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Picard–Vessiot Extensions of Real Differential Fields

Teresa CRESPO [†] *and Zbigniew HAJTO* [‡]

[†] *Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain*
E-mail: teresa.crespo@ub.edu

[‡] *Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Prof. S. Łojasiewicza 6, 30-348 Kraków, Poland*
E-mail: zbigniew.hajto@uj.edu.pl

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Abstract. For a linear differential equation defined over a formally real differential field K with real closed field of constants k, Crespo, Hajto and van der Put proved that there exists a unique formally real Picard–Vessiot extension up to K-differential automorphism. However such an equation may have Picard–Vessiot extensions which are not formally real fields. The differential Galois group of a Picard–Vessiot extension for this equation has the structure of a linear algebraic group defined over k and is a k-form of the differential Galois group H of the equation over the differential field $K\big(\sqrt{-1}\big)$. These facts lead us to consider two issues: determining the number of K-differential isomorphism classes of Picard–Vessiot extensions and describing the variation of the differential Galois group in the set of k-forms of H. We address these two issues in the cases when H is a special linear, a special orthogonal, or a symplectic linear algebraic group and conclude that there is no general behaviour.

Key words: real Picard–Vessiot theory; linear algebraic groups; group cohomology; real forms of algebraic groups

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1 Introduction

To a homogeneous linear differential equation defined over a differential field K with field of constants k, Picard–Vessiot theory associates a differential field extension L of K, differentially generated over K by a fundamental system of solutions of the equation, and with constant field equal to k, called a Picard–Vessiot extension for the given equation. When k is algebraically closed, Kolchin [18] established that a Picard–Vessiot extension for the given equation exists and is unique up to K-differential isomorphism. The differential Galois group $\operatorname{DGal}(L/K)$ is defined as the group of K-differential automorphisms of L and has the structure of a linear algebraic group defined over k.

For a homogeneous linear differential equation defined over a formally real differential field K with real closed field of constants k, Crespo, Hajto and van der Put proved in [11], the existence and unicity up to K-differential isomorphism of a formally real Picard–Vessiot extension, endowed with an ordering extending the one in K. We note that such a linear differential equation may also have Picard–Vessiot extensions which are not formally real fields. Our result was later generalized in [15], by using model-theoretic methods, to the case when K is a differential field of characteristic 0 such that its field of constants k is existentially closed in K for strongly normal extensions of K associated to logarithmic differential equations over K on algebraic groups over k. Let L be a Picard–Vessiot extension of a formally real differential field K with real closed field of constants k. Then the differential Galois group $G = \operatorname{DGal}(L|K)$ has the structure of a linear algebraic group defined over k (see Section 2).

Differential Galois theory over non-algebraically closed field of constants has been developed by several authors, see [1, 2, 19]. The inverse problem in this setting has also been considered (see [4, 13]). In particular, Dyckerhoff proved in [13] that every linear algebraic group over $\mathbb{R}$ is a differential Galois group over the field $\mathbb{R}(x)$ of rational functions.

In Picard–Vessiot theory over formally real differential fields, one may find phenomena which do not arise in the context of differential fields with algebraically closed field of constants. Given a linear differential equation

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1Y' + a_0Y = 0$$

with $a_{n-1}, \ldots, a_1, a_0$ belonging to a formally real differential field K with real closed field of constants k, one may ask the following questions which do not have a counterpart in the case when the field of constants k is algebraically closed.

– How many K-differential isomorphism classes of Picard–Vessiot extensions are there for $\mathcal{L}(Y) = 0$?

– Are the corresponding differential Galois groups k-isomorphic?

Concerning the first question, as mentioned above, it is known that there is at least one Picard–Vessiot extension of K for $\mathcal{L}(Y) = 0$, which is a formally real field. For the second one, let us note that if L and L' are Picard–Vessiot extensions of a formally real differential field K with real closed field of constants k for the same equation, and $G = \mathrm{DGal}(L/K)$, $G' = \mathrm{DGal}(L'/K)$ are the corresponding differential Galois groups, then we have $G \times_k \overline{k} \simeq G' \times_k \overline{k}$, i.e., G and G' are both k-forms of the group $H = G \times_k \overline{k}$.

In this paper, we consider a formally real Picard–Vessiot extension L of K for a linear differential equation $\mathcal{L}(Y) = 0$ defined over K and the differential Galois group $G = \mathrm{DGal}(L|K)$. We give an answer to the above questions in the case in which $H := G \times_k \overline{k}$ is a special linear, special orthogonal, or symplectic linear algebraic group. Let us note that it makes sense to start with such a Picard–Vessiot extension L/K since in [11, Proposition 3.3] Crespo, Hajto and van der Put proved that when the differential field K is real closed, given a connected semi-simple linear algebraic group G defined over k, there exists a linear differential equation defined over K and a formally real Picard–Vessiot extension $L|K$ for it such that $G = \mathrm{DGal}(L|K)$. The inspection of the different cases shows that there is no general pattern. The differential Galois group may be the same for all Picard–Vessiot extensions of $\mathcal{L}(Y) = 0$ or range over the whole set of k-forms of H. For a linear differential equation defined over a formally real differential field K the determination of its real differential group G gives more information on the behaviour of the solutions than the determination of the complexification H of G. For example, for $k = \mathbb{R}$ and K a field of real functions, the determination of G will give information on the existence of oscillating functions among the solutions of $\mathcal{L}(Y) = 0$ (see [9] and the earlier topological approach in [14]). It is then interesting to study how the real differential group $\mathrm{DGal}(L|K)$ varies as L runs over the K-isomorphism classes of Picard–Vessiot extensions.

We refer the reader to [8] for the topics on differential Galois theory, when the field of constants is algebraically closed, to [6] or [21] for those on formally real fields and to [20, 24, 25] for those on linear algebraic groups.

2 Preliminaries

For the reader's convenience, we recall the definitions of formally real field, real closed field and Picard–Vessiot extension.

Definition 2.1. A *formally real field* is a field which may be given an ordering compatible with the field operations. Equivalently, a field K is formally real if -1 is not a sum of squares in K.

A formally real field is *real closed* if it has no nontrivial algebraic extensions which are formally real fields. Equivalently, a field k is real closed if -1 is not a square in k and $k\big(\sqrt{-1}\big)$ is algebraically closed.

We note that in the literature on real algebraic geometry "real field" is frequently used for "formally real field". A field of positive characteristic is not formally real. The field $\mathbb{R}$ of real numbers is the standard example of a real closed field. The field $\mathbb{R}(x)$ of rational functions and the field of formal Laurent series $\mathbb{R}(x)$ with derivation $\mathrm{d}/\mathrm{d}x$ are examples of real differential fields with real closed field of constants.

Definition 2.2. Given a linear differential equation

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1Y' + a_0Y = 0$$

defined over a differential field K, with field of constants k, a *Picard–Vessiot extension* of K for $\mathcal{L}(Y) = 0$ is a differential field extension $L|K$ such that

a) L is differentially generated over K by a full set of solutions of $\mathcal{L}(Y) = 0$;

b) the field of constants of L is k.

Let us assume that K is a formally real differential field with real closed field of constants k, $\mathcal{L}(Y) = 0$ a linear differential equation defined over K and $L|K$ a Picard–Vessiot extension for $\mathcal{L}(Y) = 0$. In this case, we note that the set $\mathrm{DHom}_K(L, L(\mathrm{i}))$ of K-differential morphisms from L to $L(\mathrm{i})$ is in bijection with the set $\mathrm{DAut}_{K(\mathrm{i})} L(\mathrm{i})$ of $K(\mathrm{i})$-differential automorphisms of $L(\mathrm{i})$. We define the differential Galois group of $L|K$ as the set $\mathrm{DHom}_K(L, L(\mathrm{i}))$ with the group structure obtained by transferring the one of $\mathrm{DAut}_{K(\mathrm{i})} L(\mathrm{i})$ via the above bijection. The differential Galois group has the structure of a k-defined linear algebraic group (see [10, Proposition 4.1]). We note that the proof of the existence of a formally real Picard–Vessiot extension for a linear differential equation defined over a formally real differential field with real closed field of constants given in [10, Theorems 3.2 and 3.3 and Corollary 3.4] is not right. The remaining proofs in [10] are correct.

In the sequel, K will denote a formally real differential field with real closed field of constants k, $\mathcal{L}(Y) = 0$ a linear differential equation defined over K, $L|K$ a formally real Picard–Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We want to determine the number of Picard–Vessiot extensions of K for $\mathcal{L}(Y) = 0$, up to K-differential isomorphism, and the differential Galois group of each of them.

The set of K-differential isomorphism classes of Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$ is in one-to-one correspondence with the cohomology set $H^1\big(k, G\big(\overline{k}\big)\big)$, where $\overline{k}$ denotes the algebraic closure of k and G denotes the differential Galois group $\mathrm{DGal}(L/K)$. Indeed, we have a bijection between the set of K-differential isomorphism classes of Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$ and the set of isomorphism classes of fiber functors $\omega\colon \langle M\rangle_{\otimes} \to \mathrm{vect}(k)$, where $\langle M\rangle_{\otimes}$ denotes the Tannakian category generated by the K-differential module M associated to $\mathcal{L}(Y) = 0$ [11, Proposition 1]. In turn, this set of isomorphism classes of fiber functors is in bijection with the set of isomorphism classes of G-torsors, by [12, Theorem 3.2]. Finally, the set of isomorphism classes of G-torsors is in bijection with $H^1\big(k, G\big(\overline{k}\big)\big)$ (see, e.g., [27, Lemma A.5.1]). If L' is a Picard–Vessiot extension for $\mathcal{L}(Y) = 0$, $L(\mathrm{i})$ and $L'(\mathrm{i})$ are Picard–Vessiot extensions of $K(\mathrm{i})$ for $\mathcal{L}(Y) = 0$. Since the field of constants of $K(\mathrm{i})$ is $\overline{k}$, we have an isomorphism of differential fields $f\colon L(\mathrm{i}) \to L'(\mathrm{i})$, by the unicity of the Picard–Vessiot extension in the case when the field of constants is algebraically closed. The group $\mathrm{Gal}\big(\overline{k}|k\big)$ acts on the set of isomorphisms from $L(\mathrm{i})$ to $L'(\mathrm{i})$ by $s(f) = s \circ f \circ s^{-1}$, for $s \in \mathrm{Gal}\big(\overline{k}|k\big)$, where we denote also by s the automorphisms of $L(\mathrm{i})$ and $L'(\mathrm{i})$ induced by s. The 1-cocycle x corresponding to L' is determined by $x(c) = f^{-1} \circ c(f)$, where c is the nontrivial element in $\mathrm{Gal}\big(\overline{k}|k\big)$. Reciprocally, if x is a 1-cocycle from $\mathrm{Gal}\left(\overline{k}|k\right)$ to $G\left(\overline{k}\right)$, then the corresponding Picard–Vessiot extension corresponding to x is the subfield of $L(\mathrm{i})$ fixed by the automorphism $x(c) \circ c$, by Galois descent theory (see [23, Chapter III, Section 1.3]). The Galois group $\mathrm{Gal}\left(\overline{k}|k\right)$ acts on $G\left(\overline{k}\right)$, by an involution leaving G invariant, hence the cohomology set $H^1\left(k, G\left(\overline{k}\right)\right)$ depends on the k-form G of $H = G \times_k \overline{k}$.

Let now G denote a linear algebraic group defined over a real closed field k and let $H = G \times_k \overline{k}$. The set of k-forms of H is in one-to-one correspondence with the cohomology set $H^1\left(k, \mathrm{Aut}\, G\left(\overline{k}\right)\right)$, where $\mathrm{Gal}\left(\overline{k}|k\right)$ acts on $\mathrm{Aut}\, G\left(\overline{k}\right)$ by $s(f) = s \circ f \circ s^{-1}$, for $s \in \mathrm{Gal}\left(\overline{k}|k\right)$, $f \in \mathrm{Aut}\, G\left(\overline{k}\right)$, as usual. Let us note that the classification of the k-forms of H is equivalent to the classification of the real forms of the corresponding complex group.

We consider the map

$$\Phi\colon\ H^1\left(k, G\left(\overline{k}\right)\right) \to H^1\left(k, \mathrm{Aut}\, G\left(\overline{k}\right)\right)$$

induced by the morphism from $G\left(\overline{k}\right)$ to $\mathrm{Aut}\, G\left(\overline{k}\right)$ sending an element g in $G\left(\overline{k}\right)$ to conjugation by g. When G is the differential Galois group of a Picard–Vessiot extension L of K for a linear differential equation $\mathcal{L}(Y) = 0$, Φ sends the element in $H^1\left(k, G\left(\overline{k}\right)\right)$ corresponding to a Picard–Vessiot extension L' of K for $\mathcal{L}(Y) = 0$ to the element in $H^1\left(k, \mathrm{Aut}\, G\left(\overline{k}\right)\right)$ corresponding to $\mathrm{DGal}(L'|K)$ (see [11, Section 3, Observations 1]).

For c the nontrivial element of $\mathrm{Gal}\left(\overline{k}|k\right)$, we write $\overline{a} = c(a)$, for a an element in $\overline{k}$. For $v = (a_1, \dots, a_n) \in \overline{k}^n$, we shall write $\overline{v} = (\overline{a_1}, \dots, \overline{a_n})$ and for $M = (a_{ij})$ a matrix with entries in $\overline{k}$, $\overline{M} = (\overline{a_{ij}})$.

We shall consider a linear algebraic group G defined over the real closed field k, such that $H = G \times_k \overline{k}$ is either a special linear group $\mathrm{SL}(n)$, a special orthogonal group $\mathrm{SO}(n)$ or a symplectic group $\mathrm{Sp}(n)$. We will then consider the real forms of each of these groups. We refer to [23] or [16] for their determination, to [7] or [26] for a more explicit description of them.

We will determine in each case the number of K-differential isomorphism classes of Picard–Vessiot extensions of K for $\mathcal{L}(Y) = 0$ and the differential Galois group for each class.

Let i denote a square root of -1 in $\overline{k}$. For p, n integers with $0 \leq p \leq n$, we define the $n \times n$ matrices

$$I_p = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_{n-p} \end{pmatrix}, \qquad J_p = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & \mathrm{i}\,\mathrm{Id}_{n-p} \end{pmatrix}.$$

3 Forms of $\mathrm{SL}(n)$

The real forms of $\mathrm{SL}(n), n \geq 2$, are

1) $\mathrm{SL}(n, k)$;
2) $\mathrm{SL}(n/2, \mathbb{H})$ if n is even, where $\mathbb{H}$ denotes the quaternion algebra over k;
3) $\mathrm{SU}\left(n, \overline{k}, h\right)$, where h is a nondegenerate hermitian form on $\overline{k}^n$.

For G each of these real forms and K a formally real differential field, with real closed field of constants k, we consider a linear differential equation $\mathcal{L}(Y) = 0$ of order n defined over K and a Picard–Vessiot extension $L|K$ for $\mathcal{L}(Y) = 0$ such that L is formally real and $\mathrm{DGal}(L/K) \simeq G$.

3.1 $G = \mathrm{SL}(n, k)$

We have $\left|H^1\left(k, G\left(\overline{k}\right)\right)\right| = 1$ [22, Chapter X, Section 1], hence $L|K$ is the unique Picard–Vessiot extension for $\mathcal{L}(Y) = 0$, up to K-differential isomorphism.

3.2 $G = \mathrm{SL}(n/2, \mathbb{H})$

Let us denote by 1, I, J, K the basis elements of $\mathbb{H}$. We recall that $\mathrm{GL}(n/2, \mathbb{H})$ embeds into $\mathrm{GL}\left(n, \overline{k}\right)$ via the morphism $(h_{ij}) \mapsto (\mu(h_{ij}))$, where

$$\mu(a + bI + cJ + dK) = \begin{pmatrix} a + b\mathrm{i} & c + d\mathrm{i} \\ -c + d\mathrm{i} & a - b\mathrm{i} \end{pmatrix}, \qquad a, b, c, d \in k.$$

We denote by A_n the matrix $(a_{ij})_{1\le i,j\le n}$ with

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd and } j = i + 1, \\ -1 & \text{if } i \text{ is even and } j = i - 1, \\ 0 & \text{in all other cases.} \end{cases} \tag{3.1}$$

We have $\mu(\mathrm{GL}(n/2, \mathbb{H})) = \left\{M \in \mathrm{GL}\left(n, \overline{k}\right) : M = A_n \overline{M} A_n^{-1}\right\}$ and $\mu(\mathrm{SL}(n/2, \mathbb{H})) = \mathrm{SL}\left(n, \overline{k}\right) \cap \mu(\mathrm{GL}(n/2, \mathbb{H}))$.

By [17, Chapter VII, Section 29, Corollary 29.4], we have $H^1\big(k, G\big(\overline{k}\big)\big) \simeq k^*/\mathrm{Nrd}\big(\mathbb{H}^{n/2}\big)$. Since the norm of a quaternion is always positive, we obtain $\big|H^1\big(k, G\big(\overline{k}\big)\big)\big| = 2$ (see also [23, Chapter III, Section 1.4]). We have then two Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$, up to K-differential isomorphism. A nontrivial 1-cocycle x of $\mathrm{Gal}\left(\overline{k}|k\right)$ in $\mathrm{SL}\left(n, \overline{k}\right)$ is given by $x(c) = \zeta\,\mathrm{Id}$, for ζ a primitive n-th root of unity. A K-differential automorphism of $L(\mathrm{i})$ corresponding to x is given by the matrix $A := \zeta^{-1/2}\,\mathrm{Id}$ on the vector space of solutions, since A satisfies $A^{-1}c(A) = x(c)$ (see [22, Chapter X, Section 2, Proposition 4]). Conjugation by $\zeta^{-1/2}\,\mathrm{Id}$ leaves the group $\mathrm{SL}(n/2, \mathbb{H})$ stable. We obtain that the Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$ in both K-differential isomorphy classes have $\mathrm{SL}(n/2, \mathbb{H})$ as differential Galois group.

3.3 $G = \mathrm{SU}\left(n, \overline{k}, h\right)$

It is known that if h is a nondegenerate hermitian form on $\overline{k}^n$, then h is equivalent to a hermitian form with matrix I_p, for some integer p with $0 \le p \le n$, called the index of h and that two nondegenerate hermitian forms on $\overline{k}^n$ are equivalent if and only if their indices coincide (see [17, Chapter VII, Section 29, Example 29.19] and [5, Section 3.3]).

We fix $G = \left\{M \in \mathrm{SL}\left(n, \overline{k}\right) : \overline{M}^t I_p M = I_p\right\}$ and consider the action of $\mathrm{Gal}\left(\overline{k}|k\right)$ on $\mathrm{SL}\left(n, \overline{k}\right)$ given by $c(M) = I_p\big(\overline{M}^t\big)^{-1} I_p$. We shall prove

$$\big|H^1\big(k, G\big(\overline{k}\big)\big)\big| = \begin{cases} \left[\dfrac{n}{2}\right] + 1, & \text{when } n \text{ is odd or } p \text{ is even}, \\ \dfrac{n}{2}, & \text{when } n \text{ is even and } p \text{ is odd}. \end{cases} \tag{3.2}$$

To this end we shall determine a maximal set of pairwise nonequivalent 1-cocycles from $\mathrm{Gal}\left(\overline{k}|k\right)$ in $\mathrm{SL}\left(n, \overline{k}\right)$. For such a cocycle x, we may assume that the image of $\mathrm{Id} \in \mathrm{Gal}\left(\overline{k}|k\right)$ is the identity matrix and then x is determined by the image $B \in \mathrm{SL}\left(n, \overline{k}\right)$ of the unique non trivial element c in $\mathrm{Gal}\left(\overline{k}|k\right)$. By the 1-cocycle condition, B must satisfy $Bc(B) = \mathrm{Id}$. We denote by x_q the cocycle given by $c \mapsto B_q$, where $B_q = I_q I_p$, with q an integer of the same parity as p and $0 \le q \le n$. Let us see that every 1-cocycle x from $\mathrm{Gal}\left(\overline{k}|k\right)$ in $\mathrm{SL}\left(n, \overline{k}\right)$ is equivalent to some x_q. As said above, such a cocycle x is determined by $x(c) = B$ satisfying $Bc(B) = \mathrm{Id}$, i.e., $BI_p\big(\overline{B}^t\big)^{-1} I_p = \mathrm{Id}$, equivalently $BI_p = I_p\overline{B}^t = \big(\overline{BI_p}\big)^t$, so BI_p is an hermitian matrix, hence there exists an invertible matrix M such that $M^{-1}BI_p\big(\overline{M}^t\big)^{-1} = I_r$, for some integer r, $0 \le r \le n$. Equivalently

$$M^{-1}Bc(M) = I_r I_p. \tag{3.3}$$

Let us note that, taking determinants in (3.3), we obtain $\det M\,\overline{\det M} = 1$, hence there exists $\zeta \in \overline{k}$ such that $\det(\zeta M) = 1$ and ζM satisfies (3.3). We have then that there exists a matrix $M \in \mathrm{SL}\left(n, \overline{k}\right)$ satisfying (3.3), which means that x is equivalent to the 1-cocycle x_r determined by $c \mapsto I_r I_p$. Since $B \in \mathrm{SL}\left(n, \overline{k}\right)$, we have $\det(I_r I_p) = 1$, so r is an integer of the same parity as p.

Let us see now that the 1-cocycles x_q are pairwise nonequivalent. We have $x_q \sim x_{q'} \Leftrightarrow \exists\, M \in \mathrm{SL}\left(n, \overline{k}\right)$ such that $B_{q'} = M^{-1} B_q c(M)$. This equality is equivalent to $M I_{q'} \overline{M}^t = I_q$ which implies $q = q'$, so the 1-cocycles x_q are pairwise nonequivalent. We have then $H^1\big(k, G\big(\overline{k}\big)\big) = \{[x_q] \colon 0 \le q \le n,\ q \equiv p \pmod 2\}$. Then $\left|H^1\big(k, G\big(\overline{k}\big)\big)\right| = |\{q \in \mathbb{Z} \colon 0 \le q \le n, q \equiv p \pmod 2\}|$ and we obtain the values in (3.2).

We have $Z\big(\mathrm{SL}\left(n, \overline{k}\right)\big) = \mu_n\big(\overline{k}\big)$. We want to determine the image of $[x_q]$ under the map $\Phi \colon H^1\big(k, G\big(\overline{k}\big)\big) \to H^1\big(k, \mathrm{Aut}\, G\big(\overline{k}\big)\big)$. The 1-cocycle x_q corresponds to a Picard–Vessiot extension L_q of K for $\mathcal{L}(Y) = 0$ such that there is a differential isomorphism f_q from $L(\mathrm{i})$ to $L_q(\mathrm{i})$ satisfying $x_q = f_q^{-1} c(f_q)$. The differential isomorphism f_q is determined by the matrix D_q giving the images of a vector space of solutions. The isomorphism f_q satisfies $x_q = f_q^{-1} c(f_q)$ if and only if the matrix D_q satisfies $B_q = D_q^{-1} \overline{D_q}$. We may take $D_q := J_q J_p$. Since conjugation by D_q leaves the group G invariant, we obtain that all Picard–Vessiot extensions of K for $\mathcal{L}(Y) = 0$ have the same differential Galois group G.

Gathering the results in this section we may state the following theorem.

Theorem 3.1. *Let K be a formally real differential field with real closed field of constants k, $\mathcal{L}(Y) = 0$ a linear differential equation defined over K, $L|K$ a formally real Picard–Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of* $\mathrm{SL}(n)$*.*

(1) *If $G = \mathrm{SL}(n, k)$, $L|K$ is the unique Picard–Vessiot extension for the equation $\mathcal{L}(Y) = 0$, up to K-differential isomorphism.*

(2) *If $G = \mathrm{SL}(n/2, \mathbb{H})$, there are two Picard–Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K-differential isomorphism, and both of them have differential Galois group G.*

(3) *If $G = \mathrm{SU}\left(n, \overline{k}, h\right)$, there are $[n/2] + 1$ (resp. $[n/2]$) Picard–Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, if n is odd or p is even (resp. if n is even and p is odd), up to K-differential isomorphism, and all of them have differential Galois group G.*

4 Forms of $\mathrm{SO}(n)$

The real forms of $\mathrm{SO}(n)$, with n odd, are the groups $\mathrm{SO}(n, k, Q)$, where Q is a nondegenerate quadratic form on k^n. When G is one of these forms, we proved in [11, Section 3, Examples 1 and 3] that the map $\Phi \colon H^1\big(k, G\big(\overline{k}\big)\big) \to H^1\big(k, \mathrm{Aut}\, G\big(\overline{k}\big)\big)$ is a bijection. We consider now the case when n is even.

The real forms of $\mathrm{SO}(n)$, with n even, are

1) $\mathrm{SO}(n, k, Q)$, where Q is a nondegenerate quadratic form on k^n;

2) $\mathrm{SU}(n/2, \mathbb{H}, h)$, where h is a nondegenerate anti-hermitian form on $\mathbb{H}^{n/2}$ (with respect to the involution σ of $\mathbb{H}$ defined by $a + bI + cJ + dK \mapsto a - bI - cJ - dK$).

For G each of these real forms and K a formally real differential field, with real closed field of constants k, we consider a linear differential equation $\mathcal{L}(Y) = 0$ of order n defined over K and a Picard–Vessiot extension $L|K$ for $\mathcal{L}(Y) = 0$ such that L is formally real and $\mathrm{DGal}(L/K) \simeq G$.

4.1 $G = \mathrm{SO}(n,k,Q)$

The quadratic form Q is equivalent to a quadratic form with matrix I_p, for some integer p with $0 \le p \le n$, which determines the equivalence class of Q.

The cohomology set $H^1\big(k, G(\overline{k})\big)$ is in one-to-one correspondence with the set of equivalence classes of quadratic forms on k^n of rank n and index of the same parity as p (see [17, Chapter VII, Section 29, formula (29.29)]). We have then

$$\big|H^1\big(k, G(\overline{k})\big)\big| = \begin{cases} \dfrac{n}{2}+1, & \text{when } p \text{ is even,} \\ \dfrac{n}{2}, & \text{when } p \text{ is odd.} \end{cases}$$

The cocycles x_q defined by $c \mapsto B_q$, where $B_q = I_q I_p$, with q an integer of the same parity as p and $0 \le q \le n$, form a complete system of representatives of the cohomology set $H^1\big(k, G(\overline{k})\big)$. The 1-cocycle x_q corresponds to a Picard–Vessiot extension L_q of K for $\mathcal{L}(Y) = 0$ such that there is a differential isomorphism f_q from $L(\mathrm{i})$ to $L_q(\mathrm{i})$ satisfying $x_q = f_q^{-1} c(f_q)$. The differential isomorphism f_q is determined by the matrix D_q giving the images of a vector space of solutions. The isomorphism f_q satisfies $x_q = f_q^{-1} c(f_q)$ if and only if the matrix D_q satisfies $B_q = D_q^{-1}\overline{D_q}$. We may take $D_q := J_q J_p$. If the matrix M satisfies $M^t I_p M = I_p$, the conjugate matrix $N := D_q M D_q^{-1}$ satisfies $N^t I_q N = I_q$, hence the Picard–Vessiot extension corresponding to the 1-cocycle x_q has differential Galois group $\mathrm{SO}(n,k,Q_q)$, where Q_q denotes the quadratic form with index q. Let us note that $\mathrm{SO}(n,k,Q_q) = \mathrm{SO}(n,k,Q_{n-q})$, hence the Picard–Vessiot extension corresponding to x_q and x_{n-q}, $0 \le q \le (n/2)-1$, have the same differential Galois group.

4.2 $G = \mathrm{SU}(n/2, \mathbb{H}, h)$, h anti-hermitian

We have $\mathrm{U}(n/2,\mathbb{H},h) = \big\{M \in \mathrm{GL}(n/2,\mathbb{H})\colon \sigma(M)^t[h]M = [h]\big\}$, for $[h]$ the matrix of the anti-hermitian form h, in some basis of $\mathbb{H}^{n/2}$. The group $\mathrm{U}(n/2,\mathbb{H},h)$ is the group of automorphisms of the anti-hermitian vector space $\big(\mathbb{H}^{n/2}, h\big)$. The set of equivalence classes of nondegenerate anti-hermitian forms over $\mathbb{H}^{n/2}$ is in one-to-one correspondence with the cohomology set $H^1\big(k, \mathrm{U}(n/2,\mathbb{H},h)\big(\overline{k}\big)\big)$. Up to equivalence, there is one single anti-hermitian form on $\mathbb{H}^n$, hence $H^1\big(k, \mathrm{U}(n/2,\mathbb{H},h)\big(\overline{k}\big)\big) = 1$.

We consider the exact sequence

$$1 \to \mathrm{SU} \to \mathrm{U} \to \mu_2 \to 1.$$

Since the reduced norm of a quaternion is always positive, the reduced norm $\mathrm{U}(n/2,\mathbb{H},h)(k) \to \mu_2(k)$ is the trivial map. We obtain then for the cohomology sets the exact sequence

$$1 \to \mu_2 \to H^1\big(k, \mathrm{SU}(n/2,\mathbb{H},h)\big(\overline{k}\big)\big) \to H^1\big(k, \mathrm{U}(n/2,\mathbb{H},h)\big(\overline{k}\big)\big).$$

Therefore $\big|H^1\big(k, \mathrm{SU}(n/2,\mathbb{H},h)\big(\overline{k}\big)\big)\big| = 2$. We obtain then that there are two Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$, up to K-differential isomorphism. We denote by L' the non formally real one. We may check that $\mu(G)$ is the intersection of (a conjugate form of) $\mathrm{SO}\big(n, \overline{k}\big)$ with $\mu(\mathrm{GL}(n/2,\mathbb{H}))$. A nontrivial 1-cocycle of $\mathrm{Gal}\big(\overline{k}|k\big)$ in $G\big(\overline{k}\big)$ is given by $c \mapsto A_n$, for A_n the matrix defined by (3.1). The matrix $B = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} \dfrac{1}{\sqrt{2}} & \text{if } \ i = j \text{ or } i \text{ is even and } j = i-1, \\ -\dfrac{1}{\sqrt{2}} & \text{if } i \text{ is odd and } j = i+1, \\ 0 & \text{in all other cases.} \end{cases}$$

satisfies $B^{-1}c(B) = A_n$, hence the cohomology class $[x]$ corresponds to the isomorphism class of the differential isomorphism f from $L(\mathrm{i})$ to $L'(\mathrm{i})$ with matrix B on the vector space of solutions. Since conjugation by B leaves G invariant, we obtain that both Picard–Vessiot extensions have the same differential Galois group.

Gathering the results in this section we may state the following theorem. For completeness, we include the case n odd.

Theorem 4.1. *Let K be a formally real differential field with real closed field of constants k, $\mathcal{L}(Y) = 0$ a linear differential equation defined over K, $L|K$ a formally real Picard–Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of $\mathrm{SO}(n)$.*

If n is odd, there are $(n+1)/2$ Picard–Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K-differential isomorphism, and their differential Galois groups range over the whole set of real forms of $\mathrm{SO}(n)$.

Assume n even.

(1) *If $G = \mathrm{SO}(n, k, Q_p)$, where Q_p is a nondegenerate quadratic form on k^n, of index p, there are $(n/2) + 1$ (resp. $n/2$) Picard–Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K-differential isomorphism, when p is even (resp. when p is odd) and their differential Galois groups range over the whole set of groups $G = \mathrm{SO}(n, k, Q_q)$, with Q_q a nondegenerate quadratic form on k^n, of index q, $0 \leq q \leq n/2$ and q of the same parity as p.*

(2) *If $G = \mathrm{SU}(n/2, \mathbb{H}, h)$, where h is a nondegenerate anti-hermitian form on $\mathbb{H}^n$, there are two Picard–Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K-differential isomorphism, and they have both differential Galois group G.*

5 Forms of $\mathrm{Sp}(2n)$

The real forms of $\mathrm{Sp}(2n)$ are

1) $\mathrm{Sp}(2n, k)$;
2) $\mathrm{SU}(n, \mathbb{H}, h)$, where h is a nondegenerate hermitian form on $\mathbb{H}^n$ (with respect to the involution σ of $\mathbb{H}$ defined by $a + bI + cJ + dK \mapsto a - bI - cJ - dK$).

For G each of these real forms and K a formally real differential field, with real closed field of constants k, we consider a linear differential equation $\mathcal{L}(Y) = 0$ of order n defined over K and a Picard–Vessiot extension $L|K$ for $\mathcal{L}(Y) = 0$ such that L is formally real and $\mathrm{DGal}(L/K) \simeq G$.

5.1 $G = \mathrm{Sp}(2n, k)$

We have $\left|H^1\big(k, G(\overline{k})\big)\right| = 1$ [22, Chapter X, Section 2, Corollary 2], hence $L|K$ is the unique Picard–Vessiot extension for $\mathcal{L}(Y) = 0$, up to K-differential isomorphism.

5.2 $G = \mathrm{SU}(n, \mathbb{H}, h)$, h hermitian

If h is a nondegenerate hermitian form on $\mathbb{H}^n$, then h is equivalent to a hermitian form with matrix I_p, with $p \geq n - p$ [7, Section 7.5.3]. The number of equivalence classes of nondegenerate hermitian forms over $\mathbb{H}^n$ is then $\left[\frac{n}{2}\right] + 1$. We fix

$$G = \left\{M \in \mathrm{GL}(n, \mathbb{H}) \colon \sigma(M)^t I_p M = I_p\right\}.$$

The group G is the group of automorphisms of the hermitian vector space $\left(\mathbb{H}^n, h\right)$. Hence the set of equivalence classes of nondegenerate hermitian forms over $\mathbb{H}^n$ is in one-to-one correspondence with the cohomology set $H^1\big(k, G(\overline{k})\big)$ (see [22, Chapter X, Section 2, Proposition 4] or

[16, Section 2.6, Lemma 3]). Since the set of K-differential isomorphism classes of Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$ is also in one-to-one correspondence with the cohomology set $H^1\big(k, G(\overline{k})\big)$, we have that the number of K-differential isomorphism classes of Picard–Vessiot extensions for $\mathcal{L}(Y) = 0$ is equal to the number of equivalence classes of nondegenerate hermitian forms over $\mathbb{H}^n$. Let us note that for $M, N \in M_n(\mathbb{H})$, we have $\sigma(MN)^t = \sigma(N)^t\sigma(M)^t$. We determine now the image of G under the isomorphism $\mu \otimes_k \overline{k}\colon \mathrm{GL}\,\big(n, \mathbb{H} \otimes_k \overline{k}\big) \to \mathrm{GL}\,\big(2n, \overline{k}\big)$. For $M \in \mathrm{GL}\,\big(n, \mathbb{H} \otimes_k \overline{k}\big)$, we have $\big(\mu \otimes_k \overline{k}\big)\big(\sigma(M^t)\big) = A_{2n}\big(\mu \otimes_k \overline{k}\big)(M)^t A_{2n}^{-1}$, for A_{2n} the matrix $(a_{ij})_{1\le i,j\le 2n}$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd and } j = i+1, \\ -1 & \text{if } i \text{ is even and } j = i-1, \\ 0 & \text{in all other cases.} \end{cases}$$

Hence $\sigma(M)^t I_p M = I_p$ implies $\big(\mu \otimes_k \overline{k}\big)(M)^t A_{2n}^{-1} I_{2p} \big(\mu \otimes_k \overline{k}\big)(M) = A_{2n}^{-1} I_{2p}$. Now the group

$$\overline{G} := \big\{N \in \mathrm{GL}\,\big(2n, \overline{k}\big)\colon N^t A_{2n}^{-1} I_{2p} N = A_{2n}^{-1} I_{2p}\big\}$$

is a conjugate form of $\mathrm{Sp}\,\big(2n, \overline{k}\big)$ and $\mu(G) = \big\{N \in \overline{G}\colon N = A_{2n}\overline{N}A_{2n}^{-1}\big\}$. A complete set of nonequivalent 1-cocycles of $\mathrm{Gal}\,\big(\overline{k}|k\big)$ in $\mu(G)$ is given by

$$\begin{aligned} x_q\colon\ \mathrm{Gal}\,\big(\overline{k}|k\big) &\to \mu(G), \\ c &\mapsto B_q := I_{2q} I_{2p}, \end{aligned}$$

with q an integer, $0 \le q \le n$, $q \ge n-q$. The 1-cocycle x_q corresponds to a Picard–Vessiot extension L_q of K for $\mathcal{L}(Y) = 0$ such that there is a differential isomorphism f_q from $L(\mathrm{i})$ to $L_q(\mathrm{i})$ satisfying $x_q = f_q^{-1}c(f_q)$. The differential isomorphism f_q is determined by the matrix D_{2q} giving the images of a vector space of solutions. The isomorphism f_q satisfies $x_q = f_q^{-1}c(f_q)$ if and only if the matrix D_{2q} satisfies $B_q = D_{2q}^{-1}\overline{D_{2q}}$. We may take $D_{2q} := J_{2q}J_{2p}$. If N is a matrix belonging to $\mu(G)$, it satisfies $N^t A_{2n}^{-1} I_{2p} N = A_{2n}^{-1} I_{2p}$. Then the conjugate matrix P of N by D_{2q}, $P := D_{2q} N D_{2q}^{-1}$, satisfies $P^t A_{2n}^{-1} I_{2q} P = A_{2n}^{-1} I_{2q}$, hence the Picard–Vessiot extension corresponding to the 1-cocycle x_q has differential Galois group $\mathrm{SU}(n, \mathbb{H}, h_q)$, where h_q denotes the hermitian form with index q.

Gathering the results in this section we may state the following theorem.

Theorem 5.1. *Let K be a formally real differential field with real closed field of constants k, $\mathcal{L}(Y) = 0$ a linear differential equation defined over K, $L|K$ a formally real Picard–Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of* $\mathrm{Sp}(2n)$.

(1) *If $G = \mathrm{Sp}(2n, k)$, $L|K$ is the unique Picard–Vessiot extension for the equation $\mathcal{L}(Y) = 0$.*

(2) *If $G = \mathrm{SU}(n, \mathbb{H}, h_p)$, where h_p is a nondegenerate hermitian form on $\mathbb{H}^n$, of index p, $0 \le p \le n$, $p \ge n-p$, there are $[n/2]+1$ Picard–Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K-differential isomorphism, and their differential Galois groups range over the whole set of groups $G = \mathrm{SU}(n, \mathbb{H}, h_q)$, with h_q a nondegenerate hermitian form on $\mathbb{H}^n$, of index q, $0 \le q \le n$, $q \ge n-q$.*

6 Conclusions

In the preceding we have seen cases in which a linear differential equation $\mathcal{L}(Y) = 0$ defined over a formally real differential field K has Picard–Vessiot extensions which are not formally real.

The occurrence of these extensions depends on the real form of the differential Galois group of $\mathcal{L}(Y) = 0$. When the number of K-differential isomorphisms of Picard–Vessiot extensions of $\mathcal{L}(Y) = 0$ is bigger than 1, we find several situations concerning the differential Galois group, either it is the same for all Picard–Vessiot extensions or it ranges over a subset or the whole set of real forms of the group G of the formally real Picard–Vessiot extension. It would be interesting to know if, in the case when K is a field of real functions, the solutions of such an equation in a non formally real differential field and the variation of the differential Galois group have some physical interpretation. Some inspiring examples in Hamiltonian mechanics are presented in [3].

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