Gromov Rigidity of Bi-Invariant Metrics on Lie Groups and Homogeneous Spaces

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Abstract. Gromov asked if the bi-invariant metrics on a compact Lie group are extremal compared to any other metrics. In this note, we prove that the bi-invariant metrics on a compact connected semi-simple Lie group G are extremal (in fact rigid) in the sense of Gromov when compared to the left-invariant metrics. In fact the same result holds for a compact connected homogeneous manifold G/H with G compact connect and semi-simple.

Key words: extremal/rigid metrics; Lie groups; homogeneous spaces; scalar curvature

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1 Introduction

In [6], Gromov asks: are bi-invariant metrics on compact Lie groups extremal? (This is already problematic for SO(5).) Here a Riemannian metric g on a differentiable manifold M is extremal in the sense of Gromov (not to be confused with Calabi's extremal metrics in Kähler geometry) if any metric g' on M with $g' \geq g$ and $R_{g'} \geq R_g$ must have $R_{g'} = R_g$, where R_g, $R_{g'}$ denote the scalar curvature of g, g' respectively. The metric g is rigid in the sense of Gromov if in fact $g' = g$ from the conditions above.

The first result of this type is [10] in which Llarull showed that the standard metric on S^n is rigid. The work gives a positive answer to an earlier question of Gromov, which is motivated by Gromov–Lawson's famous work on the non-existence of positive scalar curvature metrics on the torus [7], later extended to more general class of manifolds, namely the enlargeable manifolds. In the same spirit, Llarull in fact proved that a metric on a compact manifold admitting a $\left(1,\Lambda^2\right)$-contracting map to S^n is rigid. Min-Oo discussed the extremality/rigidity of hermitian symmetric spaces of compact type in [12]. The extremality/rigidity of complex and quaternionic projective spaces was established by Kramer [8]. Later, Goette and Semmelmann [4] proved that compact symmetric spaces of type G/K with $\mathrm{rk}(G) - \mathrm{rk}(K) \leq 1$ are extremal (see also [3]). Then Listing improves Goette–Semmelmann's result in [9], by weakening the extremality condition.

Note that a Lie group with a bi-invariant metric is a symmetric space, but not of the types considered above. In this short note, we present a partial positive answer to Gromov's question. Namely, we show that the bi-invariant metrics on a compact connected semi-simple Lie group G are rigid among the left-invariant metrics. More generally, we show that the normal metrics

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on any compact connected homogeneous space G/H without torus factor are rigid among G-invariant metrics on G/H.

Theorem 1. *Let $M = G/H$ be a compact homogeneous space, with G a compact connected semi-simple Lie group. Then any bi-invariant metric (also known as normal homogeneous metric) g_0 on G/H is rigid among the G-invariant metrics. In other words, if g is a G-invariant metric on G/H such that $g \geq g_0$ and $R_g \geq R_{g_0}$, then $g = g_0$.*

As an immediate consequence, we have

Corollary 2. *Any bi-invariant metric on a compact connected semi-simple Lie group is rigid among the left-invariant metrics.*

According to [11], if a connected Lie group admits a bi-invariant metric, it is isomorphic to the product of a compact Lie group with an abelian one. The semi-simple condition rules out the abelian factor. On the other hand, we have the famous result of Gromov–Lawson [7] and Schoen–Yau [13, 14] which implies that the only metrics of nonnegative scalar curvatures on the torus are the flat ones.

Remark 3. The extremal/rigid metrics discussed here have positive scalar curvature. On the other hand, we would like to point out a related but different scalar curvature (local) extremality for Kähler–Einstein metrics with negative scalar curvature [2]. It is an immediate consequence of Theorem 1.5 in [2] that for a Kähler–Einstein metric g_0 with negative scalar curvature on a compact complex manifold with integrable infinitesimal complex deformations, any metric g sufficiently close to g_0 satisfying $R_g \geq R_{g_0}$ and $\mathrm{Vol}(g) \leq \mathrm{Vol}(g_0)$ must have $R_g = R_{g_0}$ (and g is also Kähler–Einstein).

2 Preliminaries

Given a Riemannian manifold (M, g), we denote by R_g the scalar curvature of g. We recall Gromov's notion of extremal/rigid metrics.

Definition 4. A metric g_0 on M is extremal (in the sense of Gromov), if any metric g on M satisfying $g \geq g_0$ and $R_g \geq R_{g_0}$ must have identical scalar curvature, $R_g = R_{g_0}$; g_0 is said to be rigid (in the sense of Gromov) if the conditions above imply that $g = g_0$.

For a Lie group G, we denote by $\mathrm{Ad}(a)$ $(a \in G)$ the adjoint action of G on its Lie algebra $\mathfrak{g}$, and by $\mathrm{ad}(X)$ $(X \in \mathfrak{g})$ the induced adjoint action of $\mathfrak{g}$ on itself. In particular,

$$\mathrm{ad}(X)Y = [X, Y], \qquad X, Y \in \mathfrak{g}.$$

A Lie group G is semi-simple if its Lie algebra $\mathfrak{g}$ is semi-simple, i.e., its Killing form

$$K(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)), \qquad X, Y \in \mathfrak{g}$$

is nondegenerate. Clearly, if $\mathfrak{g}$ is semi-simple, it has a trivial center. For a compact Lie group, the semi-simple condition is equivalent to its Lie algebra having trivial center.

If a metric on G is both left-invariant and right-invariant, then it is called bi-invariant. When G is compact, bi-invariant metrics always exist. Left-invariant metrics on G are in one-to-one correspondence with inner products on its Lie algebra $\mathfrak{g}$. The following well known result plays a crucial role in the proof of our main result.

Theorem 5 ([11, Lemma 7.2])**.** *In the case of a connected group G, a left-invariant metric is actually bi-invariant if and only if the linear transformation $\mathrm{ad}(X)$ is skew-adjoint with respect to the corresponding inner product, for every X in the Lie algebra $\mathfrak{g}$ of G.*

Now let $M = G/H$ be a compact connected homogeneous space, where G is a compact connected Lie group, H a closed subgroup, and the action of G on G/H is effective. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H. Denote by Ad_G the adjoint action of G on $\mathfrak{g}$ and $\mathrm{Ad}_H = \mathrm{Ad}_G|_H$ its restriction to H. Since Ad_H preserves $\mathfrak{h}$, it induces an action on $\mathfrak{g}/\mathfrak{h}$, which is equivalent to the isotropy representation of H. A metric g on $M = G/H$ is called G-invariant if it is invariant under the left action of G. G-invariant metrics on G/H are naturally identified with inner products on $\mathfrak{g}/\mathfrak{h}$ which are invariant under the Ad_H action, see Proposition 3.16 in [1]. In particular, a bi-invariant metric on G gives rise to a G-invariant metric on G/H. The corresponding metric on G/H, usually referred as a normal homogeneous metric on G/H in the literature, will still be called bi-invariant here.

3 Proof of the theorem

Our proof relies crucially on a simple elegant formula for the scalar curvature for G-invariant metrics, as well as another lemma, in [15]. We first recall this formula and the setup.

Let g_0 be a bi-invariant metric on G; still denote by g_0 the induced metric on G/H. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be an Ad_H invariant decomposition orthogonal with respect to g_0. Then G-invariant metrics on G/H are identified with Ad_H-invariant inner products on $\mathfrak{m}$.

Let $\langle\cdot,\cdot\rangle_0$ be the Ad_H-invariant inner product on $\mathfrak{m}$ corresponding to g_0. Let $\langle\cdot,\cdot\rangle$ be an Ad_H-invariant inner product on $\mathfrak{m}$ inducing a G-invariant metric g on G/H. Then, there is a positive self-adjoint operator S on $(\mathfrak{m}, \langle X, Y\rangle_0)$ commuting with the Ad_H-action such that

$$\langle X, Y\rangle = \langle S(X), Y\rangle_0$$

for all $X, Y \in \mathfrak{m}$.

Since any eigenspace of S is Ad_H-invariant, there are Ad_H-invariant subspaces $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ of $\mathfrak{m}$ such that

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \tag{1}$$

in orthogonal decomposition with respect to $\langle\cdot,\cdot\rangle_0$; the action of Ad_H on each $\mathfrak{m}_i$ is irreducible, and $S(X) = \lambda_i X$ for all $X \in \mathfrak{m}_i$, for some $\lambda_1, \dots, \lambda_s > 0$. Consequently,

$$\langle X, Y\rangle = \lambda_1\langle X_1, Y_1\rangle_0 + \cdots + \lambda_s\langle X_s, Y_s\rangle_0,$$

for $X = X_1 + \cdots + X_s$, $Y = Y_1 + \cdots + Y_s \in \mathfrak{m}$ decomposed with respect to (1). The metric g is called *diagonal* with respect to the decomposition in (1).

For such metrics, there is a simple elegant formula for the scalar curvature; we refer the reader to [15] for a more general discussion. Before we state this formula, let us point out the simplified situation when $M = G$. Each $\mathfrak{m}_i$ in (1) is spanned by a basis vector whenever one chooses an orthonormal basis of $\mathfrak{m} = \mathfrak{g}$ consisting of eigenvectors of S. (Thus, such decompositions are by no means unique.)

Let $\{E_\alpha\}$ be an orthonormal basis of $(\mathfrak{m}, \langle\ ,\ \rangle_0)$ adapted to the decomposition (1). We write $[E_\alpha, E_\beta]_\mathfrak{m} = \sum_\gamma C^\gamma_{\alpha\beta} E_\gamma$ for some real numbers $\left\{C^\gamma_{\alpha\beta}\right\}$ that we call structural constants. Here $[\ ,\ ]_\mathfrak{m}$ is the $\mathfrak{m}$-component of $[\ ,\ ]$. Set

$$A^k_{ij} = \sum_{\alpha,\beta,\gamma} \left(C^\gamma_{\alpha\beta}\right)^2,$$

where the summation runs over $E_\alpha \in \mathfrak{m}_i$, $E_\beta \in \mathfrak{m}_j$, $E_\gamma \in \mathfrak{m}_k$.

Let $d_i = \dim \mathfrak{m}_i$. Let B be the negative of the Killing form: $B(X, Y) = -K(X, Y)$. Then $B(X, X) \geq 0$, with equality if and only if X is central. We define the real number b_i by $B(X, Y) = b_i\langle X, Y\rangle_0$ for all $X, Y \in \mathfrak{m}_i$. Note that $b_i = 0$ if and only if $\mathfrak{m}_i$ is included in the center of $\mathfrak{g}$. The following formula is equation (1.3) in [15].

Lemma 6 ([15, equation (1.3)]). *Let g be a G-invariant metric on G/H with a corresponding decomposition* (1) *as described above. Then the scalar curvature of g is*

$$R_g = \frac{1}{2}\sum_{i=1}^{s}\frac{b_i d_i}{\lambda_i} - \frac{1}{4}\sum_{i,j,k=1}^{s} A_{ij}^k \frac{\lambda_k}{\lambda_i\lambda_j}.$$

The following lemma from [15] relates $b_i d_i$ to the structural constants. Let

$$C_{\mathfrak{m}_i, g_0|_{\mathfrak{h}}} = -\sum_{i=1}^{h} \mathrm{ad}(Z_i)\circ \mathrm{ad}(Z_i)$$

be the Casimir operator of the representation of $\mathfrak{h}$ on $\mathfrak{m}_i$, where $\{Z_1,\dots,Z_h\}$ is an orthonormal basis of $(\mathfrak{h}, g_0|_{\mathfrak{h}})$ and $\mathrm{ad}(Z_i)$ should be interpreted as its restriction on $\mathfrak{m}_i$. Since $\mathfrak{m}_i$ is Ad_H-irreducible, $C_{\mathfrak{m}_i, g_0|_{\mathfrak{h}}} = c_i \,\mathrm{Id}$ for some constant $c_i \geq 0$. Moreover, $c_i = 0$ if and only if Ad_H acts trivially on $\mathfrak{m}_i$.

Lemma 7 ([3, Lemma 1.5]). *One has, for $i = 1,\dots,s$,*

$$\sum_{j,k=1}^{s} A_{ij}^k = b_i d_i - 2c_i d_i.$$

Remark 8. Again let us look at the situation when $M = G$. In this case we choose an orthonormal basis $\{E_i\}_{i=1}^n$ of $\mathfrak{g}$ consisting of eigenvectors of S. Then $[E_i, E_j] = \sum_{k=1}^{n} C_{ij}^k E_k$ via the structure constants C_{ij}^k. The decomposition (1) is given by $\mathfrak{m}_i = \mathrm{Span}\{E_i\}$, hence $A_{ij}^k = \left(C_{ij}^k\right)^2$. Moreover $c_i = 0$ for all i. Therefore, the two lemmas above yield

$$R_g = \frac{1}{4}\sum_{i,j,k=1}^{n} \left(C_{ij}^k\right)^2 \left[\frac{2}{\lambda_i} - \frac{\lambda_k}{\lambda_i\lambda_j}\right]. \tag{2}$$

This formula can also be deduced from Koszul's formula via a direct computation.

Proof of Theorem 1. Since $\{E_\alpha\}$ is an orthonormal basis for $(\mathfrak{m}, \langle\cdot,\cdot\rangle_0)$, and $\langle\cdot,\cdot\rangle_0$ is bi-invariant, $C_{\alpha\beta}^\gamma = \langle [E_\alpha, E_\beta], E_\gamma\rangle_0$ is skew-symmetric in all three indices by Theorem 5. Hence A_{ij}^k is symmetric in all three indices.

Now the extremal conditions $\langle X, Y\rangle \geq \langle X, Y\rangle_0$ and $R_g \geq R_{g_0}$ yield $\lambda_i \geq 1$ $(i = 1,\dots,s)$ as well as $R_g - R_{g_0} \geq 0$. Lemmas 6 and 7 give

$$\begin{aligned} 0 \leq R_g - R_{g_0} &= \frac{1}{2}\sum_i \frac{b_i d_i}{\lambda_i}(1-\lambda_i) - \frac{1}{4}\sum_{i,j,k} A_{ij}^k \left(\frac{\lambda_k}{\lambda_i\lambda_j} - 1\right) \\ &= \sum_i \frac{c_i d_i}{\lambda_i}(1-\lambda_i) - \frac{1}{4}\sum_{i,j,k} A_{ij}^k \left[\frac{\lambda_k}{\lambda_i\lambda_j} + 1 - \frac{2}{\lambda_i}\right]. \end{aligned}$$

Since $c_i \geq 0$ and $d_i > 0$, each term in the first summation is less than or equal to zero, with equality if and only if either $c_i = 0$ or $\lambda_i = 1$.

For the second summation, we use the symmetry to rewrite it as

$$\begin{aligned} &-\frac{1}{12}\sum_{i,j,k} A_{ij}^k \left[\frac{\lambda_k}{\lambda_i\lambda_j} + \frac{\lambda_i}{\lambda_j\lambda_k} + \frac{\lambda_j}{\lambda_k\lambda_i} - \frac{2}{\lambda_j} - \frac{2}{\lambda_i} - \frac{2}{\lambda_k} + 3\right] \\ &\qquad = -\frac{1}{12}\sum_{i,j,k} A_{ij}^k \frac{\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_i\lambda_k - 2\lambda_k\lambda_j + 3\lambda_i\lambda_j\lambda_k}{\lambda_i\lambda_j\lambda_k}. \end{aligned}$$

For a fixed triple i, j, k, we consider the order of λ_i, λ_j, λ_k. Without loss of generality we can assume that $\lambda_k \geq \lambda_j \geq \lambda_i \geq 1$. Then the summand in the sum above can be re-organized as

$$\begin{aligned}&\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_i\lambda_k - 2\lambda_k\lambda_j + 3\lambda_i\lambda_j\lambda_k \\ &\quad = (\lambda_i - \lambda_j)^2 + (\lambda_k - \lambda_j)^2 + \lambda_j\lambda_k(\lambda_i - 1) + \lambda_j(\lambda_k - \lambda_j) + 2\lambda_i\lambda_k(\lambda_j - 1) \geq 0\end{aligned}$$

with equality if and only if $\lambda_k = \lambda_j = \lambda_i = 1$.

But then all the inequalities become equalities. Hence, either $c_i = 0$ or $\lambda_i = 1$ for each i, and, at the same time, either $A_{ij}^k = 0$ or $\lambda_k = \lambda_j = \lambda_i = 1$ for each (i, j, k). If $\lambda_i > 1$ for some i, then $c_i = 0$, and $A_{ij}^k = 0$ for all j, k. Thus $b_i = 0$ by Lemma 7. Therefore $\mathfrak{m}_i$ is in the center of $\mathfrak{g}$, which contradicts the hypotheses. We conclude that $\lambda_i = 1$ for all i, and the result follows. ■

We end with a couple of remarks.

Remark 9. From the proof, we see that if a bi-invariant metric g_0 on G/H is not rigid among the G-invariant metrics, then G/H must have a torus factor. Indeed, let $\mathfrak{z} \subset \mathfrak{g}$ be the center. If for some i, $\lambda_i > 1$, then $\mathfrak{m}_i \subset \mathfrak{z}$. Decompose $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}'$ and $\mathfrak{z} = \mathfrak{m}_i + \mathfrak{k}$. Then $\mathfrak{h} \subset \mathfrak{k} + \mathfrak{g}'$. It follows that $G/H = T^{d_i} \times (K \times G')/H$.

Remark 10. It is interesting to note that the extremal conditions $g \geq g_0$ and $R_g \geq R_{g_0}$ can not be changed to the opposite inequalities. In fact, there exist G-invariant metrics g such that $g < g_0$ and $R_g < R_{g_0}$. We illustrate the situation for $M = G = \mathrm{SU}(2)$.

The basis $E_1 = \sqrt{-1}\sigma_1$, $E_2 = \sqrt{-1}\sigma_2$, $E_3 = \sqrt{-1}\sigma_3$ of $\mathfrak{g}$ in terms of the Pauli spin matrices σ_1, σ_2, σ_3 satisfies $[E_1, E_2] = 2E_3$ as well as its cyclic permutations. We take g_0 so that $\langle X, Y\rangle_0 = \frac{1}{8}B(X, Y)$, with respect to which $\{E_1, E_2, E_3\}$ is orthonormal. Following the notations in Remark 8, we choose g so that E_1, E_2, E_3 are the eigenvectors with eigenvalues $\lambda_1 = \lambda_2 = \lambda < 1$, and $\lambda_3 = 1/2$, respectively. Then $g < g_0$. On the other hand, by (2),

$$R_g - R_{g_0} = \frac{1}{4}\sum_{i,j,k=1}^{3} (C_{ij}^k)^2 \left[\frac{2}{\lambda_i} - \frac{\lambda_k}{\lambda_i\lambda_j} - 1\right] = -\frac{1}{\lambda^2} + O\left(\frac{1}{\lambda}\right),$$

as $\lambda \to 0^+$. Thus, for λ sufficiently small, we have $R_g < R_{g_0}$.

Note that this represents the opposite rescaling of the standard sphere as compared to the example of Berger's sphere mentioned in [5, p. 34].

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