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# On the Hill Discriminant of Lamé's Differential Equation

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**Abstract.** Lamé's differential equation is a linear differential equation of the second order with a periodic coefficient involving the Jacobian elliptic function sn depending on the modulus $k$, and two additional parameters $h$ and $\nu$. This differential equation appears in several applications, for example, the motion of coupled particles in a periodic potential. Stability and existence of periodic solutions of Lamé's equations is determined by the value of its Hill discriminant $D(h, \nu, k)$. The Hill discriminant is compared to an explicitly known quantity including explicit error bounds. This result is derived from the observation that Lamé's equation with $k = 1$ can be solved by hypergeometric functions because then the elliptic function sn reduces to the hyperbolic tangent function. A connection relation between hypergeometric functions then allows the approximation of the Hill discriminant by a simple expression. In particular, one obtains an asymptotic approximation of $D(h, \nu, k)$ when the modulus $k$ tends to 1.

*Key words:* Lamé's equation; Hill's discriminant; asymptotic expansion; stability

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## 1 Introduction

Kim, Levi and Zhou [4] consider two elastically coupled particles positioned at $x(t)$, $y(t)$ in a periodic potential $V(x)$. The system is described by

$$\ddot{x} + V'(x) = \gamma(y - x), \qquad \ddot{y} + V'(y) = \gamma(x - y),$$

where $\gamma$ denotes the coupling constant. Let $x(t) = y(t) = p(t)$ be a synchronous solution. If we linearize the system around this synchronous solution, $x = p + \xi$, $y = p + \eta$, and set $u = \xi + \eta$, $w = \xi - \eta$, then we obtain

$$\begin{aligned} &\ddot{u} + V''(p)u = 0, \\ &\ddot{w} + (2\gamma + V''(p))w = 0. \end{aligned} \tag{1.1}$$

These are Hill equations [5], that is, they are of the form

$$\ddot{w} + q(t)w = 0 \tag{1.2}$$

with a periodic coefficient function $q(t)$, say of period $\sigma > 0$. In this and many other applications the Hill discriminant $D$ associated with (1.2) plays an important role. The discriminant $D$ is defined as the trace of the endomorphism $w(t) \mapsto w(t + \sigma)$ of the two-dimensional solution space of (1.2) onto itself. It is well known [5] that equation (1.2) is stable if $|D| < 2$ and unstable

This paper is a contribution to the Special Issue on Asymptotics and Applications of Special Functions in Memory of Richard Paris. The full collection is available at https://www.emis.de/journals/SIGMA/Paris.html if $|D|>2$. The condition $D=2$ is equivalent to the existence of a nontrivial solution with period $\sigma$ while $D=-2$ is equivalent to the existence of a nontrivial solution with semi-period $\sigma$.

In this work, we are interested in the special case $V(x)=-\cos x$. Then $p(t)$ is a solution of the differential equation $\ddot{p}+\sin p=0$ of the mathematical pendulum. We get [7, Section 22.19 (i)]

$$p(t,\mathcal{E})=2\,\mathrm{am}\left(\frac{t}{k},k\right),\qquad \text{where } k^2=\frac{2}{\mathcal{E}+2},$$

$\mathcal{E}$ denotes energy, and am is Jacobi's amplitude function [7, Section 22.16 (i)]. Then equation (1.1) becomes

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2}+\left(2\gamma+1-2\,\mathrm{sn}^2\left(\frac{t}{k},k\right)\right)w=0,$$

where $\mathrm{sn}(x,k)=\sin\mathrm{am}(x,k)$ is one of the Jacobian elliptic functions [7, Section 16]. If we substitute $t=ks$, we obtain Lamé's equation [7, Section 29]

$$\frac{\mathrm{d}^2w}{\mathrm{d}s^2}+\big(h-\nu(\nu+1)k^2\,\mathrm{sn}^2(s,k)\big)w=0 \tag{1.3}$$

with parameters $h=k^2(2\gamma+1)$ and $\nu=1$. There is no explicit formula for the corresponding Hill discriminant $D=D(h,\nu,k)$. However, in [4] a remarkable asymptotic formula for this Hill discriminant as $\mathcal{E}\to 0$ (or $k\to 1$) is given. It is shown that

$$D(h,1,k)=a\cos\left(\omega\ln\mathcal{E}-\phi\right)+o(\mathcal{E})\qquad \text{as } \mathcal{E}\to 0, \tag{1.4}$$

where $\omega^2=2\gamma-1$.

The main result of this paper is Theorem 5.1 which improves on (1.4) in three directions.

1. We allow any real $\nu$ in place of $\nu=1$. Since we may replace $\nu$ by $-1-\nu$ we assume $\nu\ge-\frac{1}{2}$ without loss of generality.
2. We provide explicit values for the amplitude $a$ and the phase shift $\phi$ in (1.4)
3. We give explicit error bounds. This makes it possible to prove stability of the Lamé equation in some cases.

The idea behind the proof of Theorem 5.1 is the observation that Lamé's equation (1.3) with $k=1$ can be solved in terms of the hypergeometric function $F(a,b;c,x)$. Then well-known connection relations between hypergeometric functions play a crucial role.

As a preparation, we present some elementary results on linear differential equations of the second order in Section 2. In Section 3, we give a quick review of the Lamé equation. In Section 4, we consider the special case of the Lamé equation when $k=1$. In Section 5 we combine our results to obtain Theorem 5.1.

## 2 Lemmas on second order linear equations

Let $u$ be the solution of the initial value problem

$$u''+q(t)u=r(t),\qquad u(a)=u'(a)=0,$$

where $q,r\colon[a,b]\to\mathbb{R}$ are continuous functions. By the variation of constants formula [2, Section 2.6],

$$u(t)=\int_a^t L(t,s)r(s)\,\mathrm{d}s,\qquad u'(t)=\int_a^t \partial_1 L(t,s)r(s)\,\mathrm{d}s,$$

where $y(t) = L(t,s)$ is the solution of

$$y'' + q(t)y = 0 \tag{2.1}$$

determined by the initial conditions $y(s) = 0$, $y'(s) = 1$. Let $L_1$, $L_2$ be constants such that

$$|L(t,s)| \le L_1, \quad |\partial_1 L(t,s)| \le L_2 \qquad \text{for } a \le s \le t \le b. \tag{2.2}$$

Then it follows that

$$\|u\|_\infty \le L_1 \int_a^b |r(s)|\,\mathrm{d}s, \qquad \|u'\|_\infty \le L_2 \int_a^b |r(s)|\,\mathrm{d}s, \tag{2.3}$$

where $\|f\|_\infty := \max_{t\in[a,b]} |f(t)|$.

**Lemma 2.1.** *Let $p, q\colon [a,b] \to \mathbb{R}$ be continuous. Let $L_1, L_2$ be as in (2.2). Let $y$ be a solution of (2.1) and $w$ a solution of $w'' + p(t)w = 0$ with $y(a) = w(a)$ and $y'(a) = w'(a)$. Then*

$$\|y - w\|_\infty \le L_1 \|w\|_\infty \int_a^b |p(s) - q(s)|\,\mathrm{d}s,$$

$$\|y' - w'\|_\infty \le L_2 \|w\|_\infty \int_a^b |p(s) - q(s)|\,\mathrm{d}s.$$

**Proof.** For $u = y - w$, we have

$$u''(t) + q(t)u(t) = (p(t) - q(t))w(t).$$

The desired result follows from (2.3). ■

**Lemma 2.2.** *Let $q\colon [a,b] \to (0,\infty)$ be continuously differentiable and monotone. Set*

$$m := \min_{t\in[a,b]} q(t) > 0, \qquad M := \max_{t\in[a,b]} q(t).$$

*Let $y_1$, $y_2$ be the solutions of (2.1) determined by $y_1(a) = y_2'(a) = 1$, $y_1'(a) = y_2(a) = 0$. If $q$ is nondecreasing, then*

$$\|y_1\|_\infty^2 \le 1, \qquad \|y_1'\|_\infty^2 \le M, \qquad \|y_2\|_\infty^2 \le \frac{1}{m}, \qquad \|y_2'\|_\infty^2 \le \frac{M}{m},$$

*and, if $q$ is nonincreasing,*

$$\|y_1\|_\infty^2 \le \frac{M}{m}, \qquad \|y_1'\|_\infty^2 \le M, \qquad \|y_2\|_\infty^2 \le \frac{1}{m}, \qquad \|y_2'\|_\infty^2 \le 1.$$

**Proof.** Suppose first that $q$ is nondecreasing. Set

$$u_j(t) := y_j(t)^2 + \frac{1}{q(t)} y_j'(t)^2.$$

Then

$$u_j'(t) = -\frac{q'(t)}{q(t)^2} y_j'(t)^2 \le 0,$$

so $u_j(t) \le u_j(a)$ for all $t \in [a,b]$. Now $u_1(a) = 1$ and $u_2(a) = \frac{1}{m}$ imply $y_1(t)^2 \le 1$, $y_1'(t)^2 \le M$, $y_2(t)^2 \le \frac{1}{m}$, $y_2'(t)^2 \le \frac{M}{m}$. If $q$ is nonincreasing, we argue similarly using $v_j(t) = y_j'(t)^2 + q(t)y_j(t)^2$ in place of $u_j$. ■

## 3 Lamé's equation

For $h \in \mathbb{R}$, $\nu \geq -\frac{1}{2}$, $k \in (0,1)$, we consider the Lamé equation [1, Section IX] and [3, Section XV]

$$y'' + \big(h - \nu(\nu+1)k^2 \operatorname{sn}^2(t,k)\big)y = 0. \tag{3.1}$$

This is a Hill equation with period $2K(k)$, where $K = K(k)$ is the complete elliptic integral of the first kind:

$$K = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

Equation (3.1) also makes sense for $k = 1$ [7, Section 22.5 (ii)] when it becomes

$$y'' + \big(h - \nu(\nu+1)\tanh^2 t\big)y = 0. \tag{3.2}$$

Of course, this is not a Hill equation anymore. Let

$$y_1(t) = y_1(t,s,h,\nu,k) \qquad \text{and} \qquad y_2(t) = y_2(t,s,h,\nu,k)$$

be the solutions of (3.1) determined by the initial conditions

$$y_1(s) = y_2'(s) = 1, \qquad y_1'(s) = y_2(s) = 0.$$

Set $q(t) := h - \nu(\nu+1)k^2\operatorname{sn}^2(t,k)$. This function is increasing on $[0,K]$ if $-\frac{1}{2} \leq \nu < 0$ and decreasing on $[0,K]$ if $\nu > 0$. We assume that $h > 0$ and $h > \nu(\nu+1)k^2$. Then $q(t) > 0$ for $t \in [0,K]$. We define $H := \big(h - \nu(\nu+1)k^2\big)^{1/2}$ and

$$C_1(h,\nu,k) := \begin{cases} 1 & \text{if } \nu < 0, \\ h^{1/2}H^{-1} & \text{if } \nu \geq 0, \end{cases} \qquad C_1'(h,\nu,k) := \begin{cases} H & \text{if } \nu < 0, \\ h^{1/2} & \text{if } \nu \geq 0, \end{cases}$$
$$C_2(h,\nu,k) := \begin{cases} h^{-1/2} & \text{if } \nu < 0, \\ H^{-1} & \text{if } \nu \geq 0, \end{cases} \qquad C_2'(h,\nu,k) := \begin{cases} h^{-1/2}H & \text{if } \nu < 0, \\ 1 & \text{if } \nu \geq 0. \end{cases}$$

**Lemma 3.1.** *Suppose that $h > 0$ and $h - \nu(\nu+1)k^2 > 0$. Then, for $0 \leq s \leq t \leq K$,*

$$|y_1(t,s)| \leq C_1, \qquad |y_1'(t,s)| \leq C_1', \qquad |y_2(t,s)| \leq C_2, \qquad |y_2'(t,s)| \leq C_2'.$$

*If $k = 1$, this is true for all $0 \leq s \leq t$.*

**Proof.** This follows from Lemma 2.2. ∎

In the next theorem, we use the complete elliptic integral $E = E(k)$ of the second kind:

$$E = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

**Theorem 3.2.** *Suppose that $h > 0$ and $h - \nu(\nu+1) > 0$. Then*

$$|y_1(K,0,h,\nu,k) - y_1(K,0,h,\nu,1)| \leq C_1C_2|\nu|(\nu+1)(E(k) - \tanh K(k)),$$
$$|y_2'(K,0,h,\nu,k) - y_2'(K,0,h,\nu,1)| \leq C_2C_2'|\nu|(\nu+1)(E(k) - \tanh K(k)),$$

*where the constants $C$ are formed with $k = 1$.*

**Proof.** We apply Lemma 2.1 with

$$q(t) = h - \nu(\nu+1)k^2 \operatorname{sn}^2(t,k), \qquad p(t) = h - \nu(\nu+1)\tanh^2 t,$$

and

$$y(t) = y_1(t,0,h,\nu,k), \qquad w(t) = y_1(t,0,h,\nu,1)$$

on the interval $t \in [0,K]$. We note that [8, formula (4.4)]

$$k \operatorname{sn}(t,k) \le \tanh t \le \operatorname{sn}(t,k) \qquad \text{for } t \in [0,K].$$

Therefore,

$$\begin{aligned}\int_0^K |p(s)-q(s)|\,\mathrm{d}s &= |\nu|(\nu+1)\int_0^K \big(\tanh^2 s - k^2 \operatorname{sn}^2(s,k)\big)\,\mathrm{d}s \\ &= |\nu|(\nu+1)\int_0^K \big(\operatorname{dn}^2(s,k) - 1 + \tanh^2 s\big)\,\mathrm{d}s.\end{aligned}$$

Using $\int_0^K \operatorname{dn}^2(s,k)\,\mathrm{d}s = E$ [9, p. 518], we get

$$\int_0^K |p(s)-q(s)|\,\mathrm{d}s = |\nu|(\nu+1)(E - \tanh K).$$

By Lemma 3.1, $|w(t)| \le C_1$ and we can choose $L_1 = C_2$. This gives the desired estimate for $y_1$. The estimate for $y_2'$ is proved similarly. ∎

Note that

$$\int_0^K \big(\tanh^2 s - k^2 \operatorname{sn}^2(s,k)\big)\,\mathrm{d}s \le \big(1-k^2\big)\int_0^K \operatorname{sn}^2(s,k)\,\mathrm{d}s \le k'^2 K,$$

where $k' = \sqrt{1-k^2}$, so

$$E - \tanh K \le k'^2 K.$$

Also note that [7, formula (19.9.1)]

$$K(k) \le \frac{\pi}{2} - \ln k',$$

so $E(k) - \tanh K(k) = O((1-k)\ln(1-k))$ as $k \to 1$.

## 4 The Lamé equation for $k = 1$

Let $w_1$, $w_2$ be the solutions of (3.2) determined by initial conditions $w_1(0) = w_2'(0) = 1$, $w_1'(0) = w_2(0) = 0$. Then $w_j(t) = y_j(t,0,h,\nu,1)$ using the notation of the previous section. We assume that $\nu \ge -\frac{1}{2}$ and $h > \nu(\nu+1)$, and set

$$\mu := \sqrt{\nu(\nu+1) - h} = \mathrm{i}\omega, \qquad \text{where } \omega > 0. \tag{4.1}$$

The substitution $x = \tanh t$ transforms (3.2) to the associated Legendre equation [7, formula (14.2.2)] of degree $\nu$ and order $\mu$. According to [6, Section 5, formula (15.09)], we express $w_j$ in terms of the hypergeometric function $F(a,b;c;z)$ as follows:

$$\begin{aligned} w_1(t) &= \cosh^\mu t F\big(-\tfrac{1}{2}(\mu+\nu), \tfrac{1}{2}(1-\mu+\nu); \tfrac{1}{2}; \tanh^2 t\big), \\ w_2(t) &= \tanh t \cosh^\mu t F\big(\tfrac{1}{2}(1-\mu-\nu), \tfrac{1}{2}(2-\mu+\nu); \tfrac{3}{2}; \tanh^2 t\big). \end{aligned}$$

This can be confirmed by direct computation. In order to determine the behaviour of the functions $w_j(t)$ as $\mathbb{R} \ni t \to \infty$, we use the connection formula [7, formula (15.8.4)] and find $w_j(t) = \mathrm{Re}(v_j(t))$, where

$$v_1(t) = \frac{A_1}{(2\cosh t)^{-\mu}} F\bigl(-\tfrac{1}{2}(\mu+\nu), \tfrac{1}{2}(1-\mu+\nu); 1-\mu; \cosh^{-2} t\bigr),$$
$$v_2(t) = \frac{A_2 \tanh t}{(2\cosh t)^{-\mu}} F\bigl(\tfrac{1}{2}(1-\mu-\nu), \tfrac{1}{2}(2-\mu+\nu); 1-\mu; \cosh^{-2} t\bigr),$$

and

$$A_1 = \frac{2^{1-\mu}\pi^{1/2}\Gamma(\mu)}{\Gamma\bigl(\tfrac{1}{2}(1+\mu+\nu)\bigr)\Gamma\bigl(\tfrac{1}{2}(\mu-\nu)\bigr)},$$
$$A_2 = \frac{2^{-\mu}\pi^{1/2}\Gamma(\mu)}{\Gamma\bigl(\tfrac{1}{2}(2+\mu+\nu)\bigr)\Gamma\bigl(\tfrac{1}{2}(1+\mu-\nu)\bigr)}.$$

We set

$$z_j(t) = \mathrm{Re}\bigl(A_j e^{i\omega t}\bigr), \qquad j = 1, 2.$$

**Theorem 4.1.** *Suppose $h > 0$ and $h > \nu(\nu+1)$. Then, for all $t \geq 0$,*

$$|w_1(t) - z_1(t)| \leq \omega^{-1} C_1 |\nu|(\nu+1)(1-\tanh t),$$
$$|w_2'(t) - z_2'(t)| \leq C_2 |\nu|(\nu+1)(1-\tanh t),$$

*where $C_1$, $C_2$ are formed with $k = 1$.*

**Proof.** Since $F(a, b; c; 0) = 1$, the representation $w_j(t) = \mathrm{Re}(v_j(t))$ yields

$$\lim_{t\to\infty} w_j(t) - z_j(t) = \lim_{t\to\infty} \mathrm{Re}\bigl(A_j (2\cosh t)^{i\omega} - A_j e^{i\omega t}\bigr) = 0. \tag{4.2}$$

Similarly, we have

$$\lim_{t\to\infty} w_j'(t) - z_j'(t) = 0. \tag{4.3}$$

The function $u_j = w_j - z_j$ satisfies

$$u_j'' + \omega^2 u_j = g_j(t), \qquad g_j(t) := \nu(\nu+1)\bigl(\tanh^2 t - 1\bigr) w_j(t).$$

Let $t_0, t \geq 0$. Then

$$u_j(t) = u_j(t_0)\cos(\omega(t-t_0)) + u_j'(t_0)\frac{\sin(\omega(t-t_0))}{\omega} + \int_{t_0}^{t} \frac{\sin(\omega(t-s))}{\omega} g_j(s)\, ds.$$

Letting $t_0 \to \infty$, using (4.2), (4.3) and Lemma 3.1, we obtain

$$|u_1(t)| \leq \omega^{-1} \int_t^\infty |g_1(s)|\, ds \leq \omega^{-1} C_1 |\nu|(\nu+1)(1-\tanh t)$$

as desired. The estimate for $u_2'$ is derived similarly. ■

The constant Wronskian of $z_1$, $z_2$ is

$$z_1(t) z_2'(t) - z_1'(t) z_2(t) = \omega\, \mathrm{Im}\bigl(A_1 \bar{A}_2\bigr).$$

The reflection formula for the gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

gives

$$\omega A_1 \bar{A}_2 = -\frac{\sin(\nu\pi)}{\sinh(\omega\pi)} + \mathrm{i}. \tag{4.4}$$

Therefore,

$$z_1(t)z_2'(t) - z_1'(t)z_2(t) = 1.$$

Moreover,

$$z_1(t)z_2'(t) + z_1'(t)z_2(t) = 2z_1(t)z_2'(t) - 1 = \operatorname{Re}\bigl(B\mathrm{e}^{2\mathrm{i}\omega t}\bigr), \tag{4.5}$$

where $B = \mathrm{i}\omega A_1 A_2$. Using the duplication formula for the gamma function

$$2^{x-1}\Gamma\bigl(\tfrac{1}{2}x\bigr)\Gamma\bigl(\tfrac{1}{2}(x+1)\bigr) = \pi^{1/2}\Gamma(x)$$

we see that

$$B = \frac{\Gamma(1+\mu)\Gamma(\mu)}{\Gamma(1+\mu+\nu)\Gamma(\mu-\nu)}. \tag{4.6}$$

If $\nu \in \mathbb{N}_0$, then

$$B = \frac{(\mathrm{i}\omega - 1)(\mathrm{i}\omega - 2)\cdots(\mathrm{i}\omega - \nu)}{(\mathrm{i}\omega + 1)(\mathrm{i}\omega + 2)\cdots(\mathrm{i}\omega + \nu)},$$

so $|B| = 1$. If $\nu = 1$, then

$$B = \frac{\mathrm{i}\omega - 1}{\mathrm{i}\omega + 1} = \frac{\omega^2 - 1 + \mathrm{i}2\omega}{\omega^2 + 1}$$

and

$$\operatorname{Re}\bigl(B\mathrm{e}^{2\mathrm{i}\omega t}\bigr) = \frac{1}{\omega^2+1}\bigl((\omega^2 - 1)\cos(2\omega t) - 2\omega\sin(2\omega t)\bigr).$$

By (4.4),

$$|B|^2 = \bigl|\omega A_1 \bar{A}_2\bigr|^2 = 1 + \frac{\sin^2\nu\pi}{\sinh^2\omega\pi}.$$

So $|B| > 1$ if $\nu$ is not an integer.

## 5 Hill's discriminant of Lamé's equation

The Hill discriminant $D(h,\nu,k)$ of Lamé's equation is given by [5, p. 8]

$$D(h,\nu,k) = 2(y_1(K)y_2'(K) + y_1'(K)y_2(K)) = 2(2y_1(K)y_2'(K) - 1), \tag{5.1}$$

where $y_j(t) = y_j(t,0,h,\nu,k)$ in the notation of Section 3. By combining Theorems 3.2 and 4.1, we obtain the following main theorem of this work.

**Theorem 5.1.**

(*a*) *Suppose* $\nu \geq 0$ *and* $h > \nu(\nu+1)$. *Then, for all* $k \in (0,1)$,

$$\left|D(h,\nu,k) - 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K(k)}\big)\right| \leq 8h^{1/2}\omega^{-2}\nu(\nu+1)(E(k)+1-2\tanh K(k)).$$

(*b*) *Suppose* $\nu \in [-\frac{1}{2},0)$ *and* $h > 0$. *Then, for all* $k \in (0,1)$,

$$\left|D(h,\nu,k) - 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K(k)}\big)\right| \leq 8\omega h^{-1}|\nu|(\nu+1)(E(k)+1-2\tanh K(k)).$$

*The constants* $\omega$ *and* $B$ *are given in* (4.1) *and* (4.6), *respectively.*

**Proof.** Using (4.5) and (5.1), we have

$$D(h,\nu,k) - 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K}\big) = 4(y_1(K)y_2'(K) - z_1(K)z_2'(K)).$$

Using Lemma 3.1, we estimate

$$\begin{aligned}&\left|D(h,\nu,k) - 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K(k)}\big)\right| \\ &\qquad \leq 4|y_1(K)||y_2'(K) - z_2'(K)| + 4|z_2'(K)||y_1(K) - z_1(K)| \\ &\qquad \leq 4C_1|y_2'(K) - z_2'(K)| + 4C_2'|y_1(K) - z_1(K)|.\end{aligned}$$

In fact, $|w_2'(t)| \leq C_2'$ implies $|z_2'(t)| \leq C_2'$ because of (4.3). Now we use Theorems 3.2 and 4.1 to estimate

$$\begin{aligned}|y_1(K) - z_1(K)| &\leq |y_1(K) - w_1(K)| + |w_1(K) - z_1(K)| \\ &\leq C_1C_2|\nu|(\nu+1)(E - \tanh K) + \omega^{-1}C_1|\nu|(\nu+1)(1-\tanh K),\end{aligned}$$

and

$$\begin{aligned}|y_2'(K) - z_2'(K)| &\leq |y_2'(K) - w_2'(K)| + |w_2'(K) - z_2'(K)| \\ &\leq C_2C_2'|\nu|(\nu+1)(E - \tanh K) + C_2|\nu|(\nu+1)(1-\tanh K).\end{aligned}$$

This gives the desired statements (a) and (b) substituting the values for $C_j$ and $C_j'$. ■

We may use $K(k) = \ln(4/k') + O\big(k'^2 \ln k'\big)$ [7, formula (19.12.1)] and $\left|\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{\mathrm{i}t}\right| \leq |s-t|$ for $s,t \in \mathbb{R}$ to obtain

$$D(h,\nu,k) = 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega\ln(4/k')}\big) + O((1-k)\ln(1-k)) \qquad \text{as } k \to 1.$$

As an illustration, take $h = 6$, $\nu = \frac{1}{2}$ and $k = 1 - \mathrm{e}^{-\tau}$. Figure 1 depicts the graphs of $\tau \mapsto D\big(6,\frac{1}{2},k\big)$ (red) and $\tau \mapsto 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K}\big)$ (black). These graphs are hard to distinguish for $\tau > 2$. The Hill discriminant $D\big(6,\frac{1}{2},k\big)$ is computed using (5.1). The values of $y_1(K)$ and $y_2'(K)$ are found by numerical integration of Lamé's equation (1.3) using the software Maple.

If $\tau = 5$, then $k = 0.993262\ldots$ and $2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K}\big) = -1.274528\ldots$. Theorem 5.1 gives $\left|D\big(6,\frac{1}{2},k\big) - 2\,\mathrm{Re}\big(B\mathrm{e}^{2\mathrm{i}\omega K}\big)\right| \leq 0.066641$. Therefore, $|D(6,1,k)| < 2$ and so Lamé's equation is stable for $h = 6$, $\nu = \frac{1}{2}$, $k = 1 - \mathrm{e}^{-5}$.

## 6 Discussion and further work

In Theorem 5.1, we presented an asymptotic formula describing the behavior of the discriminant of the Lamé equation (1.3) as $k \to 1$. The proof is based on the fact that the Lamé equation approaches the associated Legendre (a special case of the hypergeometric) differential equation, and the known behavior of the hypergeometric function as the independent variable tends to 1. As we know from [4] a less precise formula describing the asymptotic behavior as $\mathcal{E} \to 0$ also exists for more general potentials in (1.1). It is an interesting research question whether there exist other potentials that allow an explicit determination of the amplitude and phase shift in this asymptotic formula.

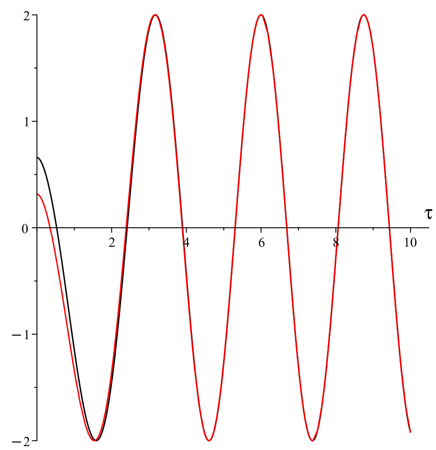


**Figure 1.** Illustration to Theorem 5.1.

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