Interplay between Opers, Quantum Curves, WKB Analysis, and Higgs Bundles

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Abstract. *Quantum curves* were introduced in the physics literature. We develop a mathematical framework for the case associated with Hitchin spectral curves. In this context, a quantum curve is a Rees $\mathcal{D}$-module on a smooth projective algebraic curve, whose semi-classical limit produces the Hitchin spectral curve of a Higgs bundle. We give a method of quantization of Hitchin spectral curves by concretely constructing one-parameter deformation families of *opers*. We propose a variant of the topological recursion of Eynard–Orantin and Mirzakhani for the context of singular Hitchin spectral curves. We show that a PDE version of topological recursion provides all-order WKB analysis for the Rees $\mathcal{D}$-modules, defined as the quantization of Hitchin spectral curves associated with meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. Topological recursion can be considered as a process of quantization of Hitchin spectral curves. We prove that these two quantizations, one via the construction of families of opers, and the other via the PDE recursion of topological type, agree for holomorphic and meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. Classical differential equations such as the Airy differential equation provides a typical example. Through these classical examples, we see that quantum curves relate Higgs bundles, opers, a conjecture of Gaiotto, and quantum invariants, such as Gromov–Witten invariants.

Key words: quantum curve; Hitchin spectral curve; Higgs field; Rees $\mathcal{D}$-module; opers; non-Abelian Hodge correspondence; mirror symmetry; Airy function; quantum invariants; WKB approximation; topological recursion

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1 Introduction

The purpose of this paper is to construct a geometric theory of *quantum curves*. The notion of quantum curves was introduced in the physics literature (see for example, [1, 19, 20, 21, 43, 44, 49, 61, 70, 72, 74]). A quantum curve is supposed to compactly capture topological invariants, such as certain Gromov–Witten invariants, Seiberg–Witten invariants, and quantum knot polynomials. Geometrically, a quantum curve is a unique quantization of the B-model

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geometry, when it is encoded in a holomorphic curve, that gives a generating function of A-model theory of genus g for all $g \geq 0$. In a broad setting, a quantum curve can be a differential operator, a difference operator, a mixture of them, or a linear operator defined by a trace-class kernel function.

The geometric theory we present here is focused on the process of *quantization* of Hitchin spectral curves [46, 47]. A concise overview of our theory is available in [25]. In Definitions 2.10 and 2.11, we introduce a quantum curve as a *Rees $\mathcal{D}$-module* on a smooth projective algebraic curve C whose *semi-classical limit* is the Hitchin spectral curve associated with a Higgs bundle on C. The process of quantization is therefore an assignment of a Rees $\mathcal{D}$-module to every Hitchin spectral curve.

The *Planck constant* $\hbar$ is a deformation parameter that appears in the definition of Rees $\mathcal{D}$-modules. For us, it has a geometric meaning, and is naturally identified with an element

$$\hbar \in H^1(C, K_C), \tag{1.1}$$

where K_C is the canonical sheaf over C. The cohomology group $H^1(C, K_C)$ controls the deformation of a classical object, i.e., a geometric object such as a Higgs bundle in our case, into a quantum object, i.e., a non-commutative quantity such as a differential operator. In our case, the result of quantization is an *oper*.

Using a fixed choice of a theta characteristic and a projective structure on C, we determine a unique quantization of the Hitchin spectral curve of a holomorphic or meromorphic $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle through a concrete construction of an $\hbar$-family of $\mathrm{SL}(r, \mathbb{C})$-opers on C, as proved in Theorem 3.10 for holomorphic case, and in Theorem 3.15 for meromorphic case. The $\hbar$-family interpolates opers and Higgs fields. We then prove, in Theorem 3.11, that the Rees $\mathcal{D}$-module as the quantization result recovers the starting Hitchin spectral curve via semi-classical limit of WKB analysis. This is our main theorem of the paper. When we choose the projective structure of C of genus $g \geq 2$ coming from the Fuchsian uniformization, our construction of opers is the same as those opers predicted by a conjecture of Gaiotto [41], as explained in Section 3.3. This conjecture has been solved in [26] (see [18] for a subsequent development.)

It has been noticed that *topological recursion* of Eynard–Orantin [36, 37] and others [17, 33, 34], and also its more recent generalizations (such as [2, 12, 57] and many papers cited in these articles), provide another aspect of quantization. A notable one is the *remodeling conjecture* of Mariño [60] and his collaborators [14, 15], and its complete solution by mathematicians [39, 40]. (For many earlier contributions to the remodeling conjecture and physics oriented discussions, we refer to the references cited in [38, 39].) From this point of view, a quantum curve is a quantization of B-model geometry that is obtained as an application of topological recursion. It then becomes a natural question:

Question 1.1. What is the relation between quantization via topological recursion and the quantization through our construction of Rees $\mathcal{D}$-modules from Hitchin spectral curves?

Topological recursion was originally developed as a computational mechanism to calculate the multi-resolvent correlation functions of random matrices (see [17, 33, 36] and references cited there). As mentioned above, it generates a mirror symmetric B-model counterpart of genus g A-model for all $g \geq 0$. This correspondence has been rigorously established for many examples (see for example, [13, 23, 28, 31, 32, 35, 39, 40, 63, 64, 65, 66, 70, 72], and others). Yet so far still no clear geometric relation between topological recursion and quantum curves (in particular, when they appear as difference operators) has been established.

Another tantalizing subject is the relation between the theory of τ-functions and topological recursion/quantum curves. The present paper does not attempt to address this relation. The subject presented in Section 3 is closely related to the work of [7, 50, 72], in terms of formalism and mathematical structures. No deep understanding is offered in this paper.

Among the earliest striking applications of topological recursion in algebraic geometry, there are *new* proofs obtained in [66] for the Witten conjecture on cotangent class intersection numbers and the λ_g-conjecture. Indeed, these celebrated formulas are straightforward consequences of the *Laplace transform* of a combinatorial formula known as the *cut-and-join equations* of [42, 75].

Applications of topological recursion to enumerative problems are effective when the spectral curve in the theory is of genus 0. In this case, the residue calculations required in the formalism of [36] can be explicitly performed. The computational aspect of the formalism as a tool is not effective in a more general context, such as when the spectral curve is a high genus non-hyperelliptic curve, or has singularities.

A novel approach proposed in [27] is the implementation of PDE recursions of topological type, which appear naturally in enumerative geometry problems, to the context of Hitchin spectral curves. It replaces the integral topological recursion formulated in terms of residue calculations at the ramification divisor of a spectral curve by a recursive set of partial differential equations that captures *local* nature of topological recursion. As we explain in Section 5, the main difference of the two recursion formulas lies in the choice of contours of integration in the original format of integral topological recursion. All other ingredients are similar. For a genus 0 spectral curve, the two sets of recursions are equivalent. In general, these two recursions aim at achieving different goals. The original choice of contours should capture some global nature of *periods* hidden in the quantum invariants. Due to the difficulties of residue calculations of higher genus curves, still we do not have a full understanding in this direction. The PDE recursion of topological type [27, 30], on the other hand, captures local nature of the functions involved, and leads to an all-order WKB analysis of quantum curves for $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. The issue of singular spectral curves is addressed in [30], in which we have developed a systematic process of normalization of singular Hitchin spectral curves associated with meromorphic rank 2 Higgs bundles.

Theorem 6.1 is our answer to Question 1.1. It states that for the case of $\mathrm{SL}(2,\mathbb{C})$, the normalization process of [30] and the PDE recursion of [27] produce an all-order WKB expansion for the meromorphic Rees $\mathcal{D}$-modules obtained by quantizing singular Hitchin spectral curves through the construction of $\hbar$-families of opers. In this sense, our result shows that quantization of Hitchin spectral curves, singular or non-singular, through the PDE recursion of topological type and construction of $\hbar$-family of opers are equivalent, for the case of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles.

We note a relation between meromorphic Higgs bundles over $\mathbb{P}^1$ and Painlevé equations [11]. An application of topological recursion to establishing new results in Painlevé theory and construction of associated quantum curves are presented in [52, 53].

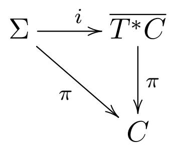
The interplay between Rees $\mathcal{D}$-modules, $\hbar$-families of opers, Hitchin spectral curves as semi-classical limit, Gaiotto's correspondence, and WKB analysis through PDE recursion of topological type, creates a sense of inevitability of the notion of quantization. Section 7 serves as an overview to this interplay, where we present the *Airy differential equation* as a prototypical example.

A totally new mathematical framework is presented in [57], in which Kontsevich and Soibelman formulate topological recursion as a special case of *deformation quantization*. They call the formalism *Airy structures*. In their work, spectral curves no longer serve as input data for topological recursion. Although construction of quantum curves is not the only purpose of the original topological recursion, what we present in our current paper is that our general procedure of quantization of Hitchin spectral curves has nothing to do with individual spectral curve, in parallel to the philosophy of [57]. As we show in (3.42), the family of spectral curves is (re)constructed from our deformation family of Rees $\mathcal{D}$-modules, not the other way around. Yet at this moment we do not have a mechanism to give the WKB expansion directly for the family of Rees $\mathcal{D}$-modules, without studying individual spectral curves. Investigating a possible connection between the Airy structures of [57] and this paper's results is a future subject. A relation between quantum curves and deformation quantization was first discussed in [71].

Let us briefly describe our quantization process of this paper now. Our geometric setting is a smooth projective algebraic curve C over $\mathbb{C}$ of an arbitrary genus $g = g(C)$ with a choice of a *spin structure*, or a *theta characteristic*, $K_C^{\frac{1}{2}}$. There are 2^{2g} choices of such spin structures. We choose any one of them. Let (E, ϕ) be an $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle on C with a meromorphic Higgs field ϕ. Denote by

$$\overline{T^*C} := \mathbb{P}(K_C \oplus \mathcal{O}_C) \overset{\pi}{\longrightarrow} C$$

the compactified cotangent bundle of C (see [5, 56]), which is a ruled surface on the base C. The *Hitchin spectral curve*



for a meromorphic Higgs bundle is defined as the divisor of zeros on $\overline{T^*C}$ of the characteristic polynomial of ϕ:

$$\Sigma := \Sigma(\phi) = \left(\det(\eta - \pi^*\phi)\right)_0, \tag{1.2}$$

where $\eta \in H^0(T^*C, \pi^*K_C)$ is the tautological 1-form on T^*C extended as a meromorphic 1-form on the compactification $\overline{T^*C}$. The morphism $\pi \colon \Sigma \longrightarrow C$ is a degree r map.

We denote by $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of holomorphic stable $\mathrm{SL}(r, \mathbb{C})$-Higgs bundles on C for $g \geq 2$. The assignment of the coefficients of the characteristic polynomial (1.2) to $(E, \phi) \in \mathcal{M}_{\mathrm{Dol}}$ defines the Hitchin fibration

$$\mu_H \colon\ \mathcal{M}_{\mathrm{Dol}} \longrightarrow B := \bigoplus_{i=2}^{r} H^0(C, K_C^{\otimes i}). \tag{1.3}$$

With the choice of a spin structure $K_C^{\frac{1}{2}}$ and Kostant's *principal three-dimensional subgroup* TDS of [58], one constructs a cross-section $\kappa \colon B \longrightarrow \mathcal{M}_{\mathrm{Dol}}$. We denote by $\langle H, X_+, X_- \rangle \subset \mathfrak{sl}(r, \mathbb{C})$ the Lie algebra of a principal TDS, where we use the standard representation as traceless matrices acting on $\mathbb{C}^r$. Thus H is diagonal, X_- is lower triangular, $X_+ = X_-^t$, and their relations are

$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = H. \tag{1.4}$$

The map κ is defined by

$$B \ni \mathbf{q} = (q_2, \dots, q_r) \longmapsto \kappa(\mathbf{q}) \in \big(E_0, \phi(\mathbf{q})\big) \in \mathcal{M}_{\mathrm{Dol}},$$

where

$$E_0 := \big(K_C^{\frac{1}{2}}\big)^H, \qquad \phi(\mathbf{q}) := X_- + \sum_{\ell=2}^{r} q_\ell X_+^{\ell-1}.$$

Clearly κ is not a *section of the fibration* μ_H in a strict sense, because $\mu_H \circ \kappa$ is *not* the identity map of B for $r \geq 3$. But it is a section in a more general sense that the image of κ always intersects with every fiber of μ_H exactly at one point. Note that B is the moduli space of Hitchin spectral curves associated with holomorphic $\mathrm{SL}(r, \mathbb{C})$-Higgs bundles on C. We use an unconventional way of defining the universal family $\mathcal{S}$ of spectral curves over B, instead of the natural family associated with (1.3), rather appealing to the Hitchin section κ, as

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & B \times \overline{T^*C} \\ & \overset{p}{\searrow} & \downarrow \mathrm{pr}_1 \\ & & B \end{array}, \qquad p^{-1}(\mathbf{q}) = \Sigma\big(\phi(\mathbf{q})\big). \tag{1.5}$$

Now we choose and fix, once and for all, a *projective coordinate system* of C subordinating the complex structure of C. This process does not depend algebraically on the moduli space of C. For a curve of genus $g \geq 2$, the Fuchsian projective structure, that appears in our solution [26] to a conjecture of Gaiotto [41], is a natural choice for our purpose. As we show in Section 3, there is a unique *filtered extension* $E_\hbar$ for every $\hbar \in H^1(C, K_C)$. For $r = 2$, $E_\hbar$ is the canonical extension

$$0 \longrightarrow K_C^{\frac{1}{2}} \longrightarrow E_\hbar \longrightarrow K_C^{-\frac{1}{2}} \longrightarrow 0$$

associated with

$$\hbar \in H^1(C, K_C) = \mathrm{Ext}^1\left(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}}\right).$$

With respect to the projective coordinate system, we can define a one-parameter family of opers

$$\big(E_\hbar, \nabla^\hbar(\mathbf{q})\big) \in \mathcal{M}_{\mathrm{deR}}$$

for $\hbar \neq 0$, where

$$\nabla^\hbar(\mathbf{q}) := \mathrm{d} + \frac{1}{\hbar}\phi(\mathbf{q}), \tag{1.6}$$

d is the exterior differentiation on C, and $\mathcal{M}_{\mathrm{deR}}$ is the moduli space of holomorphic irreducible $\mathrm{SL}(r, \mathbb{C})$-connections on C. The sum of the exterior differentiation and a Higgs field is *not* a connection in general. Here, the point is that the original vector bundle E_0 is deformed to $E_\hbar$, and we have chosen a projective coordinate system on C. Therefore, (1.6) makes sense as a global connection on C in $E_\hbar$.

Note that $\hbar\nabla^\hbar(\mathbf{q})$ is Deligne's $\hbar$-connection interpolating a connection $\mathrm{d} + \phi(\mathbf{q})$ and a Higgs field $\phi(\mathbf{q})$. We also note that $\big(E_\hbar, \hbar\nabla^\hbar(\mathbf{q})\big)$ defines a global Rees $\mathcal{D}$-module on C. Its generator is a globally defined differential operator P on C that acts on $K_C^{-\frac{r-1}{2}}$, which is what we call the *quantum curve* of the Hitchin spectral curve $\Sigma\big(\phi(\mathbf{q})\big)$ corresponding to $\mathbf{q} \in B$. The actual shape (3.40) of P is quite involved due to non-commutativity of the coordinate of C and differentiation. It is determined in the proof of Theorem 3.11. In Example 3.1 we list quantum curves P for $r = 2, 3, 4$. No matter how complicated its form is, the semi-classical limit of P recovers the spectral curve $\sigma^*\Sigma\big(\phi(\mathbf{q})\big)$ of the Higgs field $-\phi(\mathbf{q})$, where

$$\sigma\colon\ \overline{T^*C} \longrightarrow \overline{T^*C}, \qquad \sigma^2 = 1, \tag{1.7}$$

is the involution defined by the fiber-wise action of -1. This extra sign comes from the difference of conventions in the characteristic polynomial (1.2) and the connection (1.6).

The above process can be generalized in a straightforward way to *meromorphic* spectral data $\mathbf{q}$ for a curve C of arbitrary genus. The corresponding connections $\nabla^\hbar(\mathbf{q})$, and hence the Rees $\mathcal{D}$-modules, then have regular and irregular singularities.

We note that when we use the Fuchsian projective coordinate system of a curve C of genus $g \geq 2$ and holomorphic $\mathrm{SL}(r, \mathbb{C})$-Higgs bundles, our quantization process is exactly the same as the construction of $\mathrm{SL}(r,\mathbb{C})$-opers of [26] that was established by solving a conjecture of Gaiotto [41].

In Section 6, we perform a PDE variant of topological recursion for the case of meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. For this purpose, we use a normalization method of [30] for singular Hitchin spectral curves. We then show that the PDE recursion provides the WKB analysis for the quantum curve constructed through (1.6). When we deal with a singular spectral curve $\Sigma \subset \overline{T^*C}$, the key question is how to relate the singular curve with smooth ones. In terms of the Hitchin fibration, a singular spectral curve corresponds to a degenerate Abelian variety in the family. There are two different approaches to this question: one is to deform Σ locally to a non-singular curve, and the other is to blow up $\overline{T^*C}$ and obtain a resolution of singularities $\widetilde{\Sigma}$ of Σ. In this paper we will pursue the second path, and give a WKB analysis of the quantum curve using the geometric information of the desingularization.

Kostant's principal TDS plays a crucial role in our quantization through the relation (1.4). For example, it selects a particular fixed point of $\mathbb{C}^*$-action on the Hitchin section, which corresponds to the $\hbar \to \infty$ limit of (1.6). It is counterintuitive, but this limit is the connection $\mathrm{d} + X_-$ acting on $E_{\hbar=1}$, not just d which looks to be the case from the formula. This limiting connection then defines a vector space structure in the moduli space of opers.

This paper is organized as follows. The notion of quantum curves as Rees $\mathcal{D}$-modules quantizing Hitchin spectral curves is presented in Section 2. Then in Section 3, we quantize Hitchin spectral curves as Rees $\mathcal{D}$-modules through a concrete construction of $\hbar$-families of holomorphic and meromorphic $\mathrm{SL}(r,\mathbb{C})$-opers. The semi-classical limit of these resulting opers is calculated. Since our PDE recursion depends solely on the geometry of normalization of singular Hitchin spectral curves, we provide detailed study of the blow-up process in Sections 4. We give the genus formula for the normalization of the spectral curve in terms of the characteristic polynomial of the Higgs field ϕ. Then in Section 5, we define topological recursions for the case of degree 2 coverings. In Section 6, we prove that an all-order WKB analysis for quantization of meromorphic $\mathrm{SL}(2,\mathbb{C})$-Hitchin spectral curves is established through PDE recursion of topological type. We thus show that two quantizations procedures, one through $\hbar$-family of opers and the other through PDE recursion, agree for $\mathrm{SL}(2,\mathbb{C})$. The general structure of the theory is explained using the Airy differential equation as an example in Section 7. This example shows how the WKB analysis computes quantum invariants.

The current paper does not address difference equations that appear as quantum curves in knot theory, nor the mysterious spectral theory of [61].

2 Rees $\mathcal{D}$-modules as quantum curves for Higgs bundles

In this section, we give the definition of quantum curves in the context of Hitchin spectral curves. Let C be a non-singular projective algebraic curve defined over $\mathbb{C}$. The sheaf $\mathcal{D}_C$ of differential operators on C is the subalgebra of the $\mathbb{C}$-linear endomorphism algebra $\mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{O}_C)$ generated by the anti-canonical sheaf K_C^{-1} and the structure sheaf $\mathcal{O}_C$. Here, K_C^{-1} acts on $\mathcal{O}_C$ as holomorphic vector fields, and $\mathcal{O}_C$ acts on itself by multiplication. Locally every element of $\mathcal{D}_C$ is written as

$$\mathcal{D}_C \ni P(x) = \sum_{\ell=0}^{r} a_\ell(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-\ell}, \qquad a_\ell(x) \in \mathcal{O}_C$$

for some $r \geq 0$. For a fixed r, we introduce the filtration by order of differential operators into $\mathcal{D}_C$ as follows:

$$F_r\mathcal{D}_C = \left\{ P(x) = \sum_{\ell=0}^{r} a_\ell(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-\ell} \,\middle|\, a_\ell(x) \in \mathcal{O}_C \right\}.$$

The *Rees* ring $\widetilde{\mathcal{D}_C}$ is defined by

$$\widetilde{\mathcal{D}_C} = \bigoplus_{r=0}^{\infty} \hbar^r F_r \mathcal{D}_C \subset \mathbb{C}[\![\hbar]\!] \otimes_{\mathbb{C}} \mathcal{D}_C.$$

An element of $\widetilde{\mathcal{D}_C}$ on a coordinate neighborhood $U \subset C$ can be written as

$$P(x,\hbar) = \sum_{\ell=0}^{r} a_\ell(x,\hbar)\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-\ell}. \tag{2.1}$$

Definition 2.1 (Rees $\mathcal{D}$-module, cf. [59])**.** The Rees construction

$$\widetilde{\mathcal{M}} = \bigoplus_{r=0}^{\infty} \hbar^r F_r \mathcal{M}$$

associated with a filtered $\mathcal{D}_C$-module $(F_\bullet, \mathcal{M})$ is a *Rees $\mathcal{D}$-module* if the compatibility condition $F_a\mathcal{D}_C \cdot F_b\mathcal{M} \subset F_{a+b}\mathcal{M}$ holds.

Let

$$D = \sum_{j=1}^{n} m_j p_j, \qquad m_j > 0$$

be an effective divisor on C. The point set $\{p_1, \dots, p_n\} \subset C$ is the support of D. A *meromorphic Higgs bundle* with poles at D is a pair (E, ϕ) consisting of an algebraic vector bundle E on C and a Higgs field

$$\phi\colon\ E \longrightarrow K_C(D) \otimes_{\mathcal{O}_C} E.$$

Since the cotangent bundle

$$T^*C = \operatorname{Spec}\big(\operatorname{Sym}\big(K_C^{-1}\big)\big)$$

is the total space of K_C, we have the tautological 1-form $\eta \in H^0(T^*C, \pi^*K_C)$ on T^*C coming from the projection

$$\begin{array}{ccc} T^*C & \longleftarrow & \pi^*K_C \\ {\scriptstyle\pi}\big\downarrow & & \\ C & \longleftarrow & K_C. \end{array}$$

The natural holomorphic symplectic form of T^*C is given by $-\mathrm{d}\eta$. The *compactified cotangent bundle* of C is a ruled surface defined by

$$\overline{T^*C} := \mathbb{P}(K_C \oplus \mathcal{O}_C) = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty}\left(K_C^{-n}\cdot I^0 \oplus K_C^{-n+1}\cdot I \oplus \cdots \oplus K_C^0 \cdot I^n\right)\right),$$

where I represents $1 \in \mathcal{O}_C$ being considered as a degree 1 element. The divisor at infinity

$$C_\infty := \mathbb{P}(K_C \oplus \{0\})$$

is reduced in the ruled surface and supported on the subset $\mathbb{P}(K_C \oplus \mathcal{O}_C) \setminus T^*C$. The tautological 1-form η extends on $\overline{T^*C}$ as a meromorphic 1-form with simple poles along C_∞. Thus the divisor of η in $\overline{T^*C}$ is given by

$$(\eta) = C_0 - C_\infty,$$

where C_0 is the zero section of T^*C.

The relation between the sheaf $\mathcal{D}_C$ and the geometry of the compactified cotangent bundle $\overline{T^*C}$ is the following. First we have

$$\operatorname{Spec}\left(\bigoplus_{m=0}^{\infty} F_m\mathcal{D}_C/F_{m-1}\mathcal{D}_C\right) = \operatorname{Spec}\left(\bigoplus_{m=0}^{\infty} K_C^{-m}\right) = T^*C.$$

Let us denote by $\operatorname{gr}_m\mathcal{D}_C = F_m\mathcal{D}_C/F_{m-1}\mathcal{D}_C$. By writing $I = 1 \in H^0(C,\mathcal{D}_C)$, we then have

$$\overline{T^*C} = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty}\left(\operatorname{gr}_m\mathcal{D}_C\cdot I^0 \oplus \operatorname{gr}_{m-1}\mathcal{D}_C\cdot I \oplus \operatorname{gr}_{m-2}\mathcal{D}_C\cdot I^{\otimes 2}\oplus\cdots\oplus \operatorname{gr}_0\mathcal{D}_C\cdot I^{\otimes m}\right)\right).$$

Definition 2.2 (spectral curve)**.** A *spectral curve* of degree r is a divisor Σ in $\overline{T^*C}$ such that the projection $\pi\colon \Sigma \longrightarrow C$ defined by the restriction

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ i\ } & \overline{T^*C} \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle\pi} \\ & & C \end{array}$$

is a finite morphism of degree r. The *spectral curve of a Higgs bundle* (E,ϕ) is the divisor of zeros

$$\Sigma = (\det(\eta - \pi^*\phi))_0$$

on $\overline{T^*C}$ of the characteristic polynomial $\det(\eta-\pi^*\phi)$. Here,

$$\pi^*\phi\colon\ \pi^*E \longrightarrow \pi^*(K_C(D))\otimes_{\mathcal{O}_{\mathbb{P}(K_C\oplus\mathcal{O}_C)}}\pi^*E.$$

Remark 2.3. The Higgs field ϕ is holomorphic on $C\setminus\operatorname{supp}(D)$. Thus we can define the divisor of zeros

$$\Sigma^\circ = \big(\det(\eta - \pi^*(\phi|_{C\setminus\operatorname{supp}(D)}))\big)_0$$

of the characteristic polynomial on $T^*(C\setminus\operatorname{supp}(D))$. The spectral curve Σ is the complex topology closure of Σ° with respect to the compactification

$$T^*(C\setminus\operatorname{supp}(D)) \subset \overline{T^*C}.$$

A left $\mathcal{D}_C$-module $\mathcal{E}$ on C is naturally an $\mathcal{O}_C$-module with a $\mathbb{C}$-linear integrable (i.e., flat) connection $\nabla\colon \mathcal{E}\longrightarrow K_C\otimes_{\mathcal{O}_C}\mathcal{E}$. The construction goes as follows:

$$\nabla\colon\ \mathcal{E} \xrightarrow{\ \alpha\ } \mathcal{D}_C\otimes_{\mathcal{O}_C}\mathcal{E} \xrightarrow{\nabla_{\mathcal{D}}\otimes\mathrm{id}} (K_C\otimes_{\mathcal{O}_C}\mathcal{D}_C)\otimes_{\mathcal{O}_C}\mathcal{E} \xrightarrow{\beta\otimes\mathrm{id}} K_C\otimes_{\mathcal{O}_C}\mathcal{E}, \tag{2.2}$$

where

- α is the natural inclusion $\mathcal{E}\ni v\longmapsto 1\otimes v\in\mathcal{D}_C\otimes_{\mathcal{O}_C}\mathcal{E}$,
- $\nabla_{\mathcal{D}}\colon \mathcal{D}_C\longrightarrow K_C\otimes_{\mathcal{O}_C}\mathcal{D}_C$ is the connection defined by the $\mathbb{C}$-linear left-multiplication operation of K_C^{-1} on $\mathcal{D}_C$, which satisfies the derivation property

$$\nabla_{\mathcal{D}}(f\cdot P) = f\cdot\nabla_{\mathcal{D}}(P) + \mathrm{d}f\cdot P \in K_C\otimes_{\mathcal{O}_C}\mathcal{D}_C \tag{2.3}$$

  for $f\in\mathcal{O}_C$ and $P\in\mathcal{D}_C$, and

- β is the canonical right $\mathcal{D}_C$-module structure in K_C defined by the Lie derivative of vector fields.

If we choose a local coordinate neighborhood $U \subset C$ with a coordinate x, then (2.3) takes the following form. Let us denote by $P' = [\mathrm{d}/\mathrm{d}x, P]$, and define

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}(P) := P \cdot \frac{\mathrm{d}}{\mathrm{d}x} + P'.$$

Then we have

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}(f \cdot P) = f \cdot \nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}(P) + \frac{\mathrm{d}f}{\mathrm{d}x} \cdot P.$$

The connection ∇ of (2.2) is integrable because $d^2 = 0$. Actually, the statement is true for any dimensions. We note that there is no reason for $\mathcal{E}$ to be coherent as an $\mathcal{O}_C$-module.

Conversely, if an algebraic vector bundle E on C of rank r admits a holomorphic connection $\nabla\colon E \longrightarrow K_C \otimes E$, then E acquires the structure of a $\mathcal{D}_C$-module. This is because ∇ is automatically flat, and the covariant derivative ∇_X for $X \in K_C^{-1}$ satisfies

$$\nabla_X(fv) = f\nabla_X(v) + X(f)v \tag{2.4}$$

for $f \in \mathcal{O}_C$ and $v \in E$. A repeated application of (2.4) makes E a $\mathcal{D}_C$-module. The fact that every $\mathcal{D}_C$-module on a curve is principal implies that for every point $p \in C$, there is an open neighborhood $p \in U \subset C$ and a linear differential operator P of order r on U, called a generator, such that $E|_U \cong \mathcal{D}_U/\mathcal{D}_U P$. Thus on an open curve U, a holomorphic connection in a vector bundle of rank r gives rise to a differential operator of order r. The converse is true if $\mathcal{D}_U/\mathcal{D}_U P$ is $\mathcal{O}_U$-coherent.

Definition 2.4 (formal $\hbar$-connection, cf. [3])**.** A formal $\hbar$-connection on a vector bundle $E \longrightarrow C$ is a $\mathbb{C}[\![\hbar]\!]$-linear homomorphism

$$\nabla^\hbar\colon\ \mathbb{C}[\![\hbar]\!] \otimes E \longrightarrow \mathbb{C}[\![\hbar]\!] \otimes K_C \otimes_{\mathcal{O}_C} E$$

subject to the derivation condition

$$\nabla^\hbar(f \cdot v) = f\nabla^\hbar(v) + \hbar\,\mathrm{d}f \otimes v,$$

where $f \in \mathcal{O}_C \otimes \mathbb{C}[\![\hbar]\!]$ and $v \in \mathbb{C}[\![\hbar]\!] \otimes E$.

When we consider *holomorphic* dependence of a quantum curve with respect to the quantization parameter $\hbar$, we need to use a particular $\hbar$-deformation family of vector bundles. We will discuss the holomorphic case in Section 3, where we explain how (1.1) appears in our quantization.

Remark 2.5. The *classical limit* of a formal $\hbar$-connection is the evaluation $\hbar = 0$ of $\nabla^\hbar$, which is simply an $\mathcal{O}_C$-module homomorphism

$$\nabla^0\colon\ E \longrightarrow K_C \otimes_{\mathcal{O}_C} E,$$

i.e., a holomorphic Higgs field in the vector bundle E.

Remark 2.6. An $\mathcal{O}_C \otimes \mathbb{C}[\![\hbar]\!]$-coherent $\widetilde{\mathcal{D}_C}$-module is equivalent to a vector bundle on C equipped with an $\hbar$-connection.

In analysis, the *semi-classical limit* of a differential operator $P(x,\hbar)$ of the form (2.1) is a function defined by

$$\lim_{\hbar\to 0}\big(e^{-\frac{1}{\hbar}S_0(x)}P(x,\hbar)e^{\frac{1}{\hbar}S_0(x)}\big)=\sum_{\ell=0}^{r}a_\ell(x,0)(S_0'(x))^{r-\ell}, \tag{2.5}$$

where $S_0(x)\in\mathcal{O}_C(U)$. The equation

$$\lim_{\hbar\to 0}\big(e^{-\frac{1}{\hbar}S_0(x)}P(x,\hbar)e^{\frac{1}{\hbar}S_0(x)}\big)=0 \tag{2.6}$$

then determines the first term of the *singular perturbation expansion*, or the *WKB asymptotic expansion*,

$$\psi(x,\hbar)=\exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x)\right) \tag{2.7}$$

of a solution $\psi(x,\hbar)$ to the differential equation

$$P(x,\hbar)\psi(x,\hbar)=0$$

on U. We note that the expression (2.7) is never meant to be a convergent series in $\hbar$.

Since $dS_0(x)$ is a local section of T^*C on $U\subset C$, $y=S_0'(x)$ gives a local trivialization of $T^*C|_U$, with $y\in T_x^*C$ a fiber coordinate. Then (2.5) and (2.6) give an equation

$$\sum_{\ell=0}^{r}a_\ell(x,0)y^{r-\ell}=0$$

of a curve in $T^*C|_U$. This motivates us to give the following definition:

Definition 2.7 (semi-classical limit of a Rees differential operator)**.** Let $U\subset C$ be an open subset of C with a local coordinate x such that T^*C is trivial over U with a fiber coordinate y. The semi-classical limit of a local section

$$P(x,\hbar)=\sum_{\ell=0}^{r}a_\ell(x,\hbar)\left(\hbar\frac{d}{dx}\right)^{r-\ell}$$

of the Rees ring $\widetilde{\mathcal{D}_C}$ of the sheaf of differential operators $\mathcal{D}_C$ on U is the holomorphic function

$$\sum_{\ell=0}^{r}a_\ell(x,0)y^{r-\ell}$$

defined on $T^*C|_U$.

Definition 2.8 (semi-classical limit of a Rees $\mathcal{D}$-module)**.** Suppose a Rees $\widetilde{\mathcal{D}_C}$-module $\widetilde{\mathcal{M}}$ globally defined on C is written as

$$\widetilde{\mathcal{M}}(U)=\widetilde{\mathcal{D}_C}(U)\big/\widetilde{\mathcal{D}_C}(U)P_U$$

on every coordinate neighborhood $U\subset C$ with a differential operator P_U of the form (2.1). Using this expression (2.1) for P_U, we construct a meromorphic function

$$p_U(x,y)=\sum_{\ell=0}^{r}a_\ell(x,0)y^{r-\ell} \tag{2.8}$$

on $\overline{T^*C}|_U$, where y is the fiber coordinate of T^*C, which is trivialized on U. Define

$$\Sigma_U = (p_U(x,y))_0$$

as the divisor of zero of the function $p_U(x,y)$. If Σ_U's glue together to a spectral curve $\Sigma \subset \overline{T^*C}$, then we call Σ the *semi-classical limit* of the Rees $\widetilde{\mathcal{D}_C}$-module $\widetilde{\mathcal{M}}$.

Remark 2.9. For the local equation (2.8) to be consistent globally on C, the coefficients of (2.1) have to satisfy

$$a_\ell(x,0) \in \Gamma\big(U, K_C^{\otimes \ell}\big).$$

Definition 2.10 (quantum curve for holomorphic Higgs bundle)**.** A *quantum curve* associated with the spectral curve $\Sigma \subset T^*C$ of a holomorphic Higgs bundle on a projective algebraic curve C is a Rees $\widetilde{\mathcal{D}_C}$-module $\mathcal{E}$ whose semi-classical limit is Σ.

The main reason we wish to extend our framework to meromorphic connections is that there are no non-trivial holomorphic connections on $\mathbb{P}^1$, whereas many important classical examples of differential equations are naturally defined over $\mathbb{P}^1$ with regular and irregular singularities. A $\mathbb{C}$-linear homomorphism

$$\nabla\colon\ E \longrightarrow K_C(D) \otimes_{\mathcal{O}_C} E$$

is said to be a *meromorphic connection* with poles along an effective divisor D if

$$\nabla(f \cdot v) = f\nabla(v) + \mathrm{d}f \otimes v$$

for every $f \in \mathcal{O}_C$ and $v \in E$. Let us denote by

$$\mathcal{O}_C(*D) := \varinjlim \mathcal{O}_C(mD), \qquad E(*D) := E \otimes_{\mathcal{O}_C} \mathcal{O}_C(*D).$$

Then ∇ extends to

$$\nabla\colon\ E(*D) \longrightarrow K_C(*D) \otimes_{\mathcal{O}_C(*D)} E(*D).$$

Since ∇ is holomorphic on $C \setminus \operatorname{supp}(D)$, it induces a $\mathcal{D}_{C\setminus \operatorname{supp}(D)}$-module structure in $E|_{C\setminus\operatorname{supp}(D)}$. The $\mathcal{D}_C$-module direct image $\widetilde{E} = j_*\big(E|_{C\setminus\operatorname{supp}(D)}\big)$ associated with the open inclusion map $j\colon C \setminus \operatorname{supp}(D) \longrightarrow C$ is then naturally isomorphic to

$$\widetilde{E} = j_*\big(E|_{C\setminus\operatorname{supp}(D)}\big) \cong E(*D) \tag{2.9}$$

as a $\mathcal{D}_C$-module. Equation (2.9) is called the *meromorphic extension* of the $\mathcal{D}_{C\setminus\operatorname{supp}(D)}$-module $E|_{C\setminus\operatorname{supp}(D)}$.

Let us take a local coordinate x of C, this time around a pole $p_j \in \operatorname{supp}(D)$. If a generator $\widetilde{P}$ of $\widetilde{E}$ near $x=0$ has a local expression

$$\widetilde{P}\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right) = x^k \sum_{\ell=0}^{r} b_\ell(x)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-\ell}$$

around p_j with locally defined holomorphic functions $b_\ell(x)$, $b_0(0) \neq 0$, and an integer $k \in \mathbb{Z}$, then $\widetilde{P}$ has a *regular* singular point at p_j. Otherwise, p_j is an *irregular* singular point of $\widetilde{P}$.

Definition 2.11 (quantum curve for a meromorphic Higgs bundle)**.** Let (E, ϕ) be a meromorphic Higgs bundle defined over a projective algebraic curve C of any genus with poles along an effective divisor D, and $\Sigma \subset \overline{T^*C}$ its spectral curve. A *quantum curve* associated with Σ is the meromorphic extension of a Rees $\widetilde{\mathcal{D}_C}$-module $\mathcal{E}$ on $C \setminus \operatorname{supp}(D)$ such that the complex topology closure of its semi-classical limit $\Sigma^\circ \subset T^*C|_{C\setminus \operatorname{supp}(D)}$ in the compactified cotangent bundle $\overline{T^*C}$ agrees with Σ.

In Section 3, we prove that every Hitchin spectral curve associated with a holomorphic or a meromorphic $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle has a quantum curve.

Remark 2.12. We remark that several examples of quantum curves that are constructed in [13, 32, 64], for various Hurwitz numbers and Gromov–Witten theory of $\mathbb{P}^1$, do not fall into our definition in terms of Rees $\mathcal{D}$-modules. This is because in the above mentioned examples, quantum curves involve *infinite*-order differential operators, or *difference* operators, while we consider only differential operators of finite order in this paper.

3 Opers

There is a simple mechanism to construct a quantization of a Hitchin spectral curve, using a particular choice of isomorphism between a *Hitchin section* and the moduli of *opers*. The quantum deformation parameter $\hbar$, originated in physics as the Planck *constant*, is a purely formal parameter in WKB analysis. Since we will be using the PDE recursion (5.3) for the analysis of quantum curves, $\hbar$ plays the role of a formal parameter for the asymptotic expansion. This point of view motivates our definition of quantum curves as Rees D-modules in the previous section. However, the quantum curves appearing in the quantization of Hitchin spectral curves associated with G-Higgs bundles for a complex simple Lie group G always depend *holomorphically* on $\hbar$. Therefore, we need a more geometric setup for quantum curves to deal with this holomorphic dependence. The purpose of this section is to explain *holomorphic $\hbar$-connections as quantum curves*, and the geometric interpretation of $\hbar$ given in (1.1). The key concept is *opers* of Beilinson–Drinfeld [6]. Although a vast generalization of the current paper is possible, we restrict our attention to $\mathrm{SL}(r, \mathbb{C})$-opers for an arbitrary $r \geq 2$ in this paper.

In this section, most of the time C is a smooth projective algebraic curve of genus $g \geq 2$ defined over $\mathbb{C}$, unless otherwise specified.

3.1 Holomorphic $\mathrm{SL}(r, \mathbb{C})$-opers and quantization of Higgs bundles

We first recall projective structures on C following Gunning [45]. Recall that every compact Riemann surface has a projective structure subordinating the given complex structure. A *complex projective coordinate system* is a coordinate neighborhood covering

$$C = \bigcup_\alpha U_\alpha$$

with a local coordinate x_α of U_α such that for every $U_\alpha \cap U_\beta$, we have a *Möbius* coordinate transformation

$$x_\alpha = \frac{a_{\alpha\beta} x_\beta + b_{\alpha\beta}}{c_{\alpha\beta} x_\beta + \mathrm{d}_{\alpha\beta}}, \qquad \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & \mathrm{d}_{\alpha\beta} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Since we solve differential equations on C, we always assume that each coordinate neighborhood U_α is *simply connected*. In what follows, we choose and fix a projective coordinate system on C. Since

$$\mathrm{d}x_\alpha = \frac{1}{(c_{\alpha\beta} x_\beta + \mathrm{d}_{\alpha\beta})^2}\, \mathrm{d}x_\beta,$$

the transition function for the canonical line bundle K_C of C is given by the cocycle

$$\left\{(c_{\alpha\beta}x_\beta + \mathrm{d}_{\alpha\beta})^2\right\} \qquad \text{on} \quad U_\alpha \cap U_\beta.$$

We choose and fix, once and for all, a theta characteristic, or a spin structure, $K_C^{\frac{1}{2}}$ such that $\left(K_C^{\frac{1}{2}}\right)^{\otimes 2} \cong K_C$. Let $\{\xi_{\alpha\beta}\}$ denote the 1-cocycle corresponding to $K_C^{\frac{1}{2}}$. Then we have

$$\xi_{\alpha\beta} = \pm(c_{\alpha\beta}x_\beta + \mathrm{d}_{\alpha\beta}). \tag{3.1}$$

The choice of $\pm$ here is an element of $H^1(C, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$, indicating that there are 2^{2g} choices for spin structures in C.

The significance of the projective coordinate system lies in the fact that $\partial_\beta^2 \xi_{\alpha\beta} = 0$. This simple property plays an essential role in our construction of global connections on C, as we see in this section. Another way of appreciating the projective coordinate system is the vanishing of Schwarzian derivatives, as explained in [25]. A scalar valued single linear ordinary differential equation of any order can be globally defined in terms of a projective coordinate.

A holomorphic Higgs bundle (E, ϕ) is *stable* if for every vector subbundle $F \subset E$ that is invariant with respect to ϕ, i.e., $\phi\colon F \longrightarrow F \otimes K_C$, the slope condition

$$\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}$$

holds. The moduli space of stable Higgs bundles is constructed [73]. An $\mathrm{SL}(r,\mathbb{C})$-Higgs bundle is a pair (E, ϕ) with a fixed isomorphism $\det E = \mathcal{O}_C$ and $\mathrm{tr}\phi = 0$. We denote by $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of stable holomorphic $\mathrm{SL}(r,\mathbb{C})$-Higgs bundles on C. Hitchin [46] defines a holomorphic fibration

$$\mu_H\colon \quad \mathcal{M}_{\mathrm{Dol}} \ni (E,\phi) \longmapsto \det(\eta - \pi^*\phi) \in B, \qquad B := \bigoplus_{\ell=2}^{r} H^0\big(C, K_C^{\otimes \ell}\big),$$

that induces the structure of an algebraically completely integrable Hamiltonian system in $\mathcal{M}_{\mathrm{Dol}}$. With the choice of a spin structure $K_C^{\frac{1}{2}}$, we have a natural section $\kappa\colon B \hookrightarrow \mathcal{M}_{\mathrm{Dol}}$ defined by utilizing Kostant's *principal three-dimensional subgroup* (TDS) [58] as follows.

First, let

$$\mathbf{q} = (q_2, q_3, \dots, q_r) \in B = \bigoplus_{\ell=2}^{r} H^0\big(C, K_C^{\otimes \ell}\big)$$

be an arbitrary point of the Hitchin base B. Define

$$X_- := \left[\sqrt{s_{i-1}}\,\delta_{i-1,j}\right]_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \sqrt{s_1} & & & & 0 \\ 0 & \sqrt{s_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \sqrt{s_{r-1}} & 0 \end{bmatrix}, \qquad X_+ := X_-^t,$$

$$H := [X_+, X_-],$$

where $s_i := i(r-i)$. H is a diagonal matrix whose (i,i)-entry is $s_i - s_{i-1} = r - 2i + 1$, with $s_0 = s_r = 0$. The Lie algebra $\langle X_+, X_-, H\rangle \cong \mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra of the principal TDS in $\mathrm{SL}(r,\mathbb{C})$.

Lemma 3.1. *Define a Higgs bundle $(E_0, \phi(\mathbf{q}))$ consisting of a vector bundle*

$$E_0 := \left(K_C^{\frac{1}{2}}\right)^{\otimes H} = \bigoplus_{i=1}^{r} \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-2i+1)} \tag{3.2}$$

and a Higgs field

$$\phi(\mathbf{q}) := X_- + \sum_{\ell=2}^{r} q_\ell X_+^{\ell-1}. \tag{3.3}$$

Then it is a stable $\mathrm{SL}(r,\mathbb{C})$-Higgs bundle. The Hitchin section is defined by

$$\kappa\colon \ B \ni \mathbf{q} \longmapsto (E_0, \phi(\mathbf{q})) \in \mathcal{M}_{\mathrm{Dol}}, \tag{3.4}$$

which gives a biholomorphic map between B and $\kappa(B) \subset \mathcal{M}_{\mathrm{Dol}}$.

Proof. We first note that $X_-\colon E_0 \longrightarrow E_0 \otimes K_C$ is a globally defined $\mathrm{End}_0(E_0)$-valued 1-form, since it is a collection of constant maps

$$\sqrt{s_i}\colon \ \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-2i+1)} \xrightarrow{=} \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-2(i+1)+1)} \otimes K_C. \tag{3.5}$$

Similarly, since $X_+^{\ell-1}$ is an upper-diagonal matrix with non-zero entries along the $(\ell-1)$-th upper diagonal, we have

$$q_\ell\colon \ \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-2i+1)} \longrightarrow \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-2i+1+2\ell)} = \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-2(i-\ell+1)+1)} \otimes K_C.$$

Thus $\phi(\mathbf{q})\colon E_0 \longrightarrow E_0 \otimes K_C$ is globally defined as a Higgs field in E_0. The Higgs pair is stable because no subbundle of E_0 is invariant under $\phi(\mathbf{q})$, unless $\mathbf{q} = 0$. And when $\mathbf{q} = 0$, the invariant subbundles all have positive degrees, since $g \geq 2$. ■

The image $\kappa(B)$ is a holomorphic Lagrangian submanifold of a holomorphic symplectic space $\mathcal{M}_{\mathrm{Dol}}$.

To define $\hbar$-connections holomorphically depending on $\hbar$, we need to construct a one-parameter holomorphic family of deformations of vector bundles

$$\begin{array}{ccc} E_\hbar & \longrightarrow & \mathcal{E} \\ \big\downarrow & & \big\downarrow \\ C \times \{\hbar\} & \longrightarrow & C \times H^1(C, K_C) \end{array}$$

and a $\mathbb{C}$-linear first-order differential operator

$$\hbar\nabla^\hbar\colon \ E_\hbar \longrightarrow E_\hbar \otimes K_C$$

depending holomorphically on $\hbar \in H^1(C, K_C) \cong \mathbb{C}$ for $\hbar \neq 0$. Let us introduce the notion of *filtered extensions.*

Definition 3.2 (filtered extension)**.** A one-parameter family of filtered holomorphic vector bundles $\left(F_\hbar^\bullet, E_\hbar\right)$ on C with a trivialized determinant $\det(E_\hbar) \cong \mathcal{O}_C$ is a filtered extension of the vector bundle E_0 of (3.2) parametrized by $\hbar \in H^1(C, K_C)$ if the following conditions hold:

- $E_\hbar$ has a filtration

$$0 = F_\hbar^r \subset F_\hbar^{r-1} \subset F_\hbar^{r-2} \subset \cdots \subset F_\hbar^0 = E_\hbar.$$

- The second term is given by

$$F_\hbar^{r-1} = \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-1)}. \tag{3.6}$$

- For every $i = 1, 2, \dots, r-1$, there is an $\mathcal{O}_C$-module isomorphism

$$F_\hbar^i / F_\hbar^{i+1} \xrightarrow{\sim} \left(F_\hbar^{i-1}/F_\hbar^i\right) \otimes K_C. \tag{3.7}$$

Remark 3.3. Since we need to identify a deformation parameter $\hbar$ and extension classes, we make the natural identification

$$\mathrm{Ext}^1(E, F) = H^1(C, E^* \otimes F)$$

for every pair of vector bundles E and F. We also identify $\mathrm{Ext}^1(E, F)$ as the class of extensions

$$0 \longrightarrow F \longrightarrow V \longrightarrow E \longrightarrow 0$$

of E by a vector bundle V. These identifications are done by a choice of a projective coordinate system on C as explained below.

Proposition 3.4 (construction of filtered extension)**.** *For every choice of the theta characteristic $K_C^{\frac{1}{2}}$ and a non-zero element $\hbar \in H^1(C, K_C)$, there is a unique non-trivial filtered extension $\left(F_\hbar^\bullet, E_\hbar\right)$ of E_0.*

Proof. First let us examine the case of $r = 2$ to see how things work. Since

$$\hbar \in H^1(C, K_C) = \mathrm{Ext}^1\left(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}}\right) \cong \mathbb{C},$$

we have a unique extension

$$0 \longrightarrow K_C^{\frac{1}{2}} \longrightarrow E_\hbar \longrightarrow K_C^{-\frac{1}{2}} \longrightarrow 0 \tag{3.8}$$

corresponding to $\hbar$. Obviously

$$K_C^{\frac{1}{2}} \xrightarrow{\sim} \left(E_\hbar / K_C^{\frac{1}{2}}\right) \otimes K_C,$$

which proves (3.7). We also note that as a vector bundle of rank 2, we have the isomorphism

$$E_\hbar \cong \begin{cases} E_1, & \hbar \neq 0, \\ E_0, & \hbar = 0. \end{cases}$$

Now consider the general case. We use the induction on $i = r-1, r-2, r-3, \dots, 0$, in the reverse direction to construct each term of the filtration $F_\hbar^i$ subject to (3.6) and (3.7). The base case $i = r-1$ is the following. Since

$$0 \longrightarrow F_\hbar^{r-1} \longrightarrow F_\hbar^{r-2} \longrightarrow F_\hbar^{r-2}/F_\hbar^{r-1} \longrightarrow 0$$

and

$$F_\hbar^{r-2}/F_\hbar^{r-1} \cong F_\hbar^{r-1} \otimes K_C^{-1} \cong \left(K_C^{\frac{1}{2}}\right)^{\otimes(r-1)} \otimes K_C^{-1}$$

from (3.6) and (3.7), $F_\hbar^{r-2}$ is determined by the class $\hbar$ in

$$\operatorname{Ext}^1\big(F_\hbar^{r-2}/F_\hbar^{r-1}, F_\hbar^{r-1}\big) = H^1\big(C, F_\hbar^{r-1} \otimes \big(F_\hbar^{r-2}/F_\hbar^{r-1}\big)^{-1}\big) \cong H^1(C, K_C).$$

Assume that for a given $i+1$, we have

$$H^1\big(C, F_\hbar^n \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-n+r)}\big) \cong H^1(C, K_C), \tag{3.9}$$

$$H^1\big(C, F_\hbar^n \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-n+m+r)}\big) = 0, \qquad m \geq 1, \tag{3.10}$$

for every n in the range $i+1 \leq n \leq r-1$. We wish to prove that the same relation holds for $n = i$.

The sequence of isomorphisms (3.7) implies that

$$F_\hbar^{i-1}/F_\hbar^i \cong \big(F_\hbar^i/F_\hbar^{i+1}\big) \otimes K_C^{-1} \cong F_\hbar^{r-1} \otimes K_C^{\otimes(i-r)} = \Big(K_C^{\frac{1}{2}}\Big)^{\otimes(r-1)} \otimes K_C^{\otimes(i-r)} \cong K_C^{\otimes\left(i-\frac{r+1}{2}\right)}.$$

Then $F_\hbar^{i-1}$ as an extension

$$0 \longrightarrow F_\hbar^i \longrightarrow F_\hbar^{i-1} \longrightarrow F_\hbar^{r-1} \otimes K_C^{\otimes(i-r)} \longrightarrow 0 \tag{3.11}$$

is determined by a class in

$$\operatorname{Ext}^1\big(F_\hbar^{r-1} \otimes K_C^{\otimes(i-r)}, F_\hbar^i\big) = H^1\big(C, F_\hbar^i \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-i+r)}\big).$$

The exact sequence

$$0 \longrightarrow F_\hbar^n \longrightarrow F_\hbar^{n-1} \longrightarrow F_\hbar^{r-1} \otimes K_C^{\otimes(n-r)} \longrightarrow 0$$

implies that

$$0 \longrightarrow F_\hbar^n \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-n+m+r)} \longrightarrow F_\hbar^{n-1} \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-n+m+r)} \longrightarrow K_C^{\otimes m} \longrightarrow 0$$

for every $m \geq 1$. Taking the cohomology long exact sequence, we obtain

$$H^1\big(C, F_\hbar^{n-1} \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-n+1+r)}\big) \cong H^1(C, K_C)$$

for $m = 1$, which proves (3.9) for $n = i$. Similarly,

$$\begin{aligned} &H^1\big(C, F_\hbar^{n-1} \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-(n-1)+(m-1)+r)}\big) \\ &\qquad = H^1\big(C, F_\hbar^{n-1} \otimes \big(F_\hbar^{r-1}\big)^{-1} \otimes K_C^{\otimes(-n+m+r)}\big) \cong H^1(C, K_C^m) = 0 \end{aligned}$$

for $m \geq 2$, which is (3.10) for $n = i$. By induction on i in the decreasing direction, we have established that the class $\hbar$ determines the unique extension (3.11) for every i. ■

Definition 3.5 ($\mathrm{SL}(r,\mathbb{C})$-opers)**.** A point $(E, \nabla) \in \mathcal{M}_{\mathrm{deR}}$, i.e., an irreducible holomorphic $\mathrm{SL}(r,\mathbb{C})$-connection $\nabla\colon E \longrightarrow E \otimes K_C$ acting on a vector bundle E, is an $\mathrm{SL}(r,\mathbb{C})$-oper if the following conditions are satisfied.

Filtration. There is a filtration $F^\bullet$ by vector subbundles

$$0 = F^r \subset F^{r-1} \subset F^{r-2} \subset \cdots \subset F^0 = E. \tag{3.12}$$

Griffiths transversality. The connection respects the filtration:

$$\nabla|_{F^i}\colon \quad F^i \longrightarrow F^{i-1} \otimes K_C, \qquad i = 1, \dots, r. \tag{3.13}$$

Grading condition. The connection induces $\mathcal{O}_C$-module isomorphisms

$$\overline{\nabla}\colon \quad F^i/F^{i+1} \xrightarrow{\sim} \big(F^{i-1}/F^i\big) \otimes K_C, \qquad i = 1, \dots, r-1. \tag{3.14}$$

For the purpose of defining differential operators globally on the Riemann surface C, we need a projective coordinate system on C subordinating its complex structure. The coordinate also allows us to give a concrete $\hbar \in H^1(C, K_C)$-dependence in the filtered extensions. For example, the extension $E_\hbar$ of (3.8) is given by a system of transition functions

$$E_\hbar \longleftrightarrow \left\{ \begin{bmatrix} \xi_{\alpha\beta} & \hbar\sigma_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{bmatrix} \right\}$$

on each $U_\alpha \cap U_\beta$. The cocycle condition for the transition functions translates into a condition

$$\sigma_{\alpha\gamma} = \xi_{\alpha\beta}\sigma_{\beta\gamma} + \sigma_{\alpha\beta}\xi_{\beta\gamma}^{-1}. \tag{3.15}$$

The application of the exterior differentiation d to the cocycle condition $\xi_{\alpha\gamma} = \xi_{\alpha\beta}\xi_{\beta\gamma}$ yields

$$\frac{\mathrm{d}\xi_{\alpha\gamma}}{\mathrm{d}x_\gamma}\mathrm{d}x_\gamma = \frac{\mathrm{d}\xi_{\alpha\beta}}{\mathrm{d}x_\beta}\mathrm{d}x_\beta\xi_{\beta\gamma} + \xi_{\alpha\beta}\frac{\mathrm{d}\xi_{\beta\gamma}}{\mathrm{d}x_\gamma}\mathrm{d}x_\gamma.$$

Noticing that

$$\xi_{\alpha\beta}^2 = \frac{\mathrm{d}x_\beta}{\mathrm{d}x_\alpha},$$

we see that

$$\sigma_{\alpha\beta} := \frac{\mathrm{d}\xi_{\alpha\beta}}{\mathrm{d}x_\beta} = \partial_\beta \xi_{\alpha\beta} \tag{3.16}$$

solves (3.15). We note that

$$\begin{bmatrix} \xi_{\alpha\beta} & \hbar\sigma_{\alpha\beta} \\ & \xi_{\alpha\beta}^{-1} \end{bmatrix} = \exp\left(\log\xi_{\alpha\beta}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\exp\left(\hbar\partial_\beta\log\xi_{\alpha\beta}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

Therefore, in the multiplicative sense, the extension class is determined by $\partial_\beta \log\xi_{\alpha\beta}$.

Lemma 3.6. *The extension class $\sigma_{\alpha\beta}$ of (3.16) defines a non-trivial extension (3.8).*

Proof. The cohomology long exact sequence

$$\begin{array}{ccccccc}
H^1(C,\mathbb{C}) & \longrightarrow & H^1(C,\mathcal{O}_C) & \longrightarrow & H^1(C,K_C) & \xrightarrow{\sim} & H^2(C,\mathbb{C}) \\
\big\downarrow & & \big\downarrow & & \big\| & & \\
H^1(C,\mathbb{C}^*) & \longrightarrow & H^1(C,\mathcal{O}_C^*) & \xrightarrow{\mathrm{d}\log} & H^1(C,K_C) & & \\
\big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle c_1} & & \big\downarrow & & \\
H^2(C,\mathbb{Z}) & = & H^2(C,\mathbb{Z}) & \longrightarrow & 0 & &
\end{array}$$

associated with

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \big\downarrow & & \big\downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O}_C & \xrightarrow{\mathrm{d}} & K_C & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathcal{O}_C^* & \xrightarrow{\mathrm{d}\log} & K_C & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

tells us that $\{\sigma_{\alpha\beta}\}$ corresponds to the image of $\{\xi_{ab}\}$ via the map

$$H^1(C,\mathcal{O}_C^*) \xrightarrow{\mathrm{d}\log} H^1(C,K_C).$$

From the exact sequence, we see that if $\mathrm{d}\log\{\xi_{\alpha\beta}\} = 0 \in H^1(C,K_C)$, then it comes from a class in $H^1(C,\mathbb{C}^*)$, which is the moduli space of line bundles with holomorphic connections (see, for example, [9]). Hence the first Chern class of the theta characteristic $c_1\big(K_C^{\frac{1}{2}}\big)$ should be 0. But it is $g-1$, not 0, since $g(C) \ge 2$. ■

Remark 3.7. The same exact sequences in the above proof were used in [9] for constructing Bloch regulators of the algebraic K_2-group. The torsion property of the Steinberg symbol of generators of $K_2(\Sigma)$ and quantizability of a spectral curve Σ was first discussed in [44] for the case of difference equations.

The class $\{\sigma_{\alpha\beta}\}$ of (3.16) gives a natural isomorphism $H^1(C,K_C) \cong \mathbb{C}$. We identify the deformation parameter $\hbar \in \mathbb{C}$ with the cohomology class $\{\hbar\sigma_{\alpha\beta}\} \in H^1(C,K_C) = \mathbb{C}$. Let

$$\mathbf{q} = (q_2,q_3,\dots,q_r) \in B = \bigoplus_{\ell=2}^{r} H^0\big(C,K_C^{\otimes\ell}\big)$$

be an arbitrary point of the Hitchin base B. We trivialize the line bundle $K_C^{\otimes\ell}$ with respect to our projective coordinate chart $C = \bigcup_\alpha U_\alpha$, and write each q_ℓ as $\{(q_\ell)_\alpha\}$ that satisfies the transition relation

$$(q_\ell)_\alpha = (q_\ell)_\beta \xi_{\alpha\beta}^{2\ell}.$$

The transition function of the vector bundle E_0 is given by

$$\xi_{\alpha\beta}^H = \exp(H\log\xi_{\alpha\beta}).$$

Since $X_-\colon E_0 \longrightarrow E_0\otimes K_C$ is a global Higgs field, its local expressions $\{X_-\mathrm{d}x_\alpha\}$ with respect to the projective coordinate system satisfies the transition relation

$$X_-\mathrm{d}x_\alpha = \exp(H\log\xi_{\alpha\beta})X_-\mathrm{d}x_\beta\exp(-H\log\xi_{\alpha\beta}) \tag{3.17}$$

on every $U_\alpha\cap U_\beta$. The same relation holds for the Higgs field $\phi(\mathbf{q})$ as well:

$$\phi_\alpha(\mathbf{q})\mathrm{d}x_\alpha = \exp(H\log\xi_{\alpha\beta})\phi_\beta(\mathbf{q})\mathrm{d}x_\beta\exp(-H\log\xi_{\alpha\beta}). \tag{3.18}$$

We have the following:

Theorem 3.8 (construction of $\mathrm{SL}(r,\mathbb{C})$-opers)**.** *On each $U_\alpha\cap U_\beta$ define a transition function*

$$f_{\alpha\beta}^\hbar := \exp(H\log\xi_{\alpha\beta})\exp\big(\hbar\partial_\beta\log\xi_{\alpha\beta}X_+\big), \tag{3.19}$$

where $\partial_\beta = \frac{\mathrm{d}}{\mathrm{d}x_\beta}$, and $\hbar\partial_\beta\log\xi_{\alpha\beta} \in H^1(C,K_C)$. Then

- *The collection $\big\{f_{\alpha\beta}^\hbar\big\}$ satisfies the cocycle condition*

$$f_{\alpha\beta}^\hbar f_{\beta\gamma}^\hbar = f_{\alpha\gamma}^\hbar,$$

hence it defines a holomorphic bundle on C. It is exactly the filtered extension $E_\hbar$ of Proposition 3.4.

- *The locally defined differential operator*

$$\nabla_\alpha^\hbar(0) := \mathrm{d} + \frac{1}{\hbar} X_- \mathrm{d}x_a$$

for every $\hbar \neq 0$ *forms a global holomorphic connection in* $E_\hbar$*, i.e.,*

$$\frac{1}{\hbar} X_- \mathrm{d}x_\alpha = \frac{1}{\hbar} f_{\alpha\beta}^\hbar X_- \mathrm{d}x_\beta \big(f_{\alpha\beta}^\hbar\big)^{-1} - \mathrm{d}f_{\alpha\beta}^\hbar \cdot \big(f_{\alpha\beta}^\hbar\big)^{-1}. \tag{3.20}$$

- *Every point* $(E_0, \phi(\mathbf{q})) \in \kappa(B) \subset \mathcal{M}_{\mathrm{Dol}}$ *of the Hitchin section* (3.4) *gives rise to a one-parameter family of* $\mathrm{SL}(r,\mathbb{C})$*-opers* $\big(E_\hbar, \nabla^\hbar(\mathbf{q})\big) \in \mathcal{M}_{\mathrm{deR}}$*. In other words, the locally defined differential operator*

$$\nabla_\alpha^\hbar(\mathbf{q}) := \mathrm{d} + \frac{1}{\hbar} \phi_\alpha(\mathbf{q}) \mathrm{d}x_\alpha \tag{3.21}$$

for every $\hbar \neq 0$ *determines a global holomorphic connection*

$$\nabla_\alpha^\hbar(\mathbf{q}) = f_{\alpha\beta}^\hbar \nabla_\beta^\hbar(\mathbf{q}) \big(f_{\alpha\beta}^\hbar\big)^{-1} \tag{3.22}$$

in $E_\hbar$ *satisfying the oper conditions.*

- *Deligne's* $\hbar$*-connection*

$$\big(E_\hbar, \hbar\nabla^\hbar(\mathbf{q})\big) \tag{3.23}$$

interpolates the Higgs pair and the oper, i.e., at $\hbar = 0$*, the family* (3.23) *gives the Higgs pair* $(E, \phi(\mathbf{q})) \in \mathcal{M}_{\mathrm{Dol}}$*, and at* $\hbar = 1$ *it gives an* $\mathrm{SL}(r,\mathbb{C})$*-oper* $\big(E_1, \nabla^1(\mathbf{q})\big) \in \mathcal{M}_{\mathrm{deR}}$*.*

- *After a suitable gauge transformation depending on* $\hbar$*, the* $\hbar \to \infty$ *limit of the oper* $\nabla^\hbar(\mathbf{q})$ *exists and is equal to* $\nabla^{\hbar=1}(0)$*.*

Proof. Recall the Baker–Campbell–Hausdorff formula: Let A, B be elements of a Lie algebra such that $[A, B] = cB$ for a constant $c \in \mathbb{C}$. Then

$$\mathrm{e}^A \mathrm{e}^B \mathrm{e}^{-A} = \mathrm{e}^{B \exp c}. \tag{3.24}$$

From this formula, $\frac{\mathrm{d}x_\beta}{\mathrm{d}x_\alpha} = \xi_{\alpha\beta}^2$, and $[H, X_+] = 2X_+$, we calculate

$$\begin{aligned} f_{\alpha\beta}^\hbar \big(f_{\gamma\beta}^\hbar\big)^{-1} &= \exp(H \log \xi_{\alpha\beta}) \exp(\hbar\partial_\beta \log \xi_{\alpha\beta} X_+) \exp(-\hbar\partial_\beta \log \xi_{\gamma\beta} X_+) \exp(-H \log \xi_{\gamma\beta}) \\ &= \exp(H \log \xi_{\alpha\beta}) \exp(\hbar\partial_\beta \log \xi_{\alpha\gamma} X_+) \exp(-H \log \xi_{\gamma\beta}) \\ &= \exp(H \log \xi_{\alpha\beta}) \exp(\hbar\xi_{\beta\gamma}^2 \partial_\gamma \log \xi_{\alpha\gamma} X_+) \exp(-H \log \xi_{\gamma\beta}) \\ &= \exp(H \log \xi_{\alpha\gamma}) \exp(H \log \xi_{\gamma\beta}) \exp(\hbar\xi_{\beta\gamma}^2 \partial_\gamma \log \xi_{\alpha\gamma} X_+) \exp(-H \log \xi_{\gamma\beta}) \\ &= \exp(H \log \xi_{\alpha\gamma}) \exp(\hbar\xi_{\gamma\beta}^2 \xi_{\beta\gamma}^2 \partial_\gamma \log \xi_{\alpha\gamma} X_+) = f_{\alpha\gamma}^\hbar. \end{aligned}$$

Hence the cocycle condition $f_{\alpha\beta}^\hbar = f_{\alpha\gamma}^\hbar f_{\gamma\beta}^\hbar$ follows. We note that the factor $\exp c$ in (3.24) produces exactly the cocycle $\xi_{\alpha\beta}^2$ corresponding to the canonical sheaf K_C.

To prove (3.20), we use the power series expansion of the adjoint action

$$\mathrm{e}^{\hbar A} B \mathrm{e}^{-\hbar A} = \sum_{n=0}^\infty \frac{1}{n!} \hbar^n (\mathrm{ad}_A)^n (B) := \sum_{n=0}^\infty \frac{1}{n!}\, \hbar^n\, \overbrace{[A, [A, [\cdots, [A}^{n}, B] \cdots]]]. \tag{3.25}$$

We then find

$$
\begin{aligned}
&f^{\hbar}_{\alpha\beta} X_- \big(f^{\hbar}_{\alpha\beta}\big)^{-1}\\
&\qquad = \exp(H\log\xi_{\alpha\beta})\exp(\hbar\partial_\beta\log\xi_{\alpha\beta}X_+)X_-\exp(-\hbar\partial_\beta\log\xi_{\alpha\beta}X_+)\exp(-H\log\xi_{\alpha\beta})\\
&\qquad = \exp(H\log\xi_{\alpha\beta})X_-\exp(-H\log\xi_{\alpha\beta}) + \hbar\partial_\beta\log\xi_{\alpha\beta}H\\
&\qquad\quad - \hbar^2(\partial_\beta\log\xi_{\alpha\beta})^2\exp(H\log\xi_{\alpha\beta})X_+\exp(-H\log\xi_{\alpha\beta}),
\end{aligned}
$$

and

$$
\partial_\beta f^{\hbar}_{\alpha\beta}\big(f^{\hbar}_{\alpha\beta}\big)^{-1} = \partial_\beta\log\xi_{\alpha\beta}H - \hbar(\partial_\beta\log\xi_{\alpha\beta})^2\exp(H\log\xi_{\alpha\beta})X_+\exp(-H\log\xi_{\alpha\beta}).
$$

Here, we have used the formula

$$
\partial_\beta\partial_\beta\log\xi_{\alpha\beta} = \partial_\beta\big(\xi^{-1}_{\alpha\beta}\partial_\beta\xi_{\alpha\beta}\big) = -\xi^{-2}_{\alpha\beta}(\partial_\beta\xi_{\alpha\beta})^2 = -(\partial_\beta\log\xi_{\alpha\beta})^2,
$$

which follows from (3.1). Therefore, noticing (3.17), we obtain

$$
\begin{aligned}
\left(\frac{1}{\hbar}f^{\hbar}_{\alpha\beta}X_-\big(f^{\hbar}_{\alpha\beta}\big)^{-1} - \partial_\beta f^{\hbar}_{\alpha\beta}\big(f^{\hbar}_{\alpha\beta}\big)^{-1}\right)\mathrm{d}x_\beta &= \frac{1}{\hbar}\exp(H\log\xi_{\alpha\beta})X_-\mathrm{d}x_\beta\exp(-H\log\xi_{\alpha\beta})\\
&= \frac{1}{\hbar}X_-\mathrm{d}x_\alpha.
\end{aligned}
$$

To prove (3.22), we need, in addition to (3.20), the following relation:

$$
\sum_{\ell=2}^{r}(q_\ell)_\alpha X_+^{\ell-1}\mathrm{d}x_\alpha = f^{\hbar}_{\alpha\beta}\sum_{\ell=2}^{r}(q_\ell)_\beta X_+^{\ell-1}\mathrm{d}x_\beta\big(f^{\hbar}_{\alpha\beta}\big)^{-1}. \tag{3.26}
$$

But (3.26) is obvious from (3.18) and (3.19).

Noticing that $f^{\hbar}_{\alpha\beta}$ is an upper-triangular matrix, we denote by $\big(f^{\hbar}_{\alpha\beta}\big)_i$ the principal truncation of $f^{\hbar}_{\alpha\beta}$ to the first $(r-i+1)\times(r-i+1)$ upper-left corner. For example,

$$
\begin{aligned}
&\big(f^{\hbar}_{\alpha\beta}\big)_r = \big[\xi^{r-1}_{\alpha\beta}\big],\\
&\big(f^{\hbar}_{\alpha\beta}\big)_{r-1} = \begin{bmatrix}\xi^{r-1}_{\alpha\beta} & \\ & \xi^{r-3}_{\alpha\beta}\end{bmatrix}\begin{bmatrix}1 & \hbar\sqrt{s_1}\xi^{-1}_{\alpha\beta}\sigma_{\alpha\beta}\\ & 1\end{bmatrix} = \begin{bmatrix}\xi^{r-1}_{\alpha\beta} & \hbar\sqrt{s_1}\xi^{r-2}_{\alpha\beta}\sigma_{\alpha\beta}\\ & \xi^{r-3}_{\alpha\beta}\end{bmatrix},\\
&\big(f^{\hbar}_{\alpha\beta}\big)_{r-2} = \begin{bmatrix}\xi^{r-1}_{\alpha\beta} & \hbar\sqrt{s_1}\xi^{r-2}_{\alpha\beta}\sigma_{\alpha\beta} & \frac{1}{2}\hbar^2\sqrt{s_1s_2}\xi^{r-3}_{\alpha\beta}\sigma^2_{\alpha\beta}\\ & \xi^{r-3}_{\alpha\beta} & \hbar\sqrt{s_2}\xi^{r-4}_{\alpha\beta}\sigma_{\alpha\beta}\\ & & \xi^{r-5}_{\alpha\beta}\end{bmatrix}, \qquad \text{etc.}
\end{aligned}
$$

They all satisfy the cocycle condition with respect to the projective structure, and define a sequence of vector bundles $F^i_\hbar$ on C. From the shape of matrices we see that these vector bundles give the filtered extension $(F^\bullet_\hbar, E_\hbar)$ of Proposition 3.4, and hence satisfies the requirement (3.12). Since the connection $\nabla^\hbar(\mathbf{q})$ has non-zero lower-diagonal entries in its matrix representation coming from X_-, $F^i_\hbar$ is not invariant under $\nabla^\hbar(\mathbf{q})$. But since X_- has non-zero entries exactly along the first lower-diagonal, (3.13) holds. The isomorphism (3.14) is a consequence of (3.5), since $s_i = i(r-i)\neq 0$ for $i=1,\dots,r-1$.

Finally, the gauge transformation of $\nabla^\hbar(\mathbf{q})$ by a bundle automorphism

$$
\hbar^{-\frac{H}{2}} = \begin{bmatrix}\hbar^{-\frac{r-1}{2}} & & \\ & \ddots & \\ & & \hbar^{\frac{r-1}{2}}\end{bmatrix} \tag{3.27}
$$

on each coordinate neighborhood U_α gives

$$\mathrm{d} + \frac{1}{\hbar}\phi(\mathbf{q}) \longmapsto \hbar^{-\frac{H}{2}}\left(\mathrm{d} + \frac{1}{\hbar}\phi(\mathbf{q})\right)\hbar^{\frac{H}{2}} = \mathrm{d} + X_- + \sum_{\ell=2}^{r} \frac{q_\ell}{\hbar^\ell} X_+^{\ell-1}. \tag{3.28}$$

This is because

$$\hbar^{-\frac{H}{2}} X_- \hbar^{\frac{H}{2}} = \hbar X_- \qquad \text{and} \qquad \hbar^{-\frac{H}{2}} X_+^\ell \hbar^{\frac{H}{2}} = \hbar^{-\ell} X_+^\ell,$$

which follows from the adjoint formula (3.25). Therefore,

$$\lim_{\hbar\to\infty} \nabla^\hbar(\mathbf{q}) \sim \mathrm{d} + X_- = \nabla^{\hbar=1}(0),$$

where the symbol $\sim$ means gauge equivalence.

This completes the proof of the theorem. ■

Remark 3.9. In the construction theorem, our use of a projective coordinate system is essential, through (3.1). Only in such a coordinate, the definition of the global connection (3.22) makes sense. This is due to the vanishing of the second derivative of $\xi_{\alpha\beta}$.

The above construction theorem yields the following.

Theorem 3.10 (biholomorphic quantization of Hitchin spectral curves)**.** *Let C be a compact Riemann surface of genus $g \geq 2$ with a chosen projective coordinate system subordinating its complex structure. We denote by $\mathcal{M}_{\mathrm{Dol}}$ the moduli space of stable holomorphic $\mathrm{SL}(r,\mathbb{C})$-Higgs bundles over C, and by $\mathcal{M}_{\mathrm{deR}}$ the moduli space of irreducible holomorphic $\mathrm{SL}(r,\mathbb{C})$-connections on C. For a fixed theta characteristic $K_C^{\frac{1}{2}}$, we have a Hitchin section $\kappa(B) \subset \mathcal{M}_{\mathrm{Dol}}$ of (3.4). We denote by $Op \subset \mathcal{M}_{\mathrm{deR}}$ the moduli space of $\mathrm{SL}(r,\mathbb{C})$-opers with the condition that the second term of the filtration is given by $F^{r-1} = K_C^{\frac{r-1}{2}}$. Then the map*

$$\mathcal{M}_{\mathrm{Dol}} \supset \kappa(B) \ni (E_0, \phi(\mathbf{q})) \overset{\gamma}{\longmapsto} \left(E_\hbar, \nabla^\hbar(\mathbf{q})\right) \in Op \subset \mathcal{M}_{\mathrm{deR}} \tag{3.29}$$

evaluated at $\hbar = 1$ is a biholomorphic map with respect to the natural complex structures induced from the ambient spaces.

The biholomorphic quantization (3.29) is also $\mathbb{C}^*$-equivariant. The $\lambda \in \mathbb{C}^*$ action on the Hitchin section is defined by $\phi \longmapsto \lambda\phi$. The oper corresponding to $(E_0, \lambda\phi(\mathbf{q})) \in \kappa(B)$ is $\mathrm{d} + \frac{\lambda}{\hbar}\phi(\mathbf{q})$.

Proof. The $\mathbb{C}^*$-equivariance follows from the same argument of the gauge transformation (3.27), (3.28). Since the Hitchin section κ is not the section of the Hitchin fibration μ_H, we need the gauge transformation. The action $\phi \longmapsto \lambda\phi$ on the Hitchin section induces a weighted action

$$B \ni (q_2, q_3, \dots, q_r) \longmapsto \left(\lambda^2 q_2, \lambda^3 q_3, \dots, \lambda^r q_r\right) \in B$$

through μ_H. Then we have the gauge equivalence via the gauge transformation $\left(\frac{\lambda}{\hbar}\right)^{\frac{H}{2}}$:

$$\mathrm{d} + \frac{\lambda}{\hbar}\phi(\mathbf{q}) \sim \left(\frac{\lambda}{\hbar}\right)^{\frac{H}{2}}\left(\mathrm{d} + \frac{\lambda}{\hbar}\phi(\mathbf{q})\right)\left(\frac{\lambda}{\hbar}\right)^{-\frac{H}{2}} = \mathrm{d} + X_- + \sum_{\ell=2}^{r} \frac{\lambda^\ell q_\ell}{\hbar^\ell} X_+^{\ell-1}.$$ ■

3.2 Semi-classical limit of $\mathrm{SL}(r,\mathbb{C})$-opers

A holomorphic connection on a compact Riemann surface C is automatically flat. Therefore, it defines a $\mathcal{D}$-module over C. Continuing the last subsection's conventions, let us fix a projective coordinate system on C, and let $(E_0,\phi(\mathbf{q}))=\kappa(\mathbf{q})$ be a point on the Hitchin section of (3.4). It uniquely defines an $\hbar$-family of opers $\big(E_\hbar,\nabla^\hbar(\mathbf{q})\big)$.

In this subsection, we establish that the $\hbar$-connection $\hbar\nabla^\hbar(\mathbf{q})$ defines a family of Rees $\mathcal{D}$-modules on C parametrized by B such that the semi-classical limit of the family agrees with the family of spectral curves (1.5) over B.

To calculate the semi-classical limit, let us trivialize the vector bundle $E_\hbar$ on each simply connected coordinate neighborhood U_α with coordinate x_α of the chosen projective coordinate system. A flat section Ψ_α of $E_\hbar$ over U_α is a solution of

$$\hbar\nabla^\hbar_\alpha(\mathbf{q})\Psi_\alpha:=(\hbar\mathrm{d}+\phi_\alpha(\mathbf{q}))\begin{bmatrix}\psi_{r-1}\\ \psi_{r-2}\\ \vdots\\ \psi_1\\ \psi\end{bmatrix}_\alpha=0, \tag{3.30}$$

with an appropriate unknown function ψ. Since $\Psi_\alpha=f^\hbar_{\alpha\beta}\Psi_\beta$, the function ψ on U_α satisfies the transition relation $(\psi)_\alpha=\xi_{\alpha\beta}^{-r+1}(\psi)_\beta$. It means that ψ is actually a local section of the line bundle $K_C^{-\frac{r-1}{2}}$. There are r linearly independent solutions of (3.30), because $q_2,\dots,q_r$ are represented by holomorphic functions on U_α. The entries of X_+^ℓ are given by the formula

$$X_+^\ell=\big[s_{ij}^{(\ell)}\big],\qquad s_{ij}^{(\ell)}=\delta_{i+\ell,j}\sqrt{s_is_{i+1}\cdots s_{i+\ell-1}}.$$

Therefore, (3.30) is equivalent to

$$\begin{aligned}0&=\sqrt{s_{r-k-1}}\psi_{k+1}+\hbar\psi_k'+\sqrt{s_{r-k}}q_2\psi_{k-1}+\sqrt{s_{r-k}s_{r-k+1}}q_3\psi_{k-2}+\cdots\\&\quad+\sqrt{s_{r-k}s_{r-k+1}\cdots s_{r-1}}q_{k+1}\psi\\&=\sqrt{s_{k+1}}\psi_{k+1}+\hbar\psi_k'+\sqrt{s_k}q_2\psi_{k-1}+\sqrt{s_ks_{k-1}}q_3\psi_{k-2}+\cdots+\sqrt{s_ks_{k-1}\cdots s_1}q_{k+1}\psi\end{aligned} \tag{3.31}$$

for $k=0,1,\dots,r-1$, where we use $s_k=s_{r-k}$. Note that the differentiation is always multiplied by $\hbar$ as $\hbar\mathrm{d}$ in (3.30), and that $\phi(\mathbf{q})$ is independent of $\hbar$ and takes the form

$$\phi(\mathbf{q})=\begin{bmatrix}0&\sqrt{s_{r-1}}q_2&\sqrt{s_{r-2}s_{r-1}}q_3&\cdots&\cdots&\sqrt{s_2\cdots s_{r-1}}q_{r-1}&\sqrt{s_1s_2\cdots s_{r-1}}q_r\\ \sqrt{s_{r-1}}&0&\sqrt{s_{r-2}}q_2&\cdots&\cdots&\sqrt{s_2\cdots s_{r-2}}q_{r-2}&\sqrt{s_1\cdots s_{r-2}}q_{r-1}\\ &\sqrt{s_{r-2}}&0&\ddots&\cdots&\sqrt{s_2\cdots s_{r-3}}q_{r-3}&\sqrt{s_1\cdots s_{r-3}}q_{r-2}\\ &&\ddots&\ddots&\ddots&\vdots&\vdots\\ &&&\sqrt{s_3}&0&\sqrt{s_2}q_2&\sqrt{s_1s_2}q_3\\ &&&&\sqrt{s_2}&0&\sqrt{s_1}q_2\\ &&&&&\sqrt{s_1}&0\end{bmatrix}.$$

By solving (3.31) for $k=0,1,\dots,r-2$ recursively, we obtain an expression of ψ_k as a linear combination of

$$\psi=\psi_0,\qquad\hbar\psi'=\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\psi,\qquad\dots,\qquad\hbar^k\psi^{(k)}=\hbar^k\frac{\mathrm{d}^k}{\mathrm{d}x_\alpha^k}\psi,$$

with coefficients in differential polynomials of $q_2, q_3, \dots, q_k$. Moreover, the coefficients of these differential polynomials are in $\overline{\mathbb{Q}}[\hbar]$. For example,

$$\begin{aligned}
\psi_1 &= -\frac{1}{\sqrt{s_1}}\hbar\psi', \\
\psi_2 &= \frac{1}{\sqrt{s_1 s_2}}(\hbar^2\psi'' - s_1 q_2\psi), \\
\psi_3 &= \frac{1}{\sqrt{s_1 s_2 s_3}}(-\hbar^3\psi''' + \hbar(s_1+s_2)q_2\psi' + (\hbar s_1 q_2' - s_1 s_2 q_3)\psi), \\
\psi_4 &= \frac{1}{\sqrt{s_1 s_2 s_3 s_4}}\big(\hbar^4\psi'''' - \hbar^2(s_1+s_2+s_3)q_2\psi'' + (-\hbar^2(2s_1+s_2)q_2' + \hbar(s_1s_2+s_2s_3)q_3)\psi' \\
&\quad + (\hbar^2 s_1 q_2'' - \hbar s_1 s_2 q_3' + s_1 s_3 q_2^2 - s_1 s_2 s_3 q_4)\psi\big), \quad \text{etc.}
\end{aligned}$$

Since ψ_1 is proportional to ψ', inductively we can show that the linear combination expression of ψ_k by derivatives of ψ does not contain the $(k-1)$-th order differentiation of ψ. Equation (3.31) for $k = r-1$ is a differential equation

$$\hbar\psi_{r-1}' + \sqrt{s_1}q_2\psi_{r-2} + \sqrt{s_1 s_2}q_3\psi_{r-3} + \dots + \sqrt{s_1 s_2 \cdots s_{r-1}}q_r\psi = 0, \tag{3.32}$$

which is an order r differential equation for $\psi \in K_C^{-\frac{r-1}{2}}$. Its actual shape can be calculated using the procedure of the proof of the next theorem. The equation becomes (3.40). Since we are using a fixed projective coordinate system, the connection $\nabla^\hbar(\mathbf{q})$ takes the same form on each coordinate neighborhood U_α. Therefore, the shape of the differential equation of (3.32) as an equation for $\psi \in K_C^{-\frac{r-1}{2}}$ is again the same on every coordinate neighborhood.

We are now ready to calculate the semi-classical limit of the Rees $\mathcal{D}$-module corresponding to $\hbar\nabla^\hbar(\mathbf{q})$. The following is our main theorem of the paper.

Theorem 3.11 (semi-classical limit of an oper)**.** *Under the same setting of Theorem 3.10, let $\mathcal{E}(\mathbf{q})$ denote the Rees $\mathcal{D}$-module $\big(E_\hbar, \hbar\nabla^\hbar(\mathbf{q})\big)$ associated with the oper of (3.29). Then the semi-classical limit of $\mathcal{E}(\mathbf{q})$ is the spectral curve $\sigma^*\Sigma \subset T^*C$ of $-\phi(\mathbf{q})$ defined by the characteristic equation $\det(\eta + \phi(\mathbf{q})) = 0$, where σ is the involution of (1.7).*

Remark 3.12. The actual shape of the single scalar valued differential equation (3.32) of order r is given by (3.40) below. This is exactly what we usually refer to as the *quantum curve* of the spectral curve $\det(\eta + \phi(\mathbf{q})) = 0$.

Proof. Since the connection and the differential equation (3.32) are globally defined, we need to analyze the semi-classical limit only at each coordinate neighborhood U_α with a projective coordinate x_α. Let

$$\Psi_0 = \begin{bmatrix} \hbar^{r-1}\psi^{(r-1)} \\ \vdots \\ \hbar\psi' \\ \psi \end{bmatrix}.$$

From the choice of local trivialization (3.30) we have an expression

$$\Psi = \big(\Delta + A(\mathbf{q},\hbar)\big)\Psi_0, \tag{3.33}$$

where Δ is a constant diagonal matrix

$$\Delta = \left[\delta_{ij}(-1)^{r-i}\frac{1}{\sqrt{s_{r-1}s_{r-2}\cdots s_i}}\right]_{i,j=1,\dots,r} \tag{3.34}$$

with the understanding that its (r,r)-entry is 1, and $A(\mathbf{q},\hbar) = \big[a(\mathbf{q},\hbar)_{i,j}\big]$ satisfies the following properties:

Condition 3.13.

- $A(\mathbf{q},\hbar)$ is a nilpotent upper triangular matrix.
- Each entry $a(\mathbf{q},\hbar)_{i,j}$ is a differential polynomial in $q_2,\dots,q_r$ with coefficients in $\overline{\mathbb{Q}}[\hbar]$.
- All diagonal and the first upper diagonal entries of $A(\mathbf{q},\hbar)$ are 0:

$$a(\mathbf{q},\hbar)_{i,i} = a(\mathbf{q},\hbar)_{i,i+1} = 0, \qquad i = 1,\dots,r.$$

The last property is because ψ_k does not contain the $(k-1)$-th derivative of ψ, as we have noted above. The diagonal matrix Δ is designed so that

$$\Delta^{-1}X_-\Delta = \begin{bmatrix} 0 & & & & \\ -1 & 0 & & & \\ & -1 & 0 & & \\ & & \ddots & & \\ & & & -1 & 0 \end{bmatrix} =: -X.$$

Let us calculate the gauge transformation of $\hbar\nabla^\hbar(\mathbf{q})$ by $\Delta + A(\mathbf{q},\hbar)$:

$$(\Delta + A(\mathbf{q},\hbar))^{-1}(\hbar d + \phi(\mathbf{q}))(\Delta + A(\mathbf{q},\hbar)) = (\hbar d - X\mathrm{d}x_\alpha - \omega(\mathbf{q},\hbar)\mathrm{d}x_\alpha). \tag{3.35}$$

We wish to determine the matrix $\omega(\mathbf{q},\hbar)$.

Lemma 3.14. *The matrix $\omega(\mathbf{q},\hbar)$ of (3.35) consists of only the first row, and the matrix $X+\omega(\mathbf{q},\hbar)$ takes the following canonical form*

$$X+\omega(\mathbf{q},\hbar) = \begin{bmatrix} 0 & \omega_2(\mathbf{q},\hbar) & \cdots & \omega_{r-1}(\mathbf{q},\hbar) & \omega_r(\mathbf{q},\hbar) \\ 1 & 0 & \cdots & 0 & 0 \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{bmatrix}. \tag{3.36}$$

Proof of Lemma. First, define

$$B(\mathbf{q},\hbar) := \sum_{n=1}^{r-1}(-1)^n\big(\Delta^{-1}A(\mathbf{q},\hbar)\big)^n \tag{3.37}$$

so that

$$I + B(\mathbf{q},\hbar) = (I+\Delta^{-1}A(\mathbf{q},\hbar))^{-1}.$$

By definition, $B(\mathbf{q},\hbar)$ satisfies the exact same properties listed in Condition 3.13 above. We then have

$$\begin{aligned}
&(\Delta + A(\mathbf{q},\hbar))^{-1}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} + \phi(\mathbf{q})\right)(\Delta + A(\mathbf{q},\hbar))\\
&\quad = (I+\Delta^{-1}A(\mathbf{q},\hbar))^{-1}\Delta^{-1}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} + X_- + \sum_{\ell=2}^r q_\ell X_+^{\ell-1}\right)\Delta(I+\Delta^{-1}A(\mathbf{q},\hbar))\\
&\quad = (I+B(\mathbf{q},\hbar))\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} - X + \Delta^{-1}\sum_{\ell=2}^r q_\ell X_+^{\ell-1}\Delta\right)(I+\Delta^{-1}A(\mathbf{q},\hbar))\\
&\quad = \hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} - X - B(\mathbf{q},\hbar)X - X\Delta^{-1}A(\mathbf{q},\hbar) + \hbar(I+B(\mathbf{q},\hbar))\Delta^{-1}\frac{\mathrm{d}A(\mathbf{q},\hbar)}{\mathrm{d}x_\alpha}\\
&\qquad - B(\mathbf{q},\hbar)X\Delta^{-1}A(\mathbf{q},\hbar) + (I+B(\mathbf{q},\hbar))\Delta^{-1}\left(\sum_{\ell=2}^r q_\ell X_+^{\ell-1}\Delta\right)(I+\Delta^{-1}A(\mathbf{q},\hbar)).
\end{aligned}$$

Thus we define

$$\omega(\mathbf{q},\hbar) = B(\mathbf{q},\hbar)X + X\Delta^{-1}A(\mathbf{q},\hbar) - \hbar(I + B(\mathbf{q},\hbar))\Delta^{-1}\frac{\mathrm{d}A(\mathbf{q},\hbar)}{\mathrm{d}x_\alpha}$$
$$+ B(\mathbf{q},\hbar)X\Delta^{-1}A(\mathbf{q},\hbar) - (\Delta + A(\mathbf{q},\hbar))^{-1}\left(\sum_{\ell=2}^{r} q_\ell X_+^{\ell-1}\right)(\Delta + A(\mathbf{q},\hbar)). \tag{3.38}$$

Every single matrix in the right-hand side of (3.38) is either diagonal or upper triangular and nilpotent, except for X. Because of Condition 3.13, $B(\mathbf{q},\hbar)X + X\Delta^{-1}A(\mathbf{q},\hbar)$ is an upper triangular matrix with 0 along the diagonal. Obviously, so are all other terms of (3.38). Thus $\omega(\mathbf{q},\hbar)$ is upper triangular and nilpotent, and is a polynomial in $\hbar$.

Now we note that (3.30) and (3.33) yield

$$\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} - X - \omega(\mathbf{q},\hbar)\right)\Psi_0 = 0. \tag{3.39}$$

The basis vector Ψ_0 defined on a simply connected coordinate neighborhood U_α is designed so that the following equation holds for any solution ψ of (3.30):

$$\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} - X\right)\Psi_0 = \left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} - X\right)\begin{bmatrix}\hbar^{r-1}\psi^{(r-1)}\\ \hbar^{r-2}\psi^{(r-2)}\\ \vdots\\ \hbar\psi'\\ \psi\end{bmatrix} = \begin{bmatrix}\hbar^r\psi^{(r)}\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}.$$

Let $\psi_{[1]},\dots,\psi_{[r]}$ be r linearly independent solutions of (3.30). The *Wronskian matrix* is defined by

$$\mathbf{W} = \begin{bmatrix}\hbar^{r-1}\psi_{[1]}^{(r-1)} & \hbar^{r-1}\psi_{[2]}^{(r-1)} & \cdots & \hbar^{r-1}\psi_{[r]}^{(r-1)}\\ \hbar^{r-2}\psi_{[1]}^{(r-2)} & \hbar^{r-2}\psi_{[2]}^{(r-2)} & \cdots & \hbar^{r-2}\psi_{[r]}^{(r-2)}\\ \vdots & \vdots & \ddots & \vdots\\ \hbar\psi'_{[1]} & \hbar\psi'_{[2]} & \cdots & \hbar\psi'_{[r]}\\ \psi_{[1]} & \psi_{[2]} & \cdots & \psi_{[r]}\end{bmatrix}.$$

Then from (3.39), we have

$$\omega(\mathbf{q},\hbar) = \left[\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha} - X\right)\mathbf{W}\right]\cdot\mathbf{W}^{-1} = \begin{bmatrix}\hbar^r\psi_{[1]}^{(r)} & \hbar^r\psi_{[2]}^{(r)} & \cdots & \hbar^r\psi_{[r]}^{(r)}\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\mathbf{W}^{-1}.$$

Clearly we see that $\omega(\mathbf{q},\hbar)$ has non-zero entries only in the first row. The $(1,1)$-entry is also 0 because $\operatorname{tr}\omega(\mathbf{q},\hbar) = 0$. ∎

Thus (3.30) is equivalent to a single linear ordinary differential equation of order r

$$P_\alpha(x_\alpha,\hbar;\mathbf{q})\psi := \left[\hbar^r\left(\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^r - \sum_{i=2}^{r}\hbar^{r-i}\omega_i(\mathbf{q},\hbar)\left(\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^{r-i}\right]\psi = 0. \tag{3.40}$$

In other words, $P_\alpha(x_\alpha,\hbar;\mathbf{q})$ is a generator of the Rees $\mathcal{D}$-module $\left(E_\hbar,\hbar\nabla^\hbar(\mathbf{q})\right)$ on U_α. This expression of differential equation of order r is what is commonly referred to as a *quantum* *curve*. The coefficients $\omega_i(\mathbf{q},\hbar)$ are calculated by determining the matrix $A(\mathbf{q},\hbar)$ of (3.33). Then (3.34), (3.37), and (3.38) give the exact form of $P_\alpha(x_\alpha,\hbar;\mathbf{q})$. As we can see, its shape is quite involved, and is *not* obtained by simply replacing y in $\det\big(y+\phi(\mathbf{q})\big)$ by $\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}$. In particular, the coefficients contain derivatives of q_ℓ's, which never appear in the characteristic polynomial of $-\phi(\mathbf{q})$. What we are going to prove now is that nonetheless, the semi-classical limit of $P_\alpha(x_\alpha,\hbar;\mathbf{q})$ is exactly the characteristic polynomial.

As defined in Definition 2.8, the semi-classical limit of (3.40) is the limit

$$\lim_{\hbar\to 0}\mathrm{e}^{-\frac{1}{\hbar}S_0(x_\alpha)}\left[\hbar^r\left(\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^r-\sum_{i=2}^r\hbar^{r-i}\omega_i(\mathbf{q},\hbar)\left(\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^{r-i}\right]\mathrm{e}^{\frac{1}{\hbar}S_0(x_a)}y^r-\sum_{i=2}^r\omega_i(\mathbf{q},0)y^{r-i}, \tag{3.41}$$

where $S_0(x_\alpha)$ is a holomorphic function on U_α so that $\mathrm{d}S_0=y\mathrm{d}x_\alpha$ gives a local trivialization of T^*C over U_α. Since $\omega(\mathbf{q},\hbar)$ is a polynomial in $\hbar$, we can evaluate it at $\hbar=0$. Notice that (3.41) is the characteristic polynomial of the matrix $X+\omega(\mathbf{q},\hbar)$ of (3.36) at $\hbar=0$. The computation of semi-classical limit is the same as the calculation of the determinant of the connection $\hbar\nabla^\hbar(\mathbf{q})$, after taking conjugation by the scalar diagonal matrix $\mathrm{e}^{-\frac{1}{\hbar}S_0(x_\alpha)}I_{r\times r}$, and then take the limit as $\hbar$ goes to 0:

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{\hbar}S_0(x_\alpha)}I\cdot(\Delta+A(\mathbf{q},\hbar))^{-1}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}+\phi(\mathbf{q})\right)(\Delta+A(\mathbf{q},\hbar))\cdot\mathrm{e}^{\frac{1}{\hbar}S_0(x_\alpha)}I\\
&\qquad=(\Delta+A(\mathbf{q},\hbar))^{-1}\mathrm{e}^{-\frac{1}{\hbar}S_0(x_\alpha)}I\cdot\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}+\phi(\mathbf{q})\right)\cdot\mathrm{e}^{\frac{1}{\hbar}S_0(x_\alpha)}I(\Delta+A(\mathbf{q},\hbar))\\
&\qquad=(\Delta+A(\mathbf{q},\hbar))^{-1}\left(\frac{\mathrm{d}S_0(x_\alpha)}{\mathrm{d}x_\alpha}+\phi(\mathbf{q})\right)(\Delta+A(\mathbf{q},\hbar))+O(\hbar)\\
&\qquad\overset{\hbar\to 0}{\longrightarrow}(\Delta+A(\mathbf{q},0))^{-1}(y+\phi(\mathbf{q}))(\Delta+A(\mathbf{q},0)).
\end{aligned}$$

The determinant of the above matrix is the characteristic polynomial $\det\,(y+\phi(\mathbf{q}))$. Note that from (3.35), we have

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{\hbar}S_0(x_\alpha)}I\cdot(\Delta+A(\mathbf{q},\hbar))^{-1}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}+\phi(\mathbf{q})\right)(\Delta+A(\mathbf{q},\hbar))\cdot\mathrm{e}^{\frac{1}{\hbar}S_0(x_\alpha)}I\\
&\qquad=\mathrm{e}^{-\frac{1}{\hbar}S_0(x_\alpha)}I\cdot\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}-(X+\omega(\mathbf{q},\hbar))\right)\cdot\mathrm{e}^{\frac{1}{\hbar}S_0(x_\alpha)}I\\
&\qquad=\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}+y-(X+\omega(\mathbf{q},\hbar))\overset{\hbar\to 0}{\longrightarrow}y-(X+\omega(\mathbf{q},0)).
\end{aligned}$$

Taking the determinant of the above, we conclude that

$$\det\,(y+\phi(\mathbf{q}))=y^r-\sum_{i=2}^r\omega_i(\mathbf{q},0)\,y^{r-i}.$$

This completes the proof of Theorem 3.11. ∎

For every $\hbar\in H^1(C,K_C)$, the $\hbar$-connection $\big(E_\hbar,\hbar\nabla^\hbar(\mathbf{q})\big)$ of (3.23) defines a global Rees $\mathcal{D}_C$-module structure in $E_\hbar$. Thus we have constructed a universal family $\mathcal{E}_C$ of Rees $\mathcal{D}_C$-modules on a given C with a fixed spin and a projective structures:

$$\begin{array}{ccc}
\mathcal{E}_C & \overset{\supset}{\longleftarrow} & \big(E_\hbar,\nabla^\hbar(\mathbf{q})\big)\\
\big\downarrow & & \big\downarrow\\
C\times B\times H^1(C,K_C) & \overset{\supset}{\longleftarrow} & C\times\{\mathbf{q}\}\times\{\hbar\}.
\end{array}$$

The universal family $\mathcal{S}_C$ of spectral curves is defined over $C \times B$.

$$\begin{array}{ccccc} \mathbb{P}(K_C \oplus \mathcal{O}_C) \times B & \overset{\supset}{\longleftarrow} & \mathcal{S}_C & \overset{\supset}{\longleftarrow} & \big(\det\big(\eta - \phi(\mathbf{q})\big)\big)_0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ C \times B & = & C \times B & \overset{\supset}{\longleftarrow} & C \times \{\mathbf{q}\}. \end{array}$$

The semi-classical limit is thus a map of families

$$\begin{array}{ccc} \mathcal{E}_C & \longrightarrow & \mathcal{S}_C \\ \big\downarrow & & \big\downarrow \\ C \times B \times H^1(C, K_C) & \longrightarrow & C \times B. \end{array} \tag{3.42}$$

Our concrete construction (3.21) using the projective coordinate system is not restricted to the holomorphic Higgs field $\phi(\mathbf{q})$. Since the moduli spaces appearing in the correspondence (3.29) become more subtle to define due to the *wildness* of connections, we avoid the moduli problem, and state our quantization theorem as a generalization of Theorem 3.8.

Let C be a smooth projective algebraic curve of an arbitrary genus, and $C = \cup_\alpha U_\alpha$ a projective coordinate system subordinating the complex structure of C.

Theorem 3.15 (quantization of meromorphic data)**.** *Let D be an effective divisor of C, and*

$$\mathbf{q} \in B(D) := \bigoplus_{\ell=2}^{r} H^0\big(C, K_C(D)^{\otimes \ell}\big).$$

We define a meromorphic Higgs bundle $\big(E_0, \phi(\mathbf{q})\big)$ by the same formulae (3.2) and (3.3), as well as a meromorphic oper $\big(E_\hbar, \nabla^\hbar(\mathbf{q})\big)$ by (3.21). The meromorphic oper defines a meromorphic Rees $\mathcal{D}$-module

$$\mathcal{E}(\mathbf{q}) = \big(E_\hbar, \hbar\nabla^\hbar(\mathbf{q})\big)$$

on C. Then the semi-classical limit of $\mathcal{E}(\mathbf{q})$ is the spectral curve

$$\big(\det(\eta + \phi(\mathbf{q})\big)_0 \subset \overline{T^*(C)} \tag{3.43}$$

of $-\phi(\mathbf{q})$.

Proof. The proof is exactly the same as Theorem 3.11 on each simply connected coordinate neighborhood $U_\alpha \subset C \setminus \operatorname{supp}(D)$. Thus the semi-classical limit of $\mathcal{E}(\mathbf{q})|_{C \setminus \operatorname{supp}(D)}$ as a divisor is defined in $\overline{T^*(C)} \setminus \pi^{-1}(D)$. The spectral curve (3.43) is its closure in $\overline{T^*C}$ with respect to the complex topology. Thus by Definition 2.11, (3.43) is the semi-classical limit of the meromorphic extension $\mathcal{E}(\mathbf{q})$. ■

Example 3.1. Here we list characteristic polynomials and differential operators $P_\alpha(x_\alpha, \hbar; \mathbf{q})$ of (3.40) for $r = 2, 3, 4$. These formulas show that our quantization procedure is quite non-trivial.

- $r = 2$:

$$\det(y + \phi(\mathbf{q})) = y^2 - q_2,$$

$$P_\alpha(x_\alpha, \hbar; \mathbf{q}) = \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^2 - q_2.$$

- $r = 3$:

$$\det(y + \phi(\mathbf{q})) = y^3 - 4q_2 y + 4q_3,$$

$$P_\alpha(x_\alpha, \hbar; \mathbf{q}) = \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^3 - 4q_2 \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right) + 4q_3 - 2\hbar q_2'.$$

- $r = 4$:

$$\det(y + \phi(\mathbf{q})) = y^4 - 10q_2 + 24q_3 y - 36q_4 + 9q_2^2,$$

$$P_\alpha(x_\alpha, \hbar; \mathbf{q}) = \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^4 - 10q_2 \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^2 + (24q_3 - 10\hbar q_2')\left(\hbar \frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right) - 36q_4 + 9q_2^2 + 3\hbar^2 q_2'' - 12\hbar q_3'.$$

3.3 Non-Abelian Hodge correspondence and a conjecture of Gaiotto

The biholomorphic map (3.29) is defined by fixing a projective structure of the base curve C. Gaiotto [41] conjectured that such a correspondence would be canonically constructed through a *scaling limit* of non-Abelian Hodge correspondence. The conjecture has been solved in [26] for Hitchin moduli spaces $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{deR}}$ constructed over an arbitrary complex simple and simply connected Lie group G. In this subsection, we review the main result of [26] for $G = \mathrm{SL}(r, \mathbb{C})$ and compare it with our quantization.

The setting of this subsection is the following. The base curve C is a compact Riemann surface of genus $g \geq 2$. We denote by E^{top} the topologically trivial complex vector bundle of rank r on C. The prototype of the correspondence between stable holomorphic vector bundles on C and differential geometric data goes back to Narasimhan–Seshadri [69] (see also [4, 68]). Extending the classical case, the stability condition for an $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle (E, ϕ) translates into a differential geometric condition, known as *Hitchin's equations*, imposed on a set of geometric data as follows [24, 46, 73]. The data consist of a Hermitian fiber metric h on E^{top}, a unitary connection ∇ in E^{top} with respect to h, and a differentiable $\mathfrak{sl}(r, \mathbb{C})$-valued 1-form ϕ on C. The following system of nonlinear equations is called Hitchin's equations:

$$\begin{cases} F_\nabla + \left[\phi, \phi^\dagger\right] = 0, \\ \nabla^{0.1}\phi = 0. \end{cases} \tag{3.44}$$

Here, F_∇ is the curvature of ∇, $\phi^\dagger$ is the Hermitian conjugate of ϕ with respect to h, and $\nabla^{0,1}$ is the Cauchy–Riemann part of ∇. $\nabla^{0,1}$ gives rise to a natural complex structure in E^{top}, which we simply denote by E. Then ϕ becomes a holomorphic Higgs field in E because of the second equation of (3.44). The stability condition for the Higgs pair (E, ϕ) is equivalent to Hitchin's equations (3.44). Define a one-parameter family of connections

$$\nabla(\zeta) := \frac{1}{\zeta} \cdot \phi + \nabla + \zeta \cdot \phi^\dagger, \qquad \zeta \in \mathbb{C}^*. \tag{3.45}$$

Then the flatness of $\nabla(\zeta)$ for all ζ is equivalent to (3.44).

The *non-Abelian Hodge correspondence* [24, 46, 62, 73] (see also [8, 10, 77, 78]) is the following diffeomorphic correspondence

$$\nu\colon\ \mathcal{M}_{\mathrm{Dol}} \ni (E, \phi) \longmapsto \big(\widetilde{E}, \widetilde{\nabla}\big) \in \mathcal{M}_{\mathrm{deR}}.$$

First, we construct the solution (∇, ϕ, h) of Hitchin's equations corresponding to stable (E, ϕ). It induces a family of flat connections $\nabla(\zeta)$. Then define a complex structure $\widetilde{E}$ in E^{top}

by $\nabla(\zeta=1)^{0,1}$. Since $\nabla(\zeta)$ is flat, $\widetilde{\nabla} := \nabla(\zeta=1)^{1,0}$ is automatically a holomorphic connection in $\widetilde{E}$. Thus $(\widetilde{E},\widetilde{\nabla})$ becomes a holomorphic connection. Stability of (E,ϕ) implies that the resulting connection is irreducible, hence $(\widetilde{E},\widetilde{\nabla}) \in \mathcal{M}_{\mathrm{deR}}$. Since this correspondence goes through the real unitary connection ∇, the assignment $E \longmapsto \widetilde{E}$ is not a holomorphic deformation of vector bundles.

Extending the idea of the one-parameter family (3.45), Gaiotto conjectured the following:

Conjecture 3.16 (Gaiotto [41])**.** Let (∇,ϕ,h) be the solution of (3.44) corresponding to a stable Higgs bundle $(E_0,\phi(\mathbf{q}))$ on the $\mathrm{SL}(r,\mathbb{C})$-Hitchin section (3.4). Consider the following two-parameter family of connections

$$\nabla(\zeta,R) := \frac{1}{\zeta}\cdot R\phi + \nabla + \zeta\cdot R\phi^\dagger, \qquad \zeta\in\mathbb{C}^*, \qquad R\in\mathbb{R}_+.$$

Then the scaling limit

$$\lim_{\substack{R\to 0,\,\zeta\to 0\\ \zeta/R=\hbar}} \nabla(\zeta,R)$$

exists for every $\hbar\in\mathbb{C}^*$, and forms an $\hbar$-family of $\mathrm{SL}(r,\mathbb{C})$-opers.

Remark 3.17. The existence of the limit is non-trivial, because the Hermitian metric h blows up as $R\to 0$.

Remark 3.18. Unlike the case of non-Abelian Hodge correspondence, the Gaiotto limit works only for a point in the Hitchin section.

Theorem 3.19 ([26])**.** *Gaiotto's conjecture holds for an arbitrary simple and simply connected complex algebraic group G.*

The universal covering of a compact Riemann surface C is the upper-half plane $\mathbb{H}$. The fundamental group $\pi_1(C)$ acts on $\mathbb{H}$ through a representation

$$\rho\colon\ \pi_1(C) \longrightarrow \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Aut}(\mathbb{H}),$$

and generates an analytic isomorphism

$$C \cong \mathbb{H}/\rho\big(\pi_1(C)\big).$$

The representation ρ lifts to $\mathrm{SL}(2,\mathbb{R})\subset\mathrm{SL}(2,\mathbb{C})$, and induces a projective structure in C subordinating its complex structure coming from $\mathbb{H}$. This projective structure is what we call the *Fuchsian* projective structure.

Corollary 3.20 (Gaiotto correspondence and quantization)**.** *Under the same setting of Conjecture 3.16, the limit oper of [26] is given by*

$$\lim_{\substack{R\to 0,\,\zeta\to 0\\ \zeta/R=\hbar}} \nabla(\zeta,R) = d + \frac{1}{\hbar}\phi(\mathbf{q}) = \nabla^\hbar(\mathbf{q}), \qquad \hbar\neq 0, \tag{3.46}$$

with respect to the Fuchsian projective coordinate system. The operator (3.46) is a connection in the $\hbar$-filtered extension $(F_\hbar^\bullet, E_\hbar)$ of Definition 3.2. In particular, the correspondence

$$(E_0,\phi(\mathbf{q})) \overset{\gamma}{\longmapsto} \big(E_\hbar, \nabla^\hbar(\mathbf{q})\big)$$

is biholomorphic, unlike the non-Abelian Hodge correspondence.

Proof. The key point is that since E_0 is made out of K_C, the fiber metric h naturally comes from the metric of C itself. Hitchin's equations (3.44) for $\mathbf{q}=0$ then become a harmonic equation for the metric of C, and its solution is given by the constant curvature hyperbolic metric. This metric in turn defines the Fuchsian projective structure in C. For more detail, we reefer to [25, 26]. ∎

4 Geometry of singular spectral curves

The quantization mechanism of Section 3 applies to all Hitchin spectral curves, and it is not sensitive to whether the spectral curve is smooth or not. However, the local quantization mechanism of [27, 30] using the PDE version of topological recursion requires a *non-singular* spectral curve. The goal of this section is to review the systematic construction of the non-singular models of singular Hitchin spectral curves of [30]. Then in Section 6, we prove that for the case of meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles, the PDE recursion of topological type based on the non-singular model of this section provides WKB analysis for the quantum curves constructed in Section 3.

Since the quantum curve reflects the geometry of $\Sigma \subset \overline{T^*C}$ through semi-classical limit, we first need to identify the choice of the blow-up space $\mathrm{Bl}(\overline{T^*C})$ in which the non-singular model $\widetilde{\Sigma}$ of the spectral curve is realized as a smooth divisor. This geometric information determines part of the initial data for topological recursion, i.e., the spectral curve of Eynard–Orantin, and the differential form $W_{0,1}$. Geometry of spectral curve also gives us information of singularities of the quantization. For example, when we have a component of a spectral curve tangent to the divisor C_∞, the quantum curve has an irregular singular point, and the class of the irregularity is determined by the degree of tangency. We have given a classification of singularities of the quantum curves in terms of the geometry of spectral curves in [30].

In what follows, we give the construction of the canonical blow-up space $\mathrm{Bl}(\overline{T^*C})$, and determine the genus of the normalization $\widetilde{\Sigma}$. This genus is necessary to identify the Riemann prime form on it, which determines another input datum $W_{0,2}$ for the topological recursion.

There are two different ways of defining the spectral curve for Higgs bundles with meromorphic Higgs field. Our definition uses the compactified cotangent bundle. This idea also appears in [56]. The traditional definition, which assumes the pole divisor D of the Higgs field to be reduced, is suitable for the study of moduli spaces of parabolic Higgs bundles. When we deal with non-reduced effective divisors, parabolic structures do not play any role. Non-reduced divisors appear naturally when we deal with classical equations such as the Airy differential equation, which has an irregular singular point of class $\frac{3}{2}$ at $\infty \in \mathbb{P}^1$.

Our point of view of spectral curves is also closely related to considering the *stable pairs* of pure dimension 1 on $\overline{T^*C}$. Through Hitchin's abelianization idea, the moduli space of stable pairs and the moduli space of Higgs bundles are identified [51].

The geometric setting we start with is a meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle $\big(E_0,\phi(q)\big)$ defined on a smooth projective algebraic curve C of genus $g \geq 0$ with a fixed projective structure. Here,

$$q = -\det(\phi(q)) \in H^0\big(C, K_C(D)^{\otimes 2}\big), \qquad \phi(q) = \begin{bmatrix} & q \\ 1 & \end{bmatrix},$$

and D is an effective divisor of C. The spectral curve is the zero-locus in $\overline{T^*C}$ of the characteristic equation

$$\Sigma := \big(\eta^2 - \pi^*(q)\big)_0. \tag{4.1}$$

The only condition we impose here is that *the spectral curve is irreducible.* In the language of Higgs bundles, this condition corresponds to the stability of $\big(E_0,\phi(q)\big)$.

Recall that $\mathrm{Pic}(\overline{T^*C})$ is generated by the zero section C_0 of T^*C and fibers of the projection map $\pi\colon \overline{T^*C} \longrightarrow C$. Since the spectral curve Σ is a double covering of C, as a divisor it is expressed as

$$\Sigma = 2C_0 + \sum_{j=1}^{a} \pi^*(p_j) \in \mathrm{Pic}(\overline{T^*C}), \tag{4.2}$$

where $\sum_{j=1}^{a} p_j \in \mathrm{Pic}^a(C)$ is a divisor on C of degree a. As an element of the Néron–Severi group

$$\mathrm{NS}(\overline{T^*C}) = \mathrm{Pic}(\overline{T^*C})/\mathrm{Pic}^0(\overline{T^*C}),$$

it is simply

$$\Sigma = 2C_0 + aF \in \mathrm{NS}(\overline{T^*C})$$

for a typical fiber class F. Since the intersection $F \cdot C_\infty = 1$, we have $a = \Sigma \cdot C_\infty$ in $\mathrm{NS}(\overline{T^*C})$. From the genus formula

$$p_a(\Sigma) = \frac{1}{2}\Sigma \cdot (\Sigma + K_{\overline{T^*C}}) + 1$$

and

$$K_{\overline{T^*C}} = -2C_0 + (4g-4)F \in \mathrm{NS}(\overline{T^*C}),$$

we find that the arithmetic genus of the spectral curve Σ is

$$p_a(\Sigma) = 4g - 3 + a, \tag{4.3}$$

where a is the number of intersections of Σ and C_∞. Now we wish to find the geometric genus of Σ.

Recall the following from [30]:

Definition 4.1 (discriminant divisor)**.** The discriminant divisor of the spectral curve (4.1) is a divisor on C defined by

$$\Delta := (q)_0 - (q)_\infty, \tag{4.4}$$

where

$$(q)_0 = \sum_{i=1}^{m} m_i r_i, \qquad m_i > 0, \qquad q_i \in C, \tag{4.5}$$

$$(q)_\infty = \sum_{j=1}^{n} n_j p_j, \qquad n_j > 0, \qquad p_j \in C. \tag{4.6}$$

Since q is a meromorphic section of $K_C^{\otimes 2}$,

$$\deg \Delta = \sum_{i=1}^{m} m_i - \sum_{j=1}^{n} n_j = 4g - 4. \tag{4.7}$$

Theorem 4.2 (geometric genus formula)**.** *Define*

$$\delta = |\{i \mid m_i \equiv 1 \mod 2\}| + |\{j \mid n_j \equiv 1 \mod 2\}|. \tag{4.8}$$

Then the geometric genus of the spectral curve Σ of (4.1) is given by

$$p_g(\Sigma) = 2g - 1 + \frac{1}{2}\delta. \tag{4.9}$$

We note that (4.7) implies $\delta \equiv 0 \mod 2$.

Remark 4.3. If ϕ is a holomorphic Higgs field, then $m = \delta = 4g - 4$ and $n = 0$. Therefore, we recover the genus formula $g(\Sigma) = 4g - 3$ of [27, equation (2.5)]. In this case, the Hitchin fibration (1.3) is a family of *Prym* varieties, which are $(3g - 3)$-dimensional Abelian varieties associated with the ramified covering $\pi\colon \Sigma \longrightarrow C$.

Proof. Since $\Sigma \subset \overline{T^*C}$ is a double covering of C in a ruled surface, locally at every singular point p, Σ is either irreducible, or reducible and consisting of two components. When irreducible, it is locally isomorphic to

$$t^2 - s^{2m+1} = 0, \qquad m \geq 1. \tag{4.10}$$

If it has two components, then it is locally isomorphic to

$$t^2 - s^{2m} = (t - s^m)(t + s^m) = 0. \tag{4.11}$$

Note that the local form of Σ at a ramification point of $\pi\colon \Sigma \longrightarrow C$ is written as (4.10) with $m = 0$. By extending the terminology "singularity" to "critical points" of the morphism π, we include a ramification point as a cusp with $m = 0$.

Let $\nu\colon \widetilde{\Sigma} \longrightarrow \Sigma$ be the non-singular model of Σ. Then $\tilde{\pi} = \pi \circ \nu\colon \widetilde{\Sigma} \longrightarrow C$ is a double sheeted covering of C by a smooth curve $\widetilde{\Sigma}$. If Σ has two components at a singularity P as in (4.11), then $\tilde{\pi}^{-1}(P)$ consists of two points and $\tilde{\pi}$ is not ramified there. If P is a cusp (4.10), then $\tilde{\pi}^{-1}(P)$ is a ramification point of the covering $\tilde{\pi}$. Thus the invariant δ of (4.8) counts the total number of cusp singularities of Σ and the ramification points of $\pi\colon \Sigma \longrightarrow C$. Then the Riemann–Hurwitz formula gives us

$$2 - 2g\big(\widetilde{\Sigma}\big) - \delta = 2\left(2 - 2g(C) - \delta\right),$$

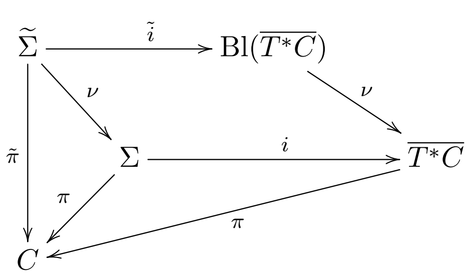
hence

$$p_g(\Sigma) = 2g(C) - 1 + \frac{1}{2} \qquad \text{(the number of cusps)}. \tag{4.12}$$

The genus formula (4.9) follows from (4.12). ∎

Our purpose is to apply topological recursion of Section 5 to a singular spectral curve of the form Σ of (4.1). To this end, we need to construct in a canonical way the normalization $\nu\colon \widetilde{\Sigma} \longrightarrow \Sigma$ through a sequence of blow-ups of the ambient space $\overline{T^*C}$. This is because we need to construct differential forms on $\widetilde{\Sigma}$ that reflect geometry of $i\colon \Sigma \longrightarrow \overline{T^*C}$. We thus proceed to analyze the local structure of Σ at each singularity using the global equation (4.1) in what follows.

Definition 4.4 (construction of the blow-up space)**.** The blow-up space $\mathrm{Bl}(\overline{T^*C})$



(4.13)

is defined by blowing up $\overline{T^*C}$ in the following way:

- At each r_i of (4.5), blow up $r_i \in \Sigma \cap C_0 \subset \overline{T^*C}$ a total of $\left\lfloor \frac{m_i}{2} \right\rfloor$ times.
- At each p_j of (4.6), blow up at the intersection $\Sigma \cap \pi^{-1}(p_j) \subset C_\infty$ a total of $\left\lfloor \frac{n_j}{2} \right\rfloor$ times.

Remark 4.5. Let us consider when both $(q)_0$ and $(q)_\infty$ are reduced. From the definition above, in this case Σ is non-singular, and the two genera (4.3) and (4.9) agree. The spectral curve is invariant under the involution $\sigma\colon \overline{T^*C} \longrightarrow \overline{T^*C}$ of (1.7). If $q \in H^0(C, K_C^{\otimes 2})$ is holomorphic, then $\pi\colon \Sigma \longrightarrow C$ is simply branched over $\Delta = (q)_0$, and Σ is a smooth curve of genus $4g-3$. This is in agreement of (4.3) because $n = 0$ in this case. If q is meromorphic, then its pole divisor is given by $(q)_\infty$ of degree n. Since $(q)_\infty$ is reduced, $\pi\colon \Sigma \longrightarrow C$ is ramified at the intersection of C_∞ and $\pi^*(q)_\infty$. The spectral curve is also ramified at its intersection with C_0. Note that $\deg(q)_0 = 4g-4+n$ because of (4.7). Thus $\pi\colon \Sigma \longrightarrow C$ is simply ramified at a total of $4g-4+2n$ points. Therefore, Σ is non-singular, and we deduce that its genus is given by

$$p_g(\Sigma) = p_a(\Sigma) = 4g - 3 + n$$

from the Riemann–Hurwitz formula. As a divisor class, we have

$$\Sigma = 2C_0 + \pi^*(q)_\infty \in \operatorname{Pic}(\overline{T^*C}),$$

in agreement of (4.2).

Theorem 4.6. *In the blow-up space* $\operatorname{Bl}(\overline{T^*C})$*, we have the following.*

- *The proper transform* $\widetilde{\Sigma}$ *of the spectral curve* $\Sigma \subset \overline{T^*C}$ *by the birational morphism* $\nu\colon \operatorname{Bl}(\overline{T^*C}) \longrightarrow \overline{T^*C}$ *is a smooth curve with a holomorphic map* $\tilde{\pi} = \pi \circ \nu\colon \widetilde{\Sigma} \longrightarrow C$.
- *The Galois action* $\sigma\colon \Sigma \longrightarrow \Sigma$ *lifts to an involution of* $\widetilde{\Sigma}$*, and the morphism* $\tilde{\pi}\colon \widetilde{\Sigma} \longrightarrow C$ *is a Galois covering with the Galois group* $\operatorname{Gal}(\widetilde{\Sigma}/C) = \langle\tilde{\sigma}\rangle \cong \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccccc} \widetilde{\Sigma} & \xrightarrow{\ \nu\ } & \Sigma & \xrightarrow{\ \pi\ } & C \\ {\scriptstyle\tilde{\sigma}}\downarrow & & \downarrow{\scriptstyle\sigma} & & \| \\ \widetilde{\Sigma} & \xrightarrow[\ \nu\ ]{} & \Sigma & \xrightarrow[\ \pi\ ]{} & C. \end{array} \tag{4.14}$$

Proof. We need to consider only when Δ is non-reduced. Let $r_i \in \operatorname{supp}(\Delta)$ be a zero of q of degree $m_i > 1$. The curve germ of Σ near $r_i \in \Sigma \cap C_0$ is given by a formula $y^2 = x^{m_i}$, where x is the base coordinate on C and y a fiber coordinate. We blow up once at $(x, y) = (0, 0)$, using a local parameter $y_1 = y/x$ on the exceptional divisor. The proper transform of the curve germ becomes $y_1^2 = x^{m_i - 2}$. Repeat this process at $(x, y_1) = (0, 0)$, until we reach the equation

$$y_\ell^2 = x^\epsilon,$$

where $\epsilon = 0$ or 1. The proper transform of the curve germ is now non-singular. We see that after a sequence of $\lfloor \frac{m_i}{2} \rfloor$ blow-ups starting at the point r_i, the proper transform of Σ is simply ramified over $r_i \in C = C_0$ if m_i is odd, and unramified if m_i is even. We apply the same sequence of blow-ups at each r_i with multiplicity greater than 1.

Let $p_j \in \operatorname{supp}(\Delta)$ be a pole of q with order $n_j > 1$. The intersection $P = \Sigma \cap \pi^{-1}(p_j)$ lies on C_∞, and Σ has a singularity at P. Let $z = 1/y$ be a fiber coordinate of $\pi^{-1}(p_j)$ at the infinity. Then the curve germ of Σ at the point P is given by

$$z^2 = x^{n_j}.$$

The involution σ in this coordinate is simply $z \longmapsto -z$. The blow-up process we apply at P is the same as before. After $\lfloor \frac{n_j}{2} \rfloor$ blow-ups starting at the point $P \in \Sigma \cap \pi^{-1}(p_j)$, the proper transform of Σ is simply ramified over $p_j \in C$ if n_j is odd, and unramified if n_j is even. Again we do this process for all p_j with a higher multiplicity.

The blow-up space $\mathrm{Bl}(\overline{T^*C})$ is defined as the application of a total of

$$\sum_{i=1}^{m}\left\lfloor\frac{m_i}{2}\right\rfloor+\sum_{j=1}^{n}\left\lfloor\frac{n_j}{2}\right\rfloor$$

times blow-ups on $\overline{T^*C}$ as described above. The proper transform $\widetilde{\Sigma}$ is the minimal resolution of Σ. Note that the morphism

$$\tilde{\pi}=\pi\circ\nu\colon\ \widetilde{\Sigma}\longrightarrow C$$

is a double covering, ramified exactly at δ points. Since $p_a(\widetilde{\Sigma})=p_g(\Sigma)$, (4.9) follows from the Riemann–Hurwitz formula applied to $\tilde{\pi}\colon\widetilde{\Sigma}\longrightarrow C$. It is also obvious that δ counts the number of cusp points of Σ, including smooth ramification points of π, in agreement of Theorem 4.2, and the fact that δ counts the total number of odd cusps on Σ.

Note that C_0 and C_∞ are point-wise invariant under the involution σ. Since $\mathrm{Bl}(\overline{T^*C})$ is constructed by blowing up points on C_0 and C_∞ and their proper transforms, we have a natural lift $\tilde{\sigma}\colon\ \mathrm{Bl}(\overline{T^*C})\longrightarrow\mathrm{Bl}(\overline{T^*C})$ of σ which induces (4.14). ■

5 A differential version of topological recursion

The Airy example of Section 7 suggests that the asymptotic expansion of a solution to a given quantum curve at its singularity contains information of quantum invariants. It also suggests that the topological recursion of [36] provides an effective tool for calculating asymptotic expansions of solutions for quantum curves. Since a linear differential equation is characterized by its solutions, topological recursion can be used as a mechanism of defining the quantization process from a spectral curve to a quantum curve. Then a natural question arises:

Question 5.1. How are the two quantizations, one with the construction of an $\hbar$-family of opers, and the other via topological recursion, related?

In Section 6, we prove that topological recursion provides WKB analysis of the quantum curves constructed through $\hbar$-families of opers, for the case of holomorphic and meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. For this purpose, in this section we review the framework of PDE recursion developed in [27, 30]. For the case of singular Hitchin spectral curves, our particular method of normalization of spectral curves of Section 4 produces the same result of quantization of Section 3.

If we consider a family of spectral curves that degenerate to a singular curve, the necessity of normalization for WKB analysis may sound unnatural. We emphasize that the semi-classical limit of the quantum curve thus obtained remains the original *singular* spectral curve, not the normalization, consistent with (3.42). Thus our quantization procedure in terms of PDE recursion is also a natural process.

Although many aspects of our current framework can be generalized to arbitrary complex simple Lie groups, since our calculation mechanism of Section 6 has been developed so far only for the $\mathrm{SL}(2,\mathbb{C})$ case, we restrict our attention to this case in this section.

We start with topological recursion for a degree 2 covering, not necessarily restricted to Hitchin spectral curves. The key ingredient of the theory is the *Riemann prime form* $E(z_1,z_2)$ on the product $\widetilde{\Sigma}\times\widetilde{\Sigma}$ of a compact Riemann surface $\widetilde{\Sigma}$ with values in a certain line bundle [67]. To define the prime form, we have to make a few more extra choices. First, we need to choose a theta characteristic $K_{\widetilde{\Sigma}}^{\frac{1}{2}}$ such that

$$\dim H^0\left(\widetilde{\Sigma},K_{\widetilde{\Sigma}}^{\frac{1}{2}}\right)=1.$$

We also need to choose a symplectic basis for $H_1(\widetilde{\Sigma}, \mathbb{Z})$, usually referred to as A-cycles and B-cycles. We follow Mumford's convention and use the unique Riemann prime form as defined in [67, p. 3.210]. See also [27, Section 2].

Definition 5.2 (topological recursion for non-singular covering)**.** Let C be a non-singular projective algebraic curve together with a choice of a symplectic basis for $H^1(C, \mathbb{Z})$, and $\tilde{\pi}\colon \widetilde{\Sigma} \longrightarrow C$ a degree 2 covering by another non-singular curve $\widetilde{\Sigma}$. We denote by R the ramification divisor of $\tilde{\pi}$. The covering $\tilde{\pi}$ is a Galois covering with the Galois group $\mathbb{Z}/2\mathbb{Z} = \langle \tilde{\sigma} \rangle$, and R is the fixed-point divisor of the involution $\tilde{\sigma}$. We also choose a labeling of points of R, and define the A-cycles of a symplectic bases for $H_1(\widetilde{\Sigma}, \mathbb{Z})$ as defined in [27, Section 2], which extend the A-cycles of C. *Topological recursion* is an inductive mechanism of constructing meromorphic differential forms $W_{g,n}$ on the Hilbert scheme $\widetilde{\Sigma}^{[n]}$ of n-points on $\widetilde{\Sigma}$ for all $g \geq 0$ and $n \geq 1$ in the *stable range* $2g - 2 + n > 0$, from given initial data $W_{0,1}$ and $W_{0,2}$. The differential form $W_{g,n}$ is a meromorphic n-linear form, i.e., a 1-form on each factor of $\widetilde{\Sigma}^{[n]}$ for $2g - 2 + n > 0$.

- $W_{0,1}$ is a meromorphic 1-form on $\widetilde{\Sigma}$ to be prescribed according to the geometric setting we have. Then define

  $$\Omega := \tilde{\sigma}^* W_{0,1} - W_{0,1},$$

  which satisfies $\tilde{\sigma}^*\Omega = -\Omega$.
- $W_{0,2}$ is defined by

  $$W_{0,2}(z_1, z_2) = \mathrm{d}_1 \mathrm{d}_2 \log E(z_1, z_2),$$

  where $E(z_1, z_2)$ is the A-cycle normalized Riemann prime form on $\widetilde{\Sigma} \times \widetilde{\Sigma}$. $W_{0,2}$ is a meromorphic differential $1 \otimes 1$-form on $\widetilde{\Sigma} \times \widetilde{\Sigma}$ with 2nd order poles along the diagonal.
- We also define the normalized Cauchy kernel on $\widetilde{\Sigma}$ by

  $$\omega^{a-b}(z) := \mathrm{d}_z \log \frac{E(a, z)}{E(b, z)}, \tag{5.1}$$

  which is a meromorphic 1-form in z with simple poles at $z = a$ of residue 1 and at $z = b$ of residue -1. We note that the ratio $E(a, z)/E(b, z)$ is a meromorphic function on the universal covering in a or b, but *not* a meromorphic function on $\widetilde{\Sigma}$. We thus choose a fundamental domain of the universal covering and restrict a and b in that domain for local calculations.

The inductive formula of the topological recursion then takes the following shape for $2g - 2 + n > 0$.

$$\begin{aligned} W_{g,n}(z_1, \dots, z_n) &= \frac{1}{2} \frac{1}{2\pi\sqrt{-1}} \oint_\Gamma \frac{\omega^{\tilde{z}-z}(z_1)}{\Omega(z)} \\ &\quad \times \Bigg[W_{g-1,n+1}(z, \tilde{z}, z_2, \dots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\dots,n\}}}^{\mathrm{No}(0,1)} W_{g_1,|I|+1}(z, z_I) W_{g_2,|J|+1}(\tilde{z}, z_J) \Bigg]. \end{aligned} \tag{5.2}$$

Here,

- The integration contour $\Gamma \subset \widetilde{\Sigma}$ is a collection of positively oriented small loops around each point $p \in \operatorname{supp}(\Omega)_0 \cup \operatorname{supp}(R)$.

- The integration is taken with respect to $z \in \Gamma$. Thus we chose a fundamental domain of the universal covering of $\widetilde{\Sigma}$ that contains $\operatorname{supp}(\Omega)_0 \cup \operatorname{supp}(R)$, and perform the integration locally as residue calculations.
- $\tilde{z} = \tilde{\sigma}(z)$ is the Galois conjugate of $z \in \widetilde{\Sigma}$.
- The expression $1/\Omega$ is a meromorphic section of K_C^{-1} which is multiplied to meromorphic quadratic differentials on $\widetilde{\Sigma}$ in z-variable.
- "No $(0,1)$" means that $g_1 = 0$ and $I = \varnothing$, or $g_2 = 0$ and $J = \varnothing$, are excluded in the summation.
- The sum runs over all partitions of g and set partitions of $\{2,\dots,n\}$, other than those containing the $(0,1)$-geometry.
- $|I|$ is the cardinality of the subset $I \subset \{2,\dots,n\}$.
- $z_I = (z_i)_{i\in I}$.

Remark 5.3. The integrand of (5.2) is not a well-defined differential form in $z \in \widetilde{\Sigma}$, due to the definition of $\omega^{\tilde{\sigma}(z)-z}(z_1)$ mentioned above. What the formula defines is a sum of residues at each point $p \in \operatorname{supp}(\Omega)_0 \cup \operatorname{supp}(R)$, which depends on the choice of the domain of $\omega^{\tilde{\sigma}(z)-z}(z_1)$ as a function in z.

Remark 5.4. Topological recursion depends on the choice of the integration contour Γ. Since the integrand of the right-hand side of (5.2) has other poles than the ramification divisor R and zeros of Ω, other choices of Γ are equally possible.

Remark 5.5. Topological recursion (5.2) can be defined for far more general situations. The bottle neck of the formalism is difficulty of integration over a high genus non-hyperelliptic Riemann surface. So the actual calculations have not been done much beyond the cases when the spectral curve $\widetilde{\Sigma}$ is of genus 0, or hyperelliptic.

When we have a non-singular Hitchin spectral curve $i\colon \Sigma \hookrightarrow T^*C$ associated with a holomorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle (E,ϕ), we apply Definition 5.2 to $\widetilde{\Sigma} = \Sigma$, $\tilde{\sigma} = \sigma$, and $W_{0,1} = i^*\eta$, where σ is the involution of (1.7) and η is the tautological 1-form on T^*C.

Under the same setting as in topological recursion, we define

Definition 5.6 (PDE recursion for a smooth covering of degree 2)**.** *PDE recursion* is the following partial differential equation for all (g,n) subject to $2g-2+n \geq 2$ defined on an open neighborhood U^n of $\widetilde{\Sigma}^{[n]}$ (or the universal covering, if the global treatment is necessary):

$$\begin{aligned}
d_1 F_{g,n}(z_1,\dots,z_n) &= \sum_{j=2}^{n}\left[\frac{\omega^{z_j-\tilde{\sigma}(z_j)}(z_1)}{\Omega(z_1)}\, d_1 F_{g,n-1}\big(z_{[\hat{j}]}\big) - \frac{\omega^{z_j-\tilde{\sigma}(z_j)}(z_1)}{\Omega(z_j)}\, d_j F_{g,n-1}\big(z_{[\hat{1}]}\big)\right]\\
&\quad + \frac{1}{\Omega(z_1)} d_{u_1} d_{u_2}\bigg[F_{g-1,n+1}\big(u_1,u_2,z_{[\hat{1}]}\big)\\
&\quad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I) F_{g_2,|J|+1}(u_2,z_J)\bigg]\bigg|_{\substack{u_1=z_1\\ u_2=z_1}}.
\end{aligned} \tag{5.3}$$

Here, the index subset $[\hat{j}]$ denotes the complement of $j \in \{1,2,\dots,n\}$. The final summation runs over all indices subject to be in the stable range, i.e., $2g_1-1+|I|>0$ and $2g_2-1+|J|>0$. The initial data for the PDE recursion are a function $F_{1,1}(z_1)$ on U and a symmetric function $F_{0,3}(z_1,z_2,z_3)$ on U^3.

Remark 5.7. The PDE recursion of topological type (5.3) is obtained from (5.2) by replacing the original contour Γ about the ramification divisor with the Cauchy kernel integration contour of [27, 30] about the diagonal poles of $\omega^{\tilde{\sigma}(z)-z}(z_1)$, $W_{0,2}(z,z_j)$, and $W_{0,2}\big(\tilde{\sigma}(z),z_j\big)$ at $z=z_j$ and $z=\tilde{\sigma}(z_j)$, $j=1,2,\dots,n$. This means the only difference comes from the alternative choice of the contour of integration Γ. All other ingredients of the formula are the same. Other than the case of $g(\widetilde{\Sigma})=0$, topological recursion and PDE recursion are not equivalent. In many enumerative problems [13, 16, 28, 31, 35, 63, 65, 66], PDE recursions are established through the Laplace transform of combinatorial relations. These PDE recursions can then be turned into the universal form of topological recursion. In these examples, the residue integral around ramification points of the spectral curve and Cauchy kernel integrations, i.e., residue calculations around the diagonals $z=z_j$, $z=\tilde{\sigma}(z_j)$, $j=1,2,\dots,n$, are equivalent by continuous deformation of the contour Γ.

6 WKB analysis of quantum curves

We consider in this section a meromorphic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle $(E_0,\phi(q))$ associated with a meromorphic quadratic differential $q\in H^0\big(C,K_C(D)^{\otimes 2}\big)$ with poles along an effective divisor D on a curve C of arbitrary genus. This includes the case of holomorphic Higgs bundles when $g(C)\geq 2$ and $D=\varnothing$. As before, we have a fixed spin structure $K_C^{\frac{1}{2}}$ and a projective coordinate system on C.

Let $\big(E_\hbar,\nabla^\hbar(q)\big)$ be the biholomorphic quantization result of Theorem 3.15. The corresponding Rees $\mathcal{D}_C$-module is generated by a single differential operator

$$P_\alpha(x_a,\hbar)=\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^2-q_\alpha \tag{6.1}$$

on each projective coordinate neighborhood U_α.

Theorem 6.1 (WKB analysis for $\mathrm{SL}(2,\mathbb{C})$-quantum curves)**.** *Let $q\in H^0\big(C,K_C(D)^{\otimes 2}\big)$ be a quadratic differential with poles along an effective divisor D on a curve C of arbitrary genus. We choose and fix a spin structure and a projective coordinate system on C. Theorem 3.15 tells us that we have a unique Rees $\mathcal{D}_C$-module $\mathcal{E}$ on C as the quantization of the possibly singular spectral curve*

$$\Sigma=(\eta^2-q)_0\subset\overline{T^*C}. \tag{6.2}$$

Then PDE recursion (5.3) with an appropriate choice of the initial data provides an all-order WKB analysis for the generator of the Rees $\mathcal{D}_C$-module $\mathcal{E}$ on a small neighborhood in C of each zero or a pole of q of odd order.

More precisely, the WKB analysis is given as follows.

- *Take an arbitrary point $p\in\operatorname{supp}(\Delta)$ in the discriminant divisor of (4.4) of odd degree. Choose a small enough simply connected coordinate neighborhood U_α of C with a projective coordinate x_α, so that $\tilde{\pi}^{-1}(U_\alpha)\subset\widetilde{\Sigma}$ in the normalization (4.13) is also simply connected. Let z be a local coordinate of $\tilde{\pi}^{-1}(U_\alpha)$ and denote by a function $x_\alpha=x_\alpha(z)$ the projection $\tilde{\pi}$.*
- *The formal WKB expansion we wish to construct is a solution to the equation*

$$P_\alpha(x_a,\hbar)\psi_\alpha(x_\alpha,\hbar)=\left[\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x_\alpha}\right)^2-q_\alpha\right]\psi_\alpha(x_\alpha,\hbar)=0 \tag{6.3}$$

  of the specific form

$$\psi_\alpha(x_\alpha,\hbar)=\exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x_\alpha)\right)=\exp F_\alpha(x_\alpha,\hbar). \tag{6.4}$$

Here, $F_\alpha(x_\alpha, \hbar)$ is a formal Laurent series in $\hbar$ starting with the power -1. In WKB analysis, this series in $\hbar$ does not converge.

- *Equation (6.3) is equivalent to*

$$\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x_\alpha^2} F_\alpha + \hbar^2 \frac{\mathrm{d}F_\alpha}{\mathrm{d}x_a}\frac{\mathrm{d}F_\alpha}{\mathrm{d}x_a} - q_\alpha = 0. \tag{6.5}$$

We interpret the above equation as an infinite sequence of ordinary differential equations for each power of $\hbar$:

$$\hbar^0\text{-terms:} \quad (S_0'(x_a))^2 - q_\alpha = 0, \tag{6.6}$$

$$\hbar^1\text{-terms:} \quad S_0''(x_\alpha) + 2S_0'(x_\alpha)S_1'(x_\alpha) = 0, \tag{6.7}$$

$$\hbar^2\text{-terms:} \quad S_1''(x_\alpha) + 2S_2'(x_\alpha)S_0'(x_\alpha) + (S_1'(x_\alpha))^2 = 0, \tag{6.8}$$

$$\hbar^{m+1}\text{-terms:} \quad S_m''(x_\alpha) + \sum_{a+b=m+1} S_a'(x_\alpha)S_b'(x_\alpha) = 0, \qquad m \geq 2. \tag{6.9}$$

The symbol $'$ denotes the x_α-derivative.

- *Solve (6.6), (6.7), and (6.8) to find $S_0(x_\alpha)$, $S_1(x_\alpha)$, and $S_2(x_\alpha)$.*
- *Construct the normalization $\tilde{\pi}\colon \widetilde{\Sigma} \longrightarrow \Sigma$ as in (4.13), and define*

$$W_{0,1} := \nu^* i^* \eta.$$

- *Define*

$$F_{1,1}(z_1) = -\int^{z_1} \frac{W_{0,2}(z_1, \tilde{\sigma}(z_1))}{\Omega(z_1)}, \tag{6.10}$$

where integration means a primitive of the meromorphic 1-form $\frac{W_{0,2}(z_1,\tilde{\sigma}(z_1))}{\Omega(z_1)}$ on $\tilde{\pi}^{-1}U_\alpha$.

- *Define*

$$F_{0,3}(z_1,z_2,z_3) = \iiint \big(-W(z_1,z_2,z_3) + 2\big(f(z_1)+f(z_2)+f(z_3)\big)\big), \tag{6.11}$$

where

$$\begin{aligned} W(z_1,z_2,z_3) = {} & \frac{1}{\Omega(z_1)}\big(W_{0,2}(z_1,z_2)W_{0,2}\big(z_1,\tilde{\sigma}(z_3)\big) + W_{0,2}(z_1,z_3)W_{0,2}\big(z_1,\tilde{\sigma}(z_2)\big)\big) \\ & + \mathrm{d}_2\left(\frac{\omega^{\tilde{\sigma}(z_2)-z_2}(z_1)W_{0,2}(z_2,\tilde{\sigma}(z_3))}{\Omega(z_2)}\right) \\ & + \mathrm{d}_3\left(\frac{\omega^{\tilde{\sigma}(z_3)-z_3}(z_1)W_{0,2}(z_2,\tilde{\sigma}(z_3))}{\Omega(z_3)}\right) \end{aligned}$$

and

$$f(z) := \widetilde{S}_2(z) - \left(F_{1,1}(z) - \frac{1}{6}\iiint^z W(z_1,z_2,z_3)\right).$$

Here, $\widetilde{S}_2(z) = S_2\big(x_\alpha(z)\big)$ is the lift of $S_2(x_a)$ to $\tilde{\pi}^{-1}(U_\alpha) \subset \widetilde{\Sigma}$.

- *Note that we have*

$$S_2(x_\alpha) = F_{1,1}\big(z(x_\alpha)\big) + \frac{1}{6}F_{0,3}\big(z(x_\alpha), z(x_\alpha), z(x_\alpha)\big)$$

for a local section $z\colon U_\alpha \longrightarrow \tilde{\pi}^{-1}(U_\alpha)$.

- *Solve PDE recursion* (5.3) *using the initial data* (6.10) *and* (6.11)*, and determine* $F_{g,n}(z_1,\dots,z_n)$ *for* $2g-2+n\geq 2$.
- *Define*

$$S_m(x_\alpha)=\sum_{2g-2+n=m-1}\frac{1}{n!}F_{g,n}\big(z(x_\alpha)\big),\qquad m\geq 3,\tag{6.12}$$

where $F_{g,n}\big(z(x_\alpha)\big)$ *is the principal specialization of* $F_{g,n}(z_1,\dots,z_n)$ *evaluated at a local section* $z=z(x_\alpha)$ *of* $\tilde{\pi}\colon\widetilde{\Sigma}\longrightarrow C$ *on* U_α.

Then (6.4) *gives the WKB expansion of the solution to the generator of the Rees* $\mathcal{D}_C$*-module.*

Remark 6.2. The WKB method is to solve (6.5) iteratively and find $S_m(x_\alpha)$ for all $m\geq 0$. Here, (6.6) is the *semi-classical limit* of (6.3), and (6.7) is the *consistency condition* we need for solving the WKB expansion. Since the 1-form $\mathrm{d}S_0(x)$ is a local section of T^*C, we identify $y=S_0'(x)$. Then (6.6) is the local expression of the spectral curve equation (6.2). This expression is the same everywhere on $C\setminus\operatorname{supp}(\Delta)$. We note q is globally defined. Therefore, we recover the spectral curve Σ from the differential operator (6.1).

Remark 6.3. $W_{1,1}:=\mathrm{d}F_{1,1}$ and $W_{0,3}:=\mathrm{d}_1\mathrm{d}_2\mathrm{d}_3F_{0,3}$ are solutions of (5.2) for $2g-2+n=1$ with respect to the contour of integration along the diagonal divisors mentioned in Remark 5.7.

The key idea of [27, 30] is the principal specialization of symmetric functions, which in our case means *restriction* of a PDE on a symmetric function to the main diagonal of the variables. Differential forms pull back, but PDEs do not. The essence of Theorem 6.1 is that the principal specialization of PDE recursion (5.3) is exactly the quantum curve equation (6.3).

Lemma 6.4 ([65, Lemma A.1])**.** *Let* $f(z_1,\dots,z_n)$ *be a symmetric function in* n *variables. Then*

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z,z,\dots,z)=n\left[\frac{\partial}{\partial u}f(u,z,\dots,z)\right]\bigg|_{u=z},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}f(z,z,\dots,z)=n\left[\frac{\partial^2}{\partial u^2}f(u,z,\dots,z)\right]\bigg|_{u=z}+n(n-1)\left[\frac{\partial^2}{\partial u_1\partial u_2}f(u_1,u_2,z,\dots,z)\right]\bigg|_{u_1=u_2=z}.$$

For a function in one variable $f(z)$*, we have*

$$\lim_{z_2\to z_1}\left[\omega^{z_2-b}(z_1)(f(z_1)-f(z_2))\right]=\mathrm{d}_1f(z_1),$$

where $\omega^{z_2-b}(z_1)$ *is the* 1*-form of* (5.1).

Proof of Theorem 6.1. First let $P\in\Sigma\cap C_\infty$ be an odd cusp singularity on the fiber $\pi^{-1}(p)$ of a point $p\in C=C_0\subset\overline{T^*C}$. The quadratic differential q has a pole of odd order at p, and the normalization $\tilde{\pi}\colon\widetilde{\Sigma}\longrightarrow C$ is simply ramified at a point $Q\in\widetilde{\Sigma}$ over p. We choose a local projective coordinate x on C centered at p. The Galois action of $\tilde{\sigma}$ on $\widetilde{\Sigma}$ fixes Q. As we have shown in Case 1 of the proof of Theorem 4.2, locally over p, the spectral curve Σ has the shape

$$z_0^2=c(x)x^{2\mu+1},$$

where $z_0=1/y$, y is the fiber coordinate on T_p^*C, and $c(x)$ is a unit $c(x)\in\mathcal{O}_{C,p}^*$. The quadratic differential q has a local expression

$$q=\frac{1}{c(x)x^{2\mu+1}}.$$

Define $z_1 = z_0/x$. The proper transform of $\widetilde{\Sigma}$ after the first blow up at P is locally written by

$$z_1^2 = c(x)x^{2\mu-1}.$$

Note that z_1 is an affine coordinate of the first exceptional divisor. Repeating this process μ-times, we end up with a coordinate $z_{\mu-1} = z_\mu x$ and an equation

$$z_\mu^2 = c(x)x.$$

Here again, z_μ is an affine coordinate of the exceptional divisor created by the μ-th blow-up. Write $z = z_\mu$ so that the proper transform of the μ-times blow-ups is given by

$$z^2 = c(x)x. \tag{6.13}$$

Note that the Galois action of $\tilde{\sigma}$ at Q is simply $z \longmapsto -z$. Solving (6.13) as a functional equation, we obtain a Galois invariant local expression

$$x = x(z) = c_Q(z^2)z^2,$$

where $c_Q \in \mathcal{O}^*_{\widetilde{\Sigma},Q}$ is a unit element. This function is precisely the local expression of the normalization $\tilde{\pi}\colon \widetilde{\Sigma} \longrightarrow C$ at $Q \in \widetilde{\Sigma}$. Since

$$z = z_\mu = \frac{z_0}{x^\mu} = \frac{1}{yx^\mu},$$

we have thus obtained the normalization coordinate z on the desingularized curve $\widetilde{\Sigma}$ near Q:

$$\begin{cases} x = x(z) = c_Q\big(z^2\big)z^2, \\ y = y(z) = \dfrac{1}{zx^\mu} = c_Q\big(z^2\big)^{-\mu} z^{-2\mu-1}. \end{cases}$$

It gives a parametric equation for the singular spectral curve Σ:

$$y^2 = \frac{1}{c(x)x^{2\mu+1}}.$$

As we have shown in the proof of Theorem 4.2, the situation is the same for a zero of q of odd order.

For the purpose of local calculation near $Q \in \widetilde{\Sigma}$, we use the following local expressions:

$$\begin{aligned} &\omega^{\tilde{\sigma}(z)-z}(z_1) = \left(\frac{1}{z_1 - \tilde{\sigma}(z)} - \frac{1}{z_1 - z} + O(1)\right)\mathrm{d}z_1, \\ &\eta = y\,\mathrm{d}x = h(z)\,\mathrm{d}z := \frac{1}{z^{2\mu}}(1 + O(z))\,\mathrm{d}z. \end{aligned} \tag{6.14}$$

Here, we adjust the normalization coordinate z by a constant factor to make (6.14) simple.

Using the notation $\partial_z = \partial/\partial z$, we have a local formula equivalent to (5.3) that is valid for $2g-2+n \geq 2$:

$$\begin{aligned} \partial_{z_1} F_{g,n}(z_1,\ldots,z_n) = {}& -\sum_{j=2}^{n}\left[\frac{\omega^{z_j-\tilde{\sigma}(z_j)}(z_1)}{2h(z_1)\,\mathrm{d}z_1}\,\partial_{z_1} F_{g,n-1}\big(z_{[\hat{j}]}\big) - \frac{\omega^{z_j-\tilde{\sigma}(z_j)}(z_1)}{\mathrm{d}z_1 \cdot 2h(z_j)}\,\partial_{z_j} F_{g,n-1}\big(z_{[\hat{1}]}\big)\right] \\ & - \frac{1}{2h(z_1)}\frac{\partial^2}{\partial u_1 \partial u_2}\Big[F_{g-1,n+1}\big(u_1,u_2,z_{[\hat{1}]}\big) \end{aligned} \tag{6.15}$$

$$+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\Bigg]\Bigg|_{\substack{u_1=z_1\\ u_2=z_1}}.$$

Let us apply principal specialization. The left-hand side becomes $\frac{1}{n}\partial_z F_{g,n}(z,\dots,z)$. To calculate the contributions from the first line of the right-hand side of (6), we choose $j>1$ and set $z_i=z$ for all i except for $i=1,j$. Then take the limit $z_j\to z_1$. In this procedure, we note that the contributions from the simple pole of $\omega^{z_j-\tilde{\sigma}(z_j)}(z_1)$ at $z_1=\tilde{\sigma}(z_j)$ cancel at $z_1=z_j$. Thus we obtain

$$\begin{aligned}
-\sum_{j=2}^{n}\frac{1}{z_1-z_j}&\left(\frac{1}{2h(z_1)}\,\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)-\frac{1}{2h(z_j)}\,\partial_{z_j}F_{g,n-1}(z_j,z,\dots,z)\right)\Bigg|_{z_1=z_j}\\
&=-\sum_{j=2}^{n}\partial_{z_1}\left(\frac{1}{2h(z_1)}\,\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)\right)\\
&=-(n-1)\partial_{z_1}\left(\frac{1}{2h(z_1)}\,\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)\right)\\
&=-(n-1)\partial_{z_1}\left(\frac{1}{2h(z_1)}\right)\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)-\frac{n-1}{2h(z_1)}\,\partial_{z_1}^2F_{g,n-1}(z_1,z,\dots,z).
\end{aligned}$$

The limit $z_1\to z$ then produces

$$\begin{aligned}
-\partial_z\frac{1}{2h(z)}\,\partial_zF_{g,n-1}(z,\dots,z)&-\frac{1}{2h(z)}\,\partial_z^2F_{g,n-1}(z,\dots,z)\\
&+\frac{(n-1)(n-2)}{2h(z)}\frac{\partial^2}{\partial u_1\partial u_2}F_{g,n-1}(u_1,u_2,z,\dots,z)\Bigg|_{u_1=u_2=z}.
\end{aligned}\tag{6.16}$$

To calculate the principal specialization of the second line of the right-hand side of (6), we note that since all points z_i's for $i\geq 2$ are set to be equal, a set partition by index sets I and J becomes a partition of $n-1$ with a combinatorial factor that counts the redundancy. The result is

$$\begin{aligned}
-\frac{1}{2h(z)}&\frac{\partial^2}{\partial u_1\partial u_2}F_{g-1,n+1}(u_1,u_2,z,\dots,z)\Bigg|_{u_1=u_2=z}\\
&-\frac{1}{2h(z)}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n-1}}^{\text{stable}}\binom{n-1}{n_1}\partial_zF_{g_1,n_1+1}(z,\dots,z)\,\partial_zF_{g_2,n_2+1}(z,\dots,z).
\end{aligned}\tag{6.17}$$

Assembling (6.16) and (6.17) together, we obtain

$$\begin{aligned}
\frac{1}{2h(z)}&\left[\partial_z^2F_{g,n-1}(z,\dots,z)+\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n-1}}^{\text{stable}}\binom{n-1}{n_1}\partial_zF_{g_1,n_1+1}(z,\dots,z)\,\partial_zF_{g_2,n_2+1}(z,\dots,z)\right]\\
&+\frac{1}{n}\partial_zF_{g,n}(z,\dots,z)+\partial_z\frac{1}{2h(z)}\,\partial_zF_{g,n-1}(z,\dots,z)\\
&=\frac{(n-1)(n-2)}{2h(z)}\frac{\partial^2}{\partial u_1\partial u_2}F_{g,n-1}(u_1,u_2,z,\dots,z)\Bigg|_{u_1=u_2=z}\\
&\quad-\frac{1}{2h(z)}\frac{\partial^2}{\partial u_1\partial u_2}F_{g-1,n+1}(u_1,u_2,z,\dots,z)\Bigg|_{u_1=u_2=z}.
\end{aligned}\tag{6.18}$$

We now apply the operation $\sum_{2g-2+n=m} \frac{1}{(n-1)!}$ to (6.18) above, and write the result in terms of

$$S_m(z) := S_m(x(z)) = \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(z, \dots, z)$$

of (6.12) to fit into the WKB formalism (6.4). We observe that summing over all possibilities of (g, n) with the fixed value of $2g - 2 + n$, the right-hand side of (6.18) exactly cancels out. Thus we have established that the functions $S_m(z)$ of (6.12) for $m \geq 2$ satisfy the recursion formula

$$\frac{1}{2h(z)} \left(\frac{\mathrm{d}^2 S_m}{\mathrm{d}z^2} + \sum_{\substack{a+b=m+1 \\ a,b \geq 2}} \frac{\mathrm{d}S_a}{\mathrm{d}z} \frac{\mathrm{d}S_b}{\mathrm{d}z} \right) + \frac{\mathrm{d}S_{m+1}}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{2h(z)} \right) \frac{\mathrm{d}S_m}{\mathrm{d}z} = 0. \tag{6.19}$$

Using (6.14) we identify the derivation

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{y}{h(z)} \frac{\mathrm{d}}{\mathrm{d}z}, \tag{6.20}$$

which is the push-forward $\tilde{\pi}_*(\mathrm{d}/\mathrm{d}z)$ of the vector field $\mathrm{d}/\mathrm{d}z$. The transformation (6.20) is singular at $z = 0$. If we allow terms $a = 0$ or $b = 0$ in (6.19), then what we have in addition is

$$\frac{1}{2h(z)} \cdot 2 \frac{\mathrm{d}S_0}{\mathrm{d}z} \frac{\mathrm{d}S_{m+1}}{\mathrm{d}z} = \frac{1}{h(z)} \frac{h(z)}{y} \frac{\mathrm{d}S_0}{\mathrm{d}x} \frac{\mathrm{d}S_{m+1}}{\mathrm{d}z} = \frac{\mathrm{d}S_{m+1}}{\mathrm{d}z},$$

since $\mathrm{d}S_0 = y\mathrm{d}x$. In other words, the $\frac{\mathrm{d}S_{m+1}}{\mathrm{d}z}$ term already there in (6.19) is absorbed in the split differentiation for $a = 0$ and $b = 0$.

From (6.20) and (6.6), we find that the second derivative with respect to x is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{S_0'}{h(z)} \frac{\mathrm{d}}{\mathrm{d}z} \right) = \frac{(S_0')^2}{h(z)^2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{S_0'}{h(z)} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{S_0'}{h(z)} \right) \frac{\mathrm{d}}{\mathrm{d}z},$$

denoting by $S_0' = \mathrm{d}S_0/\mathrm{d}x$. Then (6.9) yields

$$\frac{(S_0')^2}{h(z)^2} \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} S_m + \sum_{a+b=m+1} \frac{\mathrm{d}S_a}{\mathrm{d}z} \frac{\mathrm{d}S_b}{\mathrm{d}z} \right) + \frac{S_0'}{h(z)} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{S_0'}{h(z)} \right) \frac{\mathrm{d}S_m}{\mathrm{d}z} = 0. \tag{6.21}$$

The coefficients of $\mathrm{d}S_m/\mathrm{d}z$ in (6.21) are

$$\begin{aligned} 2\frac{(S_0')^2}{h(z)^2} \frac{\mathrm{d}S_1}{\mathrm{d}z} + \frac{S_0'}{h(z)} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{S_0'}{h(z)} \right) &= 2\frac{(S_0')^2}{h(z)^2} \frac{h(z)}{S_0'} S_1' + \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{S_0'}{h(z)} \right) \\ &= \frac{1}{h(z)} \left(2S_0' S_1' + S_0'' \right) + S_0' \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{h(z)} \right) = S_0' \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{h(z)} \right) \\ &= \frac{(S_0')^2}{h(z)^2} \cdot 2h(z) \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{2h(z)} \right) = \frac{2(S_0')^2}{h(z)} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{2h(z)} \right). \end{aligned}$$

This is exactly what the last term of (6.19) has, after adjusting overall multiplication by $\frac{2(S_0')^2}{h(z)}$. Therefore, we have established that (6.6), (6.7), and (6.8) make (6.9) equivalent to (6.19). This competed the proof of Theorem 6.1. ∎

Remark 6.5. In the examples of various Hurwitz numbers considered in [13, 65], PDE recursions can be applied to quantize the spectral curves (generalized Lambert curves) and obtain second-order linear *partial* differential equations. Since spectral curves are analytic, the direct quantization yields difference-differential equations in one variable. These different quantization results are compatible in the sense that the same asymptotic solution (6.4) satisfies both equations. A geometric interpretation is still missing for the analysis of these two different quantization mechanisms.

Remark 6.6. Geometry of normalization of the singular spectral curve leads us to the analysis of Stokes phenomena. It is beyond the scope of current paper, and will be treated elsewhere.

7 A simple classical example

Riemann and Poincaré worked on an interplay between algebraic geometry of curves in a ruled surface and the asymptotic expansion of an analytic solution to a differential equation defined on the base curve of the ruled surface. The theme of the current paper lies exactly on this link, looking at this classical subject from a modern point of view. The simple examples for $\mathrm{SL}(2,\mathbb{C})$-meromorphic Higgs bundles on $\mathbb{P}^1$ illustrate the relation between a Higgs bundle, the compactified cotangent bundle of a curve, a quantum curve, a classical differential equation, non-Abelian Hodge correspondence, and the quantum invariants that the quantum curve captures.

7.1 The Higgs bundle for the Airy function

The Higgs bundle (E,ϕ) we consider consists of the base curve $C=\mathbb{P}^1$ and a particular vector bundle

$$E_0 = K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}} = \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$$

of rank 2 on $\mathbb{P}^1$. A meromorphic Higgs field is given by

$$\phi = \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}\colon \quad E\longrightarrow E\otimes K_{\mathbb{P}^1}(m), \qquad m\geq 0. \tag{7.1}$$

Each matrix component is given by

$$\begin{aligned}
&\phi_{11}\colon \quad K_{\mathbb{P}^1}^{\frac{1}{2}} \longrightarrow K_{\mathbb{P}^1}^{\frac{1}{2}}\otimes K_{\mathbb{P}^1}(m) = K_{\mathbb{P}^1}^{\frac{3}{2}}(m), && \phi_{11}\in H^0(C,K_{\mathbb{P}^1}(m)),\\
&\phi_{12}\colon \quad K_{\mathbb{P}^1}^{-\frac{1}{2}} \longrightarrow K_{\mathbb{P}^1}^{\frac{1}{2}}\otimes K_{\mathbb{P}^1}(m) = K_{\mathbb{P}^1}^{\frac{3}{2}}(m), && \phi_{12}\in H^0\big(C,K_{\mathbb{P}^1}^{\otimes 2}(m)\big),\\
&\phi_{21}\colon \quad K_{\mathbb{P}^1}^{\frac{1}{2}} \longrightarrow K_{\mathbb{P}^1}^{-\frac{1}{2}}\otimes K_{\mathbb{P}^1}(m) = K_{\mathbb{P}^1}^{\frac{1}{2}}(m), && \phi_{21}\in H^0(C,\mathcal{O}_{\mathbb{P}^1}(m)),\\
&\phi_{22}\colon \quad K_{\mathbb{P}^1}^{-\frac{1}{2}} \longrightarrow K_{\mathbb{P}^1}^{-\frac{1}{2}}\otimes K_{\mathbb{P}^1}(m) = K_{\mathbb{P}^1}^{\frac{1}{2}}(m), && \phi_{22}\in H^0(C,K_{\mathbb{P}^1}(m)).
\end{aligned}$$

Since we are considering a point on a Hitchin section, we take $\phi_{21}=1$ to be the identity map $K_{\mathbb{P}^1}^{\frac{1}{2}} \xrightarrow{\cong} K_{\mathbb{P}^1}^{\frac{1}{2}} \hookrightarrow K_{\mathbb{P}^1}^{\frac{1}{2}}(m)$, and $\phi_{11}=\phi_{22}=0$. When we allow singularities, we can make other choices for ϕ_{21} as well.

The Planck constant $\hbar$ has a geometric meaning (1.1) as a parameter of the extension classes of line bundles. For $\mathbb{P}^1$, it is

$$\hbar\in \mathrm{Ext}^1\Big(K_{\mathbb{P}^1}^{-\frac{1}{2}}, K_{\mathbb{P}^1}^{\frac{1}{2}}\Big)\cong H^1\big(\mathbb{P}^1,K_{\mathbb{P}^1}\big)=\mathbb{C}.$$

It determines the unique extension

$$0\longrightarrow K_{\mathbb{P}^1}^{\frac{1}{2}}\longrightarrow E_\hbar\longrightarrow K_{\mathbb{P}^1}^{-\frac{1}{2}}\longrightarrow 0, \tag{7.2}$$

where

$$E_\hbar \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(1), & \hbar=0,\\ \mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}, & \hbar\neq 0\end{cases} \tag{7.3}$$

as a vector bundle, since every vector bundle on $\mathbb{P}^1$ splits.

The quantization of the Higgs field ϕ is an $\hbar$-connection of Deligne in $E_\hbar$ defined on $\mathbb{P}^1$ and is given by $\hbar\nabla^\hbar$, where

$$\nabla^\hbar = \mathrm{d}+\frac{1}{\hbar}\phi\colon\ E_\hbar \longrightarrow E_\hbar\otimes K_{\mathbb{P}^1}(m), \tag{7.4}$$

and d is the exterior differentiation operator acting on sections of the trivial bundle $E_\hbar$ for $\hbar\neq 0$. The operator $\nabla^\hbar$ is a meromorphic connection in the vector bundle $E_\hbar$. Of course $d+\phi$ is never a connection in general, because ϕ is a Higgs field belonging to a different bundle and satisfying a different transition rule with respect to coordinate changes. However, as explained in Section 3 (see also [25, 29]), a Higgs field ϕ associated with a complex simple Lie group on a Hitchin section gives rise to a connection $\nabla^\hbar=\mathrm{d}+\frac{1}{\hbar}\phi$ in $E_\hbar$ with respect to the coordinate system associated with a *projective structure* subordinating the complex structure of the base curve. Since our examples are constructed on $\mathbb{P}^1$, the affine coordinate $x\in\mathbb{A}^1=\mathbb{P}^1\setminus\{\infty\}$ is a natural coordinate representing the projective structure. Hence the expression $d+\phi$ makes sense as a connection in $E_\hbar$ for every $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle (E_0,ϕ). This is due to the vanishing of the Schwarzian derivatives for the coordinate change in a projective structure (see [25] for more detail on how the Schwarzian derivative plays a role here).

To see the effect of quantization, i.e., the passage from the Higgs bundle to an $\hbar$-family of connections (7.4), let us use the local coordinate and write everything concretely. The transition function defined on $\mathbb{C}^*=U_\infty\cap U_0$ of the vector bundle E_0 on $\mathbb{P}^1=U_\infty\cup U_0$ is given by $\begin{bmatrix} x & \\ & \frac{1}{x}\end{bmatrix}$, where $U_0=\mathbb{P}^1\setminus\{\infty\}=\mathbb{A}^1$ and $U_\infty=\mathbb{P}^1\setminus\{0\}$. With respect to the same coordinate, the extension $E_\hbar$ is given by $\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}$. The equality

$$\begin{bmatrix} 1 & \\ -\frac{1}{\hbar x} & 1\end{bmatrix}\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}\begin{bmatrix} & -\hbar\\ \frac{1}{\hbar} & x\end{bmatrix}=\begin{bmatrix} 1 & \\ & 1\end{bmatrix}$$

proves (7.3). The local expressions of the 1-form ϕ_{11} and the quadratic differential ϕ_{12} satisfy

$$(\phi_{11})_u du=(\phi_{11})_x\mathrm{d}x,\qquad (\phi_{12})_u du^2=(\phi_{12})_x\mathrm{d}x^2,$$

where $u=1/x$ is a coordinate on U_∞. Then the local expressions of the Higgs field (7.1) with $\phi_{22}=0$ satisfy the following transition relation with respect to E_0:

$$\begin{bmatrix} (\phi_{11})_u & -(\phi_{12})_u\\ -1 & \end{bmatrix}\mathrm{d}u=\begin{bmatrix} x & \\ & \frac{1}{x}\end{bmatrix}\begin{bmatrix} (\phi_{11})_x & (\phi_{12})_x\\ 1 & \end{bmatrix}\mathrm{d}x\begin{bmatrix} x & \\ & \frac{1}{x}\end{bmatrix}^{-1}.$$

The negative signs are due to $\mathrm{d}u=-\frac{1}{x^2}\mathrm{d}x$. For the case of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles, we further assume $\mathrm{tr}(\phi)=\phi_{11}=0$. In this case, we note that (7.1) is equivalent to the gauge transformation rule of connection matrices with respect to $E_\hbar$:

$$-\frac{1}{\hbar}\begin{bmatrix} & (\phi_{12})_u\\ 1 & \end{bmatrix}\mathrm{d}u=\frac{1}{\hbar}\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}\begin{bmatrix} & (\phi_{12})_x\\ 1 & \end{bmatrix}\mathrm{d}x\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}^{-1}-\mathrm{d}\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}^{-1}.$$

In other words,

$$\mathrm{d}_u-\frac{1}{\hbar}\begin{bmatrix} & (\phi_{12})_u\\ 1 & \end{bmatrix}\mathrm{d}u=\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}\left(\mathrm{d}_x+\frac{1}{\hbar}\begin{bmatrix} & (\phi_{12})_x\\ 1 & \end{bmatrix}\mathrm{d}x\right)\begin{bmatrix} x & \hbar\\ & \frac{1}{x}\end{bmatrix}^{-1}.$$

Therefore, $\nabla^\hbar = \mathrm{d} + \frac{1}{\hbar}\phi$ is a globally defined connection in $E_\hbar$ for every $\hbar \neq 0$, and

$$\left(\hbar\nabla^\hbar\right)\big|_{\hbar=0} = \phi$$

is the original Higgs field.

Let us start with a particular spectral curve, the algebraic curve

$$\Sigma \subset \mathbb{F}_2 = \mathbb{P}\big(K_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}\big) = \overline{T^*\mathbb{P}^1}$$

embedded in the Hirzebruch surface with the defining equation

$$y^2 - x = 0 \tag{7.5}$$

on U_0. Here, y is a fiber coordinate of the cotangent bundle $T^*\mathbb{P}^1 \subset \mathbb{F}^2$ over U_0. The Hirzebruch surface is the natural compactification of the cotangent bundle $T^*\mathbb{P}^1$, which is the total space of the canonical bundle $K_{\mathbb{P}^1}$. We denote by $\eta \in H^0\big(T^*\mathbb{P}^1, \pi^*K_{\mathbb{P}^1}\big)$ the tautological 1-form associated with the projection $\pi\colon T^*\mathbb{P}^1 \longrightarrow \mathbb{P}^1$. It is expressed as $\eta = y\mathrm{d}x$ in terms of the affine coordinates. The holomorphic symplectic form on $T^*\mathbb{P}^1$ is given by $-\mathrm{d}\eta = \mathrm{d}x\wedge\mathrm{d}y$. The 1-form η extends to $\mathbb{F}_2$ as a meromorphic differential form and defines a divisor

$$(\eta) = C_0 - C_\infty,$$

where C_0 is the zero-section of $T^*\mathbb{P}^1$, and C_∞ the section at infinity of $\overline{T^*\mathbb{P}^1}$. The Picard group $\mathrm{Pic}(\mathbb{F}_2)$ of the Hirzebruch surface is generated by the class C_0 and a fiber class F of π. Although (7.5) is a perfect parabola in the affine plane, it has a quintic cusp singularity at $x = \infty$. Let (u, w) be a coordinate system on another affine chart of $\mathbb{F}_2$ defined by

$$\begin{cases} x = 1/u, & \\ y\,\mathrm{d}x = v\,\mathrm{d}u, & w = 1/v. \end{cases}$$

Then Σ in the (u, w)-plane is given by

$$w^2 = u^5. \tag{7.6}$$

The expression of $\Sigma \in \mathrm{NS}(\mathbb{F}_2)$ as an element of the Néron–Severy group of $\mathbb{F}_2$, in this case the same as $\mathrm{Pic}(\mathbb{F}_2)$, is thus given by $\Sigma = 2C_0 + 5F$.

Define a stable Higgs pair $(E_0, \phi(q))$ on $\mathbb{P}^1$ with $E_0 = K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and

$$\phi(q) = \begin{bmatrix} & q \\ 1 & \end{bmatrix}\colon\ E_0 \longrightarrow E_0 \otimes K_{\mathbb{P}^1}(4).$$

Here, we choose a meromorphic quadratic differential $q \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}(2)^{\otimes 2})$ that has a simple zero at $0 \in \mathbb{P}^1$ and a pole of order 5 at $\infty \in \mathbb{P}^1$. Up to a constant factor, there is only one such differential

$$q = x(\mathrm{d}x)^2 = \frac{1}{u^5}(\mathrm{d}u)^2 \in H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}(2)^{\otimes 2}\big) = \mathbb{C}.$$

The spectral curve Σ of $(E_0, \phi(q))$ is given by the characteristic equation

$$\det(\eta - \pi^*\phi) = \eta^2 - \pi^*\mathrm{tr}(\phi) + \pi^*\det(\phi) = \eta^2 - q = 0 \tag{7.7}$$

in $\mathbb{F}_2$. As explained above, $(E_0, \phi(q))$ uniquely determines a meromorphic *oper*

$$\nabla^\hbar(q) = \mathrm{d} + \frac{1}{\hbar}\phi(q) = \mathrm{d} + \frac{1}{\hbar}\begin{bmatrix} & q \\ 1 & \end{bmatrix} \tag{7.8}$$

on the extension $E_\hbar$ of (7.2) over $\mathbb{P}^1$ [25]. Indeed, the case $q = 0$ of (7.8) for $\hbar = 1$ is the *non-Abelian Hodge correspondence* [24, 46, 73]

$$H^0(C, \mathrm{End}(E_0) \otimes K_{\mathbb{P}^1}) \ni \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Longleftrightarrow \mathrm{d} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \colon\ E_1 \longrightarrow E_1 \otimes K_{\mathbb{P}^1}.$$

The **quantization** procedure of this paper is the following $\hbar$-deformation

$$H^0(C, \mathrm{End}(E_0) \otimes K_{\mathbb{P}^1}(m)) \ni \begin{bmatrix} & q \\ 1 & \end{bmatrix} \Longleftrightarrow \mathrm{d} + \frac{1}{\hbar}\begin{bmatrix} & q \\ 1 & \end{bmatrix} \colon\ E_\hbar \longrightarrow E_\hbar \otimes K_{\mathbb{P}^1}(m),$$

where $q \in H^0\left(C, K_{\mathbb{P}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1}(m)\right)$ is a meromorphic quadratic differential on C.

Let $\psi(x, \hbar)$ denote an analytic function in x with a formal parameter $\hbar$ such that

$$\hbar \nabla_q^\hbar \begin{bmatrix} -\hbar\psi' \\ \psi \end{bmatrix} = 0, \qquad \hbar \neq 0,$$

where $'$ denotes the x differentiation. Then it satisfies a Schrödinger equation

$$\left(\left(\hbar \frac{\mathrm{d}}{\mathrm{d}x} \right)^2 - x \right) \psi(x, \hbar) = 0. \tag{7.9}$$

The differential operator

$$P(x, \hbar) := \left(\hbar \frac{\mathrm{d}}{\mathrm{d}x} \right)^2 - x$$

quantizing the spectral curve Σ of (7.5) is an example of a quantum curve. Reflecting the fact (7.6) that Σ has a quintic cusp singularity at $x = \infty$, (7.9) has an *irregular* singular point of *class* $\frac{3}{2}$ at $x = \infty$.

Let us recall the definition of regular and irregular singular points of a second-order differential equation here.

Definition 7.1. Let

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_1(x) \frac{\mathrm{d}}{\mathrm{d}x} + a_2(x) \right) \psi(x) = 0 \tag{7.10}$$

be a second-order differential equation defined around a neighborhood of $x = 0$ on a small disc $|x| < \epsilon$ with meromorphic coefficients $a_1(x)$ and $a_2(x)$ with poles at $x = 0$. Denote by k (resp. ℓ) the order of the pole of $a_1(x)$ (resp. $a_2(x)$) at $x = 0$. If $k \leq 1$ and $\ell \leq 2$, then (7.10) has a *regular singular point* at $x = 0$. Otherwise, consider the *Newton polygon* of the order of poles of the coefficients of (7.10). It is the upper part of the convex hull of three points $(0, 0)$, $(1, k)$, $(2, \ell)$. As a convention, if $a_j(x)$ is identically 0, then we assign $-\infty$ as its pole order. Let $(1, r)$ be the intersection point of the Newton polygon and the line $x = 1$. Thus

$$r = \begin{cases} k, & 2k \geq \ell, \\ \dfrac{\ell}{2}, & 2k \leq \ell. \end{cases}$$

The differential equation (7.10) has an *irregular singular point of class* $r - 1$ at $x = 0$ if $r > 1$.

The class $\frac{3}{2}$ at ∞ indicates how the asymptotic expansion of the solution ψ looks like. Indeed, any non-trivial solution has an essential singularity at ∞. We note that every solution of (7.9) is an entire function for any value of $\hbar \neq 0$. Applying our main result of this paper, we construct a particular *all-order* asymptotic expansion of this entire solution

$$\psi(x,\hbar) = \exp F(x,\hbar), \qquad F(x,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x), \tag{7.11}$$

valid for $|\mathrm{Arg}(x)| < \pi$, and $\hbar > 0$. Here, the first two terms of the asymptotic expansion are given by

$$S_0(x) = \pm\frac{2}{3}x^{\frac{3}{2}}, \tag{7.12}$$

$$S_1(x) = -\frac{1}{4}\log x. \tag{7.13}$$

Although the *classical limit* $\hbar \to 0$ of (7.9) does not make sense under the expansion (7.11), the *semi-classical limit* through the *WKB* analysis

$$\left[\mathrm{e}^{-S_1(x)}\mathrm{e}^{-\frac{1}{\hbar}S_0(x)}\left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\right)\mathrm{e}^{\frac{1}{\hbar}S_0(x)}\mathrm{e}^{S_1(x)}\right]\exp\left(\sum_{m=2}^{\infty}\hbar^{m-1}S_m(x)\right) = 0 \tag{7.14}$$

has a well-defined limit $\hbar \to 0$. The result is $S_0'(x)^2 = x$, which gives (7.12), and also (7.5) by defining $\mathrm{d}S_0 = \eta$. The vanishing of the $\hbar$-linear terms of (7.14) is $2S_0'(x)S_1'(x) + S_0''(x) = 0$, which gives (7.13) above.

The entire solution in x for $\hbar \neq 0$ and the choice of $S_0(x) = -\frac{2}{3}x^{\frac{3}{2}}$ is called the *Airy function*

$$\mathrm{Ai}(x,\hbar) = \frac{1}{2\pi}\hbar^{-\frac{1}{6}}\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}px}{\hbar^{2/3}} + \mathrm{i}\frac{p^3}{3}\right)\mathrm{d}p. \tag{7.15}$$

The surprising discovery of Kontsevich [55] (cf. [22, 76]) is that $S_m(x)$ for $m \geq 2$ has the following *closed* formula

$$S_m(x) := \sum_{2g-2+n=m-1}\frac{1}{n!}F_{g,n}^{\mathrm{Airy}}(x), \tag{7.16}$$

$$F_{g,n}^{\mathrm{Airy}}(x) := \frac{(-1)^n}{2^{2g-2+n}}x^{-\frac{(6g-6+3n)}{2}}\sum_{\substack{d_1+\cdots+d_n\\=3g-3+n}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}\prod_{i=1}^{n}|(2d_i-1)|!!, \tag{7.17}$$

where the coefficients

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}$$

are the cotangent class intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n non-singular marked points. The expansion coordinate $x^{\frac{3}{2}}$ of (7.17) indicates the class of the irregular singularity of the Airy differential equation.

Although (7.16) is not a generating function of all intersection numbers, the quantum curve (7.5) alone actually determines every intersection number $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}$. This mechanism is topological recursion of [36]. PDE recursion computes *free energies*

$$F_{g,n}^{\mathrm{Airy}}(t_1,\dots,t_n) := \frac{(-1)^n}{2^{2g-2+n}}\sum_{\substack{d_1+\cdots+d_n\\=3g-3+n}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}\prod_{i=1}^{n}\left(\frac{t_i}{2}\right)^{2d_i+1}|(2d_i-1)|!! \tag{7.18}$$

as a function in n variables from Σ through the process of blow-ups of $\mathbb{F}_2$, and the exterior derivative of free energies are the symplectic invariants of [36].

7.2 Blowing up a Hirzebruch surface

Let us now give a detailed algebraic geometry procedure for this example. We start with the spectral curve Σ of (7.5). Our goal is to come up with (7.9). The first step is to blow up $\mathbb{F}_2$ and to construct a normalization of Σ. The construction of $\mathrm{Bl}(\overline{T^*C})$ is given in Definition 4.4. It is the minimal resolution of the divisor

$$\Sigma = \left(\det(\eta - \pi^*\phi)\right)_0$$

of the characteristic polynomial. The discriminant of the defining equation (7.7) of the spectral curve is

$$-\det(\phi) = x(\mathrm{d}x)^2 = \frac{1}{u^5}(\mathrm{d}u)^2.$$

It has a simple zero at $x = 0$ and a pole of order 5 at $x = \infty$. The *geometric genus formula* (4.9) for the general base curve C reads

$$g(\widetilde{\Sigma}) = 2g(C) - 1 + \frac{1}{2}\delta,$$

where δ is the sum of the number of cusp singularities of Σ and the ramification points of $\pi\colon \Sigma \longrightarrow C$ (Theorem 4.2). In our case, it tells us that $\widetilde{\Sigma}$ is a non-singular curve of genus 0, i.e., a $\mathbb{P}^1$, after blowing up $\lfloor \frac{5}{2} \rfloor = 2$ times.

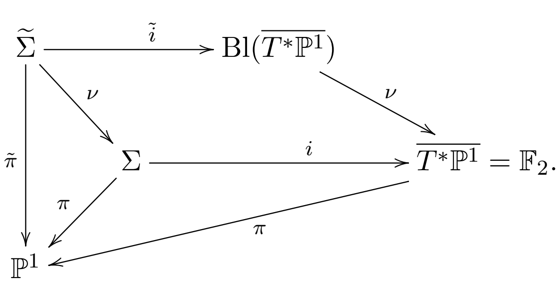
The center of blow-up is $(u, w) = (0, 0)$ for the first time. Put $w = w_1 u$, and denote by E_1 the exceptional divisor of the first blow-up. The proper transform of Σ for this blow-up, $w_1^2 = u^3$, has a cubic cusp singularity, so we blow up again at the singular point. Let $w_1 = w_2 u$, and denote by E_2 the exceptional divisor created by the second blow-up. The self-intersection of the proper transform of E_1 is -2. We then obtain the desingularized curve $\widetilde{\Sigma}$, locally given by $w_2^2 = u$. The proof of Theorem 4.2 also tells us that $\widetilde{\Sigma} \longrightarrow \mathbb{P}^1$ is ramified at two points. Choose the affine coordinate $t = 2w_2$ of the exceptional divisor added at the second blow-up. Our choice of the constant factor is to make the formula the same as in [31]. We have

$$\begin{cases} x = \dfrac{1}{u} = \dfrac{1}{w_2^2} = \dfrac{4}{t^2}, \\ y = -\dfrac{u^2}{w} = -\dfrac{u^2}{w_2 u^2} = -\dfrac{2}{t}. \end{cases} \tag{7.19}$$

In the (u, w)-coordinate, we see that the parameter t is a normalization parameter of the quintic cusp singularity:

$$\begin{cases} u = \dfrac{t^2}{4}, \\ w = \dfrac{t^5}{32}. \end{cases}$$

Note that $\widetilde{\Sigma}$ intersects transversally with the proper transform of C_∞. The blow-up space $\mathrm{Bl}(\mathbb{F}^2)$ is the result of the twice blow-ups of the Hirzebruch surface:



(7.20)

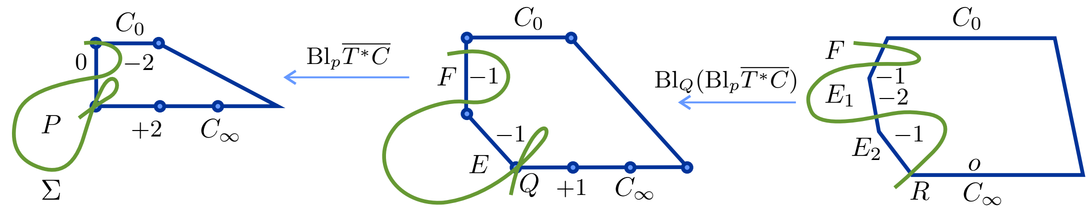


Figure 7.1.

Topological recursion (5.2) requires a globally defined meromorphic 1-form $W_{0,1}$ on $\widetilde{\Sigma}$ and a symmetric meromorphic 2-form $W_{0,2}$ on the product $\widetilde{\Sigma}\times\widetilde{\Sigma}$ as the initial data. We choose

$$\begin{cases} W_{0,1} = \tilde{i}^*\nu^*\eta, \\ W_{0,2} = \mathrm{d}_1\mathrm{d}_2 \log E_{\widetilde{\Sigma}}, \end{cases} \tag{7.21}$$

where $E_{\widetilde{\Sigma}}$ is a normalized Riemann prime form on $\widetilde{\Sigma}$ (see [27, Section 2]). The form $W_{0,2}$ depends only on the intrinsic geometry of the smooth curve $\widetilde{\Sigma}$. The geometry of (7.20) is encoded in $W_{0,1}$.

Now we apply PDE recursion (5.3) to the geometric data (7.20) and (7.21). We claim that topological recursion of [36] for the geometric data we are considering now is exactly the same as the recursive equation of [31, equation (6.12)] applied to the curve (7.19) *realized as a plane parabola* in $\mathbb{C}^2$. This is because topological recursion (5.2) has two residue contributions, one each from $t=0$ and $t=\infty$. As proved in [31, Section 6], the integrand on the right-hand side of the recursion formula [31, equation (6.12)] does not have any pole at $t=0$. Therefore, the residue contribution from this point is 0. PDE recursion is obtained by deforming the contour of integration to enclose only poles of the differential forms $W_{g,n}$. Since $t=0$ is a regular point, the two methods have no difference.

The $W_{0,2}$ of (7.21) is simply $\frac{\mathrm{d}t_1\cdot\mathrm{d}t_2}{(t_1-t_2)^2}$ because $\widetilde{\Sigma}\cong\mathbb{P}^1$. Since t of (7.19) is a normalization coordinate, we have

$$W_{0,1} = \tilde{i}^*\nu^*(\eta) = y(t)\,\mathrm{d}x(t) = \frac{16}{t^4},$$

in agreement of [31, equation (6.8)]. Noticing that the solution to topological recursion is unique from the initial data, we conclude that

$$\mathrm{d}_1\cdots\mathrm{d}_n F^{\mathrm{Airy}}_{g,n}\big(x(t_1),\dots,x(t_n)\big) = W_{g,n}.$$

By setting the constants of integration by integrating from $t=0$ for PDE recursion, we obtain the expression (7.18). Then its principal specialization gives (7.17). The equivalence of PDE recursion and the quantum curve equation Theorem 6.1 then proves (7.9) with the expression of (7.11) and (7.16).

In this process, what is truly amazing is that the single differential equation (7.9), which is our quantum curve, knows everything about the free energies (7.18). This is because we can recover the spectral curve Σ from the quantum curve. Then the procedures we need to apply, the blow-ups and PDE recursion, are canonical. Therefore, we actually recover (7.18) as explained above.

It is surprising to see that a simple entire function (7.15) contains so much geometric information. Our expansion (7.11) is an expression of this entire function viewed from its essential singularity. We can extract rich information of the solution by restricting the region where the asymptotic expansion is valid. If we consider (7.11) only as a formal expression in x and $\hbar$, then we cannot see how the coefficients are related to quantum invariants. Topological recursion [36] is a key to connect the two worlds: the world of quantum invariants, and the world of holomorphic functions and differentials. This relation is also knows as a *mirror symmetry*, or in analysis, simply as the *Laplace transform*. The intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$ belong to the A-model, while the spectral curve Σ of (7.5) and free energies belong to the B-model. We consider (7.18) as an example of the Laplace transform, playing the role of mirror symmetry [28, 31]. In the context of Hitchin theory, mirror symmetry also plays a different role through Langland duality (cf. [48, 54, 78, 79]). It is unclear to us how these two different mirror symmetries are interrelated.

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