A Note on Multi-Oriented Graph Complexes and Deformation Quantization of Lie Bialgebroids

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Abstract. Universal solutions to deformation quantization problems can be conveniently classified by the cohomology of suitable graph complexes. In particular, the deformation quantizations of (finite-dimensional) Poisson manifolds and Lie bialgebras are characterised by an action of the Grothendieck–Teichmüller group via one-colored directed and oriented graphs, respectively. In this note, we study the action of multi-oriented graph complexes on Lie bialgebroids and their "quasi" generalisations. Using results due to T. Willwacher and M. Živković on the cohomology of (multi)-oriented graphs, we show that the action of the Grothendieck–Teichmüller group on Lie bialgebras and quasi-Lie bialgebras can be generalised to quasi-Lie bialgebroids via graphs with two colors, one of them being oriented. However, this action generically fails to preserve the subspace of Lie bialgebroids. By resorting to graphs with two oriented colors, we instead show the existence of an obstruction to the quantization of a generic Lie bialgebroid in the guise of a new Lie_∞-algebra structure non-trivially deforming the "big bracket" for Lie bialgebroids. This exotic Lie_∞-structure can be interpreted as the equivalent in $d = 3$ of the Kontsevich–Shoikhet obstruction to the quantization of infinite-dimensional Poisson manifolds (in $d = 2$). We discuss the implications of these results with respect to a conjecture due to P. Xu regarding the existence of a quantization map for Lie bialgebroids.

Key words: deformation quantization; Kontsevich's graphs; Lie bialgebroids; Grothendieck–Teichmüller group

2020 Mathematics Subject Classification: 53D55; 18G85; 17B62

1 Introduction

Graph complexes play an essential rôle in the understanding of the deformation quantization of various algebraic and geometric structures, the paradigmatic example thereof being the Kontsevich graph complex and its relation to the deformation quantization problem for (finite-dimensional) Poisson manifolds [32]. In particular, the space of Kontsevich quantization maps is acted upon by the pro-unipotent group exponentiating the zeroth cohomology of the Kontsevich graph complex of directed graphs [12]. As shown by T. Willwacher [71], the latter is isomorphic to the Grothendieck–Teichmüller group GRT_1 – introduced by V. Drinfeld[1] [17] in the context of the absolute Galois group $\mathrm{Gal}\big(\bar{\mathbb{Q}}/\mathbb{Q}\big)$ and the theory of quasi-Hopf algebras – so that the space of Kontsevich maps[2] is a GRT_1-torsor [33]. Since its inception, the Grothendieck–Teichmüller group appeared in a variety of mathematical contexts such as the Kashiwara–Vergne conjecture, multiple zeta values, rational homotopy of the $\mathbf{E}_2$-operad, etc.

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[1]Based on a suggestion due to A. Grothendieck in his *Esquisse d'un Programme* [27] who proposed to study the combinatorial properties of the absolute Galois group $\mathrm{Gal}\big(\bar{\mathbb{Q}}/\mathbb{Q}\big)$ via its natural action on the tower of Teichmüller groupoids.

[2]More precisely, the space of homotopy classes [11] of stable [12] formality morphisms is a GRT_1-torsor.

In the present context of graph complexes, another incarnation of the Grothendieck–Teichmüller group can be found in relation to the deformation quantization problem for Lie bialgebras via the action of the graph complex of *oriented* graphs [72] in dimension $d = 3$.[3] The latter action generalises to Lie-quasi bialgebras[4] and furthermore provides a rationale for the classifying rôle played by the Grothendieck–Teichmüller group on the space of quantization maps for Lie and Lie-quasi bialgebras à la Etingof–Kazhdan [20, 61]. The oriented graph complex also plays a crucial rôle regarding the obstruction theory to the existence of a universal quantization of infinite-dimensional Poisson manifolds [72]. The corresponding obstruction lives in the first order cohomology of the oriented graph complex in $d = 2$ which is a one-dimensional space spanned by the so-called Kontsevich–Shoikhet cocycle. When represented on the space of (infinite-dimensional) polyvector fields, the latter yields an exotic Lie_∞-structure [65] deforming non-trivially the Schouten bracket. Further, the zeroth order cohomology of the oriented graph complex (in $d = 2$) vanishes thus preventing the Grothendieck–Teichmüller group to play a classifying rôle for quantizations of infinite-dimensional Poisson manifolds. The deformation quantization problem for infinite-dimensional Poisson manifolds thus differs essentially from the finite-dimensional case and this discrepancy can be traced back to the fact that their respective deformation theory is acted upon by a different graph complex (directed vs. oriented). Deformation quantization problems can then be partitioned into different classes according to the cohomology of the graph complex acting on them. We distinguish between three main classes:

- A *no-go* class comprising the deformation quantization problems for infinite-dimensional Poisson manifolds:

  1. The Grothendieck–Teichmüller group plays no classifying rôle regarding the universal deformations (and hence quantizations).
  2. There exists a potential obstruction to the existence of universal quantizations.

- A *yes-go* class comprising the deformation quantization problems for (finite-dimensional) Poisson manifolds and Lie-(quasi) bialgebras for which:

  1. The Grothendieck–Teichmüller group plays a classifying rôle.
  2. There is (conjecturally) no generic obstruction to the existence of universal quantizations.

- A *middle way* class comprising the deformation quantization problems for Courant algebroids:

  1. The Grothendieck–Teichmüller group plays no classifying rôle; rather deformations are generated by the triangle cocycle as well as by conformal rescalings associated with trivalent graphs.
  2. There is no generic obstruction to the existence of universal quantizations.

In the present note, we add two threads to this on-going story by introducing some new universal models for the deformation theory of Lie bialgebroids and their "quasi" versions. Lie bialgebroids have been introduced by Mackenzie–Xu [42] as linearisations of Poisson groupoids and constitute a common generalisation of the notions of Poisson manifolds and Lie bialgebras. The

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[3]Recall that the parameter d corresponds to the dimension of the source manifold of the relevant AKSZ σ-model [1]. The latter is related to the degree n of the corresponding target manifold via $d = n + 1$ and is therefore independent of the dimension of the associated algebro-geometric structure (Poisson manifold, Lie bialgebra, etc.). Consistently, it relates to the dimension of the compactified configuration spaces of points of the associated de Rham field theories [45, 46]. Therefore, any graph complex related to Poisson manifolds has dimension $d = 2$ while the ones related to Lie bialgebras and generalisations thereof have dimension $d = 3$.

[4]As well as their dual, referred to as quasi-Lie bialgebras in the following, see footnote 14 for terminology.

corresponding quantization problem unifies the quantization problems for (finite-dimensional) Poisson manifolds and Lie bialgebras. It was spelled out by P. Xu [74, 76] who then conjectured that any Lie bialgebroid is quantizable. The main result of this note consists in providing some arguments for the non-existence of *universal* quantizations of Lie bialgebroids. This is done by exhibiting a potential obstruction to the existence of a universal quantization map for Lie bialgebroids in the guise of an exotic Lie_∞-structure on the deformation complex of Lie bialgebroids. The latter is a non-trivial deformation of the so-called "big bracket" for Lie bialgebroids and can be considered as an avatar in $d = 3$ of the Kontsevich–Shoikhet obstruction to the quantization of infinite-dimensional Poisson manifolds. Our main result is stated as follows:

Theorem 1.1 (no-go). *The deformation complex of Lie bialgebroids is endowed with an exotic* Lie_∞*-structure deforming non-trivially the big bracket of Lie bialgebroids.*

It follows from the above considerations that the deformation quantization problem for Lie bialgebroids differs essentially from its Lie bialgebra counterpart and is in fact more akin to the one for infinite-dimensional Poisson manifolds, i.e., it belongs to the no-go class. The origin of this obstruction can be traced back to an action of the graph complex of *bi-oriented* graphs (i.e., graphs with two oriented colors) on the deformation theory of Lie bialgebroids. Relaxing the orientation of one of the colors yields an action on the deformation theory of Lie-quasi bialgebroids (and their dual). As a corollary, we find an action of the Grothendieck–Teichmüller group on Lie-quasi bialgebroids generalising the one on Lie-(quasi) bialgebras.

Theorem 1.2 (yes-go). *The Grothendieck–Teichmüller group acts via* Lie_∞*-automorphisms on the deformation complex of both Lie-quasi bialgebroids and quasi-Lie bialgebroids.*

Hence, the deformation quantization problem for Lie-quasi bialgebroids differs from its Lie bialgebroid counterpart and resembles more closely the one for Lie bialgebras, i.e., it belongs to the Yes-go class. We conjecture on the basis of these results the existence – given a Drinfeld associator – of a universal quantization for Lie-quasi bialgebroids (and their dual).

The graph complex approach to deformation quantization[5] is summed up in Table 1, where the original contribution of the present paper lies at $c = 2$.[6]

Organisation of this paper. The original universal model introduced by M. Kontsevich [32] takes advantage of the graded geometric interpretation of Poisson manifolds as[7] dg symplectic manifolds of degree 1. Correspondingly, Section 2 reviews the formulation of Lie bialgebras and Lie bialgebroids (as well as their generalisations Lie-quasi, quasi-Lie and proto-Lie) as particular dg symplectic manifolds of degree 2. This graded geometric description of the deformation theory of Lie bialgebroids and generalisations thereof will be instrumental in formulating associated universal models in Section 4.

The class of universal models introduced in this note involves *multi-oriented* graphs, as introduced in [77] and studied in [47] in the context of multi-oriented props and their representations on homotopy algebras with branes. The main definitions and results regarding the cohomology of multi-oriented graph complexes are reviewed in Section 3.

Following these two review sections, we introduce our main results in Section 4. We start by reviewing the known action of the (one-colored) oriented graph complex on Lie-(quasi) bialgebras in Section 4.2 and then move on to the Lie bialgebroid case in Section 4.3. Using the

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[5]Note that the case $d = 1$ is somehow special among Table 1 as there is no associated quantization problem. The relevant cohomological class $H^1(\mathsf{fcGC}_1) \simeq \mathbb{K}$ is therefore not viewed as an obstruction to quantization but rather as a non-trivial deformation of the symplectic Poisson bracket as a Lie algebra, yielding the Moyal bracket on symplectic manifolds (cf. Section 3.2 for additional details).

[6]The graph complex $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$ featured in Table 1 lives in dimension d (cf. footnote 3) and involves graphs with $c = i + j$ directed colors, i of them are oriented, cf. Section 3 for details.

[7]In the remaining of the text, we use the prefix dg to refer to differential graded objects.

Table 1. Classification of deformation quantization problems via graph cohomology.

		Graph complexes $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$		
Class name		no-go	yes-go	middle way
Model cohomology		$H^0(\mathsf{fcGC}_1) \simeq \mathbf{0}$	$H^0(\mathsf{fcGC}_2) \simeq \mathfrak{grt}_1$	$H^0(\mathsf{fcGC}_3) \simeq \mathbb{K}$
		$H^1(\mathsf{fcGC}_1) \simeq \mathbb{K}$	$H^1(\mathsf{fcGC}_2) \overset{?}{\simeq} \mathbf{0}$	$H^1(\mathsf{fcGC}_3) \simeq \mathbf{0}$
Oriented directions i		$d-1$	$d-2$	$d-3$
d	$c=i+j$	*Actions*		
$d=1$		symplectic manifolds		
$d=2$	$c=1$	Poisson $(\dim=\infty)$	Poisson $(\dim<\infty)$	
$d=3$			Lie bialgebras Lie-quasi bialgebras quasi-Lie bialgebras	proto-Lie bialgebras Courant algebroids
	$c=2$	Lie bialgebroids	Lie-quasi bialgebroids quasi-Lie bialgebroids	proto-Lie bialgebroids

cohomological results reviewed in Section 3, we prove our main results regarding the existence of an exotic Lie_∞-structure for Lie bialgebroids (Theorem 1.1) and the action of the Grothendieck–Teichmüller group on Lie-quasi bialgebroids (Theorem 1.2).

In view of the results of Section 4.3, we formulate two conjectures in Section 4.4: a no-go (Conjecture 4.11) regarding the existence of (universal) quantizations for Lie bialgebroids and a yes-go (Conjecture 4.12) regarding the one of Lie-quasi bialgebroids (and their dual).

Two appendices conclude the present note. Appendix A reviews the (ungraded) geometric formulation of Lie bialgebroids and related notions, to be compared with the graded geometric interpretation of Section 2. Appendix B contains explicit formulae and additional results regarding the graph cocycle generating the exotic Lie_∞-structure for Lie bialgebroids.

Conventions. Throughout the text, we work over a ground field $\mathbb{K}$ of characteristic zero. The operads introduced in the text live in the category of (graded) vector spaces over $\mathbb{K}$. Given a graded $\mathbb{K}$-vector space $\mathfrak{g} := \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k$, the n-suspended graded vector space $\mathfrak{g}[n]$ is defined via its homogeneous components $\mathfrak{g}[n]^k := \mathfrak{g}^{k+n}$. We will denote $s\colon \mathfrak{g}[n] \to \mathfrak{g}$ the corresponding suspension map of intrinsic degree n.

2   Graded geometry

It is a well-known result that a Lie algebra structure on a vector space $\mathfrak{g}$ yields a differential structure on the exterior algebra $\wedge^\bullet \mathfrak{g}^*$ in the guise of the Chevalley–Eilenberg differential. The exterior algebra can equivalently be recast as the algebra of functions on the shifted vector space $\mathfrak{g}[1]$ seen as a graded manifold of degree 1 on which the Chevalley–Eilenberg differential defines a homological vector field. Such a supergeometric formulation of Lie algebras was generalised by A.Yu. Vaintrob [67] who showed a bijective correspondence between dg manifolds (or NQ-manifolds) of degree 1 and Lie algebroids. On the other hand, it was shown by D. Roytenberg [58, 59] that dg symplectic manifolds (or NPQ-manifolds) of degree 1 (resp. of degree 2) are

in bijective correspondence with Poisson manifolds (resp. Courant algebroids). Our aim in this section is to review how Lie bialgebra and Lie bialgebroid structures (and generalisations[8]) can be naturally recast as Hamiltonian functions for a specific graded Poisson algebra of functions on a graded manifold (as pioneered in [39], cf. [37, 58] for details and related constructions). We start by reviewing this graded geometric construction for Lie bialgebras (including proto-Lie, Lie-quasi and quasi-Lie bialgebras) in Section 2.1 and then move on to the Lie bialgebroid counterparts of these notions in Section 2.2.

2.1 Lie bialgebras

Lie bialgebra structures (and generalisations thereof) on a vector space $\mathfrak{g}$ can be conveniently encoded into particular Hamiltonian functions on the graded manifold $T^*(\mathfrak{g}[1]) \simeq (\mathfrak{g} \oplus \mathfrak{g}^*)[1]$ with homogeneous coordinates[9] $\{\xi^a_1, \zeta_a_1\}$, with $a \in \{1, \ldots, \dim \mathfrak{g}\}$. The latter is a graded symplectic manifold with symplectic 2-form $\Omega = \mathrm{d}\xi^a \wedge \mathrm{d}\zeta_a$ of degree 2. The associated Poisson bracket of degree -2 acts on homogeneous functions in $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ as

$$\{f, g\}^{\mathfrak{g}}_\Omega = (-1)^{|f|} \left(\frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \zeta_a} + \frac{\partial f}{\partial \zeta_a} \frac{\partial g}{\partial \xi^a} \right). \tag{2.1}$$

The graded Poisson bracket (2.1) can be seen as the graded geometric formulation of the "big bracket" (introduced by Y. Kosmann–Schwarzbach [35]) acting on $\wedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*) \simeq \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$.

Upgrading the graded symplectic manifold $(\mathfrak{g} \oplus \mathfrak{g}^*)[1]$ to a dg symplectic manifold (or NPQ-manifold)[10] allows to define various algebraic structures. A differential structure on a graded symplectic manifold is given by a vector field Q of degree 1 being homological (i.e., $[\mathsf{Q}, \mathsf{Q}]_{\mathsf{Lie}} = 0$) with respect to the graded Lie bracket of vector fields and compatible with the symplectic 2-form (i.e., $\mathcal{L}_\mathsf{Q} \Omega = 0$). This last compatibility relation ensures[11] that Q is necessarily a Hamiltonian vector field, i.e., there exists a function of degree 3 called the Hamiltonian satisfying the structure equation $\{\mathscr{H}, \mathscr{H}\}^{\mathfrak{g}}_\Omega = 0$ and such that $\mathsf{Q} := \{\mathscr{H}, \cdot\}^{\mathfrak{g}}_\Omega$. The most general function of degree 3 on $(\mathfrak{g} \oplus \mathfrak{g}^*)[1]$ reads explicitly as[12]

$$\mathscr{H} = -\tfrac{1}{2} f_{ab}{}^c \xi^a \xi^b \zeta_c - \tfrac{1}{2} C_c{}^{ab} \zeta_a \zeta_b \xi^c + \tfrac{1}{6} \varphi^{abc} \zeta_a \zeta_b \zeta_c + \tfrac{1}{6} \psi_{abc} \xi^a \xi^b \xi^c, \tag{2.2}$$

where[13] $f_{ab}{}^c = f_{[ab]}{}^c$, $C_c{}^{ab} = C_c{}^{[ab]}$, $\varphi^{abc} = \varphi^{[abc]}$ and $\psi_{abc} = \psi_{[abc]}$.

The Hamiltonian condition $\{\mathscr{H}, \mathscr{H}\}^{\mathfrak{g}}_\Omega = 0$ translates as a set of 5 constraints on the defining maps $\{f, C, \varphi, \psi\}$:

- $\mathcal{D}_{1abc}{}^d := -f_{e[a}{}^d f_{bc]}{}^e - \psi_{e[ab} C_{c]}{}^{ed} = 0,$ (2.3)
- $\mathcal{D}_{2d}{}^{abc} := -C_d{}^{e[a} C_e{}^{bc]} - \varphi^{e[ab} f_{ed}{}^{c]} = 0,$ (2.4)
- $\mathcal{D}_{3ab}{}^{cd} := 2 f_{e[a}{}^{[c} C_{b]}{}^{d]e} - \tfrac{1}{2} f_{ab}{}^e C_e{}^{cd} - \tfrac{1}{2} \psi_{eab} \varphi^{ecd} = 0,$ (2.5)
- $\mathcal{D}_4{}^{abcd} := \tfrac{1}{2} \varphi^{e[ab} C_e{}^{cd]} = 0,$ (2.6)
- $\mathcal{D}_{5abcd} := \tfrac{1}{2} \psi_{e[ab} f_{cd]}{}^e = 0.$ (2.7)

A set of maps $\{f, C, \varphi, \psi\}$ satisfying the constraints (2.3)–(2.7) form the components of a *proto-Lie bialgebra* on $(\mathfrak{g}, \mathfrak{g}^*)$ (cf. Appendix A for a definition) whose deformation theory is therefore

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[8] As reviewed in Appendix A.

[9] The subscript denotes the corresponding degree.

[10] Or equivalently, endowing the graded Poisson algebra of functions $(\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]), \cdot, \{\cdot, \cdot\}^{\mathfrak{g}}_\Omega)$ with a compatible differential.

[11] Via Cartan's homotopy formula, cf. [59, Lemma 2.2].

[12] The signs and coefficients are chosen for later convenience.

[13] Here and in the following, round (resp. square) brackets of indices will denote (skew)symmetrisation.

controlled by the dg Lie algebra[14] $\left(\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]), \mathsf{Q}, \{\cdot, \cdot\}_\Omega^\mathfrak{g}\right)$. Proto-Lie bialgebras thus constitute the most general notion in the bialgebra realm and other structures (Lie-quasi, quasi-Lie and Lie bialgebras) will be defined as particular cases thereof.

The remainder of this section will therefore introduce several particular graded Poisson subalgebras of $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ whose Hamiltonian functions will encode various sub-classes of proto-Lie bialgebras. Let us start by defining the subspace $\mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g} \subset \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ as $\mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g} := \left\{f \in \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]) \,\middle|\, f|_{\zeta=0} = 0\right\}$. In plain words, the subspace $\mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g}$ is obtained by discarding all functions of the form $\psi_{a_1 \cdots a_m} \xi^{a_1} \cdots \xi^{a_m}$, for arbitrary values of $m \geq 0$. It can be easily checked that $\mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g}$ is preserved by both the pointwise product of functions and the graded Poisson bracket (2.1) and thus defines a graded Poisson subalgebra of $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$. The most general Hamiltonian function of $\mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g}$ reads as (2.2) with $\psi \equiv 0$, where the maps $\{f, C, \varphi\}$ satisfy (2.3)–(2.6) with $\psi \equiv 0$. In particular, imposing $\psi \equiv 0$ in equation (2.3) ensures that the map f defines a genuine Lie algebra structure on $\mathfrak{g}$ (while the structure defined on $\mathfrak{g}^*$ is still "quasi" due to the presence of φ). The resulting equations reproduce the defining conditions of a *Lie-quasi bialgebra* on $(\mathfrak{g}, \mathfrak{g}^*)$ as introduced by Drinfeld in [16] as semi-classicalisation of the notion of quasi-bialgebra.[15]

Dually to the previous case, one defines the graded Poisson subalgebra $\mathcal{A}_{\text{quasi-Lie}}^\mathfrak{g}$ as the subspace obtained by discarding all functions of the form $\varphi^{a_1 \cdots a_n} \zeta_{a_1} \cdots \zeta_{a_n}$, for all $n \geq 0$, i.e., $\mathcal{A}_{\text{quasi-Lie}}^\mathfrak{g} := \left\{f \in \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]) \,\middle|\, f|_{\xi=0} = 0\right\} \subset \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$. The most general Hamiltonian function of $\mathcal{A}_{\text{quasi-Lie}}^\mathfrak{g}$ reads as (2.2) with $\varphi \equiv 0$, where the functions $\{f, C, \psi\}$ satisfy (2.3)–(2.5) and (2.7) with $\varphi \equiv 0$. Dually to the Lie-quasi case, setting $\varphi \equiv 0$ in equation (2.4) ensures that the map C defines a genuine Lie algebra structure on $\mathfrak{g}^*$ (while the structure defined on $\mathfrak{g}$ is only "quasi" Lie due to the presence of ψ). The resulting equations reproduce the defining conditions of a *quasi-Lie bialgebra* on $(\mathfrak{g}, \mathfrak{g}^*)$ as introduced and studied in [3, 35].

Finally, let us define the subspace $\mathcal{A}_{\text{Lie}}^\mathfrak{g} := \left\{f \in \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]) \,\middle|\, f|_{\xi=0} = 0 \text{ and } f|_{\zeta=0} = 0\right\}$, i.e., $\mathcal{A}_{\text{Lie}}^\mathfrak{g}$ is defined as the intersection $\mathcal{A}_{\text{Lie}}^\mathfrak{g} := \mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g} \cap \mathcal{A}_{\text{quasi-Lie}}^\mathfrak{g}$ between the two previous Poisson subalgebras. The latter subspace can again be checked to be a graded Poisson subalgebra of $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ obtained by discarding all functions of the form $\psi_{a_1 \cdots a_m} \xi^{a_1} \cdots \xi^{a_m}$ and $\varphi^{a_1 \cdots a_n} \zeta_{a_1} \cdots \zeta_{a_n}$ for all $m, n \geq 0$. In particular, the most general Hamiltonian function of $\mathcal{A}_{\text{Lie}}^\mathfrak{g}$ reads as (2.2) with $\varphi \equiv 0$ and $\psi \equiv 0$, where the functions $\{f, C\}$ satisfy (2.3)–(2.5) with $\varphi \equiv 0$ and $\psi \equiv 0$. In particular, constraint (2.3) (resp. (2.4)) ensures that the map f (resp. C) defines a genuine Lie algebra structure on $\mathfrak{g}$ (resp. $\mathfrak{g}^*$). These two Lie algebras are furthermore compatible with each other due to (2.5) and hence define a *Lie bialgebra* on $(\mathfrak{g}, \mathfrak{g}^*)$ (cf. (A.1)). We sum up the previous discussion in the following proposition:

Proposition 2.1. *Let $\mathfrak{g}$ be a vector space. The following correspondences hold:*

- *Hamiltonians in $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ are in bijective correspondence with proto-Lie bialgebra structures on $(\mathfrak{g}, \mathfrak{g}^*)$.*

- " $\mathcal{A}_{\text{Lie-quasi}}^\mathfrak{g}$ "   *Lie-quasi bialgebra*   " .

- " $\mathcal{A}_{\text{quasi-Lie}}^\mathfrak{g}$ "   *quasi-Lie bialgebra*   " .

- " $\mathcal{A}_{\text{Lie}}^\mathfrak{g}$ "   *Lie bialgebra*   " .

This interpretation of the deformation theory for Lie bialgebras and generalisations as graded Poisson algebras will be put to use in Section 4 where will be discussed universal models thereof.

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[14]Note that the graded Poisson bracket has intrinsic degree -2 on $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$. To recover the usual grading, one needs to consider the 2-suspension $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])[2]$.

[15]Remark that Lie-quasi bialgebras were denoted "quasi-Lie bialgebras" in [16]. We follow the terminology used in [37] where the term Lie-quasi bialgebras is used for Lie algebras which fail to be Lie bialgebras (so that they are only "quasi" bialgebras) while the term quasi-Lie was reserved for the dual counterpart (not considered in [16]) where the Jacobi identity for f is only "quasi" satisfied.

In the next section, we turn to the generalisation of this graded geometric interpretation to the larger class of Lie bialgebroids and variations thereof.

2.2 Lie bialgebroids

Letting $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle over the smooth (finite-dimensional) manifold[16]$\mathcal{M}$, the relevant graded Poisson algebra is the algebra of functions of the graded symplectic manifold $T^*[2]E[1]$, defined as the (2-shifted) cotangent bundle of the (1-shifted[17]) vector bundle E, and denoted $\mathscr{E} \equiv T^*[2]E[1]$ in the following. The graded manifold $\mathscr{E}$ is of degree 2 and can be locally[18] coordinatised by the set of homogeneous coordinates $\left\{ \underset{0}{x^\mu}, \underset{1}{\xi^a}, \underset{1}{\zeta_a}, \underset{2}{p_\mu} \right\}$ so that the symplectic 2-form of degree 2 can be written as

$$\Omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^a \wedge \mathrm{d}\zeta_a.$$

The associated Poisson bracket of degree -2 acts as follows on homogeneous functions $f, g \in \mathscr{C}^\infty(\mathscr{E})$:

$$\{f,g\}_\Omega^E = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} + (-1)^{|f|}\left(\frac{\partial f}{\partial \xi^a}\frac{\partial g}{\partial \zeta_a} + \frac{\partial f}{\partial \zeta_a}\frac{\partial g}{\partial \xi^a} \right). \tag{2.8}$$

The latter is sometimes referred to as the "big bracket" for Lie bialgebroids. Upgrading the graded symplectic manifold $\mathscr{E}$ to a dg symplectic manifold will allow to define various geometric structures. Following the same path as in the bialgebra case, we introduce a compatible differential through a Hamiltonian function. The most general function of degree 3 on $\mathscr{E}$ reads

$$\mathscr{H} = \rho_a{}^\mu(x)\xi^a p_\mu - \tfrac{1}{2}f_{ab}{}^c(x)\xi^a\xi^b\zeta_c + R^{a|\mu}(x)\zeta_a p_\mu - \tfrac{1}{2}C_c{}^{ab}(x)\zeta_a\zeta_b\xi^c$$
$$+ \tfrac{1}{6}\varphi^{abc}(x)\zeta_a\zeta_b\zeta_c + \tfrac{1}{6}\psi_{abc}(x)\xi^a\xi^b\xi^c, \tag{2.9}$$

where $\{\rho, f, R, C, \varphi, \psi\}$ are functions on the base space $\mathcal{M}$, with symmetries $f_{ab}{}^c = f_{[ab]}{}^c$, $C_c{}^{ab} = C_c{}^{[ab]}$, $\varphi^{abc} = \varphi^{[abc]}$ and $\psi_{abc} = \psi_{[abc]}$. Imposing the Hamiltonian constraint $\left\{\mathscr{H}, \mathscr{H}\right\}_\Omega^E = 0$ yields a set of 9 conditions on the defining functions $\{\rho, f, R, C, \varphi, \psi\}$ that we denote as follows

- $\mathcal{C}_{1ab}{}^\mu := 2\rho_{[a}{}^\lambda \partial_\lambda \rho_{b]}{}^\mu - \rho_c{}^\mu f_{ab}{}^c + R^{c|\mu}\psi_{cab} = 0,$ \hfill (2.10)

- $\mathcal{C}_{2abc}{}^d := \rho_{[a}{}^\lambda \partial_\lambda f_{bc]}{}^d - f_{e[a}{}^d f_{bc]}{}^e + \tfrac{1}{3}R^{d|\lambda}\partial_\lambda \psi_{abc} - \psi_{e[ab}C_{c]}{}^{ed} = 0,$ \hfill (2.11)

- $\mathcal{C}_3{}^{ab|\mu} := 2R^{[a|\lambda}\partial_\lambda R^{b]|\mu} - R^{c|\mu}C_c{}^{ab} + \rho_c{}^\mu \varphi^{cab} = 0,$ \hfill (2.12)

- $\mathcal{C}_{4d}{}^{abc} := R^{[a|\lambda}\partial_\lambda C_d{}^{bc]} - C_d{}^{e[a}C_e{}^{bc]} + \tfrac{1}{3}\rho_d{}^\lambda \partial_\lambda \varphi^{abc} - \varphi^{e[ab}f_{ed}{}^{c]} = 0,$ \hfill (2.13)

- $\mathcal{C}_5{}^{\mu\nu} := R^{a(\mu}\rho_a{}^{\nu)} = 0,$ \hfill (2.14)

- $\mathcal{C}_{6a}{}^{b|\mu} := \rho_a{}^\lambda \partial_\lambda R^{b|\mu} - R^{b|\lambda}\partial_\lambda \rho_a{}^\mu - \rho_c{}^\mu C_a{}^{bc} - R^{c|\mu}f_{ca}{}^b = 0,$ \hfill (2.15)

- $\mathcal{C}_{7ab}{}^{cd} := \rho_{[a}{}^\lambda \partial_\lambda C_{b]}{}^{cd} + R^{[c|\lambda}\partial_\lambda f_{ab}{}^{d]} + 2f_{e[a}{}^{[c}C_{b]}{}^{d]e} - \tfrac{1}{2}f_{ab}{}^e C_e{}^{cd} - \tfrac{1}{2}\psi_{eab}\varphi^{ecd} = 0,$ \hfill (2.16)

- $\mathcal{C}_8{}^{abcd} := \tfrac{1}{3}R^{[d|\lambda}\partial_\lambda \varphi^{abc]} + \tfrac{1}{2}\varphi^{e[ab}C_e{}^{cd]} = 0,$ \hfill (2.17)

- $\mathcal{C}_{9abcd} := \tfrac{1}{3}\rho_{[d}{}^\lambda \partial_\lambda \psi_{abc]} + \tfrac{1}{2}\psi_{e[ab}f_{cd]}{}^e = 0.$ \hfill (2.18)

The latter constraints identify with the component expressions of the defining conditions of a *proto-Lie bialgebroid* on (E, E^*) (compare with (A.2)–(A.6)). The graded Poisson algebra $\mathscr{C}^\infty(\mathscr{E})$ admits several subalgebras defining in turn various sub-classes of proto-Lie bialgebroids. A convenient way to characterise these Poisson subalgebras is as vanishing ideals of

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[16]The restriction from the "algebroid" case to the "algebra" case can be done by assuming that $\mathcal{M}$ is the one-point manifold so that $E \simeq \mathfrak{g}$ becomes a $\mathbb{K}$-vector space.

[17]Here and in the following, $A[n]$ will denote the vector bundle obtained by shifting the grading of the fiber of the vector bundle A by n.

[18]Correspondingly, all formulae appearing in this note will be local.

particular Lagrangian submanifolds.[19] As noted by D. Roytenberg [60], both $E[1]$ and $E^*[1]$ are Lagrangian submanifolds of $\mathscr{E}$, thus motivating to consider the Poisson subalgebras of functions vanishing on them. We start by defining the Poisson subalgebra $\mathcal{A}^E_{\text{Lie-quasi}} \subset \mathscr{C}^\infty(\mathscr{E})$ as the vanishing ideal of $E[1]$. In plain words, the subalgebra $\mathcal{A}^E_{\text{Lie-quasi}}$ is obtained by discarding all functions of the form $\psi_{a_1 \cdots a_m}(x)\xi^{a_1} \cdots \xi^{a_m}$, for all $m \geq 0$. The most general Hamiltonian function of $\mathcal{A}^E_{\text{Lie-quasi}}$ reads as (2.9) with $\psi \equiv 0$, where the functions $\{\rho, f, R, C, \varphi\}$ satisfy (2.10)–(2.17) with $\psi \equiv 0$. In particular, setting $\psi \equiv 0$ in equations (2.10)–(2.11) ensures that the pair $\{\rho, f\}$ defines a genuine Lie algebroid structure on E. The resulting equations reproduce the defining conditions of a *Lie-quasi bialgebroid* on (E, E^*). The pair $\big((\mathscr{E}, \Omega, \mathscr{H}), E[1]\big)$ defines a Manin pair,[20] in the terminology of Roytenberg [60].

Dually to the previous case, we define the graded Poisson subalgebra $\mathcal{A}^E_{\text{quasi-Lie}} \subset \mathscr{C}^\infty(\mathscr{E})$ as the vanishing ideal of $E^*[1]$, i.e., as the subspace obtained by discarding all functions of the form $\varphi^{a_1 \cdots a_n}(x)\zeta_{a_1} \cdots \zeta_{a_n}$ with $n \geq 0$. The most general Hamiltonian function of $\mathcal{A}^E_{\text{quasi-Lie}}$ reads as (2.9) with $\varphi \equiv 0$, where the functions $\{\rho, f, R, C, \psi\}$ satisfy (2.10)–(2.16) and (2.18) with $\varphi \equiv 0$. Dually to the Lie-quasi case, setting $\varphi \equiv 0$ in equations (2.12)–(2.13) ensures that the pair $\{R, C\}$ defines a genuine Lie algebroid structure on E^* (while the structure defined on E by $\{\rho, f\}$ is still "quasi" due to the presence of ψ). The resulting equations reproduce the defining conditions of a dual structure on (E, E^*), dubbed *quasi-Lie bialgebroid* in [37]. Quasi-Lie bialgebroids are then equivalently characterised as Manin pairs of the form $\big((\mathscr{E}, \Omega, \mathscr{H}), E^*[1]\big)$.

We conclude by defining the subspace $\mathcal{A}^E_{\text{Lie}} := \mathcal{A}^E_{\text{Lie-quasi}} \cap \mathcal{A}^E_{\text{quasi-Lie}}$. The latter subspace can again be checked to be a graded Poisson subalgebra of $\mathscr{C}^\infty(\mathscr{E})$ obtained by discarding all functions of the form $\psi_{a_1 \cdots a_m}(x)\,\xi^{a_1} \cdots \xi^{a_m}$ or $\varphi^{a_1 \cdots a_n}(x)\zeta_{a_1} \cdots \zeta_{a_n}$, for all $m, n \geq 0$. In particular, the most general Hamiltonian function of $\mathcal{A}^E_{\text{Lie}}$ reads as (2.9) with $\varphi \equiv 0$ and $\psi \equiv 0$, where the functions $\{\rho, f, R, C\}$ satisfy (2.10)–(2.16) with $\varphi \equiv 0$ and $\psi \equiv 0$. Constraints (2.10)–(2.11) (resp. (2.12)–(2.13)) ensure that the pair $\{\rho, f\}$ (resp. $\{R, C\}$) defines a genuine Lie algebroid structure on E (resp. E^*). These two Lie algebroids are furthermore compatible with each other due to (2.13)–(2.16) and hence define a *Lie bialgebroid* structure on (E, E^*) (cf. Appendix A). Lie bialgebroids are furthermore equivalently characterised[21] as Manin triples[22] of the form $((\mathscr{E}, \Omega, \mathscr{H}), E[1], E^*[1])$.

As usual, gauge transformations for the Hamiltonian function (2.9) are generated by functions[23] of degree 2 in $\mathscr{C}^\infty(\mathscr{E})$ reading explicitly:

$$\mathscr{X} = X^\mu p_\mu + \lambda^a{}_b \xi^b \zeta_a - \tfrac{1}{2}\Lambda^{ab}\zeta_a \zeta_b - \tfrac{1}{2}\omega_{ab}\xi^a \xi^b, \tag{2.19}$$

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[19]Letting $\mathcal{N}$ be a (graded) manifold, the *vanishing ideal* of a (graded) submanifold $\mathcal{C} \subset \mathcal{N}$ is defined as the subalgebra of functions $I_\mathcal{C} := \big\{ f \in \mathscr{C}^\infty(\mathcal{N}) \,\big|\, f|_\mathcal{C} = 0 \big\}$. Moreover, a (graded) submanifold $\mathcal{C}$ of a (graded) symplectic manifold $(\mathcal{N}, \Omega)$ is said to be *Lagrangian* if it is maximally isotropic, i.e.,

  1. The restriction of the symplectic form Ω to $\mathcal{C}$ vanishes, i.e., $\Omega|_\mathcal{C} = 0$.
  2. The submanifold $\mathcal{C}$ has maximal dimension $\dim \mathcal{C} = \frac{1}{2}\dim \mathcal{N}$.

For graded Lagrangian submanifolds, it will be assumed that the underlying bosonic manifold of $\mathcal{C}$ (whose algebra of functions is coordinatised by coordinates of degree 0) identifies with the one of $\mathcal{N}$. Lagrangian submanifolds are coisotropic, so that the corresponding vanishing ideal is closed under the (graded) Poisson bracket associated to Ω and hence is a Poisson subalgebra of the algebra of functions on $\mathcal{N}$.

[20]A *Manin pair* of degree n is defined as a dg symplectic manifold $(\mathcal{N}, \Omega, \mathscr{H})$ of degree n supplemented with a dg Lagrangian submanifold (also called a Λ-structure [62]), i.e., a graded Lagrangian submanifold $\mathcal{C} \subset \mathcal{N}$ such that $\mathscr{H} \in I_\mathcal{C}$, with $I_\mathcal{C}$ the vanishing ideal of $\mathcal{C}$. Manin pairs of degree 0 correspond to a pair formed by a (bosonic) symplectic manifold endowed with a Lagrangian submanifold. Manin pairs of degree 1 are in bijective correspondence with pairs composed of a Poisson manifold together with a coisotropic submanifold while Manin pairs of degree 2 identify with Courant algebroids endowed with a Dirac structure [62].

[21]We refer to Appendix A for analogues of these statements within the framework of Courant algebroids.

[22]A *Manin triple* is a dg symplectic manifold supplemented with two *transverse* dg Lagrangian submanifolds $\mathcal{C}, \mathcal{D} \subset \mathcal{N}$ such that $\mathscr{H} \in I_\mathcal{C}$ and $\mathscr{H} \in I_\mathcal{D}$.

[23]Gauge transformations for Lie-quasi (resp. quasi-Lie) bialgebroids are generated by (2.19) with $\omega \equiv 0$ (resp. $\Lambda \equiv 0$), and with $\Lambda \equiv 0 \equiv \omega$ for Lie bialgebroids.

so that the gauge algebra for proto-Lie bialgebroids is isomorphic to $\Gamma(T\mathscr{M}) \oplus \mathsf{End}(\Gamma(E)) \oplus \Gamma(\wedge^2 E) \oplus \Gamma(\wedge^2 E^*)$ as a vector space. We refer to [60, Section 5] for a careful treatment of the group structure of these gauge transformations. The various parameters in (2.19) can be interpreted[24] in terms of infinitesimal morphisms of proto-Lie bialgebroids as follows:

- $X \in \Gamma(T\mathscr{M})$: diffeomorphism of $\mathscr{M}$,
- $\lambda \in \mathsf{End}(\Gamma(E))$: rotation of the fibers of (E, E^*),
- $\Lambda \in \Gamma(\wedge^2 E)$ twist of Lie-quasi bialgebroids,
- $\omega \in \Gamma(\wedge^2 E^*)$ twist of quasi-Lie bialgebroids,

where the denomination *twist* refers to the following construction: given a proto-Lie bialgebroid on (E, E^*) defined by the Hamiltonian function $\mathscr{H}$, one can define a new proto-Lie bialgebroid structure on (E, E^*) via twisting[25] by an arbitrary bivector $\Lambda \in \Gamma(\wedge^2 E)$. Explicitly, the twisted proto-Lie bialgebroid $\mathscr{H}_\Lambda$ is obtained by performing a canonical transformation generated by the flow of the Hamiltonian vector field $\mathscr{X}_\Lambda := \{-\Lambda, \cdot\}_\Omega^E$ where $\Lambda := \frac{1}{2}\Lambda^{ab}\zeta_a\zeta_b$ is a function of degree 2 in $\mathscr{C}^\infty(\mathscr{E})$. Explicitly, a twist by Λ corresponds to the canonical transformation[26]

$$x^\mu \xrightarrow{\Lambda} x^\mu, \qquad p_\mu \xrightarrow{\Lambda} p_\mu - \tfrac{1}{2}\partial_\mu \Lambda^{ab}\zeta_a\zeta_b, \qquad \xi^a \xrightarrow{\Lambda} \xi^a - \Lambda^{ab}\zeta_b, \qquad \zeta_a \xrightarrow{\Lambda} \zeta_a,$$

which amounts to a shift of the components of $\mathscr{H}$ (cf. equation (A.7) in Appendix A for explicit expressions). The previous canonical transformation maps Lie-quasi bialgebroids to Lie-quasi bialgebroids but generically fails to preserve quasi-Lie bialgebroids, as is consistent with the fact that $\Lambda \notin \mathcal{A}^E_{\text{quasi-Lie}}$ so that $\Gamma(\wedge^2 E)$ is not part of the gauge algebra for quasi-Lie bialgebroids.[27] We refer to Appendix A for more details on the twisting procedure.

We sum up the previous discussion in the following proposition, generalising Proposition 2.1 to bialgebroids:

Proposition 2.2. *Let $E \xrightarrow{\pi} \mathscr{M}$ be a vector bundle. The following correspondences hold:*

- *Hamiltonians in $\mathscr{C}^\infty(\mathscr{E})$ are in bijective correspondence with proto-Lie bialgebroid structures on (E, E^*).*
- *" $\mathcal{A}^E_{\text{Lie-quasi}}$ "   Lie-quasi bialgebroid   " .*
- *" $\mathcal{A}^E_{\text{quasi-Lie}}$ "   quasi-Lie bialgebroid   " .*
- *" $\mathcal{A}^E_{\text{Lie}}$ "   Lie bialgebroid   " .*

As noted previously, the graded geometric interpretation of algebro-geometric structures is instrumental to the construction of corresponding universal models, formulated in terms of graph complexes. The next section will introduce the relevant graph complexes which will be shown in Section 4 to act on the various (sub)-algebras previously introduced.

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[24]We refer to Appendix A for more details on this interpretation.

[25]The twisting procedure was introduced in [16] for quasi-Hopf algebras, in [35] for quasi-Lie bialgebras and in [60] for quasi-Lie bialgebroids.

[26]Dually, one can consider twisting by a 2-form field $\omega \in \Gamma(\wedge^2 E^*)$ corresponding to the canonical transformation generated by the flow of the Hamiltonian vector field $\mathscr{X}_\omega := \{-\omega, \cdot\}$, where $\omega := \frac{1}{2}\omega_{ab}\,\xi^a\xi^b$:

$$x^\mu \xrightarrow{\omega} x^\mu, \qquad p_\mu \xrightarrow{\omega} p_\mu - \frac{1}{2}\partial_\mu \omega_{ab}\xi^a\xi^b, \qquad \xi^a \xrightarrow{\omega} \xi^a, \qquad \zeta_a \xrightarrow{\omega} \zeta_a - \omega_{ab}\xi^b \qquad (2.20)$$

which amounts to a shift of the components of $\mathscr{H}$ as in equation (A.8).

[27]Dually, the canonical transformation (2.20) preserves the space of quasi-Lie bialgebroids, but generically maps Lie-quasi bialgebroids to proto-Lie bialgebroids. This is consistent with the fact that $\omega \notin \mathcal{A}^E_{\text{Lie-quasi}}$ so that $\Gamma(\wedge^2 E^*)$ is not part of the gauge algebra for quasi-Lie bialgebroids.

3   (Multi)-oriented graph complexes

The aim of this section is to review the definition and main results regarding multi-oriented graph complexes and their cohomology, as introduced and studied in [47, 77, 78] (cf. also [48] for a review). Graph complexes are most clearly defined as deformation complexes of a suitable morphism of operads [49].[28] We start by introducing the relevant graph operads of multi-directed and multi-oriented graphs from a combinatorial point of view (Section 3.1) before moving to the definition of the associated graph complexes. We conclude by discussing known results regarding the cohomology of multi-oriented graph complexes (Section 3.2) by putting the emphasis on some particular classes relevant for our purpose (cf. Section 4).

3.1   Directed, oriented and sourced graphs

We will denote $\mathsf{gra}_{N,k}$ (resp. $\mathsf{dgra}_{N,k}$) the set of multi(di)graphs[29] with N vertices and k directed edges.

   The set $\mathsf{d}_c\mathsf{gra}_{N,k}$ of *multi-directed graphs* with c colors is defined as the set of ordered pairs $(\gamma, \mathfrak{c})$ where:

- $\gamma \in \mathsf{dgra}_{N,k}$ is a multidigraph. We will denote V_γ (resp. E_γ) the set of vertices (resp. edges) of γ.

- $\mathfrak{c}$ stands for a map $\mathfrak{c}\colon E_\gamma \times [c-1] \to \{+,-\}$ where $c \in \mathbb{N}$ stands for the total number of *colors*[30] and $[c-1] := \{1, 2, \ldots, c-1\}$.

A pictorial representation of a multi-directed graph in $\mathsf{d}_c\mathsf{gra}_{N,k}$ can be given by decorating each directed edge of the underlying multidigraph in $\mathsf{dgra}_{N,k}$ with $c-1$ additional arrows of different colors (cf. Figure 1 for an example).
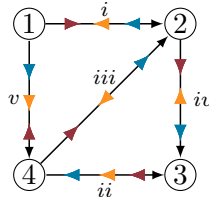

Figure 1.   Example of a multi-directed graph in $\mathsf{d}_4\mathsf{gra}_{4,5}$.

The direction of the arrow of color i on the edge e is aligned with the one of the black arrow if $\mathfrak{c}(e,i) = +$ and opposite to it if $\mathfrak{c}(e,i) = -$. There is a natural right-action of the permutation group $\mathbb{S}_N$ (resp. $\mathbb{S}_k$) on elements of $\mathsf{d}_c\mathsf{gra}_{N,k}$ by permutation of the labeling of vertices (resp. edges).

   Operads.   For all $d \in \mathbb{N}^*$, we define the collection $\{\mathsf{d}_c\mathsf{Gra}_d(N)\}_{N \geq 1}$ of $\mathbb{S}_N$-modules:

$$\mathsf{d}_c\mathsf{Gra}_d(N) := \bigoplus_{k \geq 0} \left(\mathbb{K}\left\langle \mathsf{d}_c\mathsf{gra}_{N,k} \right\rangle \otimes_{\mathbb{S}_k} \mathrm{sgn}_k^{\otimes |d-1|} \right)[k(d-1)], \tag{3.1}$$

where sgn_k denotes the 1-dimensional sign representation of $\mathbb{S}_k$. The subscript stands for taking coinvariants with respect to the diagonal right action of $\mathbb{S}_k$ and the term between brackets denotes degree suspension. In plain words, this means that edges carry an intrinsic degree

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[28]Or equivalently, as convolution Lie algebras constructed from suitable graph operads.
[29]Recall that *multi(di)graphs* are undirected (resp. directed) graphs allowed to contain both loops and multiple edges.
[30]That is, including the underlying (black) arrow of the multidigraph.

$1 - d$ and are bosonic for d odd and fermionic for d even. The $\mathbb{S}$-module $\{d_c\mathsf{Gra}_d(N)\}_{N \geq 1}$ can further be given the structure of an operad[31] by endowing it with the usual equivariant partial composition operations $\circ_i \colon d_c\mathsf{Gra}_d(M) \otimes d_c\mathsf{Gra}_d(N) \to d_c\mathsf{Gra}_d(M + N - 1)$, cf. Figure 2 for an example and, e.g., of [53, Section 4] for more details.

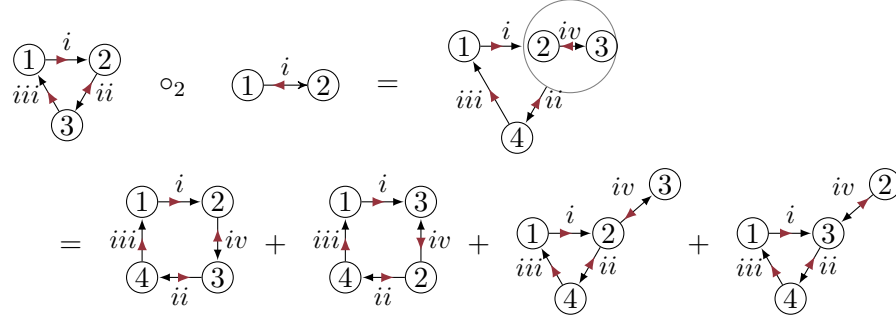**Figure 2.** Example of partial composition $\circ_2 \colon d_2\mathsf{Gra}_d(3) \otimes d_2\mathsf{Gra}_d(2) \to d_2\mathsf{Gra}_d(4)$.

There is a natural sequence of embeddings of operads[32]

$$\mathsf{Gra}_d \xrightarrow{\text{Or}} \mathsf{dGra}_d \xrightarrow{\text{Or}} \mathsf{d_2Gra}_d \xrightarrow{\text{Or}} \mathsf{d_3Gra}_d \xrightarrow{\text{Or}} \cdots \tag{3.2}$$

given by mapping each graph in $d_c\mathsf{Gra}_d$ to a sum of graphs in $d_{c+1}\mathsf{Gra}_d$ where the summation runs over all the possible ways to orient the arrow of the additional direction. We call such mapping the *orientation morphism* $\text{Or} \colon d_c\mathsf{Gra}_d \hookrightarrow d_{c+1}\mathsf{Gra}_d$ (cf. Figure 3 for an example).
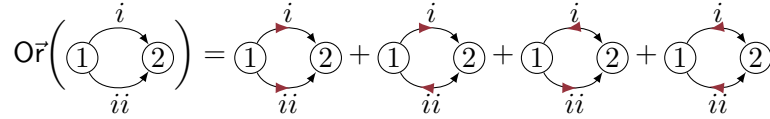

Figure 3. The orientation morphism $\text{Or} \colon \mathsf{dGra}_d \hookrightarrow \mathsf{d_2Gra}_d$.

Denoting $\mathsf{Lie}\{1 - d\}$ the $(1 - d)$-suspended Lie operad,[33] there is an operad morphism

$$\gamma_0 \colon \ \mathsf{Lie}\{1 - d\} \to d_c\mathsf{Gra}_d$$

sending the generator $\bigvee \in \mathsf{Lie}\{1 - d\}(2)$ to the graph

$$\bullet \!\!\rightarrowtail\!\! \bullet := \textcircled{1} \!\!\rightarrowtail\!\! \textcircled{2} + (-1)^d \, \textcircled{2} \!\!\rightarrowtail\!\! \textcircled{1}$$

where the graph $\textcircled{1} \!\!\rightarrowtail\!\! \textcircled{2} \in d_c\mathsf{Gra}_d(2)$ is obtained by decorating $\textcircled{1} \!\!\rightarrow\!\! \textcircled{2}$ with $c - 1$ additional colors and summing over all the possible orientations.[34]

A multidigraph in $\mathsf{dgra}_{N,k}$ will be said *oriented* (or acyclic) if it does not contain cycles.[35] Contrariwise, it will be said non-oriented (or cyclic) if it contains at least one cycle, cf. Figure 4. The subset of oriented multidigraphs will be denoted $\mathsf{ogra}_{N,k} \subset \mathsf{dgra}_{N,k}$. This definition can

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[31] The identity element $\text{id} \in d_c\mathsf{Gra}_d(1)$ is defined as the graph $\text{id} := \textcircled{1}$.

[32] Here, $\mathsf{Gra}_d \equiv d_0\mathsf{Gra}_d$ (resp. $\mathsf{dGra}_d \equiv d_1\mathsf{Gra}_d$) stands for the operad of one-colored undirected (resp. directed) graphs.

[33] Recall that representations of $\mathsf{Lie}\{1 - d\}$ on a vector space $\mathfrak{g}$ are in bijective correspondence with Lie algebra structures on $\mathfrak{g}[1 - d]$, hence the graded Lie bracket on $\mathfrak{g}$ has intrinsic degree $1 - d$.

[34] In other words, $\textcircled{1} \!\!\rightarrowtail\!\! \textcircled{2} := (\text{Or})^{c-1}(\textcircled{1} \!\!\rightarrow\!\! \textcircled{2})$. For example, $\textcircled{1} \!\!\rightarrowtail\!\! \textcircled{2} := \textcircled{1} \!\!\rightarrow\!\! \textcircled{2} + \textcircled{1} \!\!\leftarrow\!\! \textcircled{2}$ in $d_2\mathsf{Gra}_d$.

[35] Recall that a *cycle* (or wheel) is a (non-trivial) directed path from a vertex to itself.
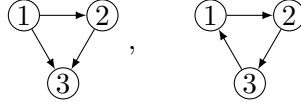
Figure 4. Example of oriented (left) and cyclic (right) graph in $\mathsf{dgra}_{3,3}$.

be extended to multi-directed graphs by defining the subset[36] $\mathsf{o}_i\mathsf{d}_j\mathsf{gra}_{N,k} \subseteq \mathsf{d}_c\mathsf{gra}_{N,k}$ of multi-directed graphs with $c = i + j$ directions for which there exists a subset of i directions – black and/or colored – such that there are no cycles made of the corresponding arrows.[37] Substituting $\mathsf{o}_i\mathsf{d}_j\mathsf{gra}_{N,k}$ in place of $\mathsf{d}_c\mathsf{gra}_{N,k}$ in (3.1) allows to define the collection of $\mathbb{S}_N$-modules $\{\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d(N)\}_{N\geq 1}$ for all $i, j, d \geq 0$. It is easy to check that the latter is closed under partial compositions and hence defines a suboperad $\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d \subseteq \mathsf{d}_c\mathsf{Gra}_d$ of multi-oriented graphs. Note that the graph $\bullet\!\!\rightarrow\!\!\bullet$ is (trivially) multi-oriented and hence defines a morphism of operads

$$\gamma_0 \colon \mathsf{Lie}\{1-d\} \to \mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d \qquad \text{for all } i, j, d \geq 0.$$

Oriented graphs belong to the larger subset of multidigraphs possessing a *source*, i.e., a vertex admitting only outgoing arrows. More generally, the suboperad of multi-directed graphs with $c = |k| + j$ directions such that $|k|$ directions are sourced will be denoted $\mathsf{s}_k\mathsf{d}_j\mathsf{Gra}_d \subseteq \mathsf{d}_c\mathsf{Gra}_d$ for all $k \in \mathbb{Z}$ and $d, j \geq 0$, where negative values of k correspond to directions admitting a *sink*, i.e., a vertex with only ingoing arrows. Finally, the suboperad of graphs such that j directions admit at least one source and one sink will be denoted $\mathsf{s}_{i,-i}\mathsf{d}_j\mathsf{Gra}_d$, for all $i, j, d \geq 0$. It is a well-known result in graph theory that oriented graphs admit at least one source and one sink (see, e.g., [78] for a statement) so that we have a sequence of inclusions

$$\mathsf{o}_{|k|}\mathsf{d}_j\mathsf{Gra}_d \subseteq \mathsf{s}_{|k|,-|k|}\mathsf{d}_j\mathsf{Gra}_d \subseteq \mathsf{s}_k\mathsf{d}_j\mathsf{Gra}_d \qquad \text{for all } k \in \mathbb{Z} \text{ and } d, j \geq 0.$$

Graph complexes. Given the multi-oriented graph operad $\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d$, one defines the dg Lie algebra of multi-oriented graphs $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$ as the deformation complex

$$\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d := \mathrm{Def}\big(\mathsf{Lie}\{1-d\} \xrightarrow{\gamma_0} \mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d\big)$$

of the morphism of operads γ_0.[38] As a graded vector space, $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$ is defined as

- d even: $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d := \bigoplus_{N\geq 1} \big(\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d(N)[d(1-N)]\big)^{\mathbb{S}_N},$ \hfill (3.3)

- d odd: $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d := \bigoplus_{N\geq 1} \big(\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d(N) \otimes \mathrm{sgn}_N[d(1-N)]\big)^{\mathbb{S}_N},$ \hfill (3.4)

where the terms between brackets denote degree suspension while the superscript stands for taking invariants with respect to the right action of $\mathbb{S}_N$, with sgn_N the 1-dimensional signature representation of $\mathbb{S}_N$. In other words, vertices are bosonic for d even and fermionic for d odd. According to the degree suspension in (3.1) and (3.3)–(3.4), the degree of an element[39] $\gamma \in \mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$ with N vertices and k edges is given by $|\gamma| = d(N-1) + k(1-d)$.

The graded Lie bracket $[\cdot, \cdot]$ of degree 0 on $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$ is defined as usual in terms of the partial composition operations (cf., e.g., [53, Section 4.2] for details). The differential $\delta := [\Upsilon_{\mathsf{S}}, \cdot]$ is

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[36]By convention, we identify $\mathsf{o}_0\mathsf{d}_0\mathsf{gra}_{N,k} \equiv \mathsf{gra}_{N,k}$, $\mathsf{o}_0\mathsf{d}_1\mathsf{gra}_{N,k} \equiv \mathsf{dgra}_{N,k}$ and $\mathsf{o}_1\mathsf{d}_0\mathsf{gra}_{N,k} \equiv \mathsf{ogra}_{N,k}$.

[37]As an example, the graph of Figure 1 belongs to $\mathsf{o}_2\mathsf{d}_2\mathsf{gra}_{4,5}$ as it does not contain cycles for black and yellow arrows (although it does so for red and blue arrows). In the right-hand side of Figure 3, the first and fourth graphs belong to $\mathsf{o}_2\mathsf{d}_0\mathsf{gra}_{2,2}$ while the second and third graphs belong to $\mathsf{o}_1\mathsf{d}_1\mathsf{gra}_{2,2}$.

[38]We refer to references [41, 49] for details.

[39]Note that the graph degree is insensitive to the number i of oriented directions.

defined by taking the adjoint action with respect to the Maurer–Cartan element[40]

$$\Upsilon_S := \bullet \blacktriangleright\!\!\blacktriangleright \bullet = \textcircled{1} \blacktriangleright\!\!\blacktriangleright \textcircled{2} + (-1)^d \textcircled{2} \blacktriangleright\!\!\blacktriangleright \textcircled{1}.$$

Note to conclude that the dg Lie algebra $\mathsf{o}_{|k|}\mathsf{d}_j\mathsf{fGC}_d$ of multi-oriented graphs is a sub-dg Lie algebra of the dg Lie algebra of multi-sourced/sinked graphs $\mathsf{s}_k\mathsf{d}_j\mathsf{fGC}_d$ defined for all $k \in \mathbb{Z}$ and $d, j \geq 0$ by[41] substituting $\mathsf{s}_k\mathsf{d}_j\mathsf{Gra}_d(N)$ in place of $\mathsf{o}_k\mathsf{d}_j\mathsf{Gra}_d(N)$ in (3.3)–(3.4).

The various graph operads and complexes discussed in the present section are summarised in Table 2.

Table 2. Summary of graph operads and complexes (and their connected variants) in dimension d.

Undirected graphs with one color (Kontsevich's graphs)				
Gra_d	cGra_d	fGC_d	fcGC_d	$d \geq 0$
Multi-directed graphs with c colors				
$\mathsf{d}_c\mathsf{Gra}_d$	$\mathsf{d}_c\mathsf{cGra}_d$	$\mathsf{d}_c\mathsf{fGC}_d$	$\mathsf{d}_c\mathsf{fcGC}_d$	$c, d \geq 0$
Multi-oriented graphs with $c = i + j$ colors and i oriented directions				
$\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d$	$\mathsf{o}_i\mathsf{d}_j\mathsf{cGra}_d$	$\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$	$\mathsf{o}_i\mathsf{d}_j\mathsf{fcGC}_d$	$i, j, d \geq 0$
Multi-sourced/sinked graphs with $c = |k| + j$ colors and $|k|$ sourced/sinked directions				
$\mathsf{s}_k\mathsf{d}_j\mathsf{Gra}_d$	$\mathsf{s}_k\mathsf{d}_j\mathsf{cGra}_d$	$\mathsf{s}_k\mathsf{d}_j\mathsf{fGC}_d$	$\mathsf{s}_k\mathsf{d}_j\mathsf{fcGC}_d$	$j, d \geq 0,\ k \in \mathbb{Z}$
Multi-sourced/sinked graphs with $c = i + j$ colors and i directions being both sourced and sinked				
$\mathsf{s}_{i,-i}\mathsf{d}_j\mathsf{Gra}_d$	$\mathsf{s}_{i,-i}\mathsf{d}_j\mathsf{cGra}_d$	$\mathsf{s}_{i,-i}\mathsf{d}_j\mathsf{fGC}_d$	$\mathsf{s}_{i,-i}\mathsf{d}_j\mathsf{fcGC}_d$	$i, j, d \geq 0$

3.2 Cohomology

Having introduced the graph complex of multi-directed graphs as well as its sub-complexes of multi-sourced and multi-oriented graphs, we conclude this review section by collecting some known facts about their respective cohomology.[42] We start by introducing the suboperad $\mathsf{o}_i\mathsf{d}_j\mathsf{cGra}_d \subset \mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d$ spanned by *connected* graphs, which in turns yields a sub-dg Lie algebra $\mathsf{o}_i\mathsf{d}_j\mathsf{fcGC}_d \subset \mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$.[43] This is justified by the fact that the cohomology of the graph complex $\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d$ is captured by its connected part,[44] as follows from

$$H^\bullet(\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d) = \widehat{\mathcal{O}}\big(H^\bullet(\mathsf{o}_i\mathsf{d}_j\mathsf{fcGC}_d)[-d]\big)[d],$$

where $\widehat{\mathcal{O}}(\mathfrak{g})$ denotes the completed symmetric algebra associated with the graded vector space $\mathfrak{g}$. As far as cohomology is concerned, one can therefore restrict the analysis to the connected part of the above complexes. Moreover, a further simplification comes from the fact that the

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[40] For example, the graph $\Upsilon_S := \bullet \blacktriangleright\!\!\blacktriangleright \bullet = \textcircled{1} \blacktriangleright\!\!\blacktriangleright \textcircled{2} + \textcircled{1} \blacktriangleleft\!\!\blacktriangleright \textcircled{2} + (-1)^d \left(\textcircled{2} \blacktriangleright\!\!\blacktriangleright \textcircled{1} + \textcircled{2} \blacktriangleleft\!\!\blacktriangleright \textcircled{1} \right)$ is a Maurer–Cartan element in $\mathsf{o}_i\mathsf{d}_j\mathsf{fGC}_d$ with $i, j, d \geq 0$ and $i + j = 2$.

[41] Equivalently, one can define $\mathsf{s}_k\mathsf{d}_j\mathsf{fGC}_d := \mathrm{Def}\big(\mathsf{Lie}\{1-d\} \xrightarrow{\gamma_0} \mathsf{s}_k\mathsf{d}_j\mathsf{Gra}_d\big)$.

[42] See [47, Section 5] and [48, Section 7] for reviews.

[43] Similarly, one can introduce the suboperads of connected multi-directed graphs $\mathsf{d}_j\mathsf{cGra}_d$ and connected multi-sourced graphs $\mathsf{s}_i\mathsf{d}_j\mathsf{cGra}_d$ as well as their corresponding sub-dg Lie algebras $\mathsf{d}_j\mathsf{fcGC}_d$ and $\mathsf{s}_i\mathsf{d}_j\mathsf{fcGC}_d$.

[44] We refer to [71] and [72] for the cases $i = 0, 1$ respectively and to [47] for a general statement.

cohomology of the various complexes previously introduced can be related to one another. This is embodied by the following important theorem, due to T. Willwacher[45] for the case $i = 0$ and to M. Živković [77, 78] for its generalisation to arbitrary $i \geq 0$.

Theorem 3.1. *For all integers $i, j, d \geq 0$ and $k \in \mathbb{Z}$:*

1. *The inclusion $\mathsf{o}_i \mathsf{d}_j \mathsf{fcGC}_d \hookrightarrow \mathsf{o}_i \mathsf{d}_{j+1} \mathsf{fcGC}_d$ is a quasi-isomorphism.*

2. *There is a quasi-isomorphism $\mathsf{o}_i \mathsf{d}_j \mathsf{fcGC}_d \longrightarrow \mathsf{o}_{i+1} \mathsf{d}_j \mathsf{fcGC}_{d+1}$.*

3. *The inclusion $\mathsf{o}_{|k|} \mathsf{d}_j \mathsf{fcGC}_d \hookrightarrow \mathsf{s}_k \mathsf{d}_j \mathsf{fcGC}_d$ is a quasi-isomorphism.*

The quasi-isomorphism of the second item preserves the additional grading provided by the first Betti number.[46]

A few comments are in order. We start by noting that the sequence of embeddings of operads (3.2) induces a sequence[47] of injective quasi-isomorphisms of complexes[48]

$$\mathsf{fcGC}_d \xrightarrow[\overrightarrow{\mathsf{or}}]{\sim} \mathsf{dfcGC}_d \xrightarrow[\overrightarrow{\mathsf{or}}]{\sim} \mathsf{d}_2 \mathsf{fcGC}_d \xrightarrow[\overrightarrow{\mathsf{or}}]{\sim} \mathsf{d}_3 \mathsf{fcGC}_d \xrightarrow[\overrightarrow{\mathsf{or}}]{\sim} \cdots .$$

Hence, adding extra colored direction does not affect the cohomology. The first item of Theorem 3.1 asserts that this result generalises to multi-oriented graphs where the number i of oriented directions is kept fixed.

The situation gets more interesting when orienting extra directions since adding oriented directions *does* change the cohomology. More precisely, the second item of the above theorem relates the cohomology of a given multi-oriented graph complex to the one of a (less oriented) complex in higher dimension. We will comment more on this important result in the next paragraph. Before doing so, let us note that the third item asserts that sourcing directions also affects the cohomology, but in a way that is completely captured by the cohomology of the (smaller) multi-oriented graph complex. Hence, the computation of the cohomology of the multi-sourced/sinked graph complex can always be reduced to computing the cohomology of the multi-oriented graph complex. In the next section, we will therefore express our results in terms of the smaller multi-oriented graph complex when possible.

For later use, Table 3 summarises the cohomology in degrees 0, 1 and 2 of the (undirected) graph complex in dimensions $d = 1, 2, 3$.[49]

**Table 3.** Cohomology in low dimension and degree.

	$H^0(\mathsf{fcGC}_d)$	$H^1(\mathsf{fcGC}_d)$	$H^2(\mathsf{fcGC}_d)$
$d = 1$	**0**	$\mathbb{K}\langle \Theta \rangle$	$\mathbb{K}\langle L_3 \rangle$
$d = 2$	$\mathfrak{grt}_1$	**0**?	?
$d = 3$	$\mathbb{K}\langle L_3 \rangle$	**0**	**0**

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[45]See [71, Appendix K] for the first item (cf. also [13]), [72] for the second and [73] for the third.

[46]The *first Betti number* is defined as $b := k - N + 1$ for a connected graph with N vertices and k edges.

[47]Here, the dg Lie algebra $\mathsf{fcGC}_d \equiv \mathsf{d}_0 \mathsf{fcGC}_d$ (resp. $\mathsf{dfcGC}_d \equiv \mathsf{d}_1 \mathsf{fcGC}_d$) stands for the usual Kontsevich graph complex of connected undirected (resp. directed) graphs.

[48]See [13, 71] for the first arrow and [47] for a general statement.

[49]The cocycle L_3 stands for the triangle loop . Regarding the Θ-cocycle and the Grothendieck–Teichmüller algebra $\mathfrak{grt}_1$, see below.

Climbing the dimension ladder. Informally, the second item of Theorem 3.1 allows to map familiar structures in low dimensions[50] to novel incarnations thereof in higher dimensions.[51] The most striking example of such hierarchy of structures stems from another important theorem of T. Willwacher [71] showing the existence of an isomorphism of Lie algebras $H^0(\mathsf{fcGC}_2) \simeq \mathfrak{grt}_1$ where $\mathfrak{grt}_1$ denotes the infinite-dimensional Grothendieck–Teichmüller algebra. Combined with the second item of Theorem 3.1, we obtain a sequence $H^0(\mathsf{fcGC}_2) \simeq H^0(\mathsf{o}_1\mathsf{d}_\bullet\mathsf{fcGC}_3) \simeq H^0(\mathsf{o}_2\mathsf{d}_\bullet\mathsf{fcGC}_4) \simeq \cdots \simeq \mathfrak{grt}_1$ of incarnations of $\mathfrak{grt}_1$ in arbitrary dimensions $d \geq 2$.[52] Different incarnations yield different actions of the Grothendieck–Teichmüller group[53] on various algebro-geometric structures. In its original incarnation via directed cocycles in $H^0(\mathsf{d}_1\mathsf{fGC}_d)$, the Grothendieck–Teichmüller group naturally acts via Lie_∞-automorphisms on the Schouten Lie algebra of polyvector fields $\mathcal{T}_{\mathrm{poly}}$ on (finite-dimensional) manifolds. This yields an action on the space of universal formality maps hence on the space of universal quantization maps for finite-dimensional Poisson manifolds [12, 29, 32, 34, 45, 71]. In dimension 3, there is a natural action of GRT_1 via oriented cocycles in $H^0(\mathsf{o}_1\mathsf{d}_0\mathsf{fcGC}_3)$ on the deformation complex of Lie bialgebras[54] hence on the space of universal formality maps related to the deformation quantization of Lie bialgebras [51, 50, 72].[55]

Another important result for our story concerns the manifold incarnations of the Θ-cocycle $\ominus \in \mathsf{dgra}_{2,3}$ spanning the first cohomology class of fcGC_1, i.e., $H^1(\mathsf{fcGC}_1) \simeq \mathbb{K}\langle\Theta\rangle$. Again, applying the second item of Theorem 3.1 results in a sequence of isomorphisms

$$H^1(\mathsf{fcGC}_1) \simeq H^1(\mathsf{o}_1\mathsf{d}_\bullet\mathsf{fcGC}_2) \simeq H^1(\mathsf{o}_2\mathsf{d}_\bullet\mathsf{fcGC}_3) \simeq \cdots \simeq \mathbb{K}.$$

In its original incarnation as the cocycle spanning $H^1(\mathsf{fcGC}_1)$, the Θ-graph can be recursively extended[56] to a non-trivial Maurer–Cartan element $\Upsilon_\Theta := \bullet\!\!\rightarrow\!\!\bullet + \ominus + \cdots$ in the graded Lie algebra $\big(\mathsf{fcGC}_1, [\cdot,\cdot]\big)$, see, e.g., [30]. This Maurer–Cartan element is mapped to the Moyal star-commutator [26, 54] via the natural action of fcGC_1 on the algebra of functions of a (bosonic) symplectic manifold. Considering the case $d = 2$ yields another important incarnation of the Θ-graph as the oriented Kontsevich–Shoikhet cocycle – denoted Θ_2 in the following, cf. Figure 5 – and spanning $H^1(\mathsf{o}_1\mathsf{d}_\bullet\mathsf{fcGC}_2)$. The latter first appeared implicitly in [56] (see also [10, 70]) as the obstruction to the existence of a cycle-less universal quantization of Poisson manifolds beyond order $\hbar^3$. It then appeared explicitly in [65] as the obstruction to formality in infinite dimension while its graph theoretical interpretation – as the avatar of the Θ-cocycle in dimension 2 – has been elucidated in [72]. The corresponding Maurer–Cartan element Υ_{Θ_2} induces an exotic (and essentially unique) universal Lie_∞-structure on polyvector fields deforming non-trivially[57] the

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[50]See footnote 3 for the interpretation of the dimension d.

[51]Since the quasi-isomorphism of the second item of Theorem 3.1 preserves both the graph degree and the first Betti number, a connected cocycle $\gamma \in \mathsf{dgra}_{N,k}$ in dimension d is mapped to a cocycle $\gamma' \in \mathsf{dgra}_{N',k'}$ in dimension $d + 1$ with $N' = k + 1$ and $k' = 2k - N + 1$. More generally, the incarnation of a cocycle $\gamma \in \mathsf{dgra}_{N,k}$ of $H^\bullet(\mathsf{o}_i\mathsf{d}_j\mathsf{fcGC}_d)$ in dimension $d' > d$ is a graph $\gamma' \in \mathsf{dgra}_{N',k'}$ with $N' = N + (d' - d)b$ and $k' = k + (d' - d)b$ with b the first Betti number.

[52]For example, the tetrahedron graph $t_2 \in H^0(\mathsf{fcGC}_2)$ is mapped to an oriented cocycle $t_3 \in H^0(\mathsf{o}_1\mathsf{d}_\bullet\mathsf{fcGC}_3)$ with $N = 7$ vertices and $k = 9$ edges (see footnote 51).

[53]Recall that the Grothendieck–Teichmüller group is defined by exponentiation of the pro-nilpotent Grothendieck–Teichmüller algebra, i.e., $\mathsf{GRT}_1 \equiv \exp(\mathfrak{grt}_1)$, see [69] for a review.

[54]As reviewed in Section 4.2.

[55]The fact that quantization of Lie bialgebras involves oriented graphs was already recognised in [18].

[56]Potential obstructions in promoting the Θ-graph to a full Maurer–Cartan element lie in $H^2(\mathsf{fcGC}_1) \simeq \mathbb{K}\langle L_3\rangle$, cf. Table 3. Since the obstruction to the prolongation of the Θ-graph at order $k \geq 2$ has Betti number $k + 2$, it never hits the loop graph L_3 (of Betti number 1) so that the prolongation is unobstructed at all orders and can be performed recursively. The argument carries identically to incarnations of the Θ-graph in higher dimensions.

[57]For finite-dimensional manifolds, this exotic Lie_∞-structure can be shown to be isomorphic to the standard Schouten bracket, although in a highly non-trivial way, cf. [50] for explicit transcendental formulae.

Schouten algebra on infinite-dimensional manifolds [65]. The latter can then be considered as the avatar in $d = 2$ of the Moyal star-commutator in $d = 1$.
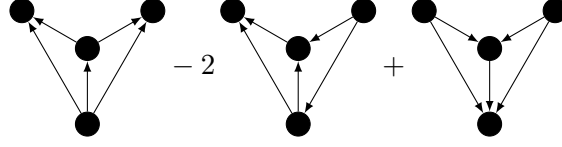

Figure 5. Kontsevich–Shoikhet cocycle $\Theta_2 \in H^1(\mathsf{o_1 d_0 fcGC_2})$.

Of special interest for our purpose is the incarnation of the Θ-cocycle in dimension 3, dubbed Θ_3 in the following. The latter is a combination of bi-oriented graphs with $N = 6$ and $k = 7$[58] (see Appendix B) that will be argued to provide an obstruction to the universal quantization of Lie bialgebroids in Section 4.

4 Universal models

The aim of the present section is to introduce universal models of multi-oriented graphs (cf. Section 3) for Lie bialgebroids – and variations thereof – using the graded geometric picture reviewed in Section 2. We start by providing an abstract characterisation of universal models and emphasise their relevance to address questions related to formality theory and deformation quantization. We then review universal models for Lie bialgebras (and their "quasi" versions) before moving on to the Lie bialgebroid case. We conclude by discussing the implications of our results regarding the deformation quantization problem for Lie bialgebroids.

4.1 Armchair formality theory

Let $(\mathfrak{g}, \delta, [\cdot, \cdot]_{\mathfrak{g}})$ be a dg Lie algebra and denote $(H(\mathfrak{g}), 0, [\cdot, \cdot]_{H(\mathfrak{g})})$ the associated cohomology endowed with the canonical dg Lie algebra structure inherited from $\mathfrak{g}$ (with trivial differential). Let furthermore $\Phi \colon (H(\mathfrak{g}), 0) \xrightarrow{\sim} (\mathfrak{g}, \delta)$ be a quasi-isomorphism of complexes. Generically, Φ fails to preserve the additional graded Lie structures – i.e., $\Phi([x, y]_{H(\mathfrak{g})}) \neq [\Phi(x), \Phi(y)]_{\mathfrak{g}}$ with $x, y \in H(\mathfrak{g})$ – so that Φ is *not* a morphism of dg Lie algebras. According to the *homotopy transfer theorem*,[59] any dg Lie algebra $\mathfrak{g}$ is quasi-isomorphic (as a Lie_∞-algebra) to its cohomology $H(\mathfrak{g})$ endowed with a certain Lie_∞-structure $(H(\mathfrak{g}), l)$ deforming the canonical dg Lie algebra structure $(H(\mathfrak{g}), 0, [\cdot, \cdot]_{H(\mathfrak{g})})$, i.e., $l_1 = 0$, $l_2 = [\cdot, \cdot]_{H(\mathfrak{g})}$ and the higher order brackets $l_{>2}$ are transferred from the dg Lie algebra structure on $\mathfrak{g}$. In other words, any quasi-isomorphism of complexes $\Phi \colon (H(\mathfrak{g}), 0) \xrightarrow{\sim} (\mathfrak{g}, \delta)$ can be upgraded to a quasi-isomorphism of Lie_∞-algebras $\mathcal{U} \colon (H(\mathfrak{g}), l) \xrightarrow{\sim} (\mathfrak{g}, \delta, [\cdot, \cdot]_{\mathfrak{g}})$ with $\mathcal{U}_1 = \Phi$. If the higher brackets $l_{>2}$ vanish, then $(H(\mathfrak{g}), 0, [\cdot, \cdot]_{H(\mathfrak{g})})$ and $(\mathfrak{g}, \delta, [\cdot, \cdot]_{\mathfrak{g}})$ are quasi-isomorphic as Lie_∞-algebras and $\mathfrak{g}$ is said to be *formal*. The homotopy transfer theorem thus allows to reduce questions regarding formality (such as existence of formality maps and their classification) to the study of the space of Lie_∞-structures deforming the canonical dg Lie structure $(H(\mathfrak{g}), 0, [\cdot, \cdot]_{H(\mathfrak{g})})$. The relevant deformation theory is controlled by the Chevalley–Eilenberg dg Lie algebra $\mathsf{CE}\big(H(\mathfrak{g})\big)$ endowed with

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[58]More generally, the Maurer–Cartan element Υ_{Θ_d} associated with the incarnation Θ_d of the Θ-graph in dimension d is a sum $\Upsilon_{\Theta_d} = \bullet \!\!\rightarrow\!\!\rightarrow\!\! \bullet + \Theta_d + \cdots = \sum_{p \geq 0} \Upsilon_{\Theta_d}^p$ where the graph $\Upsilon_{\Theta_d}^p$ has $N = 2p(d-1) + 2$ vertices and $k = 2pd + 1$ edges.

For $d = 1$, we recover the sum of graphs $\Upsilon_{\Theta_1} := \sum_{p \geq 0} \frac{1}{(2p+1)!}$ [graph: $\bullet \cdots \bullet$ with $2p+1$ edges] yielding the Moyal star-commutator

when represented on the algebra of functions on a symplectic manifold [30].

[59]See, e.g., [41, Theorem 10.3.1] for a statement as well as Chapter 10 therein for details and history.

the Nijenhuis–Richardson bracket $[\cdot,\cdot]_{\mathsf{NR}}$ and the differential $\delta_{\mathsf{S}} := \big[\,[\cdot,\cdot]_{H(\mathfrak{g})}\,,\cdot\big]_{\mathsf{NR}}$. Since we are interested in formality maps given by universal formulae, our aim is to introduce – for each deformation quantization problem at hand – a *universal model* for the deformation theory of $H(\mathfrak{g})$ in the guise of a dg Lie algebra of graphs (collectively denoted GC) together with a morphism of dg Lie algebras $\mathsf{GC} \to \mathsf{CE}\big(H(\mathfrak{g})\big)$.

Example 4.1 (universal model for polyvector fields)**.** The paradigmatic example of the above construction is due to M. Kontsevich [32] in the context of the deformation quantization problem for Poisson manifolds.

In this context, the quasi-isomorphism of complexes is provided by the HKR map $\Phi_{\mathsf{HKR}} \colon \mathcal{T}_{\mathrm{poly}} \xrightarrow{\ \sim\ } \mathcal{D}_{\mathrm{poly}}$ between:

- $\mathcal{T}_{\mathrm{poly}}$: the Schouten graded Lie algebra of polyvector fields on the affine space $\mathbb{R}^m$,
- $\mathcal{D}_{\mathrm{poly}}$: the Hochschild dg Lie algebra of multidifferential operators on $\mathbb{R}^m$.

According to the previous reasoning, the existence of a formality map $\mathcal{U} \colon \mathcal{T}_{\mathrm{poly}} \xrightarrow{\ \sim\ } \mathcal{D}_{\mathrm{poly}}$ can be probed by studying the deformation theory of the Schouten algebra $(\mathcal{T}_{\mathrm{poly}}, 0, [\cdot,\cdot]_{\mathsf{S}})$, controlled by the Chevalley–Eilenberg dg Lie algebra $\mathsf{CE}(\mathcal{T}_{\mathrm{poly}})$. In the formulation of his Formality conjecture [32], M. Kontsevich introduced a dg Lie algebra of graphs – denoted fGC_2 – together with a morphism of dg Lie algebras $\mathsf{fGC}_2 \to \mathsf{CE}(\mathcal{T}_{\mathrm{poly}})$ given by explicit local formulae.[60] The dg Lie algebra fGC_2 can therefore be interpreted as a universal model for $\mathsf{CE}(\mathcal{T}_{\mathrm{poly}})$ allowing to reduce important questions related to formality theory to the cohomology of fGC_2:

- *Existence*: Obstructions to the existence of universal formality maps live in $H^1(\mathsf{fGC}_2)$.
- *Classification*: The space of universal formality maps is classified by $H^0(\mathsf{fGC}_2)$.

This characterisation of the universal solutions to deformation quantization problems via the cohomology of suitable graph complexes can be generalised to other algebro-geometric structures. In the next sections, we will mimic the Kontsevich construction for Lie bialgebras and Lie bialgebroids by resorting to models of (multi)-oriented graphs.

4.2 Lie bialgebras

We start by reviewing some known results regarding universal models on Lie bialgebras, see, e.g., [50, 51, 72]. Let us come back to the graded symplectic manifold $T^*(\mathfrak{g}[1]) \simeq (\mathfrak{g} \oplus \mathfrak{g}^*)[1]$ of Section 2.1 endowed with a set of homogeneous local coordinates $\big\{\underset{1}{\xi^a}, \underset{1}{\zeta_a}\big\}$, with $a \in \{1, \dots, \dim\mathfrak{g}\}$. Using this set of coordinates, one can (locally) endow the graded algebra of functions $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ with a natural structure of dGra_3-algebra, with dGra_3 the operad of 1-directed graphs in dimension 3. Explicitly, we define a morphism of operads[61] $\mathsf{Rep}^{\mathfrak{g}} \colon \mathsf{dGra}_3 \to \mathsf{End}_{\mathscr{C}^\infty((\mathfrak{g}\oplus\mathfrak{g}^*)[1])}$ via the following sequence of morphisms of graded[62] vector spaces, for $N \geq 1$:

$$\mathsf{Rep}^{\mathfrak{g}}_N \colon \ \mathsf{dGra}_3(N) \otimes \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])^{\otimes N} \to \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]),$$

$$\mathsf{Rep}^{\mathfrak{g}}_N(\gamma)(f_1 \otimes \cdots \otimes f_N) = \mu_N\bigg(\prod_{e \in E_\gamma} \Delta_e (f_1 \otimes \cdots \otimes f_N) \bigg), \tag{4.1}$$

where

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[60] The explicit formulae defining the morphism $\mathsf{fGC}_2 \to \mathsf{CE}(\mathcal{T}_{\mathrm{poly}})$ take advantage of the graded geometric formulation of Poisson manifolds as dg symplectic manifolds of degree 1, see [71] for the affine space case (cf. also the earlier work [45] as well as [53] for a generalisation to dg symplectic manifolds of arbitrary degree), [29] for a globalisation to any smooth manifolds and [14] for a generalisation to the sheaf of polyvector fields on any smooth algebraic variety.

[61] Here End_V stands for the endomorphism operad associated with the (graded) vector space V.

[62] Recall that the grading of a graph in $\mathsf{o}_i\mathsf{d}_j\mathsf{Gra}_d$ is given by $|\gamma| = k(1-d)$ with k the number of edges.

- The f_i's are functions on $(\mathfrak{g} \oplus \mathfrak{g}^*)[1]$.

- The symbol μ_N denotes the multiplication map on N elements:

$$\mu_N \colon \ \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])^{\otimes N} \to \mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1]),$$
$$f_1 \otimes \cdots \otimes f_N \mapsto f_1 \cdots f_N.$$

- The product is performed over the set E_γ of edges of the graph $\gamma \in \mathsf{dGra}_3(N)$.

- For each edge $e \in E_\gamma$ connecting vertices labeled by integers i and j, the operator Δ_e is defined as

$$\Delta_{\textcircled{\scriptsize i} \to \textcircled{\scriptsize j}} = \frac{\partial}{\partial \xi^a_{(i)}} \frac{\partial}{\partial \zeta_a^{(j)}},$$

where the sub(super)scripts (i) or (j) indicate that the derivative acts on the i-th or j-th factor in the tensor product. Note that $|\Delta_e| = -2$, consistently with the grading of edges in dGra_3, and that $\Delta_e \Delta_{e'} = \Delta_{e'} \Delta_e$ as is consistent with the fact that edges are bosonic for d odd.

We refer to [53, Proposition 5.1] for a proof (in a slightly more general context) that the map $\mathsf{Rep}^{\mathfrak{g}}$ is well-defined and satisfies the axioms of a morphism of operads. The representation $\mathsf{Rep}^{\mathfrak{g}}$ yields a sequence[63] of morphisms of operads $\mathsf{Lie}\,\{-2\} \xhookrightarrow{\gamma_0} \mathsf{dGra}_3 \xrightarrow{\mathsf{Rep}^{\mathfrak{g}}} \mathsf{End}_{\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])}$ mapping the generator of $\mathsf{Lie}\,\{-2\}$ to the graded Poisson bracket (2.1) via the graph $\bullet\!\!-\!\!\!\to\!\!\bullet :=$ $\textcircled{\scriptsize 1}\!\to\!\textcircled{\scriptsize 2} - \textcircled{\scriptsize 2}\!\to\!\textcircled{\scriptsize 1}$.

Although this action of dGra_3 on $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ is well-defined, it generically fails to preserve the various subalgebras introduced in Section 2.1.

Example 4.2. Consider the following action of a cycle graph

$$\mathsf{Rep}^{\mathfrak{g}}_2\big(\textcircled{\scriptsize 1}\!\!\rightleftarrows\!\!\textcircled{\scriptsize 2}\big)(f_1 \otimes f_2) = \frac{\partial^2 f_1}{\partial \xi^a \partial \zeta_b} \frac{\partial^2 f_2}{\partial \zeta_a \partial \xi^b},$$

- on $f_1 = f_2 \sim \xi\xi\zeta \in \mathcal{A}^{\mathfrak{g}}_{\text{Lie-quasi}}$ yields $\mathsf{Rep}^{\mathfrak{g}}_2\big(\textcircled{\scriptsize 1}\!\!\rightleftarrows\!\!\textcircled{\scriptsize 2}\big)(f_1 \otimes f_2) \sim \xi\xi \notin \mathcal{A}^{\mathfrak{g}}_{\text{Lie-quasi}}$,

- on $f_1 = f_2 \sim \xi\zeta\zeta \in \mathcal{A}^{\mathfrak{g}}_{\text{quasi-Lie}}$ yields $\mathsf{Rep}^{\mathfrak{g}}_2\big(\textcircled{\scriptsize 1}\!\!\rightleftarrows\!\!\textcircled{\scriptsize 2}\big)(f_1 \otimes f_2) \sim \zeta\zeta \notin \mathcal{A}^{\mathfrak{g}}_{\text{quasi-Lie}}$,

- *A fortiori*, the action of $\textcircled{\scriptsize 1}\!\!\rightleftarrows\!\!\textcircled{\scriptsize 2}$ fails to preserve $\mathcal{A}^{\mathfrak{g}}_{\text{Lie}} := \mathcal{A}^{\mathfrak{g}}_{\text{Lie-quasi}} \cap \mathcal{A}^{\mathfrak{g}}_{\text{quasi-Lie}}$.

This defect can be cured by carefully restricting the space of graphs, as embodied in the following lemma.

Lemma 4.3. *Let $\mathfrak{g}$ be a vector space.*

- *The graded algebra of functions $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ is endowed with a structure of dGra_3-algebra via the action of 1-directed graphs.*

- *The graded subalgebra $\mathcal{A}^{\mathfrak{g}}_{\text{Lie-quasi}}$ is endowed with a structure of $\mathsf{s}_1\mathsf{d}_0\mathsf{Gra}_3$-algebra via the action of sourced 1-directed graphs.*

- *The graded subalgebra $\mathcal{A}^{\mathfrak{g}}_{\text{quasi-Lie}}$ is endowed with a structure of $\mathsf{s}_{-1}\mathsf{d}_0\mathsf{Gra}_3$-algebra via the action of sinked 1-directed graphs.*

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[63]More generally, the action of the 3-Gerstenhaber operad Ger_3 on the algebra of functions factors through dGra_3 as $\mathsf{Ger}_3 \hookrightarrow \mathsf{dGra}_3 \xrightarrow{\mathsf{Rep}^{\mathfrak{g}}} \mathsf{End}_{\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])}$, see footnote 64 for more details.

- *The graded subalgebra $\mathcal{A}_{\mathrm{Lie}}^{\mathfrak{g}}$ is endowed with a structure of $\mathsf{s}_{1,-1}\mathsf{d}_0\mathsf{Gra}_3$-algebra via the action of 1-directed graphs being both sourced and sinked.*

Proof. Let us first consider the action of a graph containing a source vertex and let f denote the function decorating the source upon the action of $\mathsf{Rep}^{\mathfrak{g}}$. Then the differential operator acting on f is of the form $\frac{\partial}{\partial \xi} \cdots \frac{\partial}{\partial \xi} f$. Assuming that f belongs to the subalgebra $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}$ ensures that f is at least linear in ζ (by definition of $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}$, cf. Section 2.1). Hence the function obtained as the outcome of the action of a sourced graph on $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}$ either vanishes or is at least linear in ζ, so that it belongs to $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}$. The subalgebra $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}$ is therefore closed under the action of sourced graphs. A similar reasoning shows that $\mathcal{A}_{\mathrm{quasi\text{-}Lie}}^{\mathfrak{g}}$ is closed under the action of sinked graphs. It follows that the intersection $\mathcal{A}_{\mathrm{Lie}}^{\mathfrak{g}} := \mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}} \cap \mathcal{A}_{\mathrm{quasi\text{-}Lie}}^{\mathfrak{g}}$ is closed under the action of graphs possessing at least one source and one sink. ∎

As noted previously (see Section 3.1) oriented graphs necessarily possess at least one source and one sink. Denoting collectively by $\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}$ the three subalgebras $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}$, $\mathcal{A}_{\mathrm{quasi\text{-}Lie}}^{\mathfrak{g}}$ and $\mathcal{A}_{\mathrm{Lie}}^{\mathfrak{g}} \subset \mathscr{C}^{\infty}((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$, the previous lemma thus ensures that $\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}$ is endowed with a structure of $\mathsf{o}_1\mathsf{d}_0\mathsf{Gra}_3$-algebra via the action of oriented 1-directed graphs, i.e., the representation $\mathsf{Rep}^{\mathfrak{g}}$ induces morphisms of operads $\mathsf{Rep}^{\mathfrak{g}} \colon \mathsf{o}_1\mathsf{d}_0\mathsf{Gra}_3 \to \mathsf{End}_{\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}}$. Since we are primarily interested in the cohomology of the associated graph complexes, it is sufficient for our purpose to resort to the smaller operad of oriented graphs as this is where the cohomology lies (cf. the third item of Theorem 3.1). Applying $\mathrm{Def}(\mathsf{Lie}\{-2\} \to \cdot)$ on both sides of the morphism $\mathsf{Rep}^{\mathfrak{g}}$ yields the following proposition:

Proposition 4.4.

- *The morphism of operads $\mathsf{Rep}^{\mathfrak{g}} \colon \mathsf{dGra}_3 \to \mathsf{End}_{\mathscr{C}^{\infty}((\mathfrak{g} \oplus \mathfrak{g}^*)[1])}$ induces a morphism of dg Lie algebras*

$$\left(\mathsf{d}_1\mathsf{fGC}_3, \delta, [\cdot, \cdot]\right) \to \left(\mathsf{CE}\left(\mathscr{C}^{\infty}((\mathfrak{g} \oplus \mathfrak{g}^*)[1])[2]\right), \delta_{\mathsf{S}}, [\cdot, \cdot]_{\mathsf{NR}}\right).$$

- *The morphisms of operads $\mathsf{Rep}^{\mathfrak{g}} \colon \mathsf{o}_1\mathsf{d}_0\mathsf{Gra}_3 \to \mathsf{End}_{\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}}$ induce morphisms of dg Lie algebras*

$$\left(\mathsf{o}_1\mathsf{d}_0\mathsf{fGC}_3, \delta, [\cdot, \cdot]\right) \to \left(\mathsf{CE}(\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}[2]), \delta_{\mathsf{S}}, [\cdot, \cdot]_{\mathsf{NR}}\right), \tag{4.2}$$

   where $\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}$ collectively denotes the subalgebras $\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^{\mathfrak{g}}, \mathcal{A}_{\mathrm{quasi\text{-}Lie}}^{\mathfrak{g}}, \mathcal{A}_{\mathrm{Lie}}^{\mathfrak{g}} \subset \mathscr{C}^{\infty}((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$.

Proof. The proposition follows straightforwardly from Lemma 4.3 together with the equivariance of $\mathsf{Rep}^{\mathfrak{g}}$. ∎

We denoted $\mathsf{CE}(\mathfrak{g})$ the Chevalley–Eilenberg cochain space (in the adjoint representation) associated with the vector space $\mathfrak{g}$ while $[\cdot, \cdot]_{\mathsf{NR}}$ stands for the Nijenhuis–Richardson bracket and $\delta_{\mathsf{S}} := \left[\{\cdot, \cdot\}_{\Omega}^{\mathfrak{g}}, \cdot\right]_{\mathsf{NR}}$ for the Chevalley–Eilenberg differential associated with the graded Poisson bracket (2.1). In plain words, Proposition 4.4 states that the graph complex $\mathsf{d}_1\mathsf{fcGC}_3$ provides a universal model for the deformation theory of proto-Lie bialgebras while $\mathsf{o}_1\mathsf{d}_0\mathsf{fcGC}_3$ can be seen as a universal model for the deformation theory of Lie-quasi, quasi-Lie and Lie bialgebras. This last fact combined with the cohomology computations reviewed in Section 3.2 yields the following well-known result:

Corollary 4.5. *The Grothendieck–Teichmüller group acts via Lie_{∞}-automorphisms on the deformation complexes of Lie-quasi, quasi-Lie and Lie bialgebras.*

Proof. Going to the zeroth cohomology in (4.2) yields a map

$$H^0\big(\mathsf{o_1d_0fGC_3}|\delta\big) \to H^0\big(\mathsf{CE}(\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}[2])|\delta_{\mathsf{S}}\big).$$

Hence any non-trivial zero degree cocycle γ in $\mathsf{o_1d_0fGC_3}$ is mapped to a Lie_∞-derivation $\mathsf{Rep}^{\mathfrak{g}}(\gamma)$ of the dg Lie algebras $\big(\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}, \{\cdot,\cdot\}_\Omega^{\mathfrak{g}}\big)$, thus ensuring that $\exp\big(\mathsf{Rep}^{\mathfrak{g}}(\gamma)\big)$ is a Lie_∞-automorphism thereof. Therefore the pro-unipotent group $\exp\big(H^0\big(\mathsf{o_1d_0fGC_3}\big)\big)$ acts on the deformation complexes $\big(\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}, \{\cdot,\cdot\}_\Omega^{\mathfrak{g}}\big)$ via Lie_∞-automorphisms. Focusing on connected graphs, the second item of Theorem 3.1 ensures that $H^0(\mathsf{o_1d_0fcGC_3}) \simeq H^0(\mathsf{fcGC_2}) \simeq \mathfrak{grt}_1$, where the last equivalence is T. Willwacher's theorem [71]. Exponentiating thus yields an action of the pro-unipotent Grothendieck–Teichmüller group $\mathsf{GRT}_1 \equiv \exp(\mathfrak{grt}_1)$ on the deformation complexes $\big(\mathcal{A}_{\mathrm{sub}}^{\mathfrak{g}}, \{\cdot,\cdot\}_\Omega^{\mathfrak{g}}\big)$ of Lie-quasi, quasi-Lie and Lie bialgebras via Lie_∞-automorphisms. ■

This result is consistent with the known action of GRT_1 (via the GRT_1-torsor of Drinfeld associators) on quantization maps for Lie bialgebras [20, 25, 28, 50, 51, 66] and Lie-quasi bialgebras [19, 61]. In comparison, there is no such action of GRT_1 on the deformation complex of proto-Lie bialgebras as the above Lie_∞-automorphisms become trivial when acting on $\mathscr{C}^\infty((\mathfrak{g} \oplus \mathfrak{g}^*)[1])$ (consistently with the fact that $H^0(\mathsf{d_1fcGC_3}) \simeq H^0(\mathsf{fcGC_3}) \simeq \mathbb{K}$, cf. Table 3). In that sense, there seems to be no true "intermediate case" between the proto-Lie bialgebra case and the Lie bialgebra case since restricting to oriented graphs allows to preserve all three subalgebras at once. As we will see, this fact is in contradistinction with the "bialgebroid" case in which Lie-quasi and quasi-Lie bialgebroids provide a true intermediate case between proto-Lie and Lie bialgebroids.

4.3 Lie bialgebroids

We now move on to the main result of this note by introducing novel universal models for the deformation complexes of the family of variations on Lie bialgebroids reviewed in Section 2.2 and Appendix A. Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle and consider the graded symplectic manifold $\big(\mathcal{E}, \Omega\big)$, with $\mathcal{E} \equiv T^*[2]E[1]$, coordinatised by the set of homogeneous coordinates $\big\{\underset{0}{x^\mu}, \underset{1}{\xi^a}, \underset{1}{\zeta_a}, \underset{2}{p_\mu}\big\}$. The algebra of functions on $\mathcal{E}$ carries a natural action $\mathsf{Rep}^E\colon \mathsf{d_2Gra_3} \to \mathsf{End}_{\mathscr{C}^\infty(\mathcal{E})}$ of the operad $\mathsf{d_2Gra_3}$ of bi-directed graphs containing both black and red directions. The action Rep^E is defined similarly as the action (4.1) where for each edge $e \in E_\gamma$ connecting vertices labeled by integers i and j, the operator Δ_e is defined as

$$\Delta_{\underset{i}{\textcircled{i}}\blacktriangleright\underset{j}{\textcircled{j}}} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_\mu^{(j)}}, \qquad \Delta_{\underset{i}{\textcircled{i}}\blacktriangleleft\underset{j}{\textcircled{j}}} = \frac{\partial}{\partial \xi_{(i)}^a} \frac{\partial}{\partial \zeta_a^{(j)}},$$

where the sub(super)scripts (i) or (j) indicate that the derivative acts on the i-th or j-th factor in the tensor product. As in the Lie bialgebra case, we note that the operator Δ_e has grading $|\Delta_e| = -2$, consistently with the grading of edges in $\mathsf{dGra_3}$, and that $\Delta_e \Delta_{e'} = \Delta_{e'} \Delta_e$ as is consistent with the fact that edges are bosonic for d odd.

The representation Rep^E maps the graph

$$\bullet\!\!\blacktriangleright\!\!\bullet = \textcircled{1}\!\!\blacktriangleright\!\!\textcircled{2} + \textcircled{1}\!\!\blacktriangleleft\!\!\textcircled{2} - \big(\textcircled{2}\!\!\blacktriangleright\!\!\textcircled{1} + \textcircled{2}\!\!\blacktriangleleft\!\!\textcircled{1}\big)$$

towards the graded Poisson bracket (2.8) and furthermore yields a sequence[64] of morphisms of

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[64]Similarly as in the Lie bialgebra case, the action of the 3-Gerstenhaber operad $\mathsf{Ger_3}$ (also called $\mathsf{e_3}$) on the algebra of functions $\mathscr{C}^\infty(\mathcal{E})$ factors through $\mathsf{d_2Gra_3}$ as $\mathsf{Ger_3} \xhookrightarrow{i_3} \mathsf{d_2Gra_3} \xrightarrow{\mathsf{Rep}^E} \mathsf{End}_{\mathscr{C}^\infty(\mathcal{E})}$ where the embedding is explicitly given by the following action on generators $a_1 \wedge a_2, \{a_1, a_2\} \in \mathsf{Ger_3}(2)$:

- $i_3(a_1 \wedge a_2) = \textcircled{1} \quad \textcircled{2}$ with $\wedge$ the graded commutative associative product of degree 0
- $i_3(\{a_1, a_2\}) = \bullet\!\!\blacktriangleright\!\!\bullet$ with $\{\cdot,\cdot\}$ the graded Lie bracket of degree -2.

operads

$$\mathsf{Lie}\{-2\} \xrightarrow{\gamma_0} \mathsf{d_2Gra_3} \xrightarrow{\mathsf{Rep}^E} \mathsf{End}_{\mathscr{C}^\infty(\mathscr{E})}.$$

Although the action of $\mathsf{d_2Gra_3}$ on $\mathscr{C}^\infty(\mathscr{E})$ is well-defined and satisfies the axioms of a morphism of operads (cf. the discussion around (4.1)), it generically fails to preserve the various subalgebras introduced in Section 2.2.

Example 4.6. Considering the following action of a cycle graph

$$\mathsf{Rep}_2^E\big(\textcircled{1}\,\textcircled{2}\big)(f_1 \otimes f_2) = \frac{\partial^2 f_1}{\partial x^\mu \partial p_\nu}\frac{\partial^2 f_2}{\partial x^\nu \partial p_\mu},$$

- on $f_1 = f_1(x)^\mu{}_a p_\mu \xi^a$ and $f_2 = f_2(x)^\mu{}_a p_\mu \xi^a \in \mathcal{A}_{\text{Lie-quasi}}^E$ yielding

$$\mathsf{Rep}_2^E\big(\textcircled{1}\,\textcircled{2}\big)(f_1 \otimes f_2) = \partial_\mu f_1(x)^\nu{}_{[a}\partial_\nu f_2(x)^\mu{}_{b]}\xi^a \xi^b \notin \mathcal{A}_{\text{Lie-quasi}}^E,$$

- on $f_1 = f_1(x)^{\mu|a} p_\mu \zeta_a$ and $f_2 = f_2(x)^{\mu|a} p_\mu \zeta_a \in \mathcal{A}_{\text{quasi-Lie}}^E$ yielding

$$\mathsf{Rep}_2^E\big(\textcircled{1}\,\textcircled{2}\big)(f_1 \otimes f_2) = \partial_\mu f_1(x)^{\nu|[a}\partial_\nu f_2(x)^{\mu|b]}\zeta_a \zeta_b \notin \mathcal{A}_{\text{quasi-Lie}}^E.$$

A fortiori, the previous graph fails to preserve the intersection $\mathcal{A}_{\text{Lie}}^E := \mathcal{A}_{\text{Lie-quasi}}^E \cap \mathcal{A}_{\text{quasi-Lie}}^E$.

Hence $\mathcal{A}_{\text{Lie-quasi}}^E$, $\mathcal{A}_{\text{quasi-Lie}}^E$ and $\mathcal{A}_{\text{Lie}}^E$ generically fail to be $\mathsf{d_2Gra_3}$-algebras. Similarly to the Lie bialgebra case, this apparent defect can be cured via a suitable restriction of the class of graphs, as performed in the following lemma:

Lemma 4.7. *Let* $E \xrightarrow{\pi} \mathscr{M}$ *be a vector bundle.*

- *The graded algebra of functions* $\mathscr{C}^\infty(\mathscr{E})$ *is endowed with a structure of* $\mathsf{d_2Gra_3}$-*algebra via the action of bi-directed graphs.*
- *The graded subalgebra* $\mathcal{A}_{\text{Lie-quasi}}^E$ *is endowed with a structure of* $\mathsf{s_1d_1Gra_3^{black}}$-*algebra via the action of bi-directed graphs with a black source.*
- *The graded subalgebra* $\mathcal{A}_{\text{quasi-Lie}}^E$ *is endowed with a structure of* $\mathsf{s_1d_1Gra_3^{red}}$-*algebra via the action of bi-directed graphs with a red source.*
- *The graded subalgebra* $\mathcal{A}_{\text{Lie}}^E$ *is endowed with a structure of* $\mathsf{s_2d_0Gra_3}$-*algebra via the action of bi-directed graphs with a black source and a red source.*

Proof. We mimic the proof in the Lie bialgebra case by considering first the action of a graph containing a black source vertex. Letting f denote the function decorating the black source upon the action of Rep^E, the differential operator acting on f is of the form $\frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x}\frac{\partial}{\partial \xi} \cdots \frac{\partial}{\partial \xi} f$. Assuming that f belongs to the subalgebra $\mathcal{A}_{\text{Lie-quasi}}^E$ ensures that f is at least linear either in p or ζ (by definition of $\mathcal{A}_{\text{Lie-quasi}}^E$, cf. Section 2.2). Hence the function obtained as the outcome of the action of a black sourced graph on $\mathcal{A}_{\text{Lie-quasi}}^E$ cannot be of the form $\xi \cdots \xi$ so that it belongs to $\mathcal{A}_{\text{Lie-quasi}}^E$. The subalgebra $\mathcal{A}_{\text{Lie-quasi}}^E$ is therefore closed under the action of black sourced graphs. A similar reasoning shows that $\mathcal{A}_{\text{quasi-Lie}}^E$ is closed under the action of red sourced graphs.[65] It follows that the intersection $\mathcal{A}_{\text{Lie}}^E := \mathcal{A}_{\text{Lie-quasi}}^E \cap \mathcal{A}_{\text{quasi-Lie}}^E$ is closed under the action of graphs possessing both a black and red source. ∎

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[65]Note that graphs with (black or red) sinks preserve neither $\mathcal{A}_{\text{Lie-quasi}}^E$ nor $\mathcal{A}_{\text{quasi-Lie}}^E$.

As in the Lie bialgebra case, one can make use of the inclusion $\mathsf{o}_\bullet \mathsf{d}_\bullet \mathsf{Gra}_3 \subset \mathsf{s}_\bullet \mathsf{d}_\bullet \mathsf{Gra}_3$ (see Section 3.1) to extract from the previous lemma an action of (suitable) operads of multi-oriented graphs on the three subalgebras at hand. Explicitly, the subalgebra $\mathcal{A}^E_{\text{Lie-quasi}}$ (resp. $\mathcal{A}^E_{\text{quasi-Lie}}$) is acted upon by the operad of bi-directed graphs with oriented black (resp. red) arrows, denoted $\mathsf{o}_1 \mathsf{d}_1 \mathsf{Gra}_3^{\text{black}}$ (resp. $\mathsf{o}_1 \mathsf{d}_1 \mathsf{Gra}_3^{\text{red}}$) in the following.[66]

Crucially, preserving the intersection $\mathcal{A}^E_{\text{Lie}} := \mathcal{A}^E_{\text{Lie-quasi}} \cap \mathcal{A}^E_{\text{quasi-Lie}}$ requires to orient *both* directions, thus yielding an action of $\mathsf{o}_2 \mathsf{d}_0 \mathsf{Gra}_3$ on $\mathcal{A}^E_{\text{Lie}}$. As pointed out in Section 3.2 (see Theorem 3.1), the number of oriented colors (in contradistinction with the number of directed colors) is the relevant factor to compute the respective cohomology. From this simple observation will then follow that the Lie bialgebroid case differs essentially from the dual cases of Lie-quasi and quasi-Lie bialgebroids. Applying $\text{Def}(\text{Lie}\{-2\} \to \cdot)$ on both sides of $\text{Rep}^E \colon \mathsf{o}_\bullet \mathsf{d}_\bullet \mathsf{Gra}_3 \to \text{End}_{\mathcal{A}^E}$ yields the following proposition:

Proposition 4.8.

- *The morphism of operads* $\text{Rep}^E \colon \mathsf{d}_2 \mathsf{Gra}_3 \to \text{End}_{\mathscr{C}^\infty(\mathscr{E})}$ *induces a morphism of dg Lie algebras*

$$\big(\mathsf{d}_2 \mathsf{fGC}_3, \delta, [\cdot, \cdot]\big) \to \big(\text{CE}\big(\mathscr{C}^\infty(\mathscr{E})[2]\big), \delta_\mathsf{S}, [\cdot, \cdot]_{\mathsf{NR}}\big). \tag{4.3}$$

- *The morphisms of operads* $\text{Rep}^E \colon \mathsf{o}_1 \mathsf{d}_1 \mathsf{Gra}_3 \to \text{End}_{\mathcal{A}^E_{\text{quasi}}}$ *induce morphisms of dg Lie algebras*

$$\big(\mathsf{o}_1 \mathsf{d}_1 \mathsf{fGC}_3, \delta, [\cdot, \cdot]\big) \to \big(\text{CE}\big(\mathcal{A}^E_{\text{quasi}}[2]\big), \delta_\mathsf{S}, [\cdot, \cdot]_{\mathsf{NR}}\big), \tag{4.4}$$

  where $\mathcal{A}^E_{\text{quasi}}$ *stands for* $\mathcal{A}^E_{\text{Lie-quasi}}$, $\mathcal{A}^E_{\text{quasi-Lie}} \subset \mathscr{C}^\infty(\mathscr{E})$.

- *The morphism of operads* $\text{Rep}^E \colon \mathsf{o}_2 \mathsf{d}_0 \mathsf{Gra}_3 \to \text{End}_{\mathcal{A}^E_{\text{Lie}}}$ *induces a morphism of dg Lie algebras*

$$\big(\mathsf{o}_2 \mathsf{d}_0 \mathsf{fGC}_3, \delta, [\cdot, \cdot]\big) \to \big(\text{CE}\big(\mathcal{A}^E_{\text{Lie}}[2]\big), \delta_\mathsf{S}, [\cdot, \cdot]_{\mathsf{NR}}\big).$$

Proof. The proposition is a direct consequence of Lemma 4.7 and the equivariance of the morphism Rep^E. ■

The conventions used are as in the bialgebra case and $\delta_\mathsf{S} := \big[\{\cdot, \cdot\}^E_\Omega, \cdot\big]_{\mathsf{NR}}$ stands for the Chevalley–Eilenberg differential associated with the graded Poisson bracket (2.8). Going into cohomology and using Theorem 3.1 allows to compute the relevant cohomology groups, as summed up in Table 4.

Table 4. Cohomology groups for Lie bialgebroid structures.

Structure	Black	Red	Cohomology
proto-Lie bialgebroids	directed	directed	$H^\bullet(\mathsf{d}_2 \mathsf{fGC}_3) \simeq H^\bullet(\mathsf{fGC}_3)$
Lie-quasi bialgebroids	oriented	directed	$H^\bullet(\mathsf{o}_1 \mathsf{d}_0 \mathsf{fGC}_3^{\text{black}}) \simeq H^\bullet(\mathsf{fGC}_2)$
quasi-Lie bialgebroids	directed	oriented	$H^\bullet(\mathsf{o}_1 \mathsf{d}_0 \mathsf{fGC}_3^{\text{red}}) \simeq H^\bullet(\mathsf{fGC}_2)$
Lie bialgebroids	oriented	oriented	$H^\bullet(\mathsf{o}_2 \mathsf{d}_0 \mathsf{fGC}_3) \simeq H^\bullet(\mathsf{fGC}_1)$

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[66]Alternatively, both actions could be written in terms of $\mathsf{o}_1 \mathsf{d}_1 \mathsf{Gra}_3^{\text{black}}$ (say) by making use of $\Delta_{\;\textcircled{i}\leftrightarrow\textcircled{j}} = \frac{\partial}{\partial \xi^a_{(i)}} \frac{\partial}{\partial \zeta_a^{(j)}}$ for $\mathcal{A}^E_{\text{Lie-quasi}}$ and $\Delta'_{\;\textcircled{i}\leftrightarrow\textcircled{j}} = \frac{\partial}{\partial \zeta_a^{(i)}} \frac{\partial}{\partial \xi^a_{(j)}}$ for $\mathcal{A}^E_{\text{quasi-Lie}}$.

The morphism (4.3) on the deformation complex of proto-Lie bialgebroids can be seen as a particular subcase of the action of fGC_3 on the deformation complex of Courant algebroids [53] when restricted to the split case.[67] The latter does not yield interesting structures in degrees 0 and 1 as the dominant level of the relevant cohomology $H^\bullet(\mathsf{fcGC}_3)$ is located in degree -3. The corresponding cohomology space $H^{-3}(\mathsf{fcGC}_3)$ is a unital commutative algebra spanned by trivalent graphs modulo the *IHX relation* where the rôle of the unit is played by the Θ-graph (see, e.g., [4]). Given a proto-Lie bialgebroid $E \xrightarrow{\pi} \mathcal{M}$ represented by the Hamiltonian function $\mathscr{H}$ (see (2.9)), each trivalent graph $\gamma \in H^{-3}(\mathsf{fcGC}_3)$ yields a cocycle function $\Omega_\gamma \in \mathscr{C}^\infty(\mathcal{M})$ (i.e., such that $\{\mathscr{H}, \Omega_\gamma\}_\Omega^E = 0$) thus yielding a conformal flow on the space of proto-Lie bialgebroids on E (cf. [53] for details).

Turning to Lie-quasi and quasi-Lie bialgebroids, the morphism (4.4) yields a natural extension of the action of the Grothendieck–Teichmüller group $\exp\left(H^0(\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3)\right) \simeq \mathsf{GRT}_1$ on the deformation complexes of Lie-quasi and quasi-Lie bialgebras to the "bialgebroid" case. This is the essence of Theorem 1.2.

Proof of Theorem 1.2. The proof is identical to the one of Corollary 4.5 upon substituting $\mathsf{o}_1\mathsf{d}_0\mathsf{fcGC}_3$ with $\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3$, both admitting the same cohomology thanks to the first item of Theorem 3.1. ∎

We refer to Section 4.4 for a discussion of the consequences of this action in the context of deformation quantization. Explicitly, the action of GRT_1 is through graphs with one oriented color (either black or red) and as such generically *fails* to preserve the sub-deformation complex $\mathcal{A}_{\mathrm{Lie}}^E$ for Lie bialgebroids. This is in contradistinction with the Lie bialgebra case whose deformation complex $\mathcal{A}_{\mathrm{Lie}}^{\mathfrak{g}}$ does carry a representation of GRT_1. Rather the action of $\mathsf{o}_2\mathsf{d}_0\mathsf{fcGC}_3$ endows $\mathcal{A}_{\mathrm{Lie}}^E$ with a new Lie_∞-structure deforming non-trivially the big bracket (2.8), as captured by Theorem 1.1, for which we are now in position to articulate the following proof:

Proof of Theorem 1.1. It follows from the third item of Proposition 4.8 that Maurer–Cartan elements for the dg Lie algebra $\left(\mathsf{o}_2\mathsf{d}_0\mathsf{fGC}_3, \delta, [\cdot,\cdot]\right)$ are mapped via Rep^E to universal deformations of the graded Lie algebra $\left(\mathcal{A}_{\mathrm{Lie}}^E, \{\cdot,\cdot\}_\Omega^E\right)$ as a Lie_∞-algebra. As recalled in Section 3.2, applying the second item of Theorem 3.1 results in a sequence of isomorphisms $H^1(\mathsf{o}_2\mathsf{d}_0\mathsf{fcGC}_3) \simeq H^1(\mathsf{o}_1\mathsf{d}_0\mathsf{fcGC}_3) \simeq H^1(\mathsf{fcGC}_1) \simeq \mathbb{K}$, thus providing a non-trivial cocycle graph $\Theta_3 \in H^1(\mathsf{o}_2\mathsf{d}_0\mathsf{fcGC}_3)$ of degree 1. The latter can be recursively extended to a non-trivial Maurer–Cartan element

$$\Upsilon_{\Theta_3} := \bullet \!\!\rightarrow\!\! \bullet + \Theta_3 + \cdots = \sum_{p \geq 0} \Upsilon_{\Theta_3}^p$$

in the graded Lie algebra $\left(\mathsf{o}_2\mathsf{d}_0\mathsf{fGC}_3, [\cdot,\cdot]\right)$ – where the graph $\Upsilon_{\Theta_3}^p$ possesses $N = 4p + 2$ vertices and $k = 6p + 1$ edges – as the corresponding obstructions vanish, cf. footnote 56. Mapping the Maurer–Cartan element Υ_{Θ_3} via the representation Rep^E yields an exotic Lie_∞-structure deforming non trivially the deformation complex $\left(\mathcal{A}_{\mathrm{Lie}}^E, \{\cdot,\cdot\}_\Omega^E\right)$ as a Lie_∞-algebra. ∎

The non-vanishing brackets of the (essentially unique[68]) exotic Lie_∞-structure of Theorem 1.1 – denoted θ in the following – take the form[69]

$$\theta_2 = \mathsf{Rep}_2^E(\Upsilon_\mathsf{S}) = \{\cdot,\cdot\}_\Omega^E, \qquad \theta_6 = \mathsf{Rep}_6^E(\Theta_3), \qquad \theta_{10} = \mathsf{Rep}_{10}^E(\Upsilon_{\Theta_3}^2), \qquad \ldots,$$
$$\theta_{4p+2} = \mathsf{Rep}_{4p+2}^E(\Upsilon_{\Theta_3}^p), \qquad \ldots.$$

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[67] Here by split Courant algebroids we mean Courant algebroids whose underlying vector bundle is a Whitney sum $E \oplus E^*$.

[68] Up to gauge transformations and rescalings.

[69] The intrinsic degree carried by each bracket is given by $|\theta_{4p+2}| = -12p - 2$. Pulling back the brackets along the suspension map $s \colon \mathcal{A}_{\mathrm{Lie}}^E[2] \to \mathcal{A}_{\mathrm{Lie}}^E$ of degree 2 yields a series of brackets $\tilde{\theta}_{4p+2}$ on $\mathcal{A}_{\mathrm{Lie}}^E[2]$ with the usual degree $-4p$.

The minimal[70] Lie_∞-structure $(\mathcal{A}^E_{\mathrm{Lie}}, \theta)$ can be interpreted as the avatar in dimension $d = 3$ both of the Moyal star-commutator in $d = 1$ and the Kontsevich–Shoikhet exotic Lie_∞-structure on infinite-dimensional manifolds in $d = 2$. It relies on bi-oriented graphs and as such possesses no counterpart in the "bialgebra" realm where only one orientable direction is available. In fact, one can explicitly check that the first non-trivial deformed bracket $\theta_6 = \mathsf{Rep}^E_6(\Theta_3)$ vanishes identically on the graded Poisson subalgebra[71] $\mathcal{A}^{\mathfrak{g}}_{\mathrm{Lie}} \subset \mathcal{A}^E_{\mathrm{Lie}}$ controlling deformations of Lie bialgebras, cf. Proposition B.3.

Remark 4.9.

- In the next section, we will consider Maurer–Cartan elements in the formal extension $\mathcal{A}^E_{\mathrm{Lie}}[[\hbar]]$ of $\mathcal{A}^E_{\mathrm{Lie}}$ by a formal parameter $\hbar$. By analogy with the $d = 2$ case [50], we will refer to Maurer–Cartan elements of $\left(\mathcal{A}^E_{\mathrm{Lie}}[[\hbar]], \theta\right)$ as formal *"quantizable Lie bialgebroids"*, to contrast with the Maurer–Cartan elements of $\left(\mathcal{A}^E_{\mathrm{Lie}}[[\hbar]], \{\cdot, \cdot\}^E_\Omega\right)$ referred to simply as *formal Lie bialgebroids*. Formal Lie bialgebroids being linear in $\hbar$ are just Lie bialgebroids and accordingly, we will refer to formal "quantizable Lie bialgebroids" linear in $\hbar$ as "quantizable Lie bialgebroids".

- Note that "quantizable Lie bialgebroids" are in particular Lie bialgebroids as they satisfy $\left\{\mathscr{H}, \mathscr{H}\right\}^E_\Omega = 0$ on top of some higher consistency conditions $\left(\theta_6\left(\mathscr{H}^{\wedge 6}\right) = 0, \text{ etc.}\right)$.

The distinction between Lie bialgebroids and "quantizable Lie bialgebroids" will become salient when applied to the quantization problem for Lie bialgebroids in Section 4.4.

4.4 Application to quantization and future directions

The quantization problem for Lie-(quasi) bialgebras was formulated by V. Drinfeld (cf. [18, Question 1.1] for Lie bialgebras and Section 5 for Lie-quasi bialgebras) and solved in [20] by Etingof–Kazhdan for the Lie bialgebra case[72] and in [19, 61] for the Lie-quasi bialgebra case. In both cases, the solution is universal and involves the use of a Drinfeld associator, yielding an action of the Grothendieck–Teichmüller group GRT_1 on the set of inequivalent universal quantization maps. The latter can be traced back to the action of $H^0(\mathsf{o}_1\mathsf{d}_0\mathsf{fcGC}_3) \simeq \mathfrak{grt}_1$ on the deformation complex of Lie and Lie-quasi bialgebras, as reviewed in Section 4.2. The situation for Lie bialgebras (and their quasi versions) is therefore much akin to the situation for finite-dimensional Poisson manifolds, in that both cases share the following important features (cf. the Introduction and Table 1):

1. The Grothendieck–Teichmüller group plays a classifying rôle.

2. There is (conjecturally) no generic obstruction to the existence of universal quantizations.

As for Lie bialgebroids, the corresponding quantization problem was formulated by P. Xu in [74, 76] as follows: Given a Lie bialgebroid structure on (E, E^*), the associated quantum object is a topological deformation (called *quantum groupoid*) of the standard (associative) bialgebroid structure on the universal enveloping algebra $\mathcal{U}_R(E)$ associated with the Lie algebroid structure on E [52] – with $R \equiv \mathscr{C}^\infty(\mathscr{M})$ – whose semi-classicalisation reproduces the original Lie bialgebroid structure. The quantization problem for Lie bialgebroids then consists in associating to each Lie bialgebroid a quantum groupoid quantizing it. Although the quantization problem for a generic Lie bialgebroid remains open, several explicit examples of quantizations for particular Lie bialgebroids have been exhibited in the literature. Apart from the above mentioned

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[70]Recall that minimal Lie_∞-algebras are characterised by a vanishing differential $\theta_1 \equiv 0$.

[71]See footnote 16.

[72]See also [25, 28, 50, 51, 66].

quantization of Lie bialgebras [20], it was shown in [76] that M. Kontsevich's solution to the quantization problem for (finite-dimensional) Poisson manifolds [34] ensures that Lie bialgebroids associated with Poisson manifolds (cf. Example A.2) constitute another example of Lie bialgebroids admitting a quantization.[73] This result was shown to extend to regular triangular Lie bialgebroids in [76] (see also [55]) using methods à la Fedosov [21] and to generic triangular Lie bialgebroids in [6] using a generalisation of Kontsevich's formality theorem for Lie algebroids. In this context, the following natural conjecture was formulated by P. Xu:

Conjecture 4.10 (Xu [76, Section 6])**.** *Every Lie bialgebroid admits a quantization as a quantum groupoid.*

Although Conjecture 4.10 might still hold true in the most general setting, we would like to argue for the non-existence of *universal*[74] quantizations of Lie bialgebroids, on the basis of the following results from Section 4.3:

1. The Grothendieck–Teichmüller group plays no classifying rôle regarding the universal deformations (and hence quantizations) of Lie bialgebroids.

2. There exists a potential obstruction to the existence of universal quantizations of Lie bialgebroids.

Contrasting these two features with their above mentioned counterparts for Lie bialgebras, one is led to conclude that the quantization problem for Lie bialgebroids differs essentially from its Lie bialgebra analogue and is in fact more akin to the quantization problem for infinite-dimensional manifolds. In view of this analogy, the obstruction appearing in Theorem 1.1 can be understood as the avatar in $d = 3$ of the Kontsevich–Shoikhet obstruction in $d = 2$. As shown in [10, 56, 65, 70], the latter obstruction is hit in $d = 2$ and thus prevents the existence of an *oriented* star product, thereby yielding a No-go result regarding the existence of universal quantizations for infinite-dimensional Poisson manifolds. Pursuing the analogy with the $d = 2$ case, we conjecture the following:

Conjecture 4.11 (no-go)**.** *There are no universal quantizations of Lie bialgebroids as quantum groupoids.*

To explicitly show that the obstruction is hit would require a better understanding of the deformation theory of (associative) bialgebroids, which goes beyond the ambition of the present note.[75] We nevertheless conclude the present discussion by outlining a strategy of proof for Conjecture 4.11 by mimicking the two-steps procedure of [50] for Poisson manifolds and Lie bialgebras and adapting it to the case at hand. Denoting $C^\bullet_{\mathsf{GS}}(\mathcal{O}_E, \mathcal{O}_E)$ the equivalent of the Gerstenhaber–Schack complex for the standard commutative co-commutative bialgebroid structure on the symmetric algebra $\mathcal{O}_E$ associated to the vector bundle E, the former should be

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[73]More precisely, the Kontsevich star product $*$ quantizing the Poisson bivector π provides a Drinfeld twistor $J_* \in \big(\mathcal{U}_R(E) \otimes_R \mathcal{U}_R(E)\big)[[\hbar]]$ for the standard bialgebroid $\mathcal{U}_R(E)$, where $R \equiv \mathscr{C}^\infty(\mathcal{M})$ and $E \equiv T\mathcal{M}$. Twisting $\mathcal{U}_R(E)$ by J_* then provides a quantization of the Lie bialgebroid associated with π on $(T\mathcal{M}, T^*\mathcal{M})$, see [76].

[74]Recall that a universal quantization admits formulae given by expansions in terms of graphs with universal coefficients.

[75]While the deformation theory for associative algebras [23] and bialgebras [49] are well understood using the frameworks of operads and properads respectively, the deformation theory for bialgebroids has for now been evading such (pr)operadic formulation. The underlying reason is that the relevant category to deal with bialgebroids is the one of $\big(R^e, R^e\big)$-bimodules – where R is a ring and R^e denotes its enveloping ring $R^e := R \otimes R^{\mathrm{op}}$ – which is *not* symmetric monoidal, as usually required to work with (pr)operads. Rather, the category of $\big(R^e, R^e\big)$-bimodules is naturally endowed with a structure of lax-oplax duoidal category (more precisely, a lax-strong duoidal category, that is the oplax structure is strong monoidal) whose bimonoids are bialgebroids, see, e.g., [5]. As such, the deformation theory for bialgebroids cannot be described using the theory of properads (at least in its standard form). We are grateful to T. Basile and D. Lejay for clarifications regarding this fact.

endowed with a Lie_∞-algebra structure μ – generalising the one of [43, 49] for the bialgebra case – whose corresponding Maurer–Cartan elements are quantum groupoids. By analogy with the bialgebra case, the cohomology of $\big(C_{\mathsf{GS}}^\bullet(\mathcal{O}_E, \mathcal{O}_E), \mu_1\big)$ should be isomorphic as a graded Lie algebra to the deformation complex $\big(\mathcal{A}_{\mathrm{Lie}}^E, \{\cdot, \cdot\}_\Omega^E\big)$ of Lie bialgebroids on E. Although these two Lie_∞-algebras should coincide in cohomology, we do not expect them to be quasi-isomorphic as Lie_∞-algebras, i.e., $\big(C_{\mathsf{GS}}^\bullet(\mathcal{O}_E, \mathcal{O}_E), \mu\big)$ is *not* formal. To show explicitly that the obstruction to formality is hit would require computing the Lie_∞-algebra structure obtained by transfer of μ on $H^\bullet\big(C_{\mathsf{GS}}^\bullet(\mathcal{O}_E, \mathcal{O}_E), \mu_1\big)$ and showing that the latter coincides with the exotic Lie_∞-structure θ on $\mathcal{A}_{\mathrm{Lie}}^E$,[76] as is the case in $d = 2$ [50, 65]. This would provide a trivial (in the sense that no Drinfeld associator is needed) formality Lie_∞ quasi-isomorphism $\big(\mathcal{A}_{\mathrm{Lie}}^E, \theta\big) \xrightarrow{\sim} \big(C_{\mathsf{GS}}^\bullet(\mathcal{O}_E, \mathcal{O}_E), \mu\big)$, yielding in turn a quantization map for (formal) "quantizable Lie bialgebroids" (the Maurer–Cartan elements of θ in $\mathcal{A}_{\mathrm{Lie}}^E[[\hbar]]$, cf. Remark 4.9). Finally, the fact that θ is not Lie_∞-isomorphic to the big bracket in $\mathcal{A}_{\mathrm{Lie}}^E$ (cf. Theorem 1.1) would prevent the existence of a formality morphism for Lie bialgebroids. We sum up these (non)-formality conjectures for Lie bialgebroids in Figure 6.

$$\big(\mathcal{A}_{\mathrm{Lie}}^E, \{\cdot, \cdot\}_\Omega^E\big) \dashrightarrow^{\times} \big(\mathcal{A}_{\mathrm{Lie}}^E, \theta\big) \xrightarrow{\ \sim\ } \big(C_{\mathsf{GS}}^\bullet(\mathcal{O}_E, \mathcal{O}_E), \mu\big)$$

$$\text{Lie bialgebroids} \dashrightarrow^{\times} \begin{array}{c}\text{"Quantizable}\\ \text{Lie bialgebroids"}\end{array} \xrightarrow{\ \sim\ } \begin{array}{c}\text{Quantum}\\ \text{groupoids}\end{array}$$

Figure 6. Conjectural (non)-formality maps for Lie bialgebroids.

Note that the situation is markedly different in the Lie-quasi (and quasi-Lie) bialgebroid case.[77] Firstly, recall that the Maurer–Cartan element Υ_{Θ_3} is *not* gauge-related to the Maurer–Cartan element Υ_{S} in $\mathsf{o}_2\mathsf{d}_0\mathsf{fcGC}_3$ – since Θ_3 is a non-trivial cocycle in $\mathsf{o}_2\mathsf{d}_0\mathsf{fcGC}_3$ – hence θ is indeed a non-trivial deformation of the big bracket in $\mathcal{A}_{\mathrm{Lie}}^E$. However, the cocycle Θ_3 *is* a coboundary in $\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3$ (cf. Appendix B) so that there exists a combination of graphs $\vartheta_3 \in \mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3$ such that $\Theta_3 = -\delta\vartheta_3 \in \mathsf{o}_2\mathsf{d}_0\mathsf{fcGC}_3$. In order for Υ_{Θ_3} and Υ_{S} to be gauge-related in $\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3$, one needs to find a degree 0 element[78] $\vartheta = \vartheta_3 + \vartheta_3^2 + \cdots + \vartheta_3^p$ of $\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3$ such that $\Upsilon_{\Theta_3} = e^{\mathrm{ad}\,\vartheta}\Upsilon_{\mathsf{S}}$. Contrarily to the problem of prolongating the cocycle Θ_3 to the Maurer–Cartan element Υ_{Θ_3} – which can be solved by a trivial induction – to display an explicit gauge map ϑ is a highly non-trivial task[79] as the higher obstructions[80] live in $H^1(\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3) \simeq H^1(\mathsf{fcGC}_2)$, i.e., the recipient of the obstructions to the universal quantization of finite-dimensional Poisson manifolds. Although this cohomological space conjecturally vanishes[81] (Drinfeld–Kontsevich), maps allowing to convert cocycles into coboundaries are highly non-trivial and necessarily involve the choice of a Drinfeld associator (consistently with the fact that two coboundaries differ by the choice of an element in $H^0(\mathsf{fcGC}_2) \simeq \mathfrak{grt}_1$). Up to the Drinfeld–Kontsevich conjecture, it is nevertheless expected that, given a Drinfeld associator, one can define a Lie_∞-isomorphism $\big(\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^E, \{\cdot, \cdot\}_\Omega^E\big) \xrightarrow{\sim} \big(\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^E, \theta\big)$. Repeating the argument laid down in the Lie bialgebroid case, one needs to find a (trivial) Lie_∞ quasi-isomorphism $\big(\mathcal{A}_{\mathrm{Lie\text{-}quasi}}^E, \theta\big) \xrightarrow{\sim} \big(C_{\mathrm{quasi}-\mathsf{GS}}^\bullet(\mathcal{O}_E, \mathcal{O}_E), \mu\big)$, where the right-hand side stands for the deformation complex of the quantum object associated to Lie-quasi bialgebroids. The relevant category

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[76]This is only possible thanks to the fact that the exotic Lie_∞-structure θ on $\mathcal{A}_{\mathrm{Lie}}^E$ of Theorem 1.1 is minimal and hence is a potential candidate for being the cohomology of another Lie_∞-structure.

[77]For definiteness, we will focus on the Lie-quasi case, keeping in mind that the arguments apply similarly to the dual case.

[78]The graph ϑ_3^p has $N = 4p + 1$ vertices and $k = 6p$ edges so that to have degree 0 in $d = 3$.

[79]We refer to [50] for an explicit construction in $d = 2$.

[80]The first obstruction vanishes since Θ_3 is exact in $\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3$. The second obstruction $\Upsilon_{\Theta_3}^2 - \frac{1}{2}[\vartheta_3, \Theta_3]$ can be checked to live in $H^1(\mathsf{o}_1\mathsf{d}_1\mathsf{fcGC}_3)$.

[81]See Table 3.

here is the one of *quasi-bialgebroids*, a common generalisation of the notions of bialgebroids and quasi-bialgebras, allowing to define the associated notion of *quasi-quantum groupoid* as topological deformation of the standard bialgebroid $\mathcal{U}_R(E)$ as a quasi-bialgebroid. Composing with the (non-trivial) Lie_∞-isomorphism $\left(\mathcal{A}^E_{\text{Lie-quasi}}, \{\cdot,\cdot\}^E_\Omega\right) \xrightarrow{\sim} \left(\mathcal{A}^E_{\text{Lie-quasi}}, \theta\right)$ would yield a formality morphism for Lie-quasi bialgebroids. We sum up the discussion by formulating the following conjecture and recap the corresponding conjectural formality maps in the Lie-quasi case in Figure 7.

Conjecture 4.12 (yes-go). *Given a Drinfeld associator, one can define a universal quantization of Lie-quasi bialgebroids as quasi-quantum groupoids.*

$$\left(\mathcal{A}^E_{\text{Lie-quasi}}, \{\cdot,\cdot\}^E_\Omega\right) \xrightarrow[\circlearrowleft_{\text{GRT}_1}]{\sim} \left(\mathcal{A}^E_{\text{Lie-quasi}}, \theta\right) \xrightarrow{\quad\sim\quad} \left(C^\bullet_{\text{quasi}-\mathsf{GS}}(\mathcal{O}_E, \mathcal{O}_E), \mu\right)$$

$$\begin{array}{ccc}
\text{Lie-quasi} & \xrightarrow[\circlearrowleft_{\text{GRT}_1}]{\sim} & \text{"Quantizable} & \xrightarrow{\quad\sim\quad} & \text{Quasi-quantum} \\
\text{bialgebroids} & & \text{Lie-quasi bialgebroids"} & & \text{groupoids}
\end{array}$$

Figure 7. Conjectural formality maps for Lie-quasi bialgebroids.

A dual conjecture can be made about quasi-Lie bialgebroids. Of particular interest in this context would be to investigate the quantization of quasi-Lie bialgebroids associated to twisted Poisson manifolds (cf. Example A.4) by means of a Drinfeld twistor [76] for the corresponding non-associative Kontsevich star product [7].

To conclude, let us note that particularising the conjectural quantization map of Conjecture 4.12 to Lie bialgebroids ensures that every Lie bialgebroid admits a quantization *as a quasi-quantum groupoid* (but generically not as a quantum groupoid as stated in Conjecture 4.10). However, for the particular subclass of "quantizable Lie bialgebroids" (which are in particular Lie bialgebroids, see Remark 4.9) – such as Lie bialgebras and coboundary Lie bialgebroids (cf. Appendix B) – the associated quantization should yield a quantum groupoid. According to the above picture, the exotic Lie_∞-structure θ of Theorem 1.1 can therefore be seen as a concrete means to delineate the subclass of Lie bialgebroids susceptible to be quantized as quantum groupoids.

A Geometry of Lie bialgebroids

Lie bialgebras. A Lie bialgebra is a vector space $\mathfrak{g}$ endowed with a Lie algebra structure on both $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ such that the cobracket $\Delta_\mathfrak{g}\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ is a cocycle for the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$, i.e., $\Delta_\mathfrak{g}([x,y]) = \text{ad}_x\Delta_\mathfrak{g}(y) - \text{ad}_y\Delta_\mathfrak{g}(x)$ where the representation used is the extension $\text{ad}\colon \mathfrak{g}\otimes(\wedge^2\mathfrak{g}) \to \wedge^2\mathfrak{g}$ of the adjoint action of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ on $\wedge^2\mathfrak{g}$ as $\text{ad}_x(y\wedge z) = [x,y]_\mathfrak{g}\wedge z + y\wedge[x,z]_\mathfrak{g}$. Letting $\{e_a\}\,|_{a\in\{1,\dots,\dim\mathfrak{g}\}}$ be a basis of $\mathfrak{g}$, one denotes $[e_a, e_b]_\mathfrak{g} = f_{ab}{}^c e_c$ and $\Delta_\mathfrak{g}(e_c) = C_c{}^{ab} e_a \otimes e_b$. The three defining conditions of a Lie bialgebra read

$$f_{e[a}{}^d f_{bc]}{}^e = 0, \qquad C_d{}^{e[a}C_e{}^{bc]} = 0, \qquad f_{ab}{}^e C_e{}^{cd} - 4f_{e[a}{}^{[c}C_{b]}{}^{d]e} = 0, \tag{A.1}$$

where $f_{ab}{}^c = f_{[ab]}{}^c$ and $C_c{}^{ab} = C_c{}^{[ab]}$ where square brackets denote skewsymmetrisation.

Lie algebroids. A Lie algebroid is a triplet $(E, \rho, [\cdot,\cdot]_E)$ where:

- $E \xrightarrow{\pi} \mathscr{M}$ is a vector bundle over the manifold $\mathscr{M}$,
- $\rho\colon E \to T\mathscr{M}$ is a morphism of vector bundles called the *anchor*,[82]
- $[\cdot,\cdot]_E \colon \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ is a $\mathbb{K}$-bilinear map called the *bracket*,

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[82]We will use the same symbol ρ to denote the induced map of sections $\rho\colon \Gamma(E) \to \Gamma(T\mathscr{M})$.

such that the following conditions are satisfied for all $f \in \mathscr{C}^{\infty}(\mathscr{M})$ and $X, Y, Z \in \Gamma(E)$:

1. *skewsymmetry*: $[X, Y]_E = -[Y, X]_E$,

2. *Leibniz rule*: $[X, f \cdot Y]_E = \rho_X[f] \cdot Y + f \cdot [X, Y]_E$,

3. *Jacobi identity*: $[X, [Y, Z]_E]_E + [Y, [Z, X]_E]_E + [Z, [X, Y]_E]_E = 0$.

The previous conditions ensure that the map $\rho \colon \Gamma(E) \to \Gamma(T\mathscr{M})$ defines a morphism of Lie algebras between the Lie algebra $(\Gamma(E), [\cdot, \cdot]_E)$ and the Lie algebra of vector fields on $\mathscr{M}$, i.e., $\rho_{[X, Y]_E} = [\rho_X, \rho_Y]$ for all $X, Y \in \Gamma(E)$.

Proposition A.1. *Let $(E, \rho, [\cdot, \cdot]_E)$ be a Lie algebroid. The following statements hold:*

1. *Let $\{x^\mu\}|_{\mu \in \{1, \ldots, \dim \mathscr{M}\}}$ be a set of coordinates of $\mathscr{M}$ and $\{e_a\}|_{a \in \{1, \ldots, \dim E\}}$ be a basis of $\Gamma(E)$. Setting $\rho_{e_a}[f] = \rho_a{}^\mu(x)\partial_\mu f$ and $[e_a, e_b]_E = f_{ab}{}^c(x)e_c$, the defining conditions of a Lie algebroid can be expressed in components as*

   $$f_{ab}{}^c = -f_{ba}{}^c, \qquad 2\rho_{[a}{}^\nu \partial_\nu \rho_{b]}{}^\mu = \rho_c{}^\mu f_{ab}{}^c, \qquad \rho_{[c}{}^\nu \partial_\nu f_{ab]}{}^d = f_{e[c}{}^d f_{ab]}{}^e.$$

   Acting on generic sections of E, the Lie algebroid bracket reads

   $$[X, Y]_E = \left(\rho_X[Y^c] - \rho_Y[X^c] + f_{ab}{}^c X^a Y^b\right)e_c.$$

2. *The exterior algebra $\Gamma(\wedge^\bullet E^*)$ is naturally endowed with a structure of dg commutative algebra with differential $\mathrm{d}_E \colon \Gamma(\wedge^\bullet E^*) \to \Gamma(\wedge^{\bullet+1} E^*)$ defined by*

   - $(\mathrm{d}_E f)(X) = \rho_X[f]$,
   - $(\mathrm{d}_E \eta)(X, Y) = \rho_X[\eta(Y)] - \rho_Y[\eta(X)] - \eta([X, Y]_E)$,
   - $\mathrm{d}_E(\alpha \wedge \beta) = (\mathrm{d}_E \alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (\mathrm{d}_E \beta)$,

   for all $X, Y \in \Gamma(E)$, $f \in \mathscr{C}^\infty(\mathscr{M})$, $\eta \in \Gamma(E^*)$ and $\alpha, \beta \in \Gamma(\wedge^\bullet E^*)$.

3. *The dual exterior algebra $\Gamma(\wedge^\bullet E)$ is endowed with a structure of Gerstenhaber algebra with graded bracket $\{\cdot, \cdot\}_E \colon \Gamma(\wedge^\bullet E) \otimes \Gamma(\wedge^\circ E) \to \Gamma(\wedge^{\bullet+\circ-1} E)$ defined as follows*

   $$\{f, g\}_E = 0, \qquad \{X, f\}_E = \rho_X[f], \qquad \{X, Y\}_E = [X, Y]_E,$$
   $$\{P, Q\}_E = -(-1)^{(|p|-1)(|q|-1)}\{Q, P\}_E,$$
   $$\{P, Q \wedge R\}_E = \{P, Q\}_E \wedge R + (-1)^{|q|(|p|-1)}Q \wedge \{P, R\}_E,$$

   for all $f, g \in \mathscr{C}^\infty(\mathscr{M})$, $X, Y \in \Gamma(E)$ and $P, Q, R \in \Gamma(\wedge^\bullet E)$.

   These conditions can be checked to ensure the graded Jacobi identity:

   $$\{\{P, Q\}_E, R\}_E + (-1)^{(|P|-1)(|Q|+|R|)}\{\{Q, R\}_E, P\}_E$$
   $$+ (-1)^{(|R|-1)(|P|+|Q|)}\{\{R, P\}_E, Q\}_E = 0.$$

Example A.2.

- A Lie algebra is a Lie algebroid whose base manifold is a point.

- Given a manifold $\mathscr{M}$, the *standard Lie algebroid* is defined as the tangent bundle $T\mathscr{M}$ together with the identity map as anchor and the usual Lie bracket of vector fields as bracket. The corresponding differential on the space of differential forms $\Omega^\bullet(\mathscr{M}) \simeq \Gamma(\wedge^\bullet T^*\mathscr{M})$ coincides with the *de Rham differential* while the induced Gerstenhaber bracket on $\Gamma(\wedge^\bullet T\mathscr{M})$ identifies with the *Schouten bracket* on polyvector fields.

- Let $(\mathcal{M}, \pi)$ be a Poisson manifold. The dual tangent bundle $T^*\mathcal{M}$ is naturally endowed with a Lie algebroid structure with anchor $\pi^\sharp \colon \Gamma(T^*\mathcal{M}) \to \Gamma(T\mathcal{M}) \colon \alpha_\mu \mathrm{d}x^\mu \mapsto \pi^{\mu\nu}\alpha_\nu \partial_\mu$ and bracket $[\alpha, \beta]_\pi = \frac{1}{2}\left(\mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha + i_{\pi^\sharp(\alpha)}\mathrm{d}\beta - i_{\pi^\sharp(\beta)}\mathrm{d}\alpha\right)$ for all $\alpha, \beta \in \Gamma(T^*\mathcal{M})$.

Lie bialgebroids. The concept of Lie bialgebroid was introduced by Mackenzie–Xu in [42] as the infinitesimal variant of a Poisson groupoid. We follow the modern definition of [36] and define a Lie bialgebroid as a vector bundle $E \xrightarrow{\pi} \mathcal{M}$ endowed with two dual Lie algebroid structures satisfying a natural compatibility condition. Denoting $(\rho, [\cdot, \cdot]_E)$ the Lie algebroid structure on E and $(R, [\cdot, \cdot]_{E^*})$ the one on E^*, the pair (E, E^*) is a Lie bialgebroid if d_{E^*} is a derivation of $\{\cdot, \cdot\}_E$, where we denoted $\{\cdot, \cdot\}_E$ the Gerstenhaber bracket on $\Gamma(\wedge^\bullet E)$ induced by $(\rho, [\cdot, \cdot]_E)$ and d_{E^*} the differential on $\Gamma(\wedge^\bullet E)$ induced by $(R, [\cdot, \cdot]_{E^*})$.[83]

Example A.3.

- A Lie bialgebra is a Lie bialgebroid whose base manifold is a point.

- Letting $\mathcal{M}$ be a Poisson manifold, the two Lie algebroid structures on $T\mathcal{M}$ and $T^*\mathcal{M}$ as defined in Example A.2 are compatible in the above sense and hence define a Lie bialgebroid structure on $(T\mathcal{M}, T^*\mathcal{M})$.

Lie bialgebroids thus generalise both Lie bialgebras and Poisson manifolds. In fact, the base manifold $\mathcal{M}$ of any Lie bialgebroid is endowed with a canonical Poisson bracket[84] defined as $\{f, g\} = \langle \mathrm{d}_{E^*}f, \mathrm{d}_E g\rangle$ (or in components as $\{f, g\} = R^{a[\mu}\rho_a{}^{\nu]}\partial_\mu f \partial_\nu g$).

The exterior algebra $\left(\Gamma(\wedge^\bullet E), \wedge\right)$ of a Lie bialgebroid is endowed with both a structure of Gerstenhaber bracket $\{\cdot, \cdot\}_E$ and of a differential d_{E^*} being a derivation for both the graded commutative product $\wedge$ and the Gerstenhaber bracket. Such a quadruplet $\left(\Gamma(\wedge^\bullet E), \wedge, \mathrm{d}_{E^*}, \{\cdot, \cdot\}_E\right)$ is called a strong differential Gerstenhaber algebra. It was in fact shown in [75] (see also [36]) that Lie bialgebroid structures on a vector bundle $E \xrightarrow{\pi} \mathcal{M}$ are in one-to-one correspondence with strong differential Gerstenhaber structures on $\left(\Gamma(\wedge^\bullet E), \wedge\right)$ (or equivalently with dg Poisson structures on $(\Gamma(\wedge^\bullet E)[1], \wedge)$).

Coboundary Lie bialgebroids. Letting E be a Lie algebroid, an *r-matrix* is a section $\Lambda \in \Gamma\left(\wedge^2 E\right)$ satisfying $\left\{X, \left\{\Lambda, \Lambda\right\}_E\right\}_E = 0$ for all $X \in \Gamma(E)$. An r-matrix endows E with a structure of Lie bialgebroid by defining $d_{E^*} = \left\{\Lambda, \cdot\right\}_E$. The defining condition on Λ is necessary and sufficient to ensure that the inner derivation d_{E^*} squares to zero. A Lie bialgebroid defined in this way is called a coboundary Lie bialgebroid. Whenever the stronger condition $\left\{\Lambda, \Lambda\right\}_E = 0$ holds, the induced Lie bialgebroid is said to be *triangular* [42] (cf. for example the Lie bialgebroid on Poisson manifolds defined in Example A.2). If furthermore, Λ is of constant rank, the triangular Lie bialgebroid is said to be *regular*.

Quasi-Lie, Lie-quasi and proto-Lie bialgebroids. The above mentioned characterisation of Lie bialgebroids as dg Poisson structures on $\left(\Gamma(\wedge^\bullet E)[1], \wedge\right)$ calls for several natural generalisations, as summarised in the following table[85] (see, e.g., [2, 22, 37, 60]):

- Lie bialgebroids on $(E, E^*) \Leftrightarrow$ dg Poisson algebras on $\Gamma(\wedge^\bullet E)[1]$,

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[83] Note that the defining condition of a Lie bialgebroid can equivalently be stated as the fact that the differential d_E on $\Gamma(\wedge^\bullet E^*)$ induced by $(\rho, [\cdot, \cdot]_E)$ is a derivation of the Gerstenhaber bracket $\{\cdot, \cdot\}_{E^*}$ induced by $(R, [\cdot, \cdot]_{E^*})$, i.e., the notion of Lie bialgebroid is self-dual.

[84] More generally, for a proto-Lie bialgebroid, the Jacobi identity for the bivector is deformed as

$$\pi^{\rho[\lambda}\partial_\rho \pi^{\mu\nu]} = \tfrac{1}{3}R^{a[\lambda}R^{b|\mu}R^{c|\nu]}\psi_{abc} + \tfrac{1}{3}\rho_a{}^{[\lambda}\rho_b{}^\mu \rho_c{}^{\nu]}\varphi^{abc}.$$

[85] As is transparent from the correspondence below, there is a series of inclusions of bialgebroids: Lie $\subset$ quasi-Lie $\subset$ proto-Lie and Lie $\subset$ Lie-quasi $\subset$ proto-Lie. Note furthermore that the notions of Lie bialgebroids and proto-Lie bialgebroids are self-dual (and thus can be defined on both $\Gamma(\wedge^\bullet E)[1]$ and $\Gamma(\wedge^\bullet E^*)[1]$) while the notions of Lie-quasi bialgebroids and quasi-Lie bialgebroids are dual to each other.

- quasi-Lie bialgebroids on $(E, E^*) \Leftrightarrow$ homotopy Poisson algebras[86] on $\Gamma(\wedge^\bullet E)[1]$,

- Lie-quasi bialgebroids on $(E, E^*) \Leftrightarrow$ homotopy Poisson algebras on $\Gamma(\wedge^\bullet E^*)[1]$,

- proto-Lie bialgebroids on $(E, E^*) \Leftrightarrow$ curved homotopy Poisson algebras on $\Gamma(\wedge^\bullet E)[1]$.

We now focus on the most general case, namely proto-Lie bialgebroids. Apart from the usual data (two anchors ρ, R and two brackets $[\cdot, \cdot]_E$, $[\cdot, \cdot]_{E^*}$), a proto-Lie bialgebroid contains two elements $\varphi \in \Gamma(\wedge^3 E)$ and $\psi \in \Gamma(\wedge^3 E^*)$ which play the rôle of various obstructions to the usual Lie bialgebroid identities. Letting $E^*[1]$ be the shifted dual bundle coordinatised by $\{x^\mu, \zeta_a\}$ of respective degree 0 and 1, the graded commutative algebra $(\mathscr{C}^\infty(E^*[1]), \cdot)$ is isomorphic to the exterior algebra of sections $(\Gamma(\wedge^\bullet E), \wedge)$. The most general curved homotopy Poisson structure l on $\Gamma(\wedge^\bullet E)[1]$ thus takes the form[87]

- $l_0 := \dfrac{1}{6}\varphi^{abc}\zeta_a\zeta_b\zeta_c,$

- $l_1(f) := \mathrm{d}_{E^*}f = R^{a|\mu}\zeta_a\dfrac{\partial f}{\partial x^\mu} - \dfrac{1}{2}C_c{}^{ab}\zeta_a\zeta_b\dfrac{\partial f}{\partial \zeta_c},$

- $l_2(f,g) := \{f,g\}_E = -\rho_a{}^\mu\left(\dfrac{\partial f}{\partial \zeta_a}\dfrac{\partial g}{\partial x^\mu} + (-1)^{|f|}\dfrac{\partial f}{\partial x^\mu}\dfrac{\partial g}{\partial \zeta_a}\right) + (-1)^{|f|}f_{ab}{}^c\zeta_c\dfrac{\partial f}{\partial \zeta_a}\dfrac{\partial g}{\partial \zeta_b},$

- $l_3(f,g,h) := (-1)^{|g|}\psi_{abc}\dfrac{\partial f}{\partial \zeta_a}\dfrac{\partial g}{\partial \zeta_b}\dfrac{\partial h}{\partial \zeta_c}.$

Imposing the defining quadratic condition $[l, l]_{\mathsf{NR}} = 0$ of a (curved) Lie_∞-algebra yields a series of identities which precisely reproduce the components conditions (2.10)–(2.18) as

- $[l_0, l_1]_{\mathsf{NR}} = 0 \; \Leftrightarrow \; \mathcal{C}_8 = 0,$ \hfill (A.2)

- $[l_0, l_2]_{\mathsf{NR}} + \frac{1}{2}[l_1, l_1]_{\mathsf{NR}} = 0 \; \Leftrightarrow \; \mathcal{C}_3 = \mathcal{C}_4 = 0,$ \hfill (A.3)

- $[l_0, l_3]_{\mathsf{NR}} + [l_1, l_2]_{\mathsf{NR}} = 0 \; \Leftrightarrow \; \mathcal{C}_5 = \mathcal{C}_6 = \mathcal{C}_7 = 0,$ \hfill (A.4)

- $[l_1, l_3]_{\mathsf{NR}} + \frac{1}{2}[l_2, l_2]_{\mathsf{NR}} = 0 \; \Leftrightarrow \; \mathcal{C}_1 = \mathcal{C}_2 = 0,$ \hfill (A.5)

- $[l_2, l_3]_{\mathsf{NR}} = 0 \; \Leftrightarrow \; \mathcal{C}_9 = 0.$ \hfill (A.6)

Imposing $\psi \equiv 0$ (resp. $\varphi \equiv 0$) yields a Lie-quasi (resp. quasi-Lie) bialgebroid and Lie bialgebroids are recovered by setting $\psi \equiv 0$, $\varphi \equiv 0$. Assuming that the base manifold $\mathscr{M}$ is the one-point manifold and denoting the vector space $\Gamma(E)$ as $\mathfrak{g}$ allows to define the counterparts of these notions in the "bialgebra" realm, cf. Section 2.1.

Twisting. The previous formulation allows us to introduce the notion of *twist* of a proto-Lie bialgebroid in terms of twist of (curved) Lie_∞-algebras.[88] Given a proto-Lie bialgebroid on (E, E^*) defined by the curved homotopy Poisson algebra $(\Gamma(\wedge^\bullet E)[1], \wedge, l)$ where $l := \{l_p\}_{0 \leq p \leq 3}$,

---

[86] Recall that a (curved) homotopy Poisson structure on a graded commutative algebra $(\mathfrak{g}, \wedge)$ is a (curved) Lie_∞-structure on $\mathfrak{g}$ such that all brackets are multi-derivations with respect to $(\mathfrak{g}, \wedge)$, see, e.g., [38, 44]. A dg Poisson algebra is thus a (flat) homotopy Poisson algebra for which the brackets of arity above 2 vanish.

[87] The bracket l_p being of degree $3 - 2p$ on $\Gamma(\wedge^\bullet E) \simeq \mathscr{C}^\infty(E^*[1])$, the fact that each bracket is a multi-derivation for the underlying graded commutative algebra constrains all brackets of arity higher than 3 to vanish. Pulling back the brackets along the suspension map $s\colon \Gamma(\wedge^\bullet E)[1] \to \Gamma(\wedge^\bullet E)$ of degree 1 yields a series of brackets on $\Gamma(\wedge^\bullet E)[1]$ with the usual degree $2 - p$.

[88] The twisting procedure has been introduced by Quillen [57] for dg Lie algebras and later generalised by Getzler [24] to Lie_∞-algebras. Letting $(\mathfrak{g}, l)$ be a nilpotent (curved) Lie_∞-algebra and $\mathfrak{m} \in \mathfrak{g}^1$ be an arbitrary element of degree 1, one defines the *twisted brackets* $l_n^{\mathfrak{m}}\colon \mathfrak{g}^{\wedge n} \to \mathfrak{g}$ of degree $2 - n$ for $n \geq 0$ as

$$l_n^{\mathfrak{m}}(a_1, \ldots, a_n) := \sum_{k \geq 0}\frac{1}{k!}l_{k+n}(\mathfrak{m}^{\wedge k}, a_1, \ldots, a_n), \qquad \text{where} \quad a_i \in \mathfrak{g}.$$

The twisted brackets can be checked to endow $\mathfrak{g}$ with a structure of *curved* Lie_∞-algebra. Whenever the curvature $l_0^{\mathfrak{m}} \in \mathfrak{g}^2$ vanishes, the element $\mathfrak{m} \in \mathfrak{g}^1$ is called a *Maurer–Cartan element* and $(\mathfrak{g}, l^{\mathfrak{m}})$ is then a *flat* Lie_∞-algebra.

one can define a new proto-Lie bialgebroid structure on (E, E^*) via twisting the (curved) Lie_∞-algebra structure l by an arbitrary bivector $\Lambda \in \Gamma(\wedge^2 E)$ (being the most general element of degree 1 in $\Gamma(\wedge^\bullet E)[1]$). In components, such a twist amounts to the following shift of the components of the proto-Lie bialgebroid:

$$
\begin{aligned}
&\rho_a{}^\mu \xrightarrow{\Lambda} \rho_a{}^\mu, \qquad f_{ab}{}^c \xrightarrow{\Lambda} f_{ab}{}^c + \psi_{abe}\Lambda^{ec}, \qquad \psi_{abc} \xrightarrow{\Lambda} \psi_{abc}, \\
&R^{a|\mu} \xrightarrow{\Lambda} R^{a|\mu} - \rho_b{}^\mu \Lambda^{ba}, \qquad C_c{}^{ab} \xrightarrow{\Lambda} C_c{}^{ab} + \rho_c{}^\mu \partial_\mu \Lambda^{ab} + 2\Lambda^{e[a} f_{ec}{}^{b]} - \psi_{cef}\Lambda^{ea}\Lambda^{fb}, \\
&\varphi^{abc} \xrightarrow{\Lambda} \varphi^{abc} - 3R^{[c|\mu}\partial_\mu \Lambda^{ab]} + 3\Lambda^{e[c}C_e{}^{ab]} + 3\rho_d{}^\mu \Lambda^{d[a}\partial_\mu \Lambda^{bc]} \\
&\qquad\qquad - 3f_{ef}{}^{[c}\Lambda^{e|a}\Lambda^{f|b]} - \psi_{def}\Lambda^{da}\Lambda^{eb}\Lambda^{fc}.
\end{aligned}
\tag{A.7}
$$

The latter can be checked to form the components of a proto-Lie bialgebroid without extra assumption on the bivector Λ. Note that, while a twist by a generic Λ does preserve the subspace of Lie-quasi bialgebroids (for which $\psi \equiv 0$), only bivectors satisfying the Maurer–Cartan equation $l_1(\Lambda) + \frac{1}{2}l_2(\Lambda, \Lambda) + \frac{1}{6}l_3(\Lambda, \Lambda, \Lambda) = 0$ in $\Gamma(\wedge^\bullet E)[1]$ preserve the subspace of quasi-Lie bialgebroids[89] (for which $\varphi \equiv 0$), see [40] for the Lie bialgebroid case and [60] for quasi-Lie bialgebroids.

Example A.4 (twisted Poisson manifolds). Let $\mathscr{M}$ be a manifold and $H \in \Omega^3(\mathscr{M})$ be a closed 3-form. There is a quasi-Lie bialgebroid structure on $(T\mathscr{M}, T^*\mathscr{M})$ defined in components as

$$
\rho_\nu{}^\mu = \delta_\nu{}^\mu, \qquad f_{\mu\nu}{}^\lambda = 0, \qquad R^{\nu|\mu} = 0, \qquad C_\lambda{}^{\mu\nu} = 0, \qquad \psi_{\lambda\mu\nu} = H_{\lambda\mu\nu}.
$$

Letting $\pi \in \Gamma(\wedge^2 T\mathscr{M})$ be a bivector, the components of the twisted proto-Lie bialgebroid read

$$
\begin{aligned}
&\tilde{\rho}_\nu{}^\mu = \delta_\nu{}^\mu, \qquad \tilde{f}_{\mu\nu}{}^\lambda = H_{\mu\nu\gamma}\pi^{\gamma\lambda}, \qquad \tilde{R}^{\nu|\mu} = -\pi^{\mu\nu}, \qquad \tilde{C}_\lambda{}^{\mu\nu} = \partial_\lambda \pi^{\mu\nu} - H_{\lambda\rho\sigma}\pi^{\rho\mu}\pi^{\sigma\nu}, \\
&\tilde{\varphi}^{\lambda\mu\nu} = 3\,\pi^{\rho[\lambda}\partial_\rho \pi^{\mu\nu]} - H_{\rho\sigma\gamma}\pi^{\rho\lambda}\pi^{\sigma\mu}\pi^{\gamma\nu}, \qquad \tilde{\psi}_{\lambda\mu\nu} = H_{\lambda\mu\nu}.
\end{aligned}
$$

The resulting proto-Lie bialgebroid is again a quasi-Lie bialgebroid if and only $\tilde{\varphi} \equiv 0$, that is, if π is a twisted Poisson bivector [60].[90] Whenever H vanishes, we recover the Lie bialgebroid structure on the (co)-tangent bundle of a Poisson manifold, cf. Example A.3.

Dually, one can consider twisting the curved homotopy Poisson algebra structure on $\Gamma(\wedge^\bullet E^*)[1]$ by a 2-form field $\omega \in \Gamma(\wedge^2 E^*)$ which amounts to the following shift of the components of the proto-Lie bialgebroid

$$
\begin{aligned}
&\rho_a{}^\mu \xrightarrow{\omega} \rho_a{}^\mu - R^{b|\mu}\omega_{ba}, \qquad f_{ab}{}^c \xrightarrow{\omega} f_{ab}{}^c + R^{c|\mu}\partial_\mu \omega_{ab} - 2\omega_{d[a}C_{b]}{}^{cd} - \varphi^{cde}\omega_{da}\omega_{eb}, \\
&R^{a|\mu} \xrightarrow{\omega} R^{a|\mu}, \qquad C_c{}^{ab} \xrightarrow{\omega} C_c{}^{ab} + \varphi^{abd}\omega_{dc}, \qquad \varphi^{abc} \xrightarrow{\omega} \varphi^{abc}, \\
&\psi_{abc} \xrightarrow{\omega} \psi_{abc} - 3\rho_{[a}{}^\mu \partial_\mu \omega_{bc]} + 3\omega_{e[c}f_{ab]}{}^e + 3\,R^{d|\mu}\omega_{d[a}\partial_\mu \omega_{bc]} \\
&\qquad\qquad - 3C_{[c}{}^{de}\omega_{d|a}\omega_{e|b} - \varphi^{def}\omega_{da}\omega_{eb}\omega_{fc}.
\end{aligned}
\tag{A.8}
$$

Twisting by a 2-form field does preserve quasi-Lie bialgebroids but fails to preserve Lie-quasi bialgebroids unless ω satisfies the associated Maurer–Cartan equation on $\Gamma(\wedge^\bullet E^*)[1]$.

Courant algebroids. A Courant algebroid structure on a pseudo-Euclidean vector bundle[91] $(\mathcal{E}, \langle \cdot, \cdot \rangle_\mathcal{E})$ is a pair $(\rho_\mathcal{E}, [\cdot, \cdot]_\mathcal{E})$ where $\rho_\mathcal{E} \colon \Gamma(\mathcal{E}) \to \Gamma(T\mathscr{M})$ is a $\mathscr{C}^\infty(\mathscr{M})$-linear map called the

---

[89]Equivalently, the Maurer–Cartan condition ensures that the twisting preserves the flatness of the Lie_∞-algebra associated to a given quasi-Lie bialgebroid, see footnote 88.

[90]Recall that *twisted Poisson bivectors* [31, 64] are bivectors satisfying the twisted Jacobi identity $[\pi, \pi]_\mathsf{S} = \frac{1}{3}H(\pi, \pi, \pi)$.

[91]We remind the reader that a *pseudo-Euclidean vector bundle* is a vector bundle $\mathcal{E} \to \mathscr{M}$ endowed with a symmetric, non-degenerate and $\mathscr{C}^\infty(\mathscr{M})$-bilinear form on the space of sections of $\mathcal{E}$, denoted $\langle \cdot, \cdot \rangle_\mathcal{E} \colon \Gamma(\mathcal{E}) \vee \Gamma(\mathcal{E}) \to \mathscr{C}^\infty(\mathscr{M})$ and referred to as the *fiber-wise metric*

anchor while $[\cdot,\cdot]_{\mathcal{E}} : \Gamma(\mathcal{E}) \otimes \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is a $\mathbb{K}$-bilinear form on the fibers of E referred to as the *Dorfman bracket*.

The latter satisfy the following minimal set of axioms:

1. The Dorfman bracket satisfies the Jacobi identity in its Leibniz form

$$[e_1, [e_2, e_3]_{\mathcal{E}}]_{\mathcal{E}} = [[e_1, e_2]_{\mathcal{E}}, e_3]_{\mathcal{E}} + [e_2, [e_1, e_3]]_{\mathcal{E}} \qquad \text{for all } e_1, e_2, e_3 \in \Gamma(\mathcal{E}).$$

2. The symmetric part of the Dorfman bracket is controlled by the anchor

$$\langle [e_1, e_1]_{\mathcal{E}}, e_2 \rangle_{\mathcal{E}} = \tfrac{1}{2} \rho_{\mathcal{E}}|_{e_2} \big[\langle e_1, e_1 \rangle_{\mathcal{E}} \big] \qquad \text{for all } e_1, e_2 \in \Gamma(\mathcal{E}).$$

3. The fiber-wise metric is compatible with the Courant algebroid structure

$$\rho_{\mathcal{E}}|_{e_1} \big[\langle e_2, e_3 \rangle_{\mathcal{E}} \big] = \langle [e_1, e_2]_{\mathcal{E}}, e_3 \rangle_{\mathcal{E}} + \langle e_2, [e_1, e_3]_{\mathcal{E}} \rangle_{\mathcal{E}} \qquad \text{for all } e_1, e_2, e_3 \in \Gamma(\mathcal{E}).$$

Letting $\big(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \rho_{\mathcal{E}}, [\cdot,\cdot]_{\mathcal{E}} \big)$ and $\big(\mathcal{E}', \langle\cdot,\cdot\rangle_{\mathcal{E}'}, \rho_{\mathcal{E}'}, [\cdot,\cdot]_{\mathcal{E}'} \big)$ be two Courant algebroids over the same manifold[92] $\mathcal{M}$, a morphism of Courant algebroids is defined as a morphism of the underlying vector bundles $\mathcal{F} \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ preserving the additional structures,[93] i.e.,

$$\langle e_1, e_2 \rangle_{\mathcal{E}} = \langle \mathcal{F}(e_1), \mathcal{F}(e_2) \rangle_{\mathcal{E}'}, \qquad \rho_{\mathcal{E}} = \rho_{\mathcal{E}'} \circ \mathcal{F},$$
$$\mathcal{F}([e_1, e_2]_{\mathcal{E}}) = [\mathcal{F}(e_1), \mathcal{F}(e_2)]_{\mathcal{E}'} \qquad \text{for all } e_1, e_2 \in \Gamma(\mathcal{E}).$$

Whenever the morphism $\mathcal{F}$ is invertible, it is called an *isomorphism* of Courant algebroids.

Courant algebroids first appeared implicitly in the work of I. Dorfman [15] and T. Courant [8, 9] on integrable Dirac structures before their precise geometric structure was abstracted away in [40] to account for the concept of double of Lie bialgebroids. More generally, the double $\mathcal{E} := E \oplus E^*$ of a proto-Lie bialgebroid (E, E^*) carries a natural structure of pseudo-Euclidean vector bundle with fiber-wise metric $\langle e_1, e_2 \rangle_{\mathcal{E}} := \alpha(Y) + \beta(X)$, where $e_1 := X \oplus \alpha$ and $e_2 := Y \oplus \beta$ for all $X, Y \in \Gamma(E)$ and $\alpha, \beta \in \Gamma(E^*)$. One can furthermore endow $\mathcal{E}$ with the anchor $\rho_{\mathcal{E}}(e_1) := \rho(X) \oplus R(\alpha)$ while the Dorfman bracket is defined through the following explicit expression

$$[e_1, e_2]_{\mathcal{E}} := \big([X, Y]_E + \mathcal{L}_{\alpha}^{E^*} Y - \iota_{\beta} \mathrm{d}_{E^*} X - \varphi(\alpha, \beta, \cdot) \big)$$
$$\oplus \big([\alpha, \beta]_{E^*} + \mathcal{L}_X^E \beta - \iota_Y d_E \alpha - \psi(X, Y, \cdot) \big),$$

where $\mathcal{L}_X^E$ stands for the unique derivative operator extending the action of $[X, \cdot]_E$ to the tensor algebra of E $\big($and similarly for $\mathcal{L}_{\alpha}^{E^*}\big)$. The axioms of a proto-Lie bialgebroid thus ensure that the pair $\big(\rho_{\mathcal{E}}, [\cdot,\cdot]_{\mathcal{E}} \big)$ defines a Courant algebroid structure on $\big(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}} \big)$, cf. [58, Theorem 3.8.2].

Example A.5 (exact Courant algebroids)**.** A Courant algebroid such that the following sequence[94]

$$0 \longrightarrow T^*\mathcal{M} \xrightarrow{\rho_{\mathcal{E}}^*} \mathcal{E} \xrightarrow{\rho_{\mathcal{E}}} T\mathcal{M} \longrightarrow 0$$

---

[92]More general notions of morphisms of Courant algebroids over different base manifolds can be defined, see [68] for a thorough treatment.

[93]A morphism of vector bundles $\mathcal{F} \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ satisfying $\langle e_1, e_2 \rangle_{\mathcal{E}} = \langle \mathcal{F}(e_1), \mathcal{F}(e_2) \rangle_{\mathcal{E}'}$ will be called a morphism of pseudo-vector bundles.

[94]The vector bundle morphism $\rho_{\mathcal{E}}^*: T^*\mathcal{M} \to \mathcal{E}$ is defined via $\langle \rho_{\mathcal{E}}^*|_{\alpha}, e \rangle_{\mathcal{E}} = \langle \alpha, \rho_{\mathcal{E}}|_e \rangle$ for all $\alpha \in \Omega^1(\mathcal{M})$ and $e \in \Gamma(\mathcal{E})$, and satisfies $\rho_{\mathcal{E}} \circ \rho_{\mathcal{E}}^* = 0$.

is exact is called an exact Courant algebroid. It is an important result due to P. Ševera [63] that exact Courant algebroids over a fixed base manifold $\mathcal{M}$ are classified[95] by the third de Rham cohomology $H^3_{\mathsf{dR}}(\mathcal{M})$ of $\mathcal{M}$.

Dirac structures. A subbundle $L \subset \mathcal{E}$ of a Courant algebroid $\mathcal{E}$ is called a Dirac structure if:

1. L is maximally isotropic with respect to the fiber-wise metric $\langle \cdot, \cdot \rangle_{\mathcal{E}}$, i.e., $\langle \Gamma(L), \Gamma(L) \rangle_{\mathcal{E}} = 0$, $\dim L = \frac{1}{2} \dim \mathcal{E}$.

2. The space of sections $\Gamma(L)$ is closed under the Courant bracket $[\cdot, \cdot]_{\mathcal{E}}$, i.e., $[\Gamma(L), \Gamma(L)]_{\mathcal{E}} \subseteq \Gamma(L)$.

These conditions ensure in particular that the restrictions of the Courant anchor and bracket to L endow the vector bundle L with a structure of Lie algebroid.

Example A.6 (Dirac structure). Let $\mathcal{M}$ be a manifold and $[H] \in H^3_{\mathsf{dR}}(\mathcal{M})$. Given a representative $H \in \Omega^3(\mathcal{M})$ and a twisted Poisson structure π with respect to H, the graph of the map $\pi^\sharp \colon T^*\mathcal{M} \to T\mathcal{M}$ can be checked to be a Dirac structure for the exact Courant algebroid associated to $(\mathcal{M}, [H])$ [63, 64].

A pair $(\mathcal{E}, L)$ where L is a Dirac structure for the Courant algebroid $\mathcal{E}$ is referred to as a *Manin pair*. A triplet $(\mathcal{E}, L, M)$ where L, M are two Dirac structures of $\mathcal{E}$ such that $\mathcal{E} = L \oplus M$ is referred to as a *Manin triple*. Both the notions of Manin pairs and triples recover their usual extension in the context of Lie algebras when the base manifold $\mathcal{M}$ is a point. Letting $(\mathcal{E}, L)$ and $(\mathcal{E}', L')$ be two Manin pairs over the same manifold $\mathcal{M}$, a morphism of Manin pairs[96] is a morphism of Courant algebroids $\mathcal{F} \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ such that $\mathcal{F}(L) \subseteq L'$.

Letting $\mathcal{E} = E \oplus E^*$ be the pseudo-Euclidean vector bundle associated to the vector bundle E over $\mathcal{M}$ and endowed with the canonical fiber-wise metric $\langle \cdot, \cdot \rangle_{\mathcal{E}}$, we have the following hierarchy of identifications:

- proto-Lie bialgebroids on $(E, E^*) \Leftrightarrow$ Courant algebroids structures on $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$,
- Lie-quasi bialgebroids on $(E, E^*) \Leftrightarrow$ Manin pairs $((\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}}), E)$,
- quasi-Lie bialgebroids on $(E, E^*) \Leftrightarrow$ Manin pairs $((\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}}), E^*)$,
- Lie bialgebroids on $(E, E^*) \Leftrightarrow$ Manin triples $((\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}}), E, E^*)$.

Such identification allows to define morphisms of proto/Lie-quasi/quasi-Lie/Lie bialgebroids as morphisms of Courant algebroids preserving additional Dirac structures. Explicitly, morphisms of pseudo-vector bundles $E \oplus E^* \xrightarrow{\mathcal{F}} E' \oplus E'^*$ with canonical fiber-wise metrics generically take the form

$$\mathcal{F}(e)^{a'} = \mathcal{F}^{a'}{}_b X^b + \mathcal{F}^{a'b} \alpha_b, \qquad \mathcal{F}(e)_{a'} = \mathcal{F}_{a'b} X^b + \mathcal{F}_{a'}{}^b \alpha_b$$

satisfying

$$\mathcal{F}_{c'(a} \mathcal{F}^{c'}{}_{b)} = 0, \qquad \mathcal{F}^{c'(a} \mathcal{F}_{c'}{}^{b)} = 0, \qquad \mathcal{F}_{c'}{}^a \mathcal{F}^{c'}{}_b + \mathcal{F}^{c'a} \mathcal{F}_{c'b} = \delta^a{}_b.$$

---

[95] More precisely, any exact Courant algebroid is isomorphic to $\mathcal{E} := T\mathcal{M} \oplus T^*\mathcal{M}$ with canonical fiber-wise metric, projection $T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M}$ as anchor and Dorfman bracket $[e_1, e_2]_{\mathcal{E}} := [X, Y] \oplus (\mathcal{L}_X \beta - \iota_Y d\alpha - H(X, Y, \cdot))$, where $H \in \Omega^3(\mathcal{M})$ is a closed 3-form. The latter Courant algebroid structure coincides with the one induced by the quasi-bialgebroid structure of Example A.4. Letting $\omega \in \Gamma(\wedge^2 E^*)$, the isomorphism $X \oplus \alpha \xrightarrow{\omega} X \oplus (\alpha + \frac{1}{2}\iota_X \omega)$ corresponding to the twist (2.20) amounts to shift the closed 3-form H by an exact 3-form as $H \xrightarrow{\omega} H - d\omega$. Hence isomorphism classes of exact Courant algebroids over $\mathcal{M}$ are in bijective correspondence with elements of $H^3_{\mathsf{dR}}(\mathcal{M})$.

[96] Similarly, letting $(\mathcal{E}, L, M)$ and $(\mathcal{E}', L'M')$ be two Manin triples over the same manifold $\mathcal{M}$, a morphism of Manin triples is a morphism of Courant algebroids $\mathcal{F} \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ such that $\mathcal{F}(L) \subseteq L'$ and $\mathcal{F}(M) \subseteq M'$.

Such morphisms map the Dirac structure E to E' if and only if $\mathcal{F}_{a'b} = 0$ and the Dirac structure E^* to E'^* if and only if $\mathcal{F}^{a'b} = 0$. Infinitesimal endomorphisms of the pseudo-Euclidean vector bundle $(E \oplus E^*, \langle \cdot, \cdot \rangle_{\mathcal{E}})$ read[97]

$$\delta\mathcal{F}(e) = \left(\lambda(X) + \tfrac{1}{2}\iota_\alpha\Lambda\right) \oplus \left(-\lambda^T(\alpha) + \tfrac{1}{2}\iota_X\omega\right),$$

where

- $\lambda \in \mathsf{End}(\Gamma(E))$ generates infinitesimal rotations of the fibers of (E, E^*),

- $\Lambda \in \Gamma(\wedge^2 E)$ generates the infinitesimal version of the twist of Lie-quasi bialgebroids (A.7),

- $\omega \in \Gamma(\wedge^2 E^*)$ generates the infinitesimal version of the twist of quasi-Lie bialgebroids (A.8).

B Incarnation of the Θ-graph in $d = 3$

The present appendix is devoted to collect some additional results regarding the exotic Lie_∞-structure θ of Theorem 1.1 generated by the cocycle class $[\Theta_3] \in H^1(\mathsf{o_2d_0fGC_3})$. For concreteness, we fix a representative of the class $[\Theta_3]$ as follows:

Proposition B.1. *There is a unique pair of combination of graphs $\Theta_3 \in \mathsf{o_2d_0fGC_3}$ and $\vartheta_3 \in \mathsf{o_1d_1fGC_3^{black}}$ such that:*

1. *$\Theta_3 = -\delta\vartheta_3$, i.e., Θ_3 is exact in $\mathsf{o_1d_1fGC_3^{black}}$.*

2. *Θ_3 contains only graphs of the shape C, cf. Figure 8.*

3. *Each graph of ϑ_3 contains at least one red cycle, i.e., $\vartheta_3 \notin \mathsf{o_2d_0fGC_3}$.*

Remark B.2.

- The combination of graphs ϑ_3 contains 68 black-oriented graphs (48 graphs of shape A and 20 graphs of shape B) while the combination Θ_3 contains 288 bi-oriented graphs of shape C.

- Although Θ_3 is exact in $\mathsf{o_1d_1fGC_3^{black}}$, it is crucial to note that Θ_3 is *not* exact in $\mathsf{o_2d_0fGC_3}$, i.e., there is no combination of graphs $\kappa_3 \in \mathsf{o_2d_0fGC_3}$ such that $\Theta_3 = -\delta\kappa_3$. Hence Θ_3 is a non-trivial cocycle in $\mathsf{o_2d_0fGC_3}$.
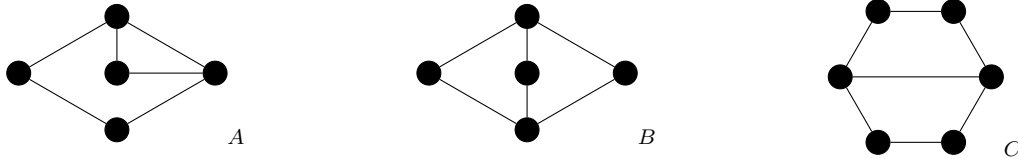
Figure 8. Shape of graphs involved in ϑ_3 (A and B) and Θ_3 (C).

Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle. To each Lie bialgebroid structure on (E, E^*) (represented by the Hamiltonian function $\mathscr{H} \in \mathcal{A}^E_{\mathrm{Lie}}$), we will associate the functions

- $\vartheta_3(\mathscr{H}) := \mathsf{Rep}^E_5(\vartheta_3)(\mathscr{H}^{\otimes 5}) \in \mathcal{A}^E_{\mathrm{Lie\text{-}quasi}}$,

- $\Theta_3(\mathscr{H}) := \mathsf{Rep}^E_6(\Theta_3)(\mathscr{H}^{\otimes 6}) \in \mathcal{A}^E_{\mathrm{Lie}}$.

---

[97] Restricting to Lie-quasi (resp. quasi-Lie) bialgebroids yields $\omega \equiv 0$ (resp. $\Lambda \equiv 0$), so that the subalgebra $\mathsf{End}(\Gamma(E))$ acts on the abelian ideal $\Gamma(\wedge^2 E)$ (resp. $\Gamma(\wedge^2 E^*)$) by rotations.

Note that the condition $\Theta_3 = -\delta\vartheta_3$ ensures that $\Theta_3(\mathscr{H}) \sim \mathsf{Q}[\vartheta_3(\mathscr{H})]$ – where the differential Q is defined as $\mathsf{Q} := \left\{\mathscr{H}, \cdot\right\}_\Omega^E$ – so that $\Theta_3(\mathscr{H})$ is a coboundary in the complex $(\mathcal{A}_{\text{Lie-quasi}}^E | \mathsf{Q})$. However, $\Theta_3(\mathscr{H})$ is generically not exact in $\mathcal{A}_{\text{Lie}}^E$ and the obstruction for $\Theta_3(\mathscr{H})$ to be a coboundary in $\mathcal{A}_{\text{Lie}}^E$ is precisely given by the component of $\vartheta_3(\mathscr{H})$ proportional to ζ^3. We will denote this obstruction as $\text{Ob}(\mathscr{H})^{abc}$, so that

$$\text{Ob}(\mathscr{H})^{abc} = 0 \;\Rightarrow\; \vartheta_3(\mathscr{H}) \in \mathcal{A}_{\text{Lie}}^E$$

and $\Theta_3(\mathscr{H})$ is a trivial cocycle in $(\mathcal{A}_{\text{Lie-quasi}}^E | \mathsf{Q})$. A straightforward computation gives

$$
\begin{aligned}
\text{Ob}(\mathscr{H})^{abc} = {}& R^{d|\mu} R^{e|\nu} \big(2\rho_f{}^\lambda \partial_\mu C_e{}^{[a|f} \partial_{\lambda\nu} C_d{}^{bc]} - \rho_d{}^\beta \partial_{\beta\nu} R^{[a|\lambda} \partial_{\mu\lambda} C_e{}^{bc]} - \rho_d{}^\lambda \partial_\mu C_e{}^{[a|f} \partial_{\lambda\nu} C_f{}^{bc]} \\
&\quad - \partial_\mu R^{f|\lambda} f_{ef}{}^{[a} \partial_{\nu\lambda} C_d{}^{bc]} - 2 f_{df}{}^{[a} \partial_\mu C_e{}^{b|g} \partial_\nu C_g{}^{c]f} + 2 f_{fg}{}^{[a} \partial_\mu C_e{}^{b|f} \partial_\nu C_d{}^{c]g}\big) \\
&+ R^{d|\mu} \partial_\mu R^{e|\nu} \big(f_{df}{}^{[a} C_e{}^{fg} \partial_\nu C_g{}^{bc]} - 2 f_{ef}{}^{[a} C_d{}^{fg} \partial_\nu C_g{}^{bc]} + \rho_d{}^\lambda \partial_\lambda C_e{}^{[a|f} \partial_\nu C_f{}^{bc]} \\
&\quad - 2\rho_e{}^\lambda \partial_\lambda C_d{}^{[a|f} \partial_\nu C_f{}^{bc]} - \rho_e{}^\lambda \partial_\lambda C_f{}^{[ab} \partial_\nu C_d{}^{c]f}\big) \\
&+ \rho_d{}^\beta \partial_\beta R^{e|\nu} R^{d|\mu} \partial_{\mu\nu} R^{[a|\lambda} \partial_\lambda C_e{}^{bc]} + 2\rho_e{}^\lambda R^{d|\mu} \partial_{\lambda\mu} R^{[a|\nu} C_d{}^{ef} \partial_\nu C_f{}^{bc]} \\
&+ \rho_g{}^\nu R^{d|\mu} C_d{}^{ef} \partial_\nu C_e{}^{[ab} \partial_\mu C_f{}^{c]g}.
\end{aligned}
$$

The latter encodes the first obstruction for $\mathscr{H}$ to define a "quantizable Lie bialgebroid". Although the obstruction does not vanish for a generic Lie bialgebroid, the following proposition displays two important examples:

Proposition B.3. *The obstruction vanishes for*

- *Lie bialgebras,*

- *coboundary Lie bialgebroids.*

Proof. Setting $R^{a|\mu} \equiv 0$ yields $\text{Ob}(\mathscr{H})^{abc} = 0$ hence the obstruction vanishes for Lie bialgebras. More generally, it can be checked that each graph appearing in the combinations ϑ_3 and Θ_3 contains at least one arrow of the type ⓘ►►ⓙ so that both $\vartheta_3(\mathscr{H})$ and $\Theta_3(\mathscr{H})$ vanish identically on the graded Poisson subalgebra[98] $\mathcal{A}_{\text{Lie}}^{\mathfrak{g}} \subset \mathcal{A}_{\text{Lie}}^E$.

For coboundary Lie bialgebroids, we perform the replacement $R^{a|\mu} = \rho_b{}^\mu \Lambda^{ba}$ and $C_c{}^{ab} = -\rho_c{}^\mu \partial_\mu \Lambda^{ab} - 2\Lambda^{d[a} f_{dc}{}^{b]}$, with $\Lambda^{ab} = \Lambda^{[ab]}$ a bivector, see Appendix A. Under this replacement, it can be checked that $\text{Ob}(\mathscr{H})^{abc}$ identically vanishes modulo the defining conditions $\mathcal{C}_1 \equiv 0$, $\mathcal{C}_2 \equiv 0$ ensuring that the maps (ρ, f) define a Lie algebroid. ∎

Acknowledgements

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[98]See footnote 16.

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