An Explicit Example of Polynomials Orthogonal on the Unit Circle with a Dense Point Spectrum Generated by a Geometric Distribution

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Abstract. We present a new explicit family of polynomials orthogonal on the unit circle with a dense point spectrum. This family is expressed in terms of q-hypergeometric function of type $_2\phi_1$. The orthogonality measure is the wrapped geometric distribution. Some "classical" properties of the above polynomials are presented.

Key words: polynomials orthogonal on the unit circle; wrapped geometric dustribution; dense point spectrum

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1 Introduction

Let $\Phi_n(z)$ be monic polynomials $\Phi_n(z) = z^n + O(z^{n-1})$ defined through the recurrence relation [17]

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n\Phi_n^*(z), \qquad \Phi_0(z) = 1,$$

where

$$\Phi_n^*(z) = z^n\bar{\Phi}_n(1/z)$$

and where $\bar{\Phi}_n(z)$ means complex conjugation of expansion coefficients of the polynomial $\Phi_n(z)$. The recursion parameters

$$a_n = -\bar{\Phi}_{n+1}(0)$$

are called the Verblunsky (sometimes also reflection, Schur etc.) parameters [17].

Under the condition

$$|a_n| < 1, \qquad n = 0, 1, 2, \dots \tag{1.1}$$

the polynomials $\Phi_n(z)$ are orthogonal on the unit circle with respect to a positive measure $d\sigma(\theta)$

$$\int_0^{2\pi} \Phi_n(e^{i\theta})\bar{\Phi}_m(e^{-i\theta})d\sigma(\theta) = h_n\delta_{nm}, \tag{1.2}$$

where

$$h_n = (1 - |a_0|^2)(1 - |a_1|^2)\cdots(1 - |a_{n-1}|^2) \tag{1.3}$$

are normalization constants (which are nonzero due to condition (1.1)). In this case $\Phi_n(z)$ are called the orthogonal polynomials on the unit circle (OPUC).

Note that orthogonality relation (1.2) is equivalent to conditions [17]

$$I_{nj} \equiv \int_0^{2\pi} \Phi_n(e^{i\theta})e^{-ij\theta}\,d\sigma(\theta) = h_n\delta_{nj}, \qquad j = 0, 1, 2, \ldots, n. \tag{1.4}$$

Equivalently, OPUC $\Phi_n(z)$ can be constructed in terms of trigonometric moments σ_n. The latter are defined as

$$\sigma_n = \int_0^{2\pi} e^{in\theta}\,d\sigma(\theta), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Then polynomials $\Phi_n(z)$ have the explicit expression

$$\Phi_n(z) = (\Delta_n)^{-1} \begin{vmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_n \\ \sigma_{-1} & \sigma_0 & \ldots & \sigma_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{1-n} & \sigma_{2-n} & \ldots & \sigma_1 \\ 1 & z & \ldots & z^n \end{vmatrix},$$

where

$$\Delta_n = \begin{vmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_{n-1} \\ \sigma_{-1} & \sigma_0 & \ldots & \sigma_{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{1-n} & \sigma_{2-n} & \ldots & \sigma_0 \end{vmatrix}$$

are Toeplitz determinants which are all positive $\Delta_n > 0$, $n = 0, 1, 2, \ldots$. Note the symmetry property of the trigonometric moments

$$\sigma_{-n} = \bar{\sigma}_n. \tag{1.5}$$

Explicit examples of polynomials orthogonal on unit circle are very interesting from different point view. By "explicit examples" we mean that all main objects: the parameters a_n, the moments σ_n, the measure $\sigma(\theta)$ and the polynomials themselves $\Phi_n(z)$ have explicit expressions in terms of special functions. Usually, in most known explicit examples the parameters a_n are given by elementary functions of n while the OPUC $\Phi_n(z)$ are expressed in terms of hypergeometric functions (either ordinary or basic). A list of known explicit examples can be found, e.g., in Simon's monograph [17].

In [21, 27] new explicit examples of OPUC were presented. In these examples polynomials $\Phi_n(z)$ are expressed in terms of elliptic hypergeometric function $_3E_2(z)$ while the moments σ_n and the recurrence parameters a_n have simple expressions in terms of elliptic functions. The most interesting property of the OPUC of these examples is that they are orthogonal on the unit circle with respect to a dense point measure. This means that the function $\sigma(\theta)$ is a step function with infinitely many points θ_s of jumps, and these points are dense on the interval $[0, 2\pi]$. In terms of the distribution function this can be presented as

$$\rho(\theta) = \sum_{s=-\infty}^{\infty} M_s\delta(\theta - \theta_s),$$

where $\rho(\theta)$ is a distribution defined as $d\sigma(\theta) = \rho(\theta)d\theta$, $\delta(\theta)$ is the Dirac delta function and M_s are concentrated masses located at points of jumps θ_s. The spectral points $z_s = \exp(i\theta_s)$ are dense on the unit circle.

Then orthogonality relation (1.2) can then be presented as

$$\sum_{s=-\infty}^{\infty} M_s \Phi_n\big(e^{i\theta_s}\big) \bar{\Phi}_m\big(e^{-i\theta_s}\big) = h_n \delta_{nm}.$$

From general considerations (see, e.g., [17]) it follows that polynomials orthogonal with respect to such dense point measures are rather generic if one assumes some natural restrictions upon behavior of the recurrence parameters a_n. On the other hand, such measures are very important from physical point of view, because they correspond to the phenomenon of the Anderson localization [10, 17].

Usually examples of OPUC with dense point spectrum are related to sequences of the parameters a_n which behave (quasi) stochastically inside the interval $|a_n| < 1$ [17]. OPUC in [21, 27] provide perhaps the first known examples of a pure point dense measure on the unit circle where both the coefficients a_n and the moments σ_n are given explicitly by analytic functions in n.

In this paper we propose a much simpler explicit example of polynomials orthogonal on the unit with respect to a (wrapped) geometric distribution which is dense on the unit circle. Polynomials themselves are expressed in terms of basic hypergeometric function $_2\phi_1(q;z)$ with $|q| = 1$.

2   Wrapped geometric distribution and corresponding OPUC

Let q be a fixed point belonging to the unit circle $|q| = 1$ and not a root of unity, i.e., we demand that $q^n \neq 1$ for all natural integer $n = 0, 1, \ldots$. Choose a real parameter p within the unit interval $0 < p < 1$. Define the measure on the unit circle as

$$\rho(\theta) = (1 - p) \sum_{s=0}^{\infty} p^s \delta(\theta - s\chi), \tag{2.1}$$

where χ is a fixed *irrational* parameter $0 < \chi < 1$ such that

$$q = \exp(2\pi i\chi). \tag{2.2}$$

Irrationality of χ means that the set of points $z_s = q^s$, $s = 0, 1, \ldots$ (i.e., the location of jumps of the measure) is dense on the unit circle. The weights (i.e., the concentrated masses) w_s at the points z_s form the geometric sequence: $w_s = p^s$, $s = 0, 1, 2, \ldots$.

Corresponding trigonometric moments are

$$\sigma_n = \sum_{s=0}^{\infty} z_s^n w_s = (1 - p) \sum_{s=0}^{\infty} q^{sn} p^s = \frac{1 - p}{1 - pq^n}, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{2.3}$$

Note that the measure (2.1) can be interpreted as the wrapped geometric distribution on the unit circle (see, e.g., [8, 15] for definition and discussion of wrapped distributions on the unit circle).

Relation $\sigma_0 = 1$ means that the measure (2.1) has the standard normalization condition.

One can present expression (2.3) as

$$\sigma_n = \frac{(p;q)_n}{(pq;q)_n}, \tag{2.4}$$

where the q-shifted factorial (q-Pochhammer symbol) is defined as [4, 9] $(a;q)_0 = 1$ and

$$(a;q)_n = (1 - a)(1 - aq) \cdots \big(1 - aq^{n-1}\big)$$

for positive $n = 1, 2, \ldots$ and

$$(a; q)_n = \frac{1}{\left(aq^n; q\right)_{-n}}$$

for negative $n = -1, -2, \ldots$.

It is known that the Laurent biorthogonal Pastro polynomials $P(z; a, b)$ [16] depending on two arbitrary parameters a, b can be uniquely defined through their moments [22]

$$\sigma_n = \frac{(a; q)_n}{(b; q)_n}, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{2.5}$$

Explicitly, these polynomials are given by [16, 22]

$$P(z; a, b) = \mu_n \, {}_2\phi_1 \left(\begin{matrix} q^{-n}, b \\ aq^{1-n} \end{matrix}; qz \right), \tag{2.6}$$

where μ_n is an appropriate normalization factor to fulfill the condition $P_n(z) = z^n + O\left(z^{n-1}\right)$. The definition and notation of the basic hypergeometric function ${}_m\phi_n(z)$ is standard (see, e.g., [4, 9]). For example, in the special case $m = n + 1$ we have the expressions

$$_{m+1}\phi_m \left(\begin{matrix} a_1, a_2 \ldots, a_{m+1} \\ b_1, b_2 \ldots, b_m \end{matrix}; z \right) = \sum_{s=0}^{\infty} \frac{(a_1; q)_s (a_2; q)_s \cdots (a_{m+1}; q)_s}{(q; q)_s (b_1; q)_s (b_2; q)_s \cdots (b_m; q)_s} z^s.$$

Note the Laurent biorthogonal polynomials (LBP) can be considered as a generalization of the OPUC. Their main distinction from OPUC is that the moments σ_n do not satisfy, in general, the symmetry condition (1.5). The LBP can also be characterized by the three-term recurrence relation of R_I type [7, 24]

$$P_{n+1}(z) + g_n P_n(z) = z(P_n(z) + d_n P_{n-1}(z)), \qquad P_0 = 1, \qquad P_{-1} = 0 \tag{2.7}$$

with some recurrence coefficients g_n, d_n.

In contrast to the case of orthogonal polynomials, the recurrence relation (2.7) can be presented in the form of the generalized eigenvalue problem [25]

$$J_1 \mathbf{P}(z) = z J_2 \mathbf{P}(z), \tag{2.8}$$

where J_1, J_2 are upper and lower bidiagonal matrices acting on the vector

$$\mathbf{P}(z) = (P_0(z), P_1(z), \ldots).$$

Comparing expressions (2.5) and (2.4) we can conclude that the OPUC corresponding to the wrappedd geometric distribution are special case of the Pastro polynomials with $a = p$, $b = qp$.

This allows us to present the main result

Theorem 2.1. *The polynomials $\Phi_n(z)$ orthogonal on the unit circle with respect to the wrapped geometric distribution* (2.1) *have the explicit expression*

$$\Phi_n(z) = \mu_n \, {}_2\phi_1 \left(\begin{matrix} q^{-n}, pq \\ pq^{1-n} \end{matrix}; zq \right), \tag{2.9}$$

where

$$\mu_n = q^{-n} \frac{(q; q)_n \left(pq^{1-n}; q\right)_n}{\left(q^{-n}; q\right)_n (pq; q)_n} = p^n \frac{\left(p^{-1}; q\right)_n}{(qp; q)_n}. \tag{2.10}$$

One can directly check that the polynomials (2.9) satisfy orthogonality relations (1.4). Indeed, one has

$$I_{nj} = (1-p)\sum_{s=0}^{\infty}\Phi_n(q^s)q^{-sj}p^s = (1-p)\mu_n\sum_{s=0}^{\infty}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(pq;q)_k}{(q;q)_k(pq^{1-n};q)_k}q^{(1+s)k}q^{-js}p^s.$$

Performing summation over s we get

$$I_{nj} = \frac{(1-p)\mu_n}{1-pq^{-j}}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(pq;q)_k(pq^{-j};q)_k}{(q;q)_k(pq^{1-n})_k(pq^{1-j};q)_k}q^k = \frac{(1-p)\mu_n}{1-pq^{-j}}{}_3\phi_2\left(\begin{matrix}q^{-n},pq,pq^{-j}\\pq^{1-n},pq^{1-j}\end{matrix};q\right).$$

The above expression can be simplified by the q-Saalschütz formula [4, 9]:

$${}_3\phi_2\left(\begin{matrix}q^{-n},a,b\\c,abc^{-1}q^{1-n}\end{matrix};q\right) = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/(ab);q)_n}.$$

We thus have

$$I_{nj} = \frac{(1-p)\mu_n}{1-pq^{-j}}\frac{(q^{-n};q)_n(q^{j+1-n};q)_n}{(pq^{1-n};q)_n(p^{-1}q^{j-n};q)_n}. \tag{2.11}$$

The factor $\left(q^{j+1-n};q\right)_n$ in (2.11) becomes zero when $j = 0,1,\dots,n-1$ and hence

$$I_{nj} = 0, \qquad j = 0,1,\dots,n-1,$$

which is equivalent to orthogonality relation (1.4). It remains to show that $I_{nn} > 0$. After simple calculations one can arrive at the expression

$$I_{nn} = h_n = \frac{|(q;q)_n|^2}{|(pq;q)_n|^2}p^n \tag{2.12}$$

from which it is clear that $h_n > 0$ for all $n = 0,1,2,\dots$ due to condition $q^n \neq 1$.

Explicit expression for the the recurrence parameters a_n follows from (2.9) and (2.10):

$$\bar{a}_{n-1} = -\Phi_n(0) = -\mu_n = -p^n\frac{(p^{-1};q)_n}{(qp;q)_n}.$$

For the square of absolute values we have rather simple expression

$$|a_{n-1}|^2 = \bar{a}_{n-1}a_{n-1} = \frac{(1-p)^2}{1+p^2-p(q^n+q^{-n})} = \frac{1}{1+\beta\sin^2(\chi\pi n)}, \tag{2.13}$$

where

$$\beta = \frac{4p}{(1-p)^2}$$

and where the parameter χ is the same as in (2.2).

It is seen from (2.13) that the values $|a_n|$ oscillate inside the interval

$$\frac{1-p}{1+p} < |a_n| < 1, \qquad n = 0,1,\dots.$$

Because of irrationality of χ the absolute value $|a_n|$ achieves the boundaries of this interval with any prescribed accuracy (never achieving exact boundary values). Note that $a_{-1} = -1$ which corresponds to the standard initial conditions for OPUC [5, 17]. It is easily verified that expression (2.12) for h_n agrees with relation (1.3).

The OPUC (2.9) can be considered as $|q| = 1$ analogs of the OPUC introduced by Askey in [1] (see also [3] for more general OPUC of Askey's type).

3 "Classicality" of the polynomials $\Phi_n(z)$

The OPUC (2.9) possess "classical" properties which make them similar to classical orthogonal polynomials.

First of all, they satisfy the three-term recurrence relation (2.7) where the recurrence coefficients are

$$g_n = \frac{q^n - p}{1 - pq^{n+1}}, \qquad d_n = -\frac{p(1-q^n)^2}{(1-pq^n)(1-pq^{n+1})}.$$

Moreover, the polynomial $\Phi_n(z)$ possess a remarkable *duality* property. Indeed, one can rewrite polynomials $\Phi_n(z)$ in a different form

$$\Phi_n(z) = p^n \frac{(q;q)_n}{(pq;q)_n} z^n \, {}_3\phi_2 \left(\begin{matrix} q^{-n}, p^{-1}, z^{-1} \\ q, 0 \end{matrix} ; q \right), \tag{3.1}$$

which can be obtained from (2.9) by standard transformation formulas [4, 9].

From this formula the duality property

$$A_s \Phi_s(q^n) = A_n \Phi_n(q^s) \tag{3.2}$$

follows, where

$$A_n = \frac{(pq;q)_n}{(q;q)_n} p^{-n}.$$

This property resembles corresponding duality properties for the classical orthogonal polynomials from the Askey scheme [2, 11, 19]. The main difference is that the polynomials $\Phi_n(z)$ satisfy the generalized eigenvalue problem (2.8) instead of the ordinary eigenvalue problem for orthogonal polynomials.

From the duality property one can derive the second-order q-difference equation

$$B_{s+1}\Phi_n\big(q^{s+1}\big) + g_s \Phi_n\big(q^s\big) = q^n\big(\Phi_n\big(q^s\big) + B_s^{-1} d_s \Phi_n\big(q^{s-1}\big)\big), \tag{3.3}$$

where

$$B_s = \frac{A_{s-1}}{A_s} = \frac{p\big(1-q^s\big)}{1 - pq^s}.$$

Equation (3.3) can also be presented in the form of the generalized eigenvalue problem

$$L\Phi_n(z) = q^n M \Phi_n(z), \tag{3.4}$$

where the first-order q-difference operators L, M act on the argument z of the polynomials.

Relations (2.8) and (3.4) mean that the polynomials possess the *bispectrality* property: they satisfy simultaneously two GEVP. Concerning definition and general theory of bispectrality see, e.g., [6]. For orthogonal polynomials from the Askey scheme this property is well known [9]. For biorthogonal polynomials and rational functions the bispectrality is known for some special families. The most general from them are elliptic biorthogonal functions [18]. However the general theory of bispectrality for systems satisfying GEVP is not yet developed (see, e.g., [20, 23] for algebraic description of bispectrality on the "lowest" level of hypergeometric functions ${}_3F_2(1)$).

The duality property implies that for $z = q^s$, $s = 0, 1, 2, \ldots$ the hypergeometric function in (3.1) reduces to a polynomial of degree s of the argument q^{-n}.

It is well known (see, e.g., [17]) that if z_0 is a point on the unit circle corresponding to a concentrated mass M_0 then the relation

$$\sum_{n=0}^{\infty} \frac{|\Phi_n(z_0)|^2}{h_n} = 1/M_0,$$

holds, where the normalization coefficient h_n is defined in (1.3).

In our case this means that for every spectral point $z_s = q^s$, $s = 0, 1, \ldots$ there exists the identity

$$\sum_{n=0}^{\infty} \frac{\left|\Phi_n\left(q^s\right)\right|^2}{h_n} = M_s^{-1} = \frac{p^{-s}}{1 - p}. \tag{3.5}$$

Identity (3.5) follows easily from the duality property (3.2) and from orthogonality relation.

So far, we have considered the case when q is not a root of unity. If, otherwise, q is a primitive root of unity

$$q = \exp\left(\frac{2\pi i M}{N}\right)$$

with coprime integers M, N, then there are only N distinct mass points on the unit circle located at $z_s = q^s$, $s = 0, 1, 2, \ldots, N - 1$. In this case the polynomials $\Phi_n(z)$ are orthogonal on vertices of a regular N-gon with respect to the finite wrapped geometric distribution:

$$\sum_{s=0}^{N-1} \Phi_n\left(q^s\right) \bar{\Phi}_m\left(q^{-s}\right)\left(1 - p^N\right)p^s = h_n \delta_{nm}, \qquad n, m = 0, 1, \ldots, N - 1.$$

See [26] for other explicit examples of polynomials orthogonal on the vertices of regular polygons.

4 Concluding remarks

In contrast to examples of OPUC obtained in [27], the polynomials (2.6) have non-real moments σ_n and hence the coefficients a_n are non-real as well. This means that it is impossible to associate with OPUC (2.6) polynomials orthogonal on an interval of the real line. In [27] explicit examples of polynomials orthogonal with dense point spectrum on an interval were presented using standard Szegő mapping from OPUC to an interval of the real line. We mention also examples of OPUC and ordinary orthogonal polynomials with dense point spectrum presented in [13, 14].

The OPUC (2.6) allow a trivial modification which shifts all spectral points on the unit circle on the same constant angle φ, i.e., we can consider the same weights $w_s = p^s(1 - p)$ located at the points

$$\theta_s = 2\pi \chi s + \varphi, \qquad s = 0, 1, 2, \ldots.$$

Equivalently, this means that the new spectral points will be $\tilde{z}_s = e^{i\varphi}q^s$, $s = 0, 1, 2, \ldots$.

Such transformation is equivalent to a simple rotation of the argument of OPUC [5, 17]:

$$\tilde{\Phi}_n(z) = e^{-i\varphi n}\Phi_n\left(e^{i\varphi}z\right).$$

Another modification of the OPUC (2.6) is more substantional. It leads to Laurent biorthogonal polynomials orthogonal on the unit circle with dense point measure.

Indeed, assume that the spectral points on the unit circle are the same: $z_s = q^s$, $s = 0, 1, 2, \ldots$. Take the weights:

$$w_s = p^s \frac{(q^k; q)_s}{(q; q)_s}, \qquad 0 < p < 1, \qquad k = 1, 2, 3, \ldots. \tag{4.1}$$

For $k = 1$ we return to the case of the wrapped geometric distribution. For $k > 1$ the moments are

$$\sigma_n = \sum_{s=0}^{\infty} \frac{(q^k; q)_s}{(q; q)_s} p^s q^{sn}. \tag{4.2}$$

By q-binomial theorem [4, 9] the above sum is simplified to

$$\sigma_n = \frac{(q^k p q^n; q)_\infty}{(p q^n; q)_\infty} = \frac{1}{(p q^n; q)_k} = \frac{(p; q)_n}{(p; q)_k (p q^k; q)_n}.$$

Remark. Usually, the convergence problem for q-series like (4.2) with $|q| = 1$ is highly nontrivial (see, e.g., [12]). In our case however this problem does not appear because for integer k there is cancellation of almost all terms (apart of a finite number of initial ones) in denominators of the coefficients in (4.2). Hence the convergence for $0 < p < 1$ still takes place.

For fixed k the moments σ_n coincide (up to a constant factor) with the moments (2.5) for the Pastro polynomials with $a = p$, $b = pq^k$. Hence from (2.6) we have explicit expression for them

$$P(z) = \mu_n {}_2\phi_1 \left(\begin{matrix} q^{-n}, pq^k \\ pq^{1-n} \end{matrix}; qz \right),$$

where

$$\mu_n = \frac{p^n (p^{-1}; q)_n}{(pq^k; q)_n}.$$

These polynomials are NOT OPUC (apart from the already considered case $k = 1$) because the weights (4.1) are not positive and hence the moments σ_n do not satisfy symmetry property (1.5).

Existence of other explicit examples of OPUC with dense point spectrum is an interesting open problem.

Acknowledgements

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