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The Weighted Ambient Metric

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Abstract. We prove the existence and uniqueness of weighted ambient metrics and weighted Poincaré metrics for smooth metric measure spaces.

Key words: ambient metric; Poincaré metric; smooth metric measure space

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1 Introduction

The Fefferman–Graham ambient space [15] is a formally Ricci flat space canonically associated to a given conformal manifold. Among its many applications are the classification of local scalar conformal invariants [1, 16] and the construction of a family of conformally covariant operators, called GJMS operators, with leading-order term a power of the Laplacian [19]. One can also use the ambient space to construct an asymptotically hyperbolic, formally Einstein space with conformal boundary a given conformal manifold. The resulting space, called a Poincaré space, likewise has many applications, among them a proof that the GJMS operators are formally self-adjoint [20] and the construction and study of variational scalar invariants generalizing the σ_2-curvatures [12, 13, 18].

A *smooth metric measure space* is a five-tuple (M^d, g, f, m, μ) formed from a Riemannian manifold (M^d, g), a positive function $f \in C^\infty(M)$, a dimensional parameter $m \in [0, \infty]$, and a curvature parameter $\mu \in \mathbb{R}$. Smooth metric measure spaces arise in many ways, including as (possibly collapsed) limits of sequences of Riemannian manifolds [14], as smooth manifolds satisfying curvature-dimension inequalities [2, 24], as the geometric framework [10] for studying curved analogues of the Caffarelli–Silvestre extension for defining the fractional Laplacian [3], and, in the limiting case $m = \infty$, as a geometric framework for the realization of the Ricci flow as a gradient flow [23].

A *weighted invariant* is tensor-valued function on smooth metric measure spaces such that $T(N, \Phi^* g, \Phi^* f, m, \mu) = \Phi^*(T(M, g, f, m, \mu))$ for any smooth metric measure space (M^d, g, f, m, μ) and any diffeomorphism $\Phi\colon N \mapsto M$. For example, the *Bakry–Émery Ricci curvature* of (M^d, g, f, m, μ) is

$$\mathrm{Ric}_\phi^m := \mathrm{Ric} - \frac{m}{f} \nabla^2 f$$

and the *weighted scalar curvature* of (M^d, g, f, m, μ) is

$$R_\phi^m := R - \frac{2m}{f} \Delta f - \frac{m(m-1)}{f^2} \big(|\nabla f|^2 - \mu\big).$$

A smooth metric measure space is often studied as an abstract analogue of the base of the warped product $M^d \times_f F^m(\mu)$ of (M^d, g) with an m-dimensional spaceform $F^m(\mu)$ of constant sectional curvature μ, in the sense that when m is a nonnegative integer, most weighted invariants are equivalent to Riemannian invariants of the warped product. For example, if $m \in \mathbb{N}_0$, then $\operatorname{Ric}_\phi^m$ is the restriction of the Ricci tensor of $M \times_f F^m(\mu)$ to horizontal (i.e., tangent to M) vectors and R_ϕ^m is the scalar curvature of $M \times_f F^m(\mu)$.

We say that two smooth metric measure spaces (M^d, g_i, f_i, m, μ), $i \in \{1,2\}$, are *pointwise conformally equivalent* if there is a function $u \in C^\infty(M)$ such that $g_2 = e^{2u} g_1$ and $f_2 = e^u f_1$. This notion has many applications. For example, the curved version of the Caffarelli–Silvestre extension [3, 9, 10] identifies a conformally covariant operator with leading-order term a fractional power of the Laplacian [20] as a generalized Dirichlet-to-Neumann operator associated to a weighted analogue of the GJMS operators [4, 10], and there is a conformally invariant analogue of the Yamabe functional on smooth metric measure spaces [5] which interpolates between the usual Yamabe functional [25] and Perelman's $\mathcal{F}$- and $\mathcal{W}$-functionals [23]. This latter perspective leads to an ad hoc construction [6, 7, 8] of some fully nonlinear analogues of Perelman's functionals.

Given the variety of applications of the Fefferman–Graham ambient space and of the conformal geometry of smooth metric measure spaces, it is natural to ask whether there is a canonical weighted ambient space associated to a smooth metric measure space. We show that this is the case. Roughly speaking, we show that if $(M^d, [g,f], m, \mu)$, $d \geq 3$ and $m < \infty$, is a conformal class of smooth metric measure spaces, then there is a unique smooth metric measure space $(\mathbb{R}_+ \times M \times (-\epsilon, \epsilon), \widetilde{g}, \widetilde{f}, m, \mu)$ of Lorentzian signature such that:

(*i*) if $d+m$ is not an even integer, then $\widetilde{\operatorname{Ric}_\phi^m}, \widetilde{F_\phi^m} = O(\rho^\infty)$, where ρ is the coordinate on $(-\epsilon, \epsilon)$ and

$$\widetilde{F_\phi^m} := \widetilde{f}\widetilde{\Delta}\widetilde{f} + (m-1)\left(\left|\widetilde{\nabla}\widetilde{f}\right|^2 - \mu\right);$$

(*ii*) if $d+m$ is an even integer, then

$$\widetilde{\operatorname{Ric}_\phi^m}, \widetilde{F_\phi^m} = O\left(\rho^{\frac{d+m}{2}-1}\right), \qquad g^{ij}\left(\widetilde{\operatorname{Ric}_\phi^m}\right)_{ij} - mf^{-2}\widetilde{F_\phi^m} = O\left(\rho^{\frac{d+m}{2}}\right),$$

where i, j denote the coordinates on M.

Note that $\widetilde{R_\phi^m} = \widetilde{g}^{IJ}\left(\widetilde{\operatorname{Ric}_\phi^m}\right)_{IJ} - mf^{-2}\widetilde{F_\phi^m}$, where I, J denote the coordinates on $\widetilde{\mathcal{G}} := \mathbb{R}_+ \times M \times (-\epsilon, \epsilon)$. The improved orders of vanishing in the case that $d+m$ is even can be interpreted as the statement that, modulo the ρ components of $\widetilde{g}^{IJ}\left(\widetilde{\operatorname{Ric}_\phi^m}\right)_{IJ}$, the weighted scalar curvature $\widetilde{R_\phi^m}$ vanishes to one higher order in ρ than the Bakry–Émery Ricci tensor $\widetilde{\operatorname{Ric}_\phi^m}$. Our arguments carry through verbatim if g has signature (p,q), in which case $\widetilde{g}$ has signature $(p+1, q+1)$.

A precise statement of the above result, using the terminology of Section 3, is as follows:

Theorem 1.1. *Let $(M^d, [g,f], m, \mu)$, $d \geq 3$ and $m < \infty$, be a conformal class of smooth metric measure spaces. Then*

(*i*) *there exists a unique, up to ambient-equivalence, weighted ambient space $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ for $(M^d, [g,f], m, \mu)$; and*

(*ii*) *if $d+m$ is an even integer, the weighted obstruction tensor,*

$$\begin{aligned} \mathcal{O}_{ij} &:= c_{d+m} \partial_\rho^{\frac{d+m}{2}-1}\Big|_{\rho=0} \left(\widetilde{\operatorname{Ric}_\phi^m}\right)_{ij}, \\ c_{d+m} &= (-2)^{(d+m)/2-1} \frac{((d+m)/2-1)!}{d+m-2}, \end{aligned} \tag{1.1}$$

is a local conformal invariant of $(M^d, [g,f], m\mu)$ which vanishes if and only if there is a weighted ambient space $(\widetilde{g}, \widetilde{f})$ such that $\widetilde{\operatorname{Ric}_\phi^m}, \widetilde{F_\phi^m} = O(\rho^\infty)$.

When $m = 0$, Theorem 1.1 recovers the Fefferman–Graham ambient space [15]. When m is a positive integer, the existence and uniqueness statement of Theorem 1.1 implies that $\widetilde{\mathcal{G}} \times_{\widetilde{f}} F^m(\mu)$ is the Fefferman–Graham ambient space of $M \times_f F^m(\mu)$. See Theorem 4.8 for additional properties of the weighted obstruction tensor.

We do not presently know if there is an analogue of Theorem 1.1 in the case $m = \infty$.

Further expansions involving fractional powers of ρ when $d + m \notin 2\mathbb{N}$ or log terms when $d + m \in \mathbb{N}$ are also possible (cf. [15]). We have not developed this because it is unnecessary for our intended applications to the construction of local invariants.

From Section 4.1 onwards, the metric measure structure $(\widetilde{g}, \widetilde{f})$ will be assumed to be of the form

$$\widetilde{g} = 2\rho\, \mathrm{d}t^2 + 2t\, \mathrm{d}\rho\, \mathrm{d}t + t^2 g_\rho, \qquad \widetilde{f} = t f_\rho, \tag{1.2}$$

where (g_ρ, f_ρ) is a one-parameter family of metric measure structures on M. Metrics of the form equation (1.2) are called *straight* and *normal*. See Sections 3 and 4 for more details.

It is in general laborious to compute the asymptotic expansions of $(\widetilde{g}, \widetilde{f})$. However, one readily computes that, in the case of a straight and normal weighted ambient space,

$$g_\rho = g + 2\rho P_\phi^m + O\big(\rho^2\big), \qquad f_\rho = f + \frac{f}{m}\rho Y_\phi^m + O\big(\rho^2\big),$$

where

$$P_\phi^m := \frac{1}{d+m-2}\big(\mathrm{Ric}_\phi^m - J_\phi^m g\big)$$

is the *weighted Schouten tensor*,

$$J_\phi^m := \frac{1}{2(d+m-1)} R_\phi^m$$

is the *weighted Schouten scalar*, and $Y_\phi^m := J_\phi^m - \mathrm{tr}_g P_\phi^m$.

We can also explicitly identify the weighted ambient space in three special cases.

Theorem 1.2. *Let (M^d, g, f, m, μ), $d \geq 3$ and $m < \infty$, be a smooth metric measure space. Set $\widetilde{\mathcal{G}} = \mathbb{R}_+ \times M^d \times (-\epsilon, \epsilon)$ and consider the smooth metric measure space $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$, where $(\widetilde{g}, \widetilde{f})$ is of the form of equation (1.2) and*

$$\begin{aligned} (g_\rho)_{ij} &= g_{ij} + 2\rho\big(P_\phi^m\big)_{ij} + \rho^2\big(P_\phi^m\big)_{ik}\big(P_\phi^m\big)_j^{\ k}, \\ f_\rho &= f + \frac{f}{m}\rho Y_\phi^m. \end{aligned}$$

Then $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ is a weighted ambient space if

(i) there is a constant $\lambda \in \mathbb{R}$ such that

$$F_\phi^m = -2(d+m-1)\lambda f^2 \qquad \text{and} \qquad \mathrm{Ric}_\phi^m = 2(d+m-1)\lambda g;$$

(ii) $Rm = P_\phi^m \wedge g$ and $\mathrm{d}P_\phi^m = 0$; or

(iii) $f = 1$ and $\mathrm{Ric} = -(d-1)\mu g$.

The first case of Theorem 1.2 is the case of a quasi-Einstein manifold [11]; the second case is the case of a smooth metric measure space which is locally conformally flat in the weighted sense [8]; and the third case is a weighted interpretation of the explicit ambient metric of Gover and Leitner [17].

As in Riemannian geometry, the weighted ambient space gives rise to a canonical weighted Poincaré space associated to a given smooth metric measure space.

Theorem 1.3. *Let $(M^d, [g, f], m, \mu)$, $d \geq 3$ and $m < \infty$, be a conformal class of smooth metric measure spaces. Then there exists an even weighted Poincaré space for it. Moreover, if $\left(g_+^1, f_+^1\right)$ and $\left(g_+^2, f_+^2\right)$ are two even weighted Poincaré spaces for $\left(M^d, g, f, m, \mu\right)$ defined on $\left(M_+^1\right)^\circ$ and $\left(M_+^2\right)^\circ$, respectively, then there are open sets $U_1 \subset M_+^1$ and $U_2 \subset M_+^2$ containing $M \times \{0\}$ and an even diffeomorphism $\phi\colon U_1 \to U_2$ such that $\phi|_{M\times\{0\}}$ is the identity map and*

(i) if $m + d$ is not an even integer, then

$$\left(g_+^1 - \phi^* g_+^2, -2\left(f_+^1 - \phi^* f_+^2\right)\right) = O(r^\infty);$$

(ii) if $d + m$ is an even integer, then

$$\left(g_+^1 - \phi^* g_+^2, -2\left(f_+^1 - \phi^* f_+^2\right)\right) = O_{\alpha\beta}^{1,+}\left(r^{d+m-2}\right).$$

See Sections 3 and 5 for the relevant definitions.

In a separate article [21], Khaitan used Theorems 1.1, 1.2 and 1.3 to give a rigorous construction of the weighted GJMS operators of all orders up to the obstruction, and to prove that they are formally self-adjoint. He also gave explicit formulas for the weighted GJMS operators for smooth metric measure spaces satisfying the first or last conditions of Theorem 1.2. This makes rigorous the factorization of the weighted GJMS operators established by Case and Chang [10] in the latter case by formally arguing via warped products.

This article is organized as follows: In Section 2, we recall some weighted invariants of smooth metric measure spaces. In Section 3, we define weighted ambient spaces and discuss weighted ambient-equivalence. In Section 4, we prove Theorem 1.1. In Section 5, we define weighted Poincaré spaces and prove Theorem 1.3. In Section 6, we prove Theorem 1.2.

2 Smooth metric measure spaces

We recall some weighted invariants relevant to the geometry of smooth metric measure spaces [8]. Let (M^d, g, f, m, μ) be a smooth metric measure space. We discuss here only the case $m < \infty$ due to the presence of this restriction to our main results. This data determines a volume element

$$\mathrm{d}v_\phi := f^m \,\mathrm{dvol}_g .$$

When $m > 0$, we set $\phi := -m \ln f$, so that

$$\mathrm{d}v_\phi = \mathrm{e}^{-\phi} \,\mathrm{dvol}_g .$$

In terms of ϕ, it holds that

$$\begin{aligned} \mathrm{Ric}_\phi^m &= \mathrm{Ric} + \nabla^2\phi - \frac{1}{m}\mathrm{d}\phi \otimes \mathrm{d}\phi, \\ R_\phi^m &= R + 2\Delta\phi - \frac{m+1}{m}|\nabla\phi|^2 + m(m-1)\mu\mathrm{e}^{2\phi/m}. \end{aligned}$$

Recall that *weighted Schouten tensor* P_ϕ^m and the *weighted Schouten scalar* J_ϕ^m of (M^d, g, f, m, μ) are

$$P_\phi^m := \frac{1}{d+m-2}\left(\mathrm{Ric}_\phi^m - J_\phi^m g\right), \qquad J_\phi^m := \frac{1}{2(d+m-1)}R_\phi^m.$$

We set

$$F_\phi^m := f\Delta f + (m-1)\left(|\nabla f|^2 - \mu\right).$$

Observe that

$$F_\phi^m = \frac{f^2}{m}\bigl[(d+m-2)\mathrm{tr}_g P_\phi^m - (d+2m-2)J_\phi^m\bigr]. \tag{2.1}$$

A *weighted Einstein manifold* is a smooth metric measure space (M^d, g, f, m, μ) such that

$$P_\phi^m = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. A weighted Einstein manifold is *quasi-Einstein* [11] if additionally

$$J_\phi^m = (d+m)\lambda.$$

It follows from equation (2.1) that this is equivalent to

$$\mathrm{Ric}_\phi^m = 2(d+m-1)\lambda g, \qquad F_\phi^m = -2(d+m-1)\lambda f^2.$$

We define the *weighted divergence* $\delta_\phi T$ of a tensor field $T \in \Gamma(\otimes^k T^*M)$ by

$$(\delta_\phi T)(X_1, \dots, X_k) := \sum_{i=1}^d \nabla_{e_i} T(e_i, X_1, \dots, X_k) - T(\nabla\phi, X_1, \dots, X_k),$$

where $X_1, \dots, X_k \in T_pM$ and $\{e_i\}$ is an orthonormal basis for T_pM.

The *weighted Weyl tensor* A_ϕ^m and the *weighted Cotton tensor* $\mathrm{d}P_\phi^m$ of a smooth metric measure space (M^d, g, f, m, μ) are

$$\begin{aligned} A_\phi^m(x,y,z,w) &:= \bigl(\mathrm{Rm} - P_\phi^m \wedge g\bigr)(x,y,z,w), \\ \mathrm{d}P_\phi^m(x,y,z) &:= \nabla_x P_\phi^m(y,z) - \nabla_y P_\phi^m(x,z), \end{aligned}$$

where $h \wedge k$ denotes the Kulkarni–Nomizu product

$$(h \wedge k)(x,y,z,w) := h(x,z)k(y,w) + h(y,w)k(x,z) - h(x,w)k(y,z) - h(y,z)k(x,w).$$

The *weighted Bach tensor* of (M^d, g, f, m, μ) is

$$B_\phi^m := \delta_\phi \mathrm{d}P_\phi^m - \frac{1}{m}\,\mathrm{tr}\,\mathrm{d}P_\phi^m \otimes \mathrm{d}\phi + \left\langle A_\phi^m, P_\phi^m - \frac{Y_\phi^m}{m} g \right\rangle,$$

where $Y_\phi^m := J_\phi^m - \mathrm{tr}\, P_\phi^m$ and the contractions are

$$\begin{aligned} (\mathrm{tr}\,\mathrm{d}P_\phi^m)(x) &:= \sum_{i=1}^d \mathrm{d}P_\phi^m(e_i, x, e_i), \\ \left\langle A_\phi^m, P_\phi^m - \frac{Y_\phi^m}{m} g \right\rangle(x,y) &:= \sum_{i,j=1}^d A_\phi^m(e_i, x, e_j, y)\left(P_\phi^m - \frac{Y_\phi^m}{m} g\right)(e_i, e_j). \end{aligned}$$

The *weighted Bianchi identity* [6] is

$$\delta_\phi \mathrm{Ric}_\phi^m - \frac{1}{2}\mathrm{d}P_\phi^m - \frac{1}{f^2} F_\phi^m \mathrm{d}\phi = 0. \tag{2.2}$$

A smooth metric measure space (M^d, g, f, m, μ) is *locally conformally flat in the weighted sense* [8] if

(i) $d+m \in \{1,2\}$;

(ii) $d+m = 3$ and $\mathrm{d}P_\phi^m = 0$; or

(iii) $d+m \notin \{1,2,3\}$ and $A_\phi^m = 0$.

Note that [8] if $d+m \geq 3$, then (M^d, g, f, m, μ) is locally conformally flat in the weighted sense if and only if $A_\phi^m, \mathrm{d}P_\phi^m = 0$.

We require some formulas satisfied by smooth metric measure spaces which are locally conformally flat in the weighted sense.

Lemma 2.1 ([8, Lemma 3.2]). *Let (M^d, g, v, m, μ) be a smooth metric measure space that is locally conformally flat in the weighted sense. Then*

$$0 = P^m_\phi(\nabla\phi) + \mathrm{d}Y^m_\phi - \frac{1}{m}Y^m_\phi \mathrm{d}\phi, \tag{2.3a}$$

$$0 = mP^m_\phi - \nabla^2\phi + \frac{1}{m}\mathrm{d}\phi \otimes \mathrm{d}\phi + Y^m_\phi g, \tag{2.3b}$$

$$0 = \mathrm{d}P^m_\phi. \tag{2.3c}$$

3 Weighted ambient spaces

Let (M^d, g, f, m, μ) be a smooth metric measure space with $d \geq 3$ and $m < \infty$. Denote by $\mathcal{E}$ the trivial line bundle over M. Let $\mathcal{G}$ be the ray subbundle of $S^2T^*M \oplus \mathcal{E}$ consisting of all triples (h, u, x) such that $h = s^2 g_x$ and $u = sf(x)$ for some $s \in \mathbb{R}_+$ and $x \in M$. We define the dilation $\delta_s \colon \mathcal{G} \mapsto \mathcal{G}$ by $(h, u, x) \mapsto (s^2h, su, x)$. Let $T := \frac{\mathrm{d}}{\mathrm{d}s}\delta_s\big|_{s=1}$ denote the infinitesimal generator of dilations. Also, let $\pi \colon (h, u, x) \mapsto x$ be the projection from $\mathcal{G}$ to M. There is a canonical metric measure structure $(\boldsymbol{g}, \boldsymbol{f})$ on $\mathcal{G}$ defined by $\boldsymbol{g}(X, Y) = h(\pi_* X, \pi_* Y)$ and $\boldsymbol{f}(h, u, x) = u(x)$.

Given (g, f), define a coordinate chart $\mathcal{G} \mapsto \mathbb{R}_+ \times M$, by $(t^2 g_x, tf(x), x) \mapsto (t, x)$. Then $T = t\partial_t$.

Consider the embedding $\iota \colon \mathcal{G} \hookrightarrow \mathcal{G} \times \mathbb{R}, (t, x) \mapsto (t, x, 0)$. Each point in $\mathcal{G} \times \mathbb{R}$ can be written as (t, x, ρ). We extend the dilation to $\mathcal{G} \times \mathbb{R}$ as $\delta_s(t, x, \rho) = (st, x, \rho)$.

We now define weighted pre-ambient spaces and special cases thereof.

Definition 3.1. $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ is called a *weighted pre-ambient space* if

(*i*) $\widetilde{\mathcal{G}}$ is a dilation-invariant neighborhood of $\mathcal{G} \times \{0\}$;

(*ii*) $\widetilde{g}$ is a smooth metric of signature $(d + 1, 1)$ and $\widetilde{f}$ is a smooth function on $\widetilde{\mathcal{G}}$;

(*iii*) $\delta_s^*\widetilde{g} = s^2\widetilde{g}$ and $\delta_s^*\widetilde{f} = s\widetilde{f}$; and

(*iv*) $(\iota^*\widetilde{g}, \iota^*\widetilde{f}) = (\boldsymbol{g}, \boldsymbol{f})$.

Remark 3.2. Our results can be extended to pseudo-Riemannian metrics in the usual way (cf. [15]).

Definition 3.3. A weighted pre-ambient space $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ is said to be in *normal form* relative to a metric measure structure (g, f) if

(*i*) for each fixed $z \in \mathcal{G}$, the set of $\rho \in \mathbb{R}$ such that $(z, \rho) \in \widetilde{\mathcal{G}}$ is an open interval I_z containing 0;

(*ii*) for each $z \in \mathcal{G}$, the curve on I_z defined as $\rho \mapsto (z, \rho)$ is a geodesic in $\widetilde{\mathcal{G}}$; and

(*iii*) on $\mathcal{G} \times \{0\}$, it holds that $\widetilde{g} = \boldsymbol{g} + 2t\,\mathrm{d}\rho\,\mathrm{d}t$.

Definition 3.4. A weighted pre-ambient space $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ for $(M^d, [g, f], m, \mu)$ is be said to be *straight* if any of the following equivalent properties hold:

(*i*) for each $p \in \widetilde{\mathcal{G}}$, the dilation orbit $s \mapsto \delta_s p$ is a geodesic for $\widetilde{g}$;

(*ii*) $\widetilde{g}(2T, \cdot) = \mathrm{d}(\widetilde{g}(T, T))$; or

(*iii*) the infinitesimal dilation field T satisfies $\widetilde{\nabla} T = \mathrm{Id}$.

Given a point $(t, x, \rho) \in \mathcal{G} \times \mathbb{R}$, we refer to the t coordinate as the 0 coordinate, and the ρ coordinate as the ∞ coordinate. Coordinates in M are denoted with lowercase latin characters $(i, j, k, \dots)$.

The metric of a normal weighted pre-ambient space takes a special form.

Lemma 3.5. *Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ be a weighted pre-ambient space such that for each $z \in \mathcal{G}$, the set of all $\rho \in \mathbb{R}$ such that $(z, \rho) \in \widetilde{\mathcal{G}}$ is an open interval containing 0. Then $(\widetilde{g}, \widetilde{f})$ is a normal metric measure structure if and only if $\widetilde{g}_{0\infty} = t$, $\widetilde{g}_{\infty i} = 0$ and $\widetilde{g}_{\infty\infty} = 0$.*

Proof. A normal metric measure structure has

$$\widetilde{g}_{\infty\infty}\big|_{\rho=0} = \widetilde{g}_{i\infty}\big|_{\rho=0} = 0 \qquad \text{and} \qquad \widetilde{g}_{0\infty}\big|_{\rho=0} = t.$$

Moreover, it also has geodesic ρ-lines. The ρ-lines are geodesics if and only if the Christoffel symbols $\widetilde{\Gamma}_{\infty\infty I}$, $I \in \{0, i, \infty\}$, vanish. Taking $I = \infty$ gives $\partial_\rho \widetilde{g}_{\infty\infty} = 0$, and hence $\widetilde{g}_{\infty\infty} = 0$. Now taking $I = i$ and $I = 0$ gives $\widetilde{g}_{\infty i} = 0$ and $\widetilde{g}_{\infty 0} = t$.

The converse follows similarly. ■

Defining a weighted ambient space required some additional notation.

Definition 3.6. Let $(\widetilde{S}_{IJ}, \widetilde{h})$ be a pair of a symmetric 2-tensor $\widetilde{S}_{IJ}$ and a smooth function $\widetilde{h}$ on an open neighborhood of $\mathcal{G} \times \{0\} \subset \mathcal{G} \times \mathbb{R}$. For $k, m \geq 0$, we write $(\widetilde{S}_{IJ}, \widetilde{h}) = O^{k,+}_{IJ}(\rho^m)$ if

(*i*) $\big(\widetilde{S}_{IJ}, \widetilde{h}\big) = O(\rho^m)$;

(*ii*) $\widetilde{S}_{00}, \widetilde{S}_{0i} = O(\rho^{m+1})$; and

(*iii*) $mf^{-k}\widetilde{h} - g^{ij}\widetilde{S}_{ij} = O(\rho^{m+1})$.

We now define a weighted ambient space.

Definition 3.7. A *weighted ambient space* for (M^d, g, f, m, μ) is a weighted pre-ambient space $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ such that

(*i*) if $d + m \notin 2\mathbb{N}$, then $\big(\widetilde{\mathrm{Ric}^m_\phi}, \widetilde{F^m_\phi}\big) = O_{IJ}(\rho^\infty)$;

(*ii*) if $d + m \in 2\mathbb{N}$, then $\big(\widetilde{\mathrm{Ric}^m_\phi}, \widetilde{F^m_\phi}\big) = O^{2,+}_{IJ}\big(\rho^{\frac{d+m}{2}-1}\big)$.

A key step in the proof of Theorem 1.1 is the construction and classification of straight and normal weighted ambient metrics [15].

Theorem 3.8. *Let (M^d, g, f, m, μ), $d \geq 3$ and $m < \infty$, be a smooth metric measure space. Then there exists a straight and normal weighted ambient space $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ for it. Moreover, if $(\widetilde{\mathcal{G}}_j, \widetilde{g}_j, \widetilde{f}_j, m, \mu)$, $j \in \{1, 2\}$, are two such weighted ambient spaces, then*

(*i*) *if $d + m \notin 2\mathbb{N}$, then $\big(\widetilde{g}_1 - \widetilde{g}_2, -2(\widetilde{f}_1 - \widetilde{f}_2)\big) = O(\rho^\infty)$; and*

(*ii*) *if $d + m \in 2\mathbb{N}$, then $\big(\widetilde{g}_1 - \widetilde{g}_2, -2(\widetilde{f}_1 - \widetilde{f}_2)\big) = O^{1,+}_{IJ}\big(\rho^{\frac{d+m}{2}-1}\big)$.*

We prove Theorem 3.8 in Section 4.

Note that if $(\widetilde{g}_1 - \widetilde{g}_2, \widetilde{f}_1 - \widetilde{f}_2) = O_{IJ}\big(\rho^{\frac{d+m}{2}-1}\big)$, then

$$\begin{aligned}
&\partial_\rho^{\frac{d+m}{2}}\big|_{\rho=0}\big(\widetilde{f}_1^m \mathrm{dvol}_{\widetilde{g}_1} - \widetilde{f}_2^m \mathrm{dvol}_{\widetilde{g}_2}\big) \\
&\qquad = \left(mf^{-1}\partial_\rho^{\frac{d+m}{2}}\big|_{\rho=0}\big(\widetilde{f}_1 - \widetilde{f}_2\big) + \frac{1}{2} g^{ij}\partial_\rho^{\frac{d+m}{2}}\big|_{\rho=0}(\widetilde{g}_1 - \widetilde{g}_2)_{ij}\right) f^m \mathrm{dvol}_g.
\end{aligned}$$

Thus, a geometric interpretation of the improved orders of vanishing in the case that $d+m \in 2\mathbb{N}$ is that the weighted volume element $\widetilde{f}^m\,\mathrm{dvol}_{\widetilde{g}}$ vanishes to one higher order in ρ than the weighted ambient space $(\widetilde{g}, \widetilde{f})$.

The following definition generalizes the notion of uniqueness from Theorem 3.8 to general pre-ambient spaces.

Definition 3.9. We say that two weighted pre-ambient spaces $(\widetilde{\mathcal{G}}_1, \widetilde{g}_1, \widetilde{f}_1, m, \mu)$ and $(\widetilde{\mathcal{G}}_2, \widetilde{g}_2, \widetilde{f}_2, m, \mu)$ for $(M^d, [g, f], m, \mu)$ are *ambient-equivalent* if there exist open sets $\mathcal{U}_1 \subset \widetilde{\mathcal{G}}_1$ and $\mathcal{U}_2 \subset \widetilde{\mathcal{G}}_2$, and a diffeomorphism $\phi\colon \mathcal{U}_1 \to \mathcal{U}_2$ such that

(*i*) $\mathcal{U}_1$ and $\mathcal{U}_2$ both contain $\mathcal{G} \times \{0\}$;

(*ii*) $\mathcal{U}_1$ and $\mathcal{U}_2$ are dilation-invariant and ϕ commutes with dilations;

(*iii*) the restriction of ϕ to $\mathcal{G} \times \{0\}$ is the identity map; and

(*iv*) (*a*) if $d + m \notin 2\mathbb{N}$, then $\big(\widetilde{g}_1 - \phi^*\widetilde{g}_2, -2(\widetilde{f}_1 - \phi^*\widetilde{f}_2)\big) = O(\rho^\infty)$;

(*b*) if $d + m \in 2\mathbb{N}$, then $\big(\widetilde{g}_1 - \phi^*\widetilde{g}_2, -2(\widetilde{f}_1 - \phi^*\widetilde{f}_2)\big) = O^{1,+}_{IJ}\big(\rho^{\frac{d+m}{2}-1}\big)$.

Using Theorem 4.4 below, we see that every weighted pre-ambient space is ambient-equivalent to a straight and normal weighted pre-ambient space.

Theorem 3.10. *Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ be a weighted ambient space for the conformal class $(M^d, [g, f], m, \mu)$, $d \geq 3$ and $m < \infty$, of smooth metric measure spaces. Then it is ambient-equivalent to a weighted ambient space in straight and normal form relative to (g, f).*

Proof. Following the proof in [15, Proposition 2.8], we observe that any weighted pre-ambient space is ambient-equivalent to a normal weighted pre-ambient space. Moreover, it follows from Theorem 4.4 below that a normal weighted ambient space is ambient-equivalent to a straight and normal ambient space through the identity map. ∎

The uniqueness of Theorem 3.8 implies the uniqueness of weighted ambient spaces up to ambient-equivalence (cf. [15, Theorem 2.9]).

Theorem 3.11. *Any two weighted ambient spaces for $(M, [g, f], m, \mu)$, $d \geq 3$ and $m < \infty$, are ambient-equivalent.*

Proof. We pick a representative (g, f) and invoke Theorem 3.8. Then there exists a straight and normal weighted ambient space $(\widetilde{\mathcal{G}}_1, \widetilde{g}_1, \widetilde{f}_1, m, \mu)$ for (M, g, f, m, μ). Now let $(\widetilde{\mathcal{G}}_2, \widetilde{g}_2, \widetilde{f}_2, m, \mu)$ be a weighted ambient space for $(M, [g, f], m, \mu)$. Applying Theorem 3.10, we find that $(\widetilde{\mathcal{G}}_2, \widetilde{g}_2, \widetilde{f}_2, m, \mu)$ is ambient-equivalent to a weighted ambient space in straight and normal form relative to (g, f). By Theorem 3.8, this space is ambient-equivalent to $(\widetilde{\mathcal{G}}_1, \widetilde{g}_1, \widetilde{f}_1, m, \mu)$. ∎

4 Formal theory

We prove Theorem 3.8 by iteratively constructing a power series solution to $\widetilde{R}(\widetilde{g}), \widetilde{F^m_\phi} = O(\rho^j)$. This process is only obstructed when $d + m \in 2\mathbb{N}$; in this case the obstruction is at order $O\big(\rho^{\frac{d+m}{2}-1}\big)$.

We first give a necessary condition for a weighted ambient space to be normal.

Lemma 4.1. *Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ be a normal weighted ambient space. Then*

$$\widetilde{g} = a\,\mathrm{d}t^2 + 2b_i\,\mathrm{d}x^i\,\mathrm{d}t + 2t\,\mathrm{d}\rho\,\mathrm{d}t + t^2(g_{ij})_\rho, \tag{4.1}$$

with $a = 2\rho + O(\rho^2)$ and $b_i = O(\rho^2)$.

Proof. Lemma 3.5 implies that $\widetilde{g}$ has the form of equation (4.1). Since $\widetilde{g}$ is normal, $a|_{\rho=0} = 0$ and $b_i|_{\rho=0} = 0$. Denote $\widetilde{R}_{IJ} := \big(\widetilde{\mathrm{Ric}^m_\phi}\big)_{IJ}$. By definition,

$$\begin{aligned}\widetilde{R}_{IJ} = {}& \frac{1}{2}\widetilde{g}^{KL}\big(\partial^2_{IL}\widetilde{g}_{JK} + \partial^2_{JK}\widetilde{g}_{IL} - \partial^2_{KL}\widetilde{g}_{IJ} - \partial^2_{IJ}\widetilde{g}_{KL}\big) \\ & + \widetilde{g}^{KL}\widetilde{g}^{PQ}\big(\widetilde{\Gamma}_{ILP}\widetilde{\Gamma}_{JKQ} - \widetilde{\Gamma}_{IJP}\widetilde{\Gamma}_{KLQ}\big) - \frac{m}{\widetilde{f}}\big(\partial^2_{IJ}\widetilde{f} - \widetilde{\Gamma}^K_{IJ}\partial_K\widetilde{f}\big).\end{aligned}$$

Evaluating at $\rho = 0$ yields

$$\widetilde{R}_{00} = \frac{d+m}{2t^2}(2 - \partial_\rho a), \qquad \widetilde{R}_{0i} = \frac{1}{2t}\partial^2_{i\rho}a - \frac{d+m}{2t}\partial_\rho b_i.$$

We conclude that $a = 2\rho + O(\rho^2)$ and $b_i = O(\rho^2)$. ∎

We now take a brief digression to discuss straight weighted ambient spaces. First, notice that a simple choice of certain components of $\widetilde{g}$ makes $\widetilde{R}_{0I}$ vanish.

Lemma 4.2. *Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ be a weighted pre-ambient space. If $\widetilde{g}$ has the form*

$$\widetilde{g}_{IJ} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_\rho & 0 \\ t & 0 & 0 \end{pmatrix}, \tag{4.2}$$

where g_ρ is a one-parameter family of metrics on M, then $\widetilde{R}_{0I} = 0$.

Proof. This follows by direct computation (cf. [15, Lemma 3.2]). ∎

The relevance of Lemma 4.2 stems from the following equivalent characterizations of straight normal pre-ambient metrics.

Proposition 4.3. *Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$, $d \geq 3$ and $m < \infty$, be in normal form relative to (g, f). Then the following conditions are equivalent:*

(i) $\widetilde{g}_{00} = 2\rho$ *and* $\widetilde{g}_{0i} = 0$*;*

(ii) for each $p \in \widetilde{\mathcal{G}}$, the dilation orbit $s \mapsto \delta_s p$ is a geodesic for $\widetilde{g}$;

(iii) $\widetilde{g}(2T, \cdot) = \mathrm{d}(\widetilde{g}(T,T))$*;*

(iv) the infinitesimal dilation field T satisfies $\widetilde{\nabla} T = \mathrm{Id}$.

Proof. The proof is identical to that of [15, Proposition 3.4]. ∎

The following result implies that, up to ambient-equivalence, we may restrict our attention to metrics of the form of equation (4.2).

Theorem 4.4. *A normal weighted ambient space is ambient-equivalent to a straight and normal ambient space.*

Proof. Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ be a normal weighted ambient space. By Lemma 4.1, we may write

$$\widetilde{g} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_\rho & 0 \\ t & 0 & 0 \end{pmatrix} + \rho^n \begin{pmatrix} a & tb_i & 0 \\ tb_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $n \geq 2$ and some $a = a(x, \rho)$ and $b_i = b_i(x, \rho)$. Direct computation using Lemma 4.2 yields

$$\begin{aligned} t^2 \widetilde{R}_{00} &= n\left(n - 1 - \frac{d+m}{2}\right)\rho^{n-1} a + O(\rho^n), \\ t\widetilde{R}_{0i} &= n\left(n - 1 - \frac{d+m}{2}\right)\rho^{n-1} b_i + \frac{n}{2}\rho^{n-1}\partial_i a + O(\rho^n), \end{aligned}$$

(cf. [15, equation (3.11)]). We conclude that if $d+m \notin 2\mathbb{N}$, then $a, b_i = O(\rho^\infty)$; and if $d+m \in 2\mathbb{N}$, then $n \geq \frac{d+m}{2}$. Therefore $\widetilde{g}$ is ambient-equivalent to a metric of the form of equation (4.2). The conclusion follows from Proposition 4.3. ∎

We now iteratively construct a straight and normal weighted ambient space. Let $n \in \mathbb{N}$ be such that there is a metric

$$\widetilde{g}^{(n-1)}_{IJ} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_\rho & 0 \\ t & 0 & 0 \end{pmatrix}$$

and a function $\widetilde{f}^{(n-1)} = t f_\rho$ such that

$$\widetilde{R}^{(n-1)} = O\bigl(\rho^{n-1}\bigr), \qquad \widetilde{F^m_\phi}^{(n-1)} = O\bigl(\rho^{n-1}\bigr).$$

Note that the existence of $\widetilde{g}^{(1)}_{IJ}$ and $\widetilde{f}^{(1)}$ trivially holds. Let $\widetilde{g}^{(n)}_{IJ} = \widetilde{g}^{(n-1)}_{IJ} + \Phi_{IJ}$ and $\widetilde{f}^{(n)} = \widetilde{f}^{(n-1)} + \rho^n t \upsilon$, where

$$\Phi_{IJ} = \rho^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & t^2 \psi_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and ψ_{ij}, υ depend only on x and ρ. We seek ψ_{IJ} and υ such that

$$\widetilde{R}^{(n)} = O(\rho^n), \qquad \widetilde{F^m_\phi}^{(n)} = O(\rho^n).$$

Note that the inverse of $\widetilde{g}^{(n)}$, calculated modulo $O(\rho^n)$, is

$$\widetilde{g}^{IJ} = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & t^{-2} g^{ij} & 0 \\ t^{-1} & 0 & -2\rho t^{-2} \end{pmatrix}.$$

The Christoffel symbols for $\widetilde{g}^{(n)}$ modulo $O(\rho^n)$ are (cf. [15, equation (3.16)]).

$$\widetilde{\Gamma}^0_{IJ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} t\bigl(\partial_\rho g_{ij} + n\rho^{n-1}\psi_{ij}\bigr) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\widetilde{\Gamma}^\ell_{IJ} = \begin{pmatrix} 0 & t^{-1}\delta^\ell_j & 0 \\ t^{-1}\delta^\ell_i & \Gamma^\ell_{ij} & \frac{1}{2} g^{\ell k}\partial_\rho g_{ik} + \frac{n}{2}\rho^{n-1}\psi^\ell_i \\ 0 & \frac{1}{2} g^{\ell k}\partial_\rho g_{jk} + \frac{n}{2}\rho^{n-1}\psi^\ell_j & 0 \end{pmatrix},$$
$$\widetilde{\Gamma}^\infty_{IJ} = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & \rho\partial_\rho g_{ij} - g_{ij} & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}.$$

It follows that (cf. [15, equation (3.11)])

$$\widetilde{R}^{(n)}_{ij} = \widetilde{R}^{(n-1)}_{ij} + n\rho^{n-1}\left[\left(n - \frac{d+m}{2}\right)\psi_{ij} - \frac{1}{2} g^{kl}\psi_{kl} g_{ij} - \frac{m}{f}\upsilon g_{ij}\right] + O\bigl(\rho^n\bigr), \tag{4.3a}$$

$$\widetilde{F^m_\phi}^{(n)} = \widetilde{F^m_\phi}^{(n-1)} + n\rho^{n-1} f\left[\frac{1}{2} f g^{kl}\psi_{kl} + \upsilon(d + 2m - 2n)\right] + O\bigl(\rho^n\bigr), \tag{4.3b}$$

$$\widetilde{R}^{(n)}_{\infty\infty} = \widetilde{R}^{(n-1)}_{\infty\infty} - n(n-1)\rho^{n-2}\left[\frac{1}{2} g^{kl}\psi_{kl} + m f^{-1}\upsilon\right] + O\bigl(\rho^{n-1}\bigr). \tag{4.3c}$$

4.1 Solving $\widetilde{R}_{ij}, \widetilde{F^m_\phi} = O(\rho^n)$

The following theorem specifies the result of recursively determining $(\widetilde{g}^{(n)}, \widetilde{f}^{(n)})$ by the requirements $\widetilde{R}^{(n)}_{ij}, (\widetilde{F^m_\phi})^{(n)} = O(\rho^n)$.

Theorem 4.5. *Let (M^d, g, f, m, μ), $d \geq 3$ and $m < \infty$, be a smooth metric measure space. Set $\widetilde{\mathcal{G}} := \mathbb{R}_+ \times M \times (-\epsilon, \epsilon)$.*

(i) If $d + m \notin \mathbb{N}$, then there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ on $\widetilde{\mathcal{G}}$, unique modulo $O(\rho^\infty)$, such that $\widetilde{R}_{ij}, \widetilde{F^m_\phi} = O(\rho^\infty)$.

(ii) If $d + m \in 2\mathbb{N}$, then

(a) there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ on $\widetilde{\mathcal{G}}$, unique modulo $O^+\big(\rho^{\frac{d+m}{2}}\big)$, such that $(\widetilde{R}_{ij}, \widetilde{F^m_\phi}) = O^{2,+}_{ij}\big(\rho^{\frac{d+m}{2}-1}\big)$;

(b) if

$$\partial_\rho^{(d+m)/2-1}\big|_{\rho=0}\widetilde{R}_{ij} = 0,$$

then there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ on $\widetilde{\mathcal{G}}$ such that $\widetilde{R}_{ij}, \widetilde{F^m_\phi} = O(\rho^{d+m-1})$;

(c) if

$$\frac{m}{f^2}\widetilde{F^m_\phi}^{(d+m-1)} - g^{ij}\widetilde{R}^{(d+m-1)}_{ij} = O\big(\rho^{d+m}\big), \tag{4.4}$$

then there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ on $\widetilde{\mathcal{G}}$ such that $\widetilde{R}_{ij}, \widetilde{F^m_\phi} = O(\rho^\infty)$.

(iii) If $d + m \in \mathbb{N} \setminus 2\mathbb{N}$, then

(a) there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ on $\widetilde{\mathcal{G}}$, unique modulo $O(\rho^{d+m})$, such that $\widetilde{R}_{ij}, \widetilde{F^m_\phi} = O(\rho^{d+m-1})$;

(b) if equation (4.4) holds, then there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ on $\widetilde{\mathcal{G}}$ such that $\widetilde{R}_{ij}, \widetilde{F^m_\phi} = O(\rho^\infty)$.

Remark 4.6. We say that $(\widetilde{g}, \widetilde{f})$ is unique modulo $O^+(\rho^k)$ if it is unique modulo $O(\rho^k)$ and $\frac{1}{2}g^{ij}\widetilde{g}_{ij} + \frac{m}{f}\widetilde{f}$ is unique modulo $O(\rho^{k+1})$.

Proof. On studying equations (4.3a) and (4.3b), we observe that if $n \neq \frac{d+m}{2}$, then there is a unique choice of the trace-free part $\psi_{ij} - \frac{1}{d}g^{kl}\psi_{kl}g_{ij}$ of ψ_{ij} which makes the trace-free part of $\widetilde{R}^{(n)}_{ij}$ vanish modulo $O(\rho^n)$. However, if $n = \frac{d+m}{2}$, then we may not be able to make the trace-free part of $\widetilde{R}^{(n)}_{ij}$ vanish modulo $O(\rho^n)$. Additionally, equations (4.3a) and (4.3b) imply that

$$\begin{aligned} g^{ij}\widetilde{R}^{(n)}_{ij} &= g^{ij}\widetilde{R}^{(n-1)}_{ij} + n\rho^{n-1}\frac{(2n-2d-m)}{2}g^{kl}\psi_{kl} - \frac{dmn}{f}\upsilon\rho^{n-1} + O\big(\rho^n\big), \\ \widetilde{F^m_\phi}^{(n)} &= \widetilde{F^m_\phi}^{(n-1)} + n\rho^{n-1}f\left[\frac{1}{2}fg^{kl}\psi_{kl} + \upsilon(d+2m-2n)\right] + O\big(\rho^n\big). \end{aligned} \tag{4.5}$$

The determinant of the coefficient matrix of $g^{kl}\psi_{kl}$ and υ is

$$(2n-d-m)(n-d-m).$$

Therefore if $n \notin \{\frac{d+m}{2}, d+m\}$, then there are unique $g^{kl}\psi_{kl}$ and υ such that $g^{ij}\widetilde{R}_{ij}^{(n)}, (\widetilde{F_\phi^m})^{(n)} = O(\rho^n)$. However, if $n \in \{\frac{d+m}{2}, d+m\}$, then we may not be able to simultaneously solve $g^{ij}\widetilde{R}_{ij}^{(n)} = O(\rho^n)$ and $\widetilde{F_\phi^m} = O(\rho^n)$.

Suppose first that $d+m \notin \mathbb{N}$. Combining the above discussion with Borel's lemma yields a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ such that $\widetilde{R}_{ij}, \widetilde{F_\phi^m} = O(\rho^\infty)$.

Suppose next that $d+m \in 2\mathbb{N}$. Set $n = \frac{d+m}{2}$. Equations (4.5) imply that

$$\begin{aligned} \frac{m}{f^2}\widetilde{F_\phi^m}^{(\frac{d+m}{2})} - g^{ij}\widetilde{R}_{ij}^{(\frac{d+m}{2})} &= \frac{m}{f^2}\widetilde{F_\phi^m}^{(\frac{d+m}{2}-1)} - g^{ij}\widetilde{R}_{ij}^{(\frac{d+m}{2}-1)} \\ &\quad + \frac{(d+m)^2}{2}\rho^{\frac{d+m}{2}-1}\left(\frac{1}{2}g^{kl}\psi_{kl} + \frac{m}{f}\upsilon\right) + O\big(\rho^{\frac{d+m}{2}}\big). \end{aligned} \tag{4.6}$$

In particular, there is a unique choice of $\frac{1}{2}g^{kl}\psi_{kl} + \frac{m}{f}\upsilon$ such that the left hand side of equation (4.6) is $O(\rho^{\frac{d+m}{2}})$. Making this choice yields a unique straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ such that $(\widetilde{R}_{ij}, \widetilde{F_\phi^m}) = O_{ij}^{2,+}(\rho^{\frac{d+m}{2}-1})$. Moreover, if the weighted obstruction is zero, then $\widetilde{R}_{ij}^{(n)} = O(\rho^{\frac{d+m}{2}})$. Hence, we may iteratively solve equations (4.3a) and (4.3b) to obtain a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ such that $\widetilde{R}_{ij}, \widetilde{F_\phi^m} = O(\rho^{d+m-1})$. The possible obstruction is the same as that discussed in the next case.

Suppose finally that $d+m \in \mathbb{N} \setminus 2\mathbb{N}$. Set $n = d+m$. From the discussion of the first paragraph, we may choose the trace-free part of ψ_{ij} such that $\widetilde{R}_{ij}^{(d+m)}$ is pure trace. Additionally, by equations (4.5),

$$\begin{aligned} g^{ij}\widetilde{R}_{ij}^{(d+m)} &= g^{ij}\widetilde{R}_{ij}^{(d+m-1)} + m(d+m)\rho^{d+m-1}\left(\frac{1}{2}g^{kl}\psi_{kl} - \frac{d}{f}\upsilon\right) + O\big(\rho^{d+m}\big), \\ \widetilde{F_\phi^m}^{(d+m)} &= \widetilde{F_\phi^m}^{(d+m-1)} + f^2(d+m)\rho^{d+m-1}\left(\frac{1}{2}g^{kl}\psi_{kl} - \frac{d}{f}\upsilon\right) + O\big(\rho^{d+m}\big). \end{aligned}$$

Therefore we may choose $g^{ij}\psi_{ij}$ and υ such that $g^{ij}\widetilde{R}_{ij}^{(d+m)}, \widetilde{F_\phi^m}^{(d+m)} = O(\rho^{d+m})$ if and only if equation (4.4) holds. If equation (4.4) holds, then we may continue iteratively improving $(\widetilde{g}, \widetilde{f})$ as in the first paragraph. Hence, by Borel's lemma, there is a straight and normal metric measure structure $(\widetilde{g}, \widetilde{f})$ such that $\widetilde{R}_{ij}, \widetilde{F_\phi^m} = O(\rho^\infty)$. ∎

4.2 Solving $\widetilde{R}_{I\infty} = 0$

In this subsection, we show that if $(\widetilde{g}, \widetilde{f})$ is as in Theorem 4.5, then the components $\widetilde{R}_{I\infty}$ can be made to vanish to the appropriate order and that equation (4.4) automatically holds. This proves Theorem 3.8.

Proof of Theorem 3.8. Let $(\widetilde{g}, \widetilde{f})$ be as in Theorem 4.5. Equation (2.2) implies that

$$\widetilde{g}^{JK}\widetilde{\partial}_J\widetilde{R}_{KI} - \widetilde{g}^{JK}\widetilde{\Gamma}_{JK}^{Q}\widetilde{R}_{QI} - \widetilde{g}^{JK}\widetilde{\Gamma}_{JI}^{Q}\widetilde{R}_{KQ} - \widetilde{g}^{JK}\widetilde{R}_{IJ}\partial_K\widetilde{\phi} - \frac{1}{2}\partial_I\widetilde{R_\phi^m} - \frac{1}{f^2}\widetilde{F}_\phi^m\partial_I\widetilde{\phi} = 0. \tag{4.7}$$

Set

$$n = \begin{cases} \infty & \text{if} \quad d+m \notin \mathbb{N}, \\ \frac{d+m}{2} - 1 & \text{if} \quad d+m \in 2\mathbb{N}, \\ d+m-1 & \text{if} \quad d+m \in \mathbb{N} \setminus 2\mathbb{N}. \end{cases}$$

Taking $I = l$ and $I = \infty$ in equation (4.7) and computing mod $O(\rho^n)$ yields

$$[d+m-2-2\rho\partial_\rho]\widetilde{R}_{\infty l} - \rho g^{ks}g'_{sl}\widetilde{R}_{\infty k} + 2\rho\widetilde{R}_{l\infty}\partial_\rho\phi + \rho\partial_l\widetilde{R}_{\infty\infty} = O\bigl(\rho^n\bigr), \tag{4.8a}$$

$$\begin{aligned}[d+m-2-\rho\partial_\rho]\widetilde{R}_{\infty\infty} &- \frac{1}{2}\partial_\rho\left(g^{ij}\widetilde{R}_{ij} - \frac{m}{f^2}\widetilde{F_\phi^m}\right) + g^{ij}\bigl(\widetilde{\nabla}_\phi\bigr)_i\widetilde{R}_{j\infty} \\ &+\rho\bigl(2\partial_\rho\phi - g^{ij}g'_{ij}\bigr)\widetilde{R}_{\infty\infty} = O(\rho^n).\end{aligned} \tag{4.8b}$$

First, suppose that $d+m \notin \mathbb{N}$. We know that $\widetilde{R}_{ij}, \widetilde{F_\phi^m} = O(\rho^\infty)$. From equations (4.8a) and (4.8b) we conclude that $\widetilde{R}_{\infty l}, \widetilde{R}_{\infty\infty} = O(\rho^\infty)$.

Second, suppose that $d+m \in \mathbb{N}\setminus 2\mathbb{N}$. Equation (4.8) implies that $\widetilde{R}_{\infty l}, \widetilde{R}_{\infty\infty} = O(\rho^n)$, and that $g^{ij}\widetilde{R}_{ij} - mf^{-2}\widetilde{F_\phi^m} = O(\rho^{n+1})$. This verifies equation (4.4). Equation (4.3c) gives us the unique value of $\frac{1}{2}g^{kl}\psi_{kl} + \frac{m}{f}\upsilon$ such that $\widetilde{R}_{\infty\infty} = O(\rho^n)$. Hence, we can now uniquely determine the values of $g^{kl}\psi_{kl}$ and υ, and solve $\widetilde{R}_{IJ}, \widetilde{F_\phi^m} = O(\rho^\infty)$.

Third, suppose that $d+m \in 2\mathbb{N}$. Equation (4.8a) tells us that $\widetilde{R}_{\infty l} = O(\rho^n)$ and that $\widetilde{R}_{\infty\infty} = O(\rho^{n-1})$. Now recall that $g^{ij}\widetilde{R}_{ij} - mf^{-2}\widetilde{F_\phi^m} = O\bigl(\rho^{\frac{d+m}{2}}\bigr)$. Hence, equation (4.8b) tells us that $\widetilde{R}_{\infty\infty} = O(\rho^n)$.

Finally, since the terms of $(\widetilde{g}, \widetilde{f})$ are uniquely determined using the process described above, any two straight and normal weighted ambient structures will be ambient-equivalent to the orders stated in the theorem. ■

4.3 The weighted obstruction tensor

We conclude this section by establishing some properties of the weighted obstruction tensor. To that end, we first introduce notation for the corresponding term in the asymptotic expansion of $\widetilde{F_\phi^m}$.

Definition 4.7. We define

$$\mathcal{F} := c_{d+m}\partial_\rho^{\frac{d+m}{2}-1}\big|_{\rho=0}\widetilde{F_\phi^m},$$

where c_{d+m} is defined in equation (1.1).

There is a simple relationship between $\mathcal{F}$ and the trace and divergence of $\mathcal{O}_{ij}$ (cf. [8]).

Theorem 4.8. *Let (M^d, g, f, m, μ), $d \geq 3$ and $m < \infty$, be a smooth metric measure space with $d+m \in 2\mathbb{N}$. Then*

(i) $\mathcal{O}_i^i = \frac{m}{f^2}\mathcal{F}$;

(ii) $\delta_\phi\mathcal{O}_i = \frac{1}{f^2}\mathcal{F}\partial_i\phi$; *and*

(iii) $\mathcal{O}_{ij}$ *and* $\mathcal{F}$ *are local weighted conformal invariants of weight* $2-d-m$.

Proof. We can use the recursive construction of $(\widetilde{g}, \widetilde{f})$ to express $\mathcal{O}_{ij}$ and $\mathcal{F}$ as polynomials in g, g^{-1}, f, Rm and ∇. Hence $\mathcal{O}_{ij}$ and $\mathcal{F}$ are local weighted invariants.

Since $g^{ij}\widetilde{R}_{ij} - mf^{-2}\widetilde{F_\phi^m} = O\bigl(\rho^{\frac{d+m}{2}}\bigr)$, we see that $\mathcal{O}_i^i = \frac{m}{f^2}\mathcal{F}$.

Writing equation (4.8a) modulo $O\bigl(\rho^{\frac{d+m}{2}}\bigr)$ and recalling that $\widetilde{R}_{\infty l} = O\bigl(\rho^{\frac{d+m}{2}-1}\bigr)$, we get

$$\frac{1}{t^2}\widetilde{\nabla}^j\widetilde{R}_{jl} - \frac{1}{t^2}g^{ij}\widetilde{R}_{li}\partial_j\widetilde{\phi} - \frac{1}{f^2}\widetilde{F_\phi^m}\partial_l\widetilde{\phi} = O\bigl(\rho^{\frac{d+m}{2}}\bigr).$$

Hence, $\delta_\phi\mathcal{O}_i = \frac{1}{f^2}\mathcal{F}\partial_i\phi$.

The conformal invariance of the weighted ambient space implies that $\mathcal{O}_{ij}$ is conformally invariant of weight $2-d-m$. The conformal invariance of $\mathcal{F}$ follows from the identity $\mathcal{O}_i^i = \frac{m}{f^2}\mathcal{F}$. ■

4.4 The first few terms in the expansions of g_ρ and f_ρ

Although it is difficult in general to compute the terms of $(\widetilde{g},\widetilde{f})$, we are able to compute the first few terms by hand.

Let $(\widetilde{g},\widetilde{f})$ be as in equation (1.2). Then

$$\begin{aligned}\widetilde{R}_{ij} &= \rho g''_{ij} - \rho g^{kl}g'_{ik}g'_{jl} + \frac{1}{2}\rho g^{kl}g'_{kl}g'_{ij} + \rho\frac{m}{\widetilde{f}}g'_{ij}\widetilde{f}' - \left(\frac{d+m}{2}-1\right)g'_{ij} - \frac{1}{2}g^{kl}g'_{kl}g_{ij}\\ &\quad - \frac{m}{\widetilde{f}}g_{ij}\widetilde{f}' + \left(\mathrm{Ric}_\phi^m\right)_{ij},\\ \widetilde{F_\phi^m} &= -2\rho f f'' - \rho f f' g^{ij}g'_{ij} - 2(m-1)\rho(f')^2 + \frac{1}{2}f^2g^{ij}g'_{ij} + (2m+d-2)ff' + F_\phi^m. \end{aligned} \tag{4.9}$$

Using these, we readily compute that

$$g'_{ij}(\cdot,0) = 2\left(P_\phi^m\right)_{ij}, \qquad f'(\cdot,0) = \frac{f}{m}Y_\phi^m.$$

Moreover, differentiating equation (4.9) once with respect to ρ and evaluating at $\rho=0$ yields

$$\begin{aligned}(d+m-4)g''_{ij} &= -2\left(B_\phi^m\right)_{ij} + 2(d+m-4)\left(P_\phi^m\right)_i^{\ k}\left(P_\phi^m\right)_{jk},\\ (d+m-4)f'' &= \frac{f}{m}g^{ij}\left(B_\phi^m\right)_{ij}.\end{aligned}$$

In particular, if $d+m=4$, then $\mathcal{O}_{ij} = (B_\phi^m)_{ij}$.

5 Weighted Poincaré spaces

In this section, we construct a weighted Poincaré space for a smooth metric measure space.

Let M_+ be a smooth manifold with compact boundary M. Denote by M_+^0 the interior of M_+. A smooth metric measure space $(M_+^0, g_+, r_+, m, \mu)$, $m<\infty$, is *conformally compact* if there is a defining function r for M such that

(*i*) (r^2g_+, rf_+) extends smoothly to M_+; and

(*ii*) $(M, r^2g_+|_M, rf_+|_M, m, \mu)$ is a smooth metric measure space.

We call $(M, [r^2g_+|_M, rf_+|_M], m, \mu)$ the conformal infinity of $(M_+^0, g_+, r_+, m, \mu)$.

In the following, we identify M_+ with an open neighborhood of $M\times\{0\}$ in $M\times[0,\infty)$ and choose r to be the standard coordinate on $[0,\infty)$.

Definition 5.1. A *weighted Poincaré space* for $(M^d, [g,f], m, \mu)$, $m<\infty$, is a metric measure structure (g_+, f_+) on $M\times[0,\epsilon)$ such that

(*i*) g_+ has signature $(d+1,0)$;

(*ii*) (g_+, f_+) has $(M^d, [g,f], m, \mu)$ as conformal infinity; and

(*iii*) (*a*) if $d+m\notin 2\mathbb{N}$, then

$$\left(\mathrm{Ric}_\phi^m(g_+) + (d+m)g_+, F_\phi^m(g_+) - (d+m)f_+^2\right) = O\left(r^\infty\right);$$

(*b*) if $d+m\in 2\mathbb{N}$, then

$$\left(\mathrm{Ric}_\phi^m(g_+) + (d+m)g_+, F_\phi^m(g_+) - (d+m)f_+^2\right) = O_{\alpha\beta}^{2,+}\left(r^{d+m-2}\right).$$

If (M_+, g_+, f_+, m, μ) is conformally compact, then $|\mathrm{d}r/r|_{g_+} = |\mathrm{d}r|_{r^2 g_+}$ extends smoothly to the boundary. We call (M_+, g_+, f_+, m, μ) *asymptotically hyperbolic* if $|\mathrm{d}r/r|_{g_+} = 1$ on M. In this case, all the sectional curvatures of g_+ approach -1 at a boundary point [22].

Definition 5.2. A weighted Poincaré space (M_+, g_+, f_+, m, μ), $m < \infty$, is in *normal form relative to* (g, f) if $g_+ = r^{-2}(\mathrm{d}r^2 + g_r)$ and $f_+ = r^{-1} f_r$. Here g_r is a one-parameter family of metrics on M such that $g_0 = g$, and f_r is a one-parameter family of functions such that $f_0 = f$.

Proposition 5.3. *Let (M_+, g_+, f_+, m, μ) be asymptotically hyperbolic. Then there exists a neighborhood U of $M \times \{0\}$ in $M \times [0, \infty)$ on which there is a unique diffeomorphism ϕ from U into M_+ such that $\phi|_M$ is the identity map and $(\phi^* g_+, \phi^* f_+)$ is in normal form relative to (g, f) on U.*

Proof. This follows as in the conformal case [15, Proposition 4.3]. ■

Definition 5.4. An asymptotically hyperbolic smooth metric measure space, denoted as (M_+, g_+, f_+, m, μ), is *even* if $(r^2 g_+, r f_+)$ is the restriction to M_+ of a metric measure structure (g'_+, f'_+) on an open set $V \subset M \times (-\infty, \infty)$ containing M_+ such that V and (g'_+, f'_+) are invariant under $r \mapsto -r$.

A diffeomorphism $\psi \colon M_+ \mapsto M \times [0, \infty)$ satisfying $\psi"_{M \times \{0\}} = \mathrm{Id}$ is *even* if ψ is the restriction of a diffeomorphism on an open set V as above which commutes with $r \mapsto -r$.

It is easily seen that if ψ if an even diffeomorphism and (M_+, g_+, f_+, m, μ) is an even asymptotically hyperbolic smooth metric measure, then $\psi^* g_+$ is also even.

Theorem 5.5. *Let $(M^d, [g, f], m, \mu)$, $d \geq 3$ and $m < \infty$, be a conformal class of smooth metric measure spaces. Then there exists an even weighted Poincaré space corresponding to it. Moreover, if (g^1_+, f^1_+) and (g^2_+, f^2_+) are two even weighted Poincaré structures defined on $(M^1_+)^\circ$ and $(M^2_+)^\circ$ respectively, then there exist open sets $U_1 \subset M^1_+$ and $U_2 \subset M^2_+$ containing $M \times \{0\}$, and an even diffeomorphism $\phi \colon U_1 \mapsto U_2$, such that $\phi|_{M \times \{0\}}$ is the identity map and*

(i) if $d + m \notin 2\mathbb{N}$, then $(g^1_+ - \phi^* g^2_+, -2(f^1_+ - \phi^* f^2_+)) = O(\rho^\infty)$;

(ii) if $d + m \in 2\mathbb{N}$, then $(g^1_+ - \phi^* g^2_+, -2(f^1_+ - \phi^* f^2_+)) = O^+_{\alpha\beta}(r^{d+m-2})$.

Theorem 5.5 will be proved at the end of this section.

Let $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ be a straight weighted pre-ambient space for (M, g, f, m, μ). Then $\widetilde{g}(T, T)$ vanishes to first order on $\mathcal{G} \times \{0\} \subset \widetilde{\mathcal{G}}$. Therefore, shrinking $\widetilde{\mathcal{G}}$ if necessary, the hypersurface $H := \widetilde{\mathcal{G}} \cap \{\widetilde{g}(T, T) = -1\}$ lies on one side of $\mathcal{G} \times \{0\}$. Also, since $\widetilde{g}(T, T)$ is homogeneous of degree 2, each δ_s-orbit on this side of $\mathcal{G} \times \{0\}$ intersects H exactly once. We extend the projection $\pi \colon \mathcal{G} \mapsto M$ to $\widetilde{\mathcal{G}} \subset \mathcal{G} \times \mathbb{R} \mapsto M \times \mathbb{R}$ by projecting only the first factor. Define $\chi \colon M \times \mathbb{R} \mapsto M \times [0, \infty)$ by $(x, \rho) \mapsto (x, \sqrt{2|\rho|})$. Then there is an open set $M_+ \subset M \times [0, \infty)$ containing $M \times \{0\}$ such that $\chi \circ \pi|_H \colon H \mapsto M^\circ_+$ is a diffeomorphism.

Proposition 5.6. *If $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ is a straight and normal weighted pre-ambient space of the form of equation (1.2), and if H and M°_+ are as above, then*

$$g_+ := \big((\chi \circ \pi|_H)^{-1}\big)^* \widetilde{g}, \qquad f_+ := \big((\chi \circ \pi|_H)^{-1}\big)^* \widetilde{f},$$

defines an even, asymptotically hyperbolic, normal smooth metric measure space (M_+, g_+, f_+, m, μ) with conformal infinity (M^d, g, f, m, μ).

Proof. We have $H = \{2\rho t^2 = -1\}$. Introduce new variables $r > 0$ and $s > 0$ by $-2\rho = r^2$ and $s = rt$. Then

$$\widetilde{g} = \frac{s^2}{r^2}\big(g_{-\frac{1}{2}r^2} + \mathrm{d}r^2\big) - \mathrm{d}s^2, \qquad \widetilde{f} = \frac{s}{r} f_{-\frac{1}{2}r^2},$$

Note that $H = s^{-1}(\{1\})$. Thus the restriction of $\widetilde{g}$ to TH is $r^{-2}(g_{-\frac{1}{2}r^2} + \mathrm{d}r^2)$. Similarly, the restriction of $\widetilde{f}$ to H is $r^{-1}f_{-\frac{1}{2}r^2}$. By the definition of χ, we see that r is the coordinate in the second factor of $M \times [0,\infty)$. Therefore (g_+, f_+) is an even asymptotically hyperbolic smooth metric measure space with conformal infinity $(M^d, [g,f], m, \mu)$ in normal form relative to (g,f). ∎

We now show that weighted Poincaré spaces and weighted ambient spaces are closely related.

Proposition 5.7. *Let $(M_+^{d+1}, g_+, f_+, m, \mu)$ be a smooth metric measure space. Consider $(M_+ \times \mathbb{R}_+, \widetilde{g}, \widetilde{f}, m, \mu)$, where $\widetilde{g} = s^2 g_+ - \mathrm{d}s^2$ and $\widetilde{f} = sf_+$. Then*

$$\mathrm{Ric}_\phi^m(\widetilde{g}) = \mathrm{Ric}(g_+) + (d+m)g_+, \qquad \widetilde{F_\phi^m} = F_\phi^m(g_+) - (d+m)f_+^2.$$

Proof. The Christoffel symbols of the metric $\widetilde{g} = -\mathrm{d}s^2 + s^2 g_+$ are

$$\widetilde{\Gamma}_{IJ}^0 = \begin{pmatrix} 0 & 0 \\ 0 & s(g_+)_{ij} \end{pmatrix}, \qquad \widetilde{\Gamma}_{IJ}^k = \begin{pmatrix} 0 & \frac{1}{s}\delta_i^k \\ \frac{1}{s}\delta_i^k & (\Gamma_{g_+})_{ij}^k \end{pmatrix}. \tag{5.1}$$

Using the formula

$$\mathrm{Ric}_{ij} = \partial_L \Gamma_{ij}^L - \partial_i \Gamma_{Lj}^L + \Gamma_{ij}^P \Gamma_{LP}^L - \Gamma_{Lj}^P \Gamma_{iP}^L$$

and equation (5.1), we readily compute that $\mathrm{Ric}_\phi^m(\widetilde{g}) = \mathrm{Ric}(g_+) + (d+m)g_+$. Using the formula

$$\widetilde{F_\phi^m} = sf_+\left(-\frac{d+1}{s}f_+ + \frac{1}{s^2}\Delta_{g_+} sf_+\right) + (m-1)\big(-f_+^2 + |\nabla_{g_+} f_+|^2 - \mu\big),$$

we readily compute that $\widetilde{F_\phi^m} = F_\phi^m(g_+) - (d+m)f_+^2$. ∎

Proof of Theorem 5.5. Pick a representative (M^d, g, f, m, μ). Theorem 3.8 implies that there is a weighted ambient metric in straight and normal form relative to (g,f). By Proposition 5.6, the metric measure structure (g_+, f_+) is even and in normal form relative to (g,f). Proposition 5.7 implies that conditions (3) and (4) of Definition 5.1 are satisfied, so that $(M_+^\circ, g_+, f_+, m, \mu)$ is a weighted Poincaré metric space for (M^d, g, f, m, μ). This proves the first part of the theorem. The second part follows similarly from the argument in Theorem 3.8. ∎

6 Explicit examples of weighted ambient structures

We conclude this article by proving Theorem 1.2. The following three subsections handle the three separate cases mentioned in the theorem.

6.1 Quasi-Einstein space

Suppose that (M^d, g, f, m, μ), $m < \infty$, is quasi-Einstein. Then

$$\mathrm{Ric}_\phi^m = \lambda g, \qquad F_\phi^m = -\lambda f^2.$$

Set

$$g_\rho := (1+\lambda\rho)^2 g, \qquad f_\rho := (1+\lambda\rho) f.$$

Direct computation yields $\widetilde{\mathrm{Ric}_\phi^m} = 0$ and $\widetilde{F_\phi^m} = 0$. Thus $(\mathcal{G} \times (-\epsilon,\epsilon), \widetilde{g}, \widetilde{f}, m, \mu)$ is a weighted ambient metric for (M^d, g, f, m, μ). This verifies Theorem 1.2(*i*).

6.2 Weighted locally conformally flat space

Recall from Section 2 that the locally conformally flat condition is equivalent to $A^m_\phi = 0$ and $\mathrm{d}P^m_\phi = 0$. Therefore $(P^m_\phi)_{ij;k} = (P^m_\phi)_{(ij;k)}$. We use this fact repeatedly in the rest of the section.

Let (g_ρ, f_ρ) be as in the statement of Theorem 1.2. Note that

$$(g_\rho)_{ij} = g_{il}\big(U^m_\phi\big)^l_k\big(U^m_\phi\big)^k_j = \big(U^m_\phi\big)_{ik}\big(U^m_\phi\big)^k_j,$$

where

$$\big(U^m_\phi\big)^i_j := \delta^i_j + \rho\big(P^m_\phi\big)^i_j.$$

Set $V^m_\phi := (U^m_\phi)^{-1}$. Then $(V^m_\phi)^i_k(U^m_\phi)^k_j = \delta^i_j$. Note that $(U^m_\phi)_{ij}$ and $(V^m_\phi)_{ij}$ are both symmetric. Additionally,

$$\big(V^m_\phi\big)^k_i(g_\rho)_{kj} = \big(U^m_\phi\big)_{ij}, \qquad (g_\rho)'_{ij} = 2\big(P^m_\phi\big)_{ik}\big(U^m_\phi\big)^k_j. \tag{6.1}$$

We now relate the Levi-Civita connections ${}^{g_\rho}\nabla$ and ${}^{g}\nabla$, and the curvature tensors ${}^{g_\rho}R_{ijkl}$ and ${}^{g}R_{ijkl}$.

Lemma 6.1. *Let (g_ρ, f_ρ) and (g, f) be defined as in Theorem 1.2. The Levi-Civita connections of g_ρ and g are related by*

$${}^{g_\rho}\nabla_i\eta_j = {}^{g}\nabla_i\eta_j - \rho\big(V^m_\phi\big)^k_l\big(P^m_\phi\big)^l_{i;j}\eta_k, \tag{6.2}$$

and the curvature tensors by

$${}^{g_\rho}R_{ijkl} = {}^{g}R_{abkl}\big(U^m_\phi\big)^a_i\big(U^m_\phi\big)^b_j. \tag{6.3}$$

Proof. First note that, since $\mathrm{d}P^m_\phi = 0$, the right hand side of equation (6.2) defines a torsion-free connection.

We now show that g_ρ is parallel with respect to the connection determined by the right side of equation (6.2). On differentiating the metric, we get

$${}^{g}\nabla_k(g_\rho)_{ij} = 2\rho\big(P^m_\phi\big)_{ij;k} + 2\rho^2\big(P^m_\phi\big)^l_{(i}\big(P^m_\phi\big)_{j)l;k}. \tag{6.4}$$

Using the definitions of U^m_ϕ and V^m_ϕ and the symmetry of $(P^m_\phi)_{ij;k}$, we get

$$(V^m_\phi)^m_l\big(P^m_\phi\big)^l_{k;(i}(g_\rho)_{j)m} = \big(P^m_\phi\big)^l_{k;(i}\big(U^m_\phi\big)_{j)l} = \big(P^m_\phi\big)_{ij;k} + \rho\big(P^m_\phi\big)^l_{k;(i}\big(P^m_\phi\big)_{j)l}.$$

Therefore

$${}^{g}\nabla_k(g_\rho)_{ij} - 2\rho\big(V^m_\phi\big)^m_l\big(P^m_\phi\big)^l_{k;(i}(g_\rho)_{j)m} = 0.$$

Combining the previous two paragraphs verifies equation (6.2).

Now set

$$\big(D^m_\phi\big)^i_{jk} := \rho\big(V^m_\phi\big)^i_l\big(P^m_\phi\big)^l_{j;k},$$

so that $(D^m_\phi)^i_{jk}$ is the difference of ${}^{g}\nabla$ and ${}^{g_\rho}\nabla$. The difference of the curvature tensors of the connections is given in terms of D^m_ϕ by

$${}^{g_\rho}R_{mjkl}(g_\rho)^{im} - {}^{g}R^i_{jkl} = 2\big(D^m_\phi\big)^i_{j[l;k]} + 2\big(D^m_\phi\big)^c_{j[l}\big(D^m_\phi\big)^i_{k]c};$$

see [15, p. 70]. We compute that

$$
\begin{aligned}
\left(D_\phi^m\right)^i_{jl;k} &= \rho\left(\left(V_\phi^m\right)^i_{a;k}\left(P_\phi^m\right)^a_{j;l} + \left(V_\phi^m\right)^i_a\left(P_\phi^m\right)^a_{j;lk}\right)\\
&= -\rho\left(V_\phi^m\right)^i_b\left(U_\phi^m\right)^b_{c;k}\left(V_\phi^m\right)^c_a\left(P_\phi^m\right)^a_{j;l} + \rho\left(V_\phi^m\right)^i_a\left(P_\phi^m\right)^a_{j;lk}\\
&= -\rho^2\left(V_\phi^m\right)^i_b\left(P_\phi^m\right)^b_{c;k}\left(V_\phi^m\right)^c_a\left(P_\phi^m\right)^a_{j;l} + \rho\left(V_\phi^m\right)^i_a\left(P_\phi^m\right)^a_{j;lk}\\
&= -\left(D_\phi^m\right)^i_{ck}\left(D_\phi^m\right)^c_{jl} + \rho\left(V_\phi^m\right)^i_a\left(P_\phi^m\right)^a_{j;lk}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{}^{g_\rho}R_{mjkl}(g_\rho)^{im} &= {}^gR^i_{jkl} + 2\rho\left(V_\phi^m\right)^{ia}\left(P_\phi^m\right)_{aj;[lk]}\\
&= {}^gR^i_{jkl} + \rho\left(V_\phi^m\right)^{ia}\left({}^gR^b_{alk}\left(P_\phi^m\right)_{bj} + {}^gR^b_{jlk}\left(P_\phi^m\right)_{ab}\right).
\end{aligned}
$$

Since $(V_\phi^m)^k_i(g_\rho)_{kj} = (U_\phi^m)_{ij}$, we conclude that

$$
\begin{aligned}
{}^{g_\rho}R_{ijkl} &= {}^gR^b_{jkl}(g_\rho)_{bi} + \rho\left(U_\phi^m\right)^a_i\left({}^gR^b_{alk}\left(P_\phi^m\right)_{bj} + {}^gR^b_{jlk}\left(P_\phi^m\right)_{ab}\right)\\
&= {}^gR^b_{jkl}\left(U_\phi^m\right)_{ba}\left(U_\phi^m\right)^a_i + \rho\left(U_\phi^m\right)^a_i\left({}^gR^b_{alk}\left(P_\phi^m\right)_{bj} + {}^gR^b_{jlk}\left(P_\phi^m\right)_{ab}\right)\\
&= \left[{}^gR_{abkl}\left(\delta^b_j + \rho\left(P_\phi^m\right)^b_j\right)\right]\left(U_\phi^m\right)^a_i\\
&= {}^gR_{abkl}\left(U_\phi^m\right)^b_j\left(U_\phi^m\right)^a{}_i.
\end{aligned}
$$

■

The metric $\widetilde{g}$ in Theorem 1.2(*ii*) is in fact flat.

Proposition 6.2. *Let (g_ρ, f_ρ) and (g, f) be as in Theorem 1.2. Then $\widetilde{g}$ is flat.*

Proof. The curvature tensor of $\widetilde{g} = 2\rho\,\mathrm{d}\rho\,\mathrm{d}t + t^2 g_\rho + 2t\,\mathrm{d}\rho\,\mathrm{d}t$ is [15, equation (6.1)]

$$
\begin{aligned}
&\widetilde{R}_{IJK0} = 0,\\
&\widetilde{R}_{ijkl} = t^2\left[{}^{g_\rho}R_{ijkl} + (g_\rho)_{i[l}(g_\rho)'_{k]j} + (g_\rho)_{j[k}(g_\rho)'_{l]i} - \rho(g_\rho)'_{i[l}(g_\rho)'_{k]j}\right],\\
&\widetilde{R}_{\infty jkl} = \frac{1}{2}t^2\left[\nabla_l(g_\rho)'_{jk} - \nabla_k(g_\rho)'_{jl}\right],\\
&\widetilde{R}_{\infty jk\infty} = \frac{1}{2}t^2\left[(g_\rho)''_{jk} - \frac{1}{2}(g_\rho)^{pq}(g_\rho)'_{jp}(g_\rho)'_{kq}\right].
\end{aligned}
$$

Hence, $\widetilde{g}$ is flat if and only if

$$0 = {}^{g_\rho}R_{ijkl} + (g_\rho)_{i[l}(g_\rho)'_{k]j} + (g_\rho)_{j[k}(g_\rho)'_{l]i} - \rho(g_\rho)'_{i[l}(g_\rho)'_{k]j}, \tag{6.5a}$$

$$0 = {}^{g_\rho}\nabla_k(g_\rho)'_{ij} - {}^{g_\rho}\nabla_j(g_\rho)'_{ik}, \tag{6.5b}$$

$$0 = (g_\rho)''_{ij} - \frac{1}{2}(g_\rho)^{pq}(g_\rho)'_{ip}(g_\rho)'_{jq}. \tag{6.5c}$$

Equation (6.5c) follows directly from the definition of g_ρ.

We next verify equation (6.5b). Differentiating equation (6.4) with respect to ρ yields

$${}^g\nabla_k(g_\rho)'_{ij} = 2\left(P_\phi^m\right)_{ij;k} + 4\rho\left(P_\phi^m\right)^l_{(i}\left(P_\phi^m\right)_{j)l;k}. \tag{6.6}$$

Using equation (6.1) gives

$$\left(V_\phi^m\right)^s_l\left(P_\phi^m\right)^l_{k;(i}(g_\rho)'_{j)s} = 2\left(P_\phi^m\right)^l_{k;(i}\left(P_\phi^m\right)_{j)l}. \tag{6.7}$$

Combining equations (6.2), (6.6) and (6.7) yields

$$ {}^{g_\rho}\nabla_k (g_\rho)'_{ij} = 2\big(P^m_\phi\big)_{ij;k}. $$

Since $\mathrm{d}P^m_\phi = 0$, we conclude that equation (6.5b) holds.

We now verify equation (6.5a). Note that

$$ g_\rho - \frac{1}{2}\rho g'_\rho = U^m_\phi. $$

Thus

$$ (g_\rho)_{i[l}(g_\rho)'_{k]j} + (g_\rho)_{j[k}(g_\rho)'_{l]i} - \rho(g_\rho)'_{i[l}(g_\rho)'_{k]j} = \big(U^m_\phi\big)_{i[l}(g_\rho)'_{k]j} + \big(U^m_\phi\big)_{j[k}(g_\rho)'_{l]i}. $$

Using equation (6.1), we deduce that

$$ (g_\rho)_{i[l}(g_\rho)'_{k]j} + (g_\rho)_{j[k}(g_\rho)'_{l]i} - \rho(g_\rho)'_{i[l}(g_\rho)'_{k]j} = 2\big(U^m_\phi\big)_i^{\ a}\big(U^m_\phi\big)_j^{\ b}\big(g_{a[l}\big(P^m_\phi\big)_{k]b} + g_{b[k}\big(P^m_\phi\big)_{l]a}\big). $$

Since $A^m_\phi = 0$, it holds that

$$ {}^{g}R_{abkl} = -2\big(g_{a[l}\big(P^m_\phi\big)_{k]b} + g_{b[k}\big(P^m_\phi\big)_{l]a}\big). \tag{6.8} $$

Combining equations (6.3), (6.8) and (6.9) yields equation (6.5a). ■

Proof of Theorem 1.2(*ii*). Proposition 6.2 implies that $\widetilde{\mathrm{Ric}^m_\phi} = -\frac{m}{\widetilde{f}}\widetilde{\nabla}^2\widetilde{f}$.

We now prove that $\widetilde{\nabla}^2\widetilde{f} = 0$. Recall [15, equation (3.16)] that the Christoffel symbols of $\widetilde{g} = 2\rho\,\mathrm{d}t^2 + t^2 g_\rho + 2\rho\,\mathrm{d}t^2$ are

$$ \widetilde{\Gamma}^0_{IJ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}t(g_\rho)'_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\Gamma}^k_{IJ} = \begin{pmatrix} 0 & t^{-1}\delta^k_j & 0 \\ t^{-1}\delta^k_i & \Gamma^k_{ij} & \frac{1}{2}(g_\rho)^{kl}(g_\rho)'_{il} \\ 0 & \frac{1}{2}(g_\rho)^{kl}(g_\rho)'_{jl} & 0 \end{pmatrix}, $$
$$ \widetilde{\Gamma}^\infty_{IJ} = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & -(g_\rho)_{ij} + \rho(g_\rho)'_{ij} & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}. \tag{6.9} $$

Direct computation immediately yields

$$ \big(\widetilde{\nabla}^2\widetilde{f}\big)_{0I} = \big(\widetilde{\nabla}^2\widetilde{f}\big)_{\infty\infty} = 0. $$

We next prove that $(\widetilde{\nabla}^2\widetilde{f})_{i\infty} = 0$. Observe that, by Lemma 2.1,

$$ \frac{1}{m}\partial_i\big(fY^m_\phi\big) = \big(P^m_\phi\big)_i^k\partial_k f, \tag{6.10} $$

and, by equation (6.1),

$$ \frac{1}{2}g^{kl}g'_{li} = \big(V^m_\phi\big)_s^k\big(P^m_\phi\big)_i^s. $$

We then compute that

$$ \begin{aligned} \frac{1}{t}\big(\widetilde{\nabla}^2\widetilde{f}\big)_{i\infty} &= \partial_i\left(\frac{Y^m_\phi}{m}f\right) - \widetilde{\Gamma}^k_{i\infty}\partial_k\left(f + \rho\frac{Y^m_\phi}{m}f\right) \\ &= \big(P^m_\phi\big)_i^k\partial_k f - \big(V^m_\phi\big)_s^k\big(P^m_\phi\big)_i^s\partial_k f - \rho\big(V^m_\phi\big)_s^k\big(P^m_\phi\big)_i^s\big(P^m_\phi\big)_k^l\partial_l f. \end{aligned} $$

The conclusion follows from the identity $-\rho(P^m_\phi)^l_k = \delta^l_k - (U^m_\phi)^l_k$.

We now prove that $(\widetilde{\nabla}^2\widetilde{f})_{ij} = 0$. Note that

$$\frac{1}{t}\widetilde{\nabla}^2_{ij}f = {}^{g_\rho}\nabla^2_{ij}\left(f + \frac{f}{m}\rho Y^m_\phi\right) - \frac{1}{t}\Gamma^0_{ij}\left(f + \frac{f}{m}\rho Y^m_\phi\right) - \frac{1}{m}\Gamma^\infty_{ij} f Y^m_\phi.$$

Now applying equations (6.1) and (6.2) yields

$$\begin{aligned}\frac{1}{t}\big(\widetilde{\nabla}^2\widetilde{f}\big)_{ij} &= {}^{g}\nabla^2_{ij}\left(f + \frac{f}{m}Y^m_\phi\rho\right) - \rho\big(V^m_\phi\big)_l^k\big(P^m_\phi\big)^l_{i;j}\partial_k\left(f + \frac{f}{m}Y^m_\phi\rho\right)\\ &\quad + f\left(\big(P^m_\phi\big)_{ik} + \frac{1}{m}Y^m_\phi g_{ik}\right)\big(U^m_\phi\big)^k_j.\end{aligned} \tag{6.11}$$

Equation (6.10) implies that

$${}^{g}\nabla^2_{ij}\big(fY^m_\phi\big) = m\big[\big(P^m_\phi\big)^k_{i;j}\partial_k f + \big(P^m_\phi\big)^k_i\nabla^2_{jk}f\big]. \tag{6.12}$$

Also, Lemma 2.1 implies that

$${}^{g}\nabla^2_{ij}f = -f\big(P^m_\phi\big)_{ij} - \frac{f}{m}Y^m_\phi g_{ij}. \tag{6.13}$$

Equations (6.11), (6.12) and (6.13), put together, yield $(\widetilde{\nabla}^2\widetilde{f})_{ij} = 0$. Hence $\widetilde{R}_{IJ} = 0$.

We now prove that $\widetilde{F^m_\phi} = 0$. Since $(\widetilde{\nabla}^2\widetilde{f})_{IJ} = 0$, it holds that $\widetilde{\Delta}\widetilde{f} = 0$. For $m = 1$, we have $\widetilde{F^m_\phi} = 0$. Suppose now that $m \neq 1$. Since $(\widetilde{\nabla}^2\widetilde{f})_{I\infty} = 0$, it holds that $\partial_\rho(|\widetilde{\nabla}\widetilde{f}|^2 - \mu) = 0$. Hence, it suffices to show that $(|\widetilde{\nabla}\widetilde{f}|^2 - \mu)\big|_{\rho=0} = 0$. A direct computation yields

$$\big(|\widetilde{\nabla}\widetilde{f}|^2 - \mu\big)\big|_{\rho=0} = 2\frac{f^2}{m}Y^m_\phi + |\nabla f|^2 - \mu.$$

On the one hand, it holds that [8, Lemma 3.1]

$$\frac{f^2}{m}Y^m_\phi = -\frac{1}{d+m-2}\big[(m-1)\big(|\nabla f|^2 - \mu\big) + f\Delta f + f^2 J^m_\phi\big]. \tag{6.14}$$

On the other hand, the trace of equation (2.3b) gives

$$f\Delta f + f^2 J^m_\phi + \frac{d-m}{m}f^2 Y^m_\phi = 0. \tag{6.15}$$

Combining equation (6.14) with equation (6.15) yields $\widetilde{F^m_\phi} = 0$.

As both $\widetilde{\mathrm{Ric}^m_\phi}_{IJ} = 0$ and $\widetilde{F^m_\phi} = 0$, we conclude that $(\widetilde{\mathcal{G}}, \widetilde{g}, \widetilde{f}, m, \mu)$ is a weighted ambient metric for (M^d, g, f, m, μ). ∎

6.3 The Gover–Leitner conditions

Let $(M^d, g, 1, m, \mu)$, $d \geq 3$ and $m < \infty$, be a smooth metric measure space such that

$$\mathrm{Ric}^m_\phi = -(d-1)\mu g. \tag{6.16}$$

Then $F^m_\phi = -(m-1)\mu$.

Set $\lambda = -\mu/2$ and $g_\rho = (1+\lambda\rho)^2 g$ and $f_\rho = 1 - \lambda\rho$. Define $(\widetilde{g}, \widetilde{f})$ as in equation (1.2). It follows immediately from equation (4.9) that $\widetilde{\mathrm{Ric}^m_\phi} = 0$ and $\widetilde{F^m_\phi} = 0$. Hence, $(\widetilde{G}, \widetilde{g}, \widetilde{f}, m, \mu)$ is the weighted ambient metric of $(M, g, 1, m, \mu)$. Moreover, equations (2.1) and (6.16) imply that

$$P^m_\phi = \lambda g, \qquad Y^m_\phi = -m\lambda.$$

This verifies Theorem 1.2*(iii)*.

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References

[1] Bailey T.N., Eastwood M.G., Graham C.R., Invariant theory for conformal and CR geometry, *Ann. of Math.* **139** (1994), 491–552.

[2] Bakry D., Émery M., Diffusions hypercontractives, in Séminaire de probabilités, XIX, 1983/84, *Lecture Notes in Math.*, Vol. 1123, Springer, Berlin, 1985, 177–206.

[3] Caffarelli L., Silvestre L., An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245–1260, arXiv:math.AP/0608640.

[4] Case J.S., Smooth metric measure spaces, quasi-Einstein metrics, and tractors, *Cent. Eur. J. Math.* **10** (2012), 1733–1762, arXiv:1110.3009.

[5] Case J.S., A Yamabe-type problem on smooth metric measure spaces, *J. Differential Geom.* **101** (2015), 467–505, arXiv:1306.4358.

[6] Case J.S., A notion of the weighted σ_k-curvature for manifolds with density, *Adv. Math.* **295** (2016), 150–194, arXiv:1409.4455.

[7] Case J.S., A weighted renormalized curvature for manifolds with density, *Proc. Amer. Math. Soc.* **145** (2017), 4031–4040, arXiv:1603.02989.

[8] Case J.S., The weighted σ_k-curvature of a smooth metric measure space, *Pacific J. Math.* **299** (2019), 339–399, arXiv:1608.01663.

[9] Case J.S., Sharp weighted Sobolev trace inequalities and fractional powers of the Laplacian, *J. Funct. Anal.* **279** (2020), 108567, 33 pages, arXiv:1901.09843.

[10] Case J.S., Chang S.-Y.A., On fractional GJMS operators, *Comm. Pure Appl. Math.* **69** (2016), 1017–1061, arXiv:1406.1846.

[11] Case J.S., Shu Y.-J., Wei G., Rigidity of quasi-Einstein metrics, *Differential Geom. Appl.* **29** (2011), 93–100, arXiv:0805.3132.

[12] Chang S.-Y.A., Fang H., A class of variational functionals in conformal geometry, *Int. Math. Res. Not.* **2008** (2008), Art ID rnn008, 16 pages, arXiv:0803.0333.

[13] Chang S.-Y.A., Fang H., Graham C.R., A note on renormalized volume functionals, *Differential Geom. Appl.* **33** (2014), suppl., 246–258, arXiv:1211.6422.

[14] Cheeger J., Colding T.H., On the structure of spaces with Ricci curvature bounded below. II, *J. Differential Geom.* **54** (2000), 13–35, arXiv:1805.07988.

[15] Fefferman C., Graham C.R., The ambient metric, *Ann. of Math. Stud.*, Vol. 178, Princeton University Press, Princeton, NJ, 2012.

[16] Gover A.R., Invariant theory and calculus for conformal geometries, *Adv. Math.* **163** (2001), 206–257.

[17] Gover A.R., Leitner F., A sub-product construction of Poincaré–Einstein metrics, *Internat. J. Math.* **20** (2009), 1263–1287, arXiv:math.DG/0608044.

[18] Graham C.R., Extended obstruction tensors and renormalized volume coefficients, *Adv. Math.* **220** (2009), 1956–1985, arXiv:0810.4203.

[19] Graham C.R., Jenne R., Mason L.J., Sparling G.A.J., Conformally invariant powers of the Laplacian. I. Existence, *J. London Math. Soc.* **46** (1992), 557–565.

[20] Graham C.R., Zworski M., Scattering matrix in conformal geometry, *Invent. Math.* **152** (2003), 89–118, arXiv:math.DG/0109089.

[21] Khaitan A., GJMS operators on smooth metric measure spaces, arXiv:2203.04719.

[22] Mazzeo R.R., Melrose R.B., Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, *J. Funct. Anal.* **75** (1987), 260–310.

[23] Perelman G., The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.

[24] Wei G., Wylie W., Comparison geometry for the Bakry–Emery Ricci tensor, *J. Differential Geom.* **83** (2009), 377–405, arXiv:0706.1120.

[25] Yamabe H., On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12** (1960), 21–37.