernels on spheres play a fundamental role in several other applications. For instance, two recent papers authored by Beatson and Zu Castell [3, 4] provide new families of strictly positive definite functions on spheres via the so-called half-step operators, a spherical analogue of Matheron's montée and descente operators on $\mathbb{R}^{m+1}$. Additional applications are mentioned in [5, 10].

The spheres belong to a much larger class of metric spaces, that is, they are compact two-point homogeneous spaces. Positive definiteness on these spaces in the same sense we explained above was investigated by Gangolli [8] while strict positive definiteness was completely characterized in [1].

In 2011, the paper [7] considered strictly positive definite functions on compact abelian groups taking into account a paper on strict positive definiteness on S^1 previously written by X. Sun [19]. Among other things, the paper included abstract characterizations for strict positive definiteness on a torsion group and on a product of a finite group and a torus.

In the past two years, the attention shifted all the way to positive definiteness on a product of spaces, the main motivation coming from problems involving random fields on spaces across time. The first important reference we would like to mention along this line is [5], where the authors investigated positive definite kernels on a product of the form $G \times S^m$, in which G is an arbitrary locally compact group, keeping both the group structure of G and the isotropy of S^m in the setting. Let us denote by e the neutral element of G, $*$ the operation of the group G and by u^{-1} the inverse of $u \in G$ with respect to $*$. The main contribution in [5] states that a continuous kernel of the form $((u, x), (v, y)) \in (G \times S^m)^2 \to f(u^{-1} * v, x \cdot y)$ is positive definite if, and only if, the function f has a representation in the form

$$f(u,t) = \sum_{n=0}^{\infty} f_n^m(u) P_n^m(t), \qquad (u,t) \in G \times [-1,1],$$

in which $\{f_n^m\}$ is a sequence of continuous functions on G for which $\sum f_n^m(e) P_n^m(1) < \infty$, with uniform convergence of the series for $(u,t) \in G \times [-1,1]$. As a matter of fact, the functions f_n^m are positive definite on G in the sense that the kernel $(u, v) \in G^2 \to f_n^m(u^{-1} * v)$ is positive definite as previously defined. The paper [5] did not considered any strict positive definiteness issues. It is worth to mention that [5] may be linked to [9] where the reader can find a possible reason for considering positive definiteness on a product of a locally compact abelian group with a classical space.

Simultaneously, positive definiteness and strict positive definiteness on a product of spheres was investigated in [11, 12, 13]. A characterization for the isotropic and positive definite kernels on $S^m \times S^M$ was deduced in [12] and that agreed with the characterization mentioned in [5] and also with [18, Chapter 4]. By the way, a real, continuous and isotropic kernel $((x,z),(y,w)) \in (S^m \times S^M)^2 \to f(x\cdot y, z\cdot w)$ is positive definite if, and only if, the function f has a double series representation in the form

$$f(t,s) = \sum_{k,l=0}^{\infty} a_{k,l}^{m,M} P_k^m(t) P_l^M(s), \qquad t,s \in [-1,1], \tag{1.1}$$

in which $a_{k,l}^{m,M} \geq 0$, $k,l \in \mathbb{Z}_+$ and $\sum_{k,l=0}^{\infty} a_{k,l}^{m,M} P_k^m(1) P_l^M(1) < \infty$. As before, we will call f the generating function of the kernel.

One of the main theorems in [11] reveals that, in the case in which $m, M \geq 2$, a positive definite kernel as above is strictly positive definite if, and only if, the set $\{(k,l)\colon a_{k,l}^{m,M} > 0\}$ contains sequences from each one of the sets $2\mathbb{Z}_+ \times 2\mathbb{Z}_+$, $2\mathbb{Z}_+ \times (2\mathbb{Z}_+ + 1)$, $(2\mathbb{Z}_+ + 1) \times 2\mathbb{Z}_+$, and $(2\mathbb{Z}_+ + 1) \times (2\mathbb{Z}_+ + 1)$, all of them having both component sequences unbounded. The very same paper contains some other intriguing results related to strict positive definiteness, including a notion of strict positive definiteness that holds in product spaces only. In the case $m = M = 1$, the condition becomes this one [13]: the set $\{(k,l)\colon a_{|k|,|l|}^{1,1} > 0\}$ intersects all the translations of each subgroup of $\mathbb{Z}^2$ having the form $\{(pa, qb)\colon q,p \in \mathbb{Z}\}$, $a, b > 0$. Even with the completion of these papers, it became clear very soon that a similar characterization for the remaining case, that is, strict positive definiteness on $S^1 \times S^m$, $m \geq 2$, would demand much more work, perhaps a mix of the techniques used in [11, 13].

This is the point where we explain what the contributions in this paper are. In the next section, we present two abstract results that describe how to obtain continuous, isotropic and strictly positive kernels on $S^1 \times S^m$ via the characterization for strict positive definiteness of continuous, isotropic and positive definite kernels on either S^1 or S^m separately. In Section 3, we present necessary and sufficient conditions in order that a real, continuous, isotropic and positive definite kernel on $S^1 \times S^m$, $m \geq 2$, be strictly positive definite, thus filling in the missing gap in the previous papers on the subject. In Section 4, we indicate how to obtain a similar characterization after we replace the sphere S^m with an arbitrary compact two-point homogeneous space.

2 Abstract sufficient conditions

In this section, we describe two abstract sufficient conditions for strict positive definiteness on $S^1 \times S^m$, both derived from strict positive definiteness on single spheres. The results show that transferring strict positive definiteness from the factors S^1 and S^m to strict positive definiteness of the product $S^1 \times S^m$ is not so obvious as it seems.

Here and in the next sections, *we will assume all the generating functions of the kernels are real-valued continuous functions and that the dimension m in S^m is at least* 2. But the reader is advised that the results hold true for complex kernels after some obvious modifications.

The results to be presented here will depend upon a technical lemma that provides an alternative formulation for the strict positive definiteness of a continuous, isotropic and positive definite kernel on $S^1 \times S^m$, to be described below. Let $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ be distinct points on $S^1 \times S^m$ and represent the components in S^1 in polar form:

$$x_\mu = (\cos\theta_\mu, \sin\theta_\mu), \qquad \theta_\mu \in [0, 2\pi), \qquad \mu = 1, 2, \ldots, n.$$

Write A to denote the interpolation matrix of $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ at $\{(x_1,z_1),(x_2,z_2),\ldots,(x_n,z_n)\}$ and consider the quadratic form c^tAc, where c^t indicates the transpose of c. If the kernel is positive definite, then the addition theorem for spherical harmonics [15, p. 18] and the representation (1.1) imply that

$$c^tAc = \sum_{k,l=0}^{\infty} C(k,l,m)a_{k,l}^{1,m} \sum_{j=1}^{d(l,m)} \left| \sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} Y_{l,j}^m(z_\mu) \right|^2,$$

in which $c=(c_1,c_2,\ldots,c_n)$, $\{Y_{l,j}^m\colon j=1,2,\ldots,d(l,m)\}$ is a basis for the space of all spherical harmonics of degree l in $m+1$ variables and $C(k,l,m)$ is a positive constant that depends upon k, l and m. The deduction of the equality above requires the formulas $P_0^1 \equiv 1$ and

$$P_k^1(\cos\theta) = \frac{2}{k}\cos k\theta, \qquad t\in[0,2\pi), \qquad k=1,2,\ldots.$$

The equality $c^tAc=0$ is equivalent to

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} Y_{l,j}^m(z_\mu) = 0, \qquad j=1,2,\ldots,d(l,m), \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}.$$

If this last piece of information holds, we can invoke the addition theorem once again in order to see that

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} P_l^m(z_\mu\cdot z) = 0, \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}, \qquad z\in S^m.$$

Finally, if the condition above holds, we can multiply the equality in it by $e^{-i\theta}$ and split the equation once again via the addition theorem to obtain

$$\sum_{j=1}^{d(l,m)} \left[\sum_{\mu=1}^{n} c_\mu e^{ik(\theta_\mu-\theta)} Y_{l,j}^m(z_\mu)\right] Y_{l,j}^m(z) = 0, \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}, \qquad z\in S^m.$$

Since $\{Y_{l,j}^m\colon j=1,2,\ldots,d(l,m)\}$ is linearly independent, we reach

$$\sum_{\mu=1}^{n} c_\mu Y_{l,j}^m(z_\mu) e^{ik(\theta_\mu-\theta)} = 0, \qquad j=1,2,\ldots,d(l,m), \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\},$$

for $\theta\in[0,2\pi)$. Multiplying the real part of the equality in the previous formula by $\cos k\theta$ and integrating in $[0,2\pi]$ with respect to θ, we obtain

$$\sum_{\mu=1}^{n} c_\mu Y_{l,j}^m(z_\mu)\cos k\theta_\mu = 0, \qquad j=1,2,\ldots,d(l,m), \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}.$$

Similarly, it is easily seen that

$$\sum_{\mu=1}^{n} c_\mu Y_{l,j}^m(z_\mu)\sin k\theta_\mu = 0, \qquad j=1,2,\ldots,d(l,m), \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}.$$

The above computations justify the following lemma.

Lemma 2.1. *Let f be the generating function of an isotropic and positive definite kernel on $S^1\times S^m$ and consider its series representation (1.1). The following statements are equivalent:*

(i) *the kernel* $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ *is strictly positive definite;*

(ii) *if* n *is a positive integer and* $(x_1,z_1),(x_2,z_2),\ldots,(x_n,z_n)$ *are distinct points on* $S^1 \times S^m$, *then the only solution* $c=(c_1,c_2,\ldots,c_n)$ *of the system*

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} P_l^m(z_\mu\cdot z)=0, \qquad (k,l)\in\big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}, \qquad z\in S^m,$$

is $c=0$.

A particular case of the lemma is pertinent (see Theorem 2 in [6]).

Lemma 2.2. *Let* g *be the generating function of an isotropic and positive definite kernel on* S^m *and consider its series representation according to Schoenberg. The following statements are equivalent:*

(i) *the kernel* $(z,w)\in(S^m)^2\to g(z\cdot w)$ *is strictly positive definite;*

(ii) *if* n *is a positive integer and* $z_1,z_2,\ldots,z_n$ *are distinct points on* S^m, *then the only solution* $c=(c_1,c_2,\ldots,c_n)$ *of the system*

$$\sum_{\mu=1}^{n} c_\mu P_k^m(z_\mu\cdot z)=0, \qquad k\in\big\{k\colon a_k^m>0\big\}, \qquad z\in S^m,$$

is $c=0$.

Another consequence is this one (see Theorem 5.1 in [16]).

Lemma 2.3. *Let* g *be the generating function of an isotropic and positive definite kernel on* S^1 *and consider its series representation according to Schoenberg. The following statements are equivalent:*

(i) *the kernel* $(x,y)\in(S^1)^2\to g(x\cdot y)$ *is strictly positive definite;*

(ii) *if* n *is a positive integer and* $x_1,x_2,\ldots,x_n$ *are distinct points on* S^1 *given in polar form* $x_\mu=(\cos\theta_\mu,\sin\theta_\mu)$, $\mu=1,2,\ldots,n$, *then the only solution* $c=(c_1,c_2,\ldots,c_n)$ *of the system*

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu}=0, \qquad k\in\big\{k\colon a_k^1>0\big\},$$

is $c=0$.

If f generates an isotropic positive definite kernel on $S^1\times S^m$, we will adopt the following notation attached to its double series representation (1.1):

$$J_f := \big\{(k,l)\colon a_{k,l}^{1,m}>0\big\}.$$

If I is a subset of $\mathbb{Z}_+$, we will write $I\in \mathrm{SPD}(S^m)$ to indicate that there exists a continuous and positive definite kernel $(z,w)\in(S^m)^2\to g(z\cdot w)$ for which the set $\{l\colon a_l^m>0\}$ attached to the series representation for g in Schoenberg's result is precisely I. This definition is well posed once strict positive definiteness does not depend upon the actual values of the numbers a_l^m in the series representation for the generating function but only on the set $\{l\colon a_l^m>0\}$ itself.

The first important contribution in this section is as follows.

Theorem 2.4. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider its series representation (1.1). If*

$$\big\{k\colon \{l\colon (k,l) \in J_f\} \in \mathrm{SPD}\big(S^m\big)\big\} \in \mathrm{SPD}\big(S^1\big),$$

then the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite on $S^1 \times S^m$.

Proof. We will show that, under the assumption

$$\big\{k\colon \{l\colon (k,l) \in J_f\} \in \mathrm{SPD}\big(S^m\big)\big\} \in \mathrm{SPD}\big(S^1\big),$$

the alternative condition in Lemma 2.1 holds. In particular, the notation employed in that lemma will be adopted here. Let $(x_1,z_1),(x_2,z_2),\ldots,(x_n,z_n)$ be distinct points in $S^1 \times S^m$ and suppose that

$$\sum_{\mu=1}^{n} c_\mu e^{ik\theta_\mu} P_l^m(z_\mu \cdot z) = 0, \qquad (k,l) \in J_f, \qquad z \in S^m.$$

Let M be a maximal subset of $\{1,2,\ldots,n\}$ that indexes the distinct elements among the z_j. Writing $M_j := \{\mu\colon z_\mu = z_j\}$, $j \in M$, the previous equality becomes

$$\sum_{j\in M}\left(\sum_{\mu\in M_j} c_\mu e^{ik\theta_\mu}\right) P_l^m(z_j\cdot z) = 0, \qquad (k,l) \in J_f, \qquad z \in S^m.$$

In particular,

$$\sum_{j\in M}\left(\sum_{\mu\in M_j} c_\mu e^{ik\theta_\mu}\right) P_l^m(z_j\cdot z) = 0, \qquad l \in \{l\colon (k,l) \in J_f\}, \qquad z \in S^m,$$

whenever $k \in \{k\colon \{l\colon (k,l) \in J_f\} \in \mathrm{SPD}(S^m)\}$. Since the z_j in the expression above are all distinct, Lemma 2.2 yields that

$$\sum_{\mu\in M_j} c_\mu e^{ik\theta_\mu} = 0, \qquad k \in \big\{k\colon \{l\colon (k,l) \in J_f\} \in \mathrm{SPD}\big(S^m\big)\big\},$$

for every $j \in M$. An application of Lemma 2.3 for each j plus the help of our original assumption leads to $c_\mu = 0$, $\mu \in M_j$, $j \in M$. But this corresponds to $c = 0$. ∎

In a similar manner, but with slightly easier arguments, the following cousin theorem can be proved.

Theorem 2.5. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider its series representation (1.1). If*

$$\big\{l\colon \{k\colon (k,l) \in J_f\} \in \mathrm{SPD}\big(S^1\big)\big\} \in \mathrm{SPD}\big(S^m\big),$$

then the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite on $S^1 \times S^m$.

We close this section presenting realizations for the previous theorems. They follow from the characterizations for strict positive definiteness on singles spheres described in the Introduction.

Corollary 2.6. *Let f be the generating function for an isotropic and positive definite kernel on $S^1 \times S^m$ and consider its series representation (1.1). Either condition below is sufficient for the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ to be strictly positive definite:*

(*i*) *J_f contains sequences $\{(k_\mu, 2l_{\mu\nu})\colon \mu,\nu \in \mathbb{Z}_+\}$ and $\{(k'_\mu, 2l'_{\mu\nu}+1)\colon \mu,\nu \in \mathbb{Z}_+\}$ so that*

$$\{\pm k_\mu\colon \mu \in \mathbb{Z}_+\} \cap (n\mathbb{Z}+j) \neq \varnothing, \qquad j=0,1,\ldots,n-1, \qquad n \geq 1,$$
$$\{\pm k'_\mu\colon \mu \in \mathbb{Z}_+\} \cap (n\mathbb{Z}+j) \neq \varnothing, \qquad j=0,1,\ldots,n-1, \qquad n \geq 1,$$

and

$$\lim_{\nu\to\infty} l_{\mu\nu} = \lim_{\nu\to\infty} l'_{\mu\nu} = \infty, \qquad \mu \in \mathbb{Z}_+.$$

(*ii*) *J_f contains sequences $\{(k_{\mu\nu}, 2l_\mu)\colon \mu,\nu \in \mathbb{Z}_+\}$ and $\{(k'_{\mu\nu}, 2l'_\mu+1)\colon \mu,\nu \in \mathbb{Z}_+\}$ so that*

$$\{\pm k_{\mu\nu}\colon \nu \in \mathbb{Z}_+\} \cap (n\mathbb{Z}+j) \neq \varnothing, \qquad j=0,1,\ldots,n-1, \qquad n \geq 1, \qquad \mu \in \mathbb{Z}_+,$$
$$\{\pm k'_{\mu\nu}\colon \nu \in \mathbb{Z}_+\} \cap (n\mathbb{Z}+j) \neq \varnothing, \qquad j=0,1,\ldots,n-1, \qquad n \geq 1, \qquad \mu \in \mathbb{Z}_+,$$

and

$$\lim_{\mu\to\infty} l_\mu = \lim_{\mu\to\infty} l'_\mu = \infty.$$

Finally, we would like to point that the previous theorems can be reproduced in other settings, with or without the presence of isotropy (for example $S^m \times S^m$ and the product of S^m and a torus). Details will not be included here.

3 Characterizations for strict positive definiteness on $S^1 \times S^m$

In order to obtain a characterization for the strict positive definiteness of an isotropic positive definite kernel on $S^1 \times S^m$, we need to look at the concept of strict positive definiteness in an enhanced form. We begin this section explaining what we mean by that and introducing the additional concepts needed.

It is an obvious matter to see that we can write the generating function f of a positive definite kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ in the form

$$f(t,s) = \sum_{l=0}^{\infty} f_l(t) P_l^m(s), \qquad t,s \in [-1,1], \tag{3.1}$$

in which

$$f_l(t) := \sum_{k=0}^{\infty} a_{k,l}^{1,m} P_k^1(t), \qquad t \in [-1,1], \qquad l = 0,1,\ldots.$$

Since $P_l^m(1) \geq 1$, $m \geq 2$, $l = 0,1,\ldots$, the series $\sum_{k=0}^{\infty} a_{k,l}^{1,m} P_k^1(1) < \infty$ converges. In particular, each f_l is the generating function of a continuous, isotropic and positive definite kernel on S^1.

In the statement of the next lemma, we will employ the following additional notation related to another one we have previously introduced:

$$J_f^k := \{l\colon (k,l) \in J_f\}.$$

In particular,

$$\cup_k J_f^k = \{l\colon f_l \neq 0\}.$$

Among other things, the lemma suggests how a characterization for the strict positive definiteness of an isotropic and positive definite kernel on $S^1 \times S^m$ should look like.

Lemma 3.1. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider the alternative series representation (3.1) for f. If p is a positive integer, $x_1, x_2, \ldots, x_p$ are distinct points on S^1 and c is a real vector in $\mathbb{R}^p$, then the continuous function g given by*

$$g(s) = \sum_{l \in \cup_k J_f^k} \left\{ c^t [f_l(x_i \cdot x_j)]_{i,j=1}^p c \right\} P_l^m(s), \qquad s \in [-1,1],$$

generates an isotropic and positive definite kernel on S^m. If c is nonzero and the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ is strictly positive definite, then $(z,w) \in (S^m)^2 \to g(z \cdot w)$ is strictly positive definite as well.

Proof. Write $c = (c_1, c_2, \ldots, c_p)$. If $z_1, z_2, \ldots, z_q$ are distinct points on S^m and $d_1, d_2, \ldots, d_q$ are real numbers, then

$$\sum_{\mu,\nu=1}^{q} d_\mu d_\nu g(z_\mu \cdot z_\nu) = \sum_{\mu,\nu=1}^{q} \sum_{i,j=1}^{p} (d_\mu c_i)(d_\nu c_j) f(x_i \cdot x_j, z_\mu \cdot z_\nu).$$

But, the last expression above corresponds to a quadratic form involving f and the pq distinct points (x_i, z_μ), $i = 1, 2, \ldots, p$, $\mu = 1, 2, \ldots, q$, of $S^1 \times S^m$. In particular,

$$\sum_{\mu,\nu=1}^{q} d_\mu d_\nu g(z_\mu \cdot z_\nu) \geq 0.$$

If the real numbers d_μ are not all zero and $c \neq 0$, then at least one of the scalars $d_\mu c_i$ is likewise nonzero. Further, if $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ is strictly positive definite, then the quadratic form above is, in fact, positive. In particular, $(z,w) \in (S^m)^2 \to g(z \cdot w)$ is strictly positive definite. ■

Another useful technical result is as follows.

Lemma 3.2. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider the alternative series representation (3.1) for f. If $(x_1, z_1), (x_2, z_2), \ldots, (x_n, z_n)$ are distinct points in $S^1 \times S^m$ and $c_1, c_2, \ldots, c_n$ are real scalars, then the following assertions are equivalent:*

(i) $\sum_{i,j=1}^{n} c_i c_j f(x_i \cdot x_j, z_i \cdot z_j) = 0$;

(ii) $\sum_{i,j=1}^{n} c_i c_j f_l(x_i \cdot x_j) P_l^m(z_i \cdot z_j) = 0$, $l \in \cup_k J_f^k$.

Proof. One direction is immediate while the other follows simply from the observation that $(t,s) \in [-1,1]^2 \to f_l(t) P_l^m(s)$ is the generating function of a positive definite kernel on $S^1 \times S^m$. ■

Next, we formalize the definition of enhancement we use in this section.

Definition 3.3. Let p and q be positive integers, $\{x_1, x_2, \ldots, x_p\} \subset S^1$ and $\{z_1, z_2, \ldots, z_q\}$ an antipodal free subset of S^m, that is, a set containing no pairs of antipodal points. An *enhanced subset* of $S^1 \times S^m$ generated by them is the set

$$\begin{aligned} \big\{ &(x_1, z_1), (x_2, z_1), \ldots, (x_p, z_1), (x_1, z_2), (x_2, z_2), \ldots, (x_p, z_2), \ldots, \\ &(x_1, z_q), (x_2, z_q), \ldots, (x_p, z_q), (x_1, -z_1), (x_2, -z_1), \ldots, (x_p, -z_1), \\ &(x_1, -z_2), (x_2, -z_2), \ldots, (x_p, -z_2), \ldots, (x_1, -z_q), (x_2, -z_q), \ldots, (x_p, -z_q) \big\}. \end{aligned}$$

The positive numbers p and q in the previous definition may have no connection at all. The order in which the elements in an enhanced subset of $S^1 \times S^m$ are displayed is not relevant, but the writing of the upcoming results will take into account the order adopted above and inherited from those in the subsets of S^1 and S^m involved. An enhanced set as above contains $2pq$ distinct points.

The following lemma is concerned with the existence of enhanced sets.

Lemma 3.4. *If $B' = \{(x_1', z_1'), (x_2', z_2'), \ldots, (x_n', z_n')\}$ is a subset of $S^1 \times S^m$, then there exists an enhanced subset B of $S^1 \times S^m$ that contains B'.*

Proof. It suffices to consider the enhanced subset of $S^1 \times S^m$ generated by the set $\{x_1, x_2, \ldots, x_p\}$ that encompasses the distinct elements among the x_i' and an antipodal free subset $\{z_1, z_2, \ldots, z_q\}$ of S^m satisfying $z_i' \in \{z_1, z_2, \ldots, z_q\}$, $i = 1, 2, \ldots, n$. ■

If f is the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and B is an enhanced set as previously described, we will write $\mathcal{E}(f, B)$ to denote the interpolation matrix of $((x, z), (y, w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ at B, keeping the order for the points of B. If A is a subset of $S^1 \times S^m$ and B is an enhanced subset of $S^1 \times S^m$ containing A, then the interpolation matrix of $((x, z), (y, w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ at A is a principal sub-matrix of $\mathcal{E}(f, B)$. In particular, if $\mathcal{E}(f, B)$ is positive definite, so is A. These comments justify the following lemma.

Lemma 3.5. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider the alternative series representation (3.1) for f. The following assertions are equivalent:*

(*i*) *the kernel $((x, z), (y, w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ is strictly positive definite;*

(*ii*) *If B is an enhanced subset of $S^1 \times S^m$, then the matrix $\mathcal{E}(f, B)$ is positive definite.*

Due to the decomposition for the generating function f of an isotropic positive definite kernel as described in (3.1), we can write

$$\mathcal{E}(f, B) = \sum_{l=0}^{\infty} \mathcal{E}(f, B, l)$$

in which $\mathcal{E}(f, B, l)$ is the interpolation matrix of the kernel $(t, s) \in S^1 \times S^m \to f_l(t) P_l^m(s)$ at B. The order in which the elements of an enhanced subset of $S^1 \times S^m$ appears forces the matrix $\mathcal{E}(f, B, l)$ to have a very distinctive block representation. Precisely,

$$\mathcal{E}(f, B, l) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where each block $M_{\rho\sigma} = M_{\rho\sigma}(f, B, l)$ has its own block structure

$$M_{\rho\sigma} = [M_{\rho\sigma}^{\mu\nu}]_{\mu,\nu=1}^{q}, \qquad \rho, \sigma = 1, 2,$$

defined by

$$M_{\rho\sigma}^{\mu\nu} = [f_l(x_i \cdot x_j)]_{i,j=1}^{p} (-1)^{l(\sigma+\rho)} P_l^m(z_\mu \cdot z_\nu), \qquad \mu, \nu = 1, 2, \ldots, q.$$

Implicitly used in the writing of the block decomposition above is the fact that Gegenbauer polynomials of even degree are even functions while those of odd degree are odd functions. In particular, since $M_{22} = M_{11}$ and $M_{12} = M_{21} = (-1)^l M_{11}$, the matrix $\mathcal{E}(f, B, l)$ depends upon M_{11} only.

Keeping all the notation introduced so far, Lemmas 3.2 and 3.5 and the comments above lead to the following characterization for the strict positive definiteness of the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ via

$$M_{11} = \left[[f_l(x_i\cdot x_j)]_{i,j=1}^p P_l^m(z_\mu \cdot z_\nu)\right]_{\mu,\nu=1}^q.$$

Lemma 3.6. *Let f be the generating function of an isotropic and positive definite kernel on $S^1\times S^m$ and consider the alternative series representation (3.1) for f. The following assertions are equivalent:*

- *(i) the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite;*
- *(ii) if B is an enhanced subset of $S^1\times S^m$ generated by a subset $\{x_1,x_2,\ldots,x_p\}$ of S^1 and the antipodal free subset $\{z_1,z_2,\ldots,z_q\}$ of S^m, then the only solution $(c_1,c_2)\in(\mathbb{R}^{pq})^2$ of the system*

$$\left[c_1+(-1)^l c_2\right]^t M_{11}\left[c_1+(-1)^l c_2\right]=0, \qquad l\in \cup_k J_f^k,$$

is the trivial one, that is, $c_1=c_2=0$.

Introducing components for the vectors c_i in the previous lemma, we obtain the following reformulation.

Lemma 3.7. *Let f be as in the previous lemma. The following assertions are equivalent:*

- *(i) the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite;*
- *(ii) if B is an enhanced subset of $S^1\times S^m$ generated by a subset $\{x_1,x_2,\ldots,x_p\}$ of S^1 and the antipodal free subset $\{z_1,z_2,\ldots,z_q\}$ of S^m, then the only solution $(c_1^1,c_1^2,\ldots,c_1^q,c_2^1,c_2^2,\ldots,c_2^q)\in(\mathbb{R}^p)^{2q}$ of the system*

$$\sum_{\mu,\nu=1}^q \left\{(c_1^\mu+(-1)^l c_2^\mu)^t[f_l(x_i\cdot x_j)]_{i,j=1}^p(c_1^\nu+(-1)^l c_2^\nu)\right\}P_l^m(z_\mu\cdot z_\nu)=0, \qquad l\in\cup_k J_f^k,$$

is the trivial one.

Next, we break up the system in the previous lemma, according to the parity of the elements in $\cup_k J_f^k$.

Proposition 3.8. *Let f be as in the previous lemma. The following assertions are equivalent:*

- *(i) the kernel $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite;*
- *(ii) if p and q are positive integers, $x_1,x_2,\ldots,x_p$ are distinct points on S^1 and $\{z_1,z_2,\ldots,z_q\}$ is an antipodal free subset of S^m, then the only solution $(d_1^1,d_1^2,\ldots,d_1^q,d_2^1,d_2^2,\ldots,d_2^q)$ in $(\mathbb{R}^p)^{2q}$ of the system*

$$\sum_{\mu,\nu=1}^q \left\{(d_1^\mu)^t[f_l(x_i\cdot x_j)]_{i,j=1}^p d_1^\nu\right\}P_l^m(z_\mu\cdot z_\nu)=0, \qquad l\in(2\mathbb{Z}_++1)\cap\left(\cup_k J_f^k\right),$$

$$\sum_{\mu,\nu=1}^q \left\{(d_2^\mu)^t[f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\nu\right\}P_l^m(z_\mu\cdot z_\nu)=0, \qquad l\in(2\mathbb{Z}_+)\cap\left(\cup_k J_f^k\right),$$

is the trivial one.

Proof. Assume that for some distinct points $x_1, x_2, \ldots, x_p$ in S^1 and some antipodal free subset $\{z_1, z_2, \ldots, z_q\}$ of S^m, the system in (ii) has a nontrivial solution $(d_1^1, d_1^2, \ldots, d_1^q, d_2^1, d_2^2, \ldots, d_2^q)$. If $d_1^\mu \neq 0$ for some μ, we define $c_1^\mu = -c_2^\mu = 2^{-1}d_1^\mu$, $\mu = 1, 2, \ldots, q$. Otherwise, we define $c_1^\mu = c_2^\mu = 2^{-1}d_2^\mu$, $\mu = 1, 2, \ldots, q$. In both cases, the vector $(c_1^1, c_1^2, \ldots, c_1^q, c_2^1, c_2^2, \ldots, c_2^q)$ is nonzero and, in addition,

$$\sum_{\mu,\nu=1}^{q} \left\{ \left(c_1^\mu + (-1)^l c_2^\mu\right)^t [f_l(x_i \cdot x_j)]_{i,j=1}^p \left(c_1^\nu + (-1)^l c_2^\nu\right) \right\} P_l^m(z_\mu \cdot z_\nu) = 0,$$

for $l \in [(2\mathbb{Z}_+ + 1) \cap (\cup_k J_f^k)] \cup [(2\mathbb{Z}_+) \cap (\cup_k J_f^k)] = \cup_k J_F^k$. In other words, Condition (ii) in Lemma 3.7 does not hold for the enhanced set B generated by the subset $\{x_1, x_2, \ldots, x_p\}$ of S^1 and the antipodal free subset $\{z_1, z_2, \ldots, z_q\}$ of S^m. Thus, $((x, z), (y, w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ is not strictly positive definite. Conversely, if (i) does not hold, the previous lemma assures the existence of a subset $\{x_1, x_2, \ldots, x_p\}$ of S^1, an antipodal free subset $\{z_1, z_2, \ldots, z_q\}$ of S^m, an enhanced subset A of $S^1 \times S^m$ generated by them and a nonzero vector $(c_1^1, c_1^2, \ldots, c_1^q, c_2^1, c_2^2, \ldots, c_2^q) \in (\mathbb{R}^p)^{2q}$ so that

$$\sum_{\mu,\nu=1}^{q} \left\{ \left(c_1^\mu + (-1)^l c_2^\mu\right)^t [f_l(x_i \cdot x_j)]_{i,j=1}^p \left(c_1^\nu + (-1)^l c_2^\nu\right) \right\} P_l^m(z_\mu \cdot z_\nu) = 0, \qquad l \in \cup_k J_f^k.$$

However, this last piece of information corresponds to

$$\sum_{\mu,\nu=1}^{q} \left\{ \left(c_1^\mu - c_2^\mu\right)^t [f_l(x_i \cdot x_j)]_{i,j=1}^p \left(c_1^\nu - c_2^\nu\right) \right\} P_l^m(z_\mu \cdot z_\nu) = 0, \qquad l \in (2\mathbb{Z}_+ + 1) \cap \left(\cup_k J_f^k\right),$$

and

$$\sum_{\mu,\nu=1}^{q} \left\{ \left(c_1^\mu + c_2^\mu\right)^t [f_l(x_i \cdot x_j)]_{i,j=1}^p \left(c_1^\nu + c_2^\nu\right) \right\} P_l^m(z_\mu \cdot z_\nu) = 0, \qquad l \in (2\mathbb{Z}_+) \cap \left(\cup_k J_f^k\right).$$

On the other hand, it is easily verifiable that the vector

$$\left(c_1^1 - c_2^1, c_1^2 - c_2^2, \ldots, c_1^q - c_2^q, c_1^1 + c_2^1, c_1^2 + c_2^2, \ldots, c_1^q + c_2^q\right) \in \left(\mathbb{R}^p\right)^{2q}$$

is nonzero. Thus, (ii) does not hold due to Lemma 3.7. ■

We are about ready to prove the crucial result in this section.

Theorem 3.9. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider the alternative series representation (3.1) for f. The following assertions are equivalent:*

(i) the kernel $((x, z), (y, w)) \in (S^1 \times S^m)^2 \to f(x \cdot y, z \cdot w)$ is strictly positive definite;

(ii) if p is a positive integer, $x_1, x_2, \ldots, x_p$ are distinct points in S^1 and $c \in \mathbb{R}^p \setminus \{0\}$, then the set

$$\left\{ l \in \cup_k J_f^k \colon c^t [f_l(x_i \cdot x_j)]_{i,j=1}^p c > 0 \right\}$$

contains infinitely many even and infinitely many odd integers.

Proof. Lemma 3.1 justifies one implication. As for the other, assume the condition in the statement of the theorem holds but $((x,z),(y,w)) \in (S^1 \times S^m)^2 \to f(x\cdot y, z\cdot w)$ is not strictly positive definite. Hence, we can find a positive integer p, distinct points $x_1, x_2, \ldots, x_p$ in S^1, an antipodal free subset $\{y_1, y_2, \ldots, y_n\}$ of S^m and a nonzero vector $(d_1^1, d_1^2, \ldots, d_1^q, d_2^1, d_2^2, \ldots, d_2^q)$ in $(\mathbb{R}^p)^{2q}$ so that the two equations in Proposition 3.8(ii) hold. We will proceed assuming that $(d_2^1, d_2^2, \ldots, d_2^q)$ is a nonzero vector and that

$$\sum_{\mu,\nu=1}^{q} \left\{(d_2^\mu)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\nu\right\} P_l^m(z_\mu \cdot z_\nu) = 0, \qquad l \in 2\mathbb{Z}_+ \cap \left(\cup_k J_f^k\right)$$

and will reach a contradiction. The other possibility can be handled similarly but the details will be omitted. Without loss of generality, we can assume that the vector d_2^1 is nonzero. Since the set

$$\left\{l \in \cup_k J_f^k \colon \left(d_2^1\right)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^1 > 0\right\} \cap 2\mathbb{Z}$$

is infinite by assumption, we can select an infinite subset Q of it and a number $\theta = \theta(l)$ in $\{1, 2, \ldots, q\}$ so that

$$\left(d_2^\theta\right)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\theta \geq \left(d_2^\mu\right)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\mu, \qquad \mu = 1, 2, \ldots, q, \qquad l \in Q.$$

In particular,

$$\left(d_2^\theta\right)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\theta > 0, \qquad l \in Q.$$

Returning to the initial equality we can write

$$\begin{aligned} 0 = 1 &+ \sum_{\substack{\mu=1\\ \mu\neq\theta}}^{q} \frac{(d_2^\mu)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\mu}{(d_2^\theta)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\theta} \frac{P_l^m(z_\mu\cdot z_\mu)}{P_l^m(1)} \\ &+ \sum_{\mu\neq\nu} \frac{(d_2^\mu)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\nu}{(d_2^\theta)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\theta} \frac{P_l^m(z_\mu\cdot z_\nu)}{P_l^m(1)}, \qquad l \in Q. \end{aligned}$$

Since each f_l is the continuous and isotropic part of a positive definite kernel on S^1, we have that $(d_2^\mu)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\mu \geq 0$, $\mu = 1, 2, \ldots, q$. In particular,

$$\sum_{\substack{\mu=1\\ \mu\neq\theta}}^{q} \frac{(d_2^\mu)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\mu}{(d_2^\theta)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\theta} \in [0, q-1],$$

while the Cauchy–Schwarz inequality implies that

$$0 \leq \left| \frac{(d_2^\mu)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\nu}{(d_2^\theta)^t [f_l(x_i\cdot x_j)]_{i,j=1}^p d_2^\theta} \right| \leq 1, \qquad \mu \neq \nu.$$

Since $z_\mu \cdot z_\nu \in (-1,1)$, $\mu \neq \nu$, a well-known property of the Gegenbauer polynomials provides the limit formula [20, p. 196]

$$\lim_{\substack{l\to\infty\\ l\in Q}} \frac{P_l^m(z_\mu\cdot z_\nu)}{P_l^m(1)} = 0, \qquad \mu \neq \nu.$$

Consequently, we may apply the definition of limit conveniently (l large enough), to conclude that $0 \geq 1 + 0 - 1/2 = 1/2$, a contradiction. ■

The next theorem demands the truncated sum functions ($\gamma \geq 0$)

$$f_\gamma^{\rm o} = \sum_{2l+1\geq \gamma} f_{2l+1} \qquad \text{and} \qquad f_\gamma^{\rm e} = \sum_{2l\geq\gamma} f_{2l}$$

attached to the generating function of an isotropic and positive definite kernel. Since $P_l^m(1) \geq 1$, $m \geq 2$, $l \geq 0$, it follows that

$$|f_l(t)| \leq \sum_{k=0}^{\infty} a_{k,l}^{1,m} P_k^1(1) \leq \sum_{k=0}^{\infty} a_{k,l}^{1,m} P_k^1(1) P_l^m(1), \qquad l \geq \gamma.$$

In particular, since $\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} a_{k,l}^{1,m} P_k^1(1)P_l^m(1) < \infty$, the functions $f_\gamma^{\rm o}$ and $f_\gamma^{\rm e}$ are continuous.

Theorem 3.10. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times S^m$ and consider the alternative series representation (3.1) for f. The following assertions are equivalent:*

- (*i*) *the kernel $((x,z),(y,w)) \in (S^1\times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite;*
- (*ii*) *for each $\gamma \geq 0$, the functions $f_\gamma^{\rm o}$ and $f_\gamma^{\rm e}$ are the generating functions of isotropic and strictly positive definite kernels on S^1.*

Proof. If (i) holds, we can apply Lemma 3.1 in order to see that

$$c^t f_\gamma^{\rm o}(x_i\cdot x_j)c = \sum_{2l+1\geq\gamma} c^t f_{2l+1}(x_i\cdot x_j)c > 0,$$

whenever $c \in \mathbb{R}^p\setminus\{0\}$, $\gamma \geq 0$ and $x_1, x_2, \ldots, x_p$ are distinct points in S^1. Obviously, a similar property is valid for $f_\gamma^{\rm e}$. Thus, (ii) follows. Conversely, if (ii) holds, then Condition (ii) in the previous theorem holds as well, due to the fact that (ii) is valid for all $\gamma \geq 0$. Thus, Theorem 3.9 guarantees the strict positive definiteness of $((x,z),(y,w)) \in (S^1\times S^m)^2 \to f(x\cdot y, z\cdot w)$. ■

The coefficient in the series expansion of $f_\gamma^{\rm o}$ attached to the polynomial P_k^1 is

$$\sum_{\gamma\leq 2l+1\in J_f^k} a_{k,2l+1}^{1,m}.$$

It is positive if, and only if, J_f^k contains an odd integer greater than or equal to γ. A similar remark applies to the coefficients in the series of $f_\gamma^{\rm e}$. Taking into account the characterization for strict positive definiteness on S^1 quoted at the introduction, we have the following consequence of the previous theorem and our final characterization for strict positive definiteness on $S^1\times S^m$.

Theorem 3.11. *Let f be the generating function of an isotropic and positive definite kernel on $S^1\times S^m$. The following assertions are equivalent:*

- (*i*) *the kernel $((x,z),(y,w)) \in (S^1\times S^m)^2 \to f(x\cdot y, z\cdot w)$ is strictly positive definite;*
- (*ii*) *for each $\gamma\geq 0$, the sets*

$$\left\{k\in\mathbb{Z}\colon J_f^{|k|}\cap\{\gamma,\gamma+1,\ldots\}\cap(2\mathbb{Z}+1)\neq\varnothing\right\}$$

and

$$\left\{k\in\mathbb{Z}\colon J_f^{|k|}\cap\{\gamma,\gamma+1,\ldots\}\cap 2\mathbb{Z}_+\neq\varnothing\right\}$$

intersect every arithmetic progression in $\mathbb{Z}$.

4 Replacing S^m with a compact two-point homogeneous space

The results obtained so far in the paper can be adapted to hold for kernels on a product of the form $S^1 \times \mathbb{M}^d$, in which $\mathbb{M}^d$ is a compact two-point homogeneous space. The case $S^1 \times S^1$ was covered in [13] while the case $S^d \times \mathbb{M}^d$, $d \geq 3$, was covered in [2, Theorem 4.5]. The results sketched in this section complement these two cases.

Let us write $|zw|$ to denote the usual normalized surface distance between z and w in $\mathbb{M}^d$. As described in [2], an isotropic kernel $((x,z),(y,w)) \in (S^1 \times \mathbb{M}^d)^2 \to f(x \cdot y, \cos(|zw|/2))$ is positive definite if, and only if, the generating function f has a double series representation in the form

$$f(t,s) = \sum_{k,l=0}^{\infty} a_{k,l}^d P_k^1(t) P_l^{d,\beta}(s), \qquad t,s \in [-1,1]^2,$$

in which $a_{k,l}^d \geq 0$, $k,l \in \mathbb{Z}_+$, $P_l^{d,\beta}$ is the Jacobi polynomial associated to the pair $((d-2)/2, \beta)$, β is a number from the list $-1/2, 0, 1, 3$, depending on the respective category $\mathbb{M}^d$ belongs to, that is, the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2,3,\ldots$, the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4,6,\ldots$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8,12,\ldots$, and the Cayley projective plane $\mathbb{P}^d(\mathrm{Cay})$, $d = 16$, and $\sum_{k,l=0}^{\infty} a_{k,l} P_k^1(1) P_l^{d,\beta}(1) < \infty$.

In this setting, the procedure adopted in Section 3 can be considerably simplified. An alternative series representation for the generating function of the kernel can be likewise defined and the fact that a point in $\mathbb{M}^d$ possesses infinitely many antipodal points permits the deduction of a version of Theorem 3.9 without considering any enhancements and augmentations. Precisely, we have the following result.

Theorem 4.1. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times \mathbb{M}^d$ and consider the alternative series representation for f. The following assertions are equivalent:*

(i) the kernel $((x,z),(y,w)) \in (S^1 \times \mathbb{M}^d)^2 \to f(x \cdot y, z \cdot w)$ is strictly positive definite;

(ii) if p is a positive integer, $x_1, x_2, \ldots, x_p$ are distinct points in S^1 and $c \in \mathbb{R}^p \setminus \{0\}$, then the set

$$\left\{ l \in \cup_k J_f^k \colon c^t [f_l(x_i \cdot x_j)]_{i,j=1}^p c > 0 \right\}$$

contains infinitely many integers.

Taking into account the characterization for strict positive definiteness obtained in [1], the final characterization in these remaining cases is this one.

Theorem 4.2. *Let f be the generating function of an isotropic and positive definite kernel on $S^1 \times \mathbb{M}^d$. Assume $\mathbb{M}^d$ is not a sphere. The following assertions are equivalent:*

(i) the kernel $((x,z),(y,w)) \in (S^1 \times \mathbb{M}^d)^2 \to f(x \cdot y, z \cdot w)$ is strictly positive definite;

(ii) for each $\gamma \geq 0$, the set

$$\left\{ k \in \mathbb{Z} \colon \; J_f^{|k|} \cap \{\gamma, \gamma+1, \ldots\} \neq \varnothing \right\}$$

intersects every arithmetic progression in $\mathbb{Z}$.

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions. Second author acknowledges partial financial support from FAPESP, grant 2014/00277-5. Likewise, the third author acknowledges support from the same foundation, under grants 2014/25796-5 and 2016/03015-7.

References

[1] Barbosa V.S., Menegatto V.A., Strictly positive definite kernels on compact two-point homogeneous spaces, *Math. Inequal. Appl.* **19** (2016), 743–756, arXiv:1505.00591.

[2] Barbosa V.S., Menegatto V.A., Strict positive definiteness on a product of compact two-point homogeneous spaces, *Integral Transform. Spec. Funct.*, to appear, arXiv:1605.07071.

[3] Beatson R.K., zu Castell W., Dimension hopping and families of strictly positive definite zonal basis functions on spheres, arXiv:1510.08658.

[4] Beatson R.K., zu Castell W., One-step recurrences for stationary random fields on the sphere, *SIGMA* **12** (2016), 043, 19 pages, arXiv:1601.07743.

[5] Berg C., Porcu E., From Schoenberg coefficients to Schoenberg functions, *Constr. Approx.*, to appear, arXiv:1505.05682.

[6] Chen D., Menegatto V.A., Sun X., A necessary and sufficient condition for strictly positive definite functions on spheres, *Proc. Amer. Math. Soc.* **131** (2003), 2733–2740.

[7] Emonds J., Führ H., Strictly positive definite functions on compact abelian groups, *Proc. Amer. Math. Soc.* **139** (2011), 1105–1113, arXiv:1002.3017.

[8] Gangolli R., Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters, *Ann. Inst. H. Poincaré Sect. B* **3** (1967), 121–226.

[9] Gneiting T., Nonseparable, stationary covariance functions for space-time data, *J. Amer. Statist. Assoc.* **97** (2002), 590–600.

[10] Gneiting T., Strictly and non-strictly positive definite functions on spheres, *Bernoulli* **19** (2013), 1327–1349, arXiv:1111.7077.

[11] Guella J.C., Menegatto V.A., Strictly positive definite kernels on a product of spheres, *J. Math. Anal. Appl.* **435** (2016), 286–301, arXiv:1505.03695.

[12] Guella J.C., Menegatto V.A., Peron A.P., An extension of a theorem of Schoenberg to products of spheres, *Banach J. Math. Anal.* **10** (2016), 671–685, arXiv:1503.08174.

[13] Guella J.C., Menegatto V.A., Peron A.P., Strictly positive definite kernels on a product of circles, *Positivity*, to appear, arXiv:1505.01169.

[14] Menegatto V.A., Oliveira C.P., Peron A.P., Strictly positive definite kernels on subsets of the complex plane, *Comput. Math. Appl.* **51** (2006), 1233–1250.

[15] Müller C., Analysis of spherical symmetries in Euclidean spaces, *Applied Mathematical Sciences*, Vol. 129, Springer-Verlag, New York, 1998.

[16] Ron A., Sun X., Strictly positive definite functions on spheres in Euclidean spaces, *Math. Comp.* **65** (1996), 1513–1530.

[17] Schoenberg I.J., Positive definite functions on spheres, *Duke Math. J.* **9** (1942), 96–108.

[18] Shapiro V.L., Fourier series in several variables with applications to partial differential equations, *Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series*, CRC Press, Boca Raton, FL, 2011.

[19] Sun X., Strictly positive definite functions on the unit circle, *Math. Comp.* **74** (2005), 709–721.

[20] Szegő G., Orthogonal polynomials, *Colloquium Publications*, Vol. 23, 4th ed., Amer. Math. Soc., Providence, RI, 1975.