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Reflection Vectors and Quantum Cohomology of Blowups

Todor MILANOV and Xiaokun XIA

Kavli IPMU (WPI), UTIAS, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
E-mail: *todor.milanov@ipmu.jp*, *xia.xiaokun@ipmu.jp*

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Abstract. Let X be a smooth projective variety with a semisimple quantum cohomology. It is known that the blowup $\mathrm{Bl}_{\mathrm{pt}}(X)$ of X at one point also has semisimple quantum cohomology. In particular, the monodromy group of the quantum cohomology of $\mathrm{Bl}_{\mathrm{pt}}(X)$ is a reflection group. We found explicit formulas for certain generators of the monodromy group of the quantum cohomology of $\mathrm{Bl}_{\mathrm{pt}}(X)$ depending only on the geometry of the exceptional divisor.

Key words: Frobenius structures; Gromov–Witten invariants; quantum cohomology

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1 Introduction

The notion of a Frobenius manifold was invented by Dubrovin in order to give a geometric formulation of the properties of quantum cohomology (see [9]). Later on, it was discovered by Dubrovin and Zhang (see [12]) that if the Frobenius manifold is in addition semisimple, then the corresponding Frobenius structure has very important applications to the theory of integrable hierarchies of KdV type. Our main interest is in a certain system of vectors which we call *reflection vectors*, associated to any semisimple Frobenius manifold. The most general problem is to obtain a classification of the set of reflection vectors corresponding to a semisimple Frobenius manifold. In fact, the set of reflection vectors contain the information about the monodromy group of the so-called *second structure connection*, so by solving an appropriate classical Riemann–Hilbert problem, the reflection vectors uniquely determine the corresponding semisimple Frobenius structure.

1.1 Period vectors

The main motivation to define period vectors for semisimple Frobenius manifolds comes from the work of Givental [19]. Using the period integrals of a simple singularity of type A, Givental was able to construct an integrable hierarchy in the form of Hirota bilinear equations. This result generalizes to simple singularities too (see [21]). The key to the constructions in these two papers are the period integrals of K. Saito (see [33]). Given a singularity $f \in \mathcal{O}_{\mathbb{C}^n}$, Saito has invented an extension of the classical residue pairing, called *higher residue pairing*, and used it to construct a period map for the hypersurface $\{f(x) = \lambda\} \subset \mathbb{C}^n$, where λ is a regular value. Since the hypersurface is a Stein manifold, the vector space of holomorphic forms is infinite dimensional and it is not clear at all that a good notion of a period map exists. Saito's remarkable idea was to ask for a set of holomorphic forms, called *good basis*, $(\omega_1, \dots, \omega_N)$, where N is the Milnor number of the singularity, such that, the higher-residue pairing of ω_i and ω_j vanish for all $1 \le i, j \le N$. The existence of a good basis was proved by M. Saito [34]. The existence of a good basis implies that the space of miniversal deformations of the singularity f has a *flat structure* which turns out to be semisimple Frobenius structure in the sense of Dubrovin (see [23]). Moreover, the Gauss–Manin connection in vanishing cohomology is identified with the second structure connection (2.1)–(2.2) with some $m \in \mathbb{Z}$ or $m \in \frac{1}{2} + \mathbb{Z}$ depending on whether the number n of variables of the singularity f is odd or even. Therefore, the second structure connection of the Frobenius structure in singularity theory admits solutions in terms of period integrals, see [14, Section 3.1] for a nice overview of the construction of such solutions. The period integrals used by Givental and Milanov correspond to singularities with odd number of variables, i.e., they are solutions to the second structure connection $\nabla^{(m)}$ for some $m \in \mathbb{Z}$. Let us point out that Dubrovin has introduced the so-called *Gauss–Manin* and *extended Gauss–Manin systems* (see [11, equations (5.9), (5.11) and (5.12)]). In our notation, the Gauss–Manin system is the system of linear differential equations (2.1) with $m = -1$ and $\lambda = 0$, while the extended Gauss–Manin system is the system (2.1)–(2.2) with $m = 0$. However, in order to obtain a more satisfactory characterization of the period integrals, we need to consider the entire sequence of second structure connections (2.1)–(2.2) with $m \in \mathbb{Z}$!

Remark 1.1. The entire set of connections $\nabla^{(m)}$ $(m \in \mathbb{C})$ was introduced by Manin and Merkulov [27] in order to classify semisimple Frobenius manifolds as special solutions to the Schlesinger equations.

It is easy to check that if $I^{(m)}(t,\lambda)$ is a solution to $\nabla^{(m)}$, then $I^{(m+1)}(t,\lambda) := \partial_\lambda I^{(m)}(t,\lambda)$ is a solution to $\nabla^{(m+1)}$. Moreover, if $m + \frac{1}{2}$ is not an eigenvalue of the grading operator θ, then the differential operator ∂_λ defines an isomorphism between the solutions of $\nabla^{(m)}$ and $\nabla^{(m+1)}$. In particular, the monodromy representations of these two connections are isomorphic. In singularity theory, even if $m + \frac{1}{2}$ is an eigenvalue of θ and ∂_λ might fail to be surjective, the period integrals are always in the image of ∂_λ! The reason is the following. If $I^{(m+1)}$ is a solution to $\nabla^{(m+1)}$ defined in terms of period integrals, then by *stabilizing* the singularity once, i.e., adding a square of a new variable to f, we get a new period integral $I^{(m+1/2)}$ which is a solution to $\nabla^{(m+1/2)}$. Moreover, by stabilizing twice we get a period integral $I^{(m)}$ which is a solution to $\nabla^{(m)}$ satisfying $I^{(m+1)} = \partial_\lambda I^{(m)}$. By stabilizing twice we get that if a period integral is a solution to $\nabla^{(m+1)}$ for some $m \in \mathbb{Z}$, then it must be a derivative of a solution of $\nabla^{(m)}$ which itself is also a period integral. The above discussion motivates the following definition.

Definition 1.2. For a given semisimple Frobenius manifold, a sequences $I^{(m)}(t,\lambda)$ $(m \in \mathbb{Z})$ satisfying the following two conditions:

(i) flatness: $I^{(m)}(t,\lambda)$ is a solution to $\nabla^{(m)}$ for all $m \in \mathbb{Z}$,

(ii) translation invariance: $\partial_\lambda I^{(m)}(t,\lambda) = I^{(m+1)}(t,\lambda)$ for all $m \in \mathbb{Z}$

is said to be a *period vector* of the Frobenius manifold.

1.2 Reflection vectors

The notion of a reflection vector was suggested by the first author in [28]. The definition depends on the choice of a *calibration* and it will be recalled in Section 2.2. In this section we would like to give an alternative definition which has the advantage of being independent of the choice of a calibration. The relation to the original definition will be explained in Section 1.3 below.

Suppose that M is a semisimple Frobenius manifold and let $\mathcal{A}$ be the set of all period vectors of M. Given $\alpha \in \mathcal{A}$, we denote by $I^{(m)}_\alpha(t,\lambda)$ $(m \in \mathbb{Z})$ the corresponding sequence of solutions. To be more precise, we fix a reference point $(t^\circ, \lambda^\circ) \in M \times \mathbb{C}$ in the complement of the discriminant (see Section 2.2) and let each $I^{(m)}_\alpha(t,\lambda)$ be an analytic solution to $\nabla^{(m)}$ defined in a neighborhood of (t°, λ°). Clearly, $\mathcal{A}$ has a vector space structure: if $\alpha, \beta \in \mathcal{A}$, then we define $I^{(m)}_{\alpha+\beta}(t,\lambda) := I^{(m)}_\alpha(t,\lambda) + I^{(m)}_\beta(t,\lambda)$ and $I^{(m)}_{c\alpha}(t,\lambda) := cI^{(m)}_\alpha(t,\lambda)$ where $c \in \mathbb{C}$ is a scalar. As we already explained above, if m is a sufficiently negative integer, then the map $\alpha \mapsto I^{(m)}_\alpha(t,\lambda)$ gives an isomorphism between $\mathcal{A}$ and the space of solutions to $\nabla^{(m)}$. In particular, $\mathcal{A}$ is a finite dimensional vector space and the analytic continuation along closed loops in the complement of the discriminant defines a representation of $\pi_1((M\times\mathbb{C})', (t^\circ,\lambda^\circ))$ on $\mathcal{A}$ isomorphic to the monodromy representation of $\nabla^{(m)}$. Here $(M\times\mathbb{C})'$ denotes the complement of the discriminant in $M\times\mathbb{C}$ (see Section 2.2). The monodromy group, i.e., the image of the fundamental group of this representation will be called the *stable monodromy group* of M, or just the *monodromy group* of M.

Remark 1.3. Dubrovin has defined the monodromy group of a Frobenius manifold to be the monodromy group of the Gauss–Manin system (see [11, Definition 5.6]), i.e., the monodromy group of the second structure connection with $m=-1$. In this paper, by monodromy of a Frobenius manifold we mean the monodromy group of the connection $\nabla^{(m)}$, where m is a sufficiently negative integer, such that, $m+\frac{1}{2}$ is not an eigenvalue of the grading operator θ. Our definition of a monodromy group will coincide with Dubrovin's one if the eigenvalues of the grading operator θ do not take values in $\frac{1}{2}+\mathbb{Z}$. Since passing to more negative values of m corresponds to stabilization in singularity theory, if one needs to make a clear distinction between our definition and Dubrovin's one, we suggest to refer to our monodromy group as the *stable* monodromy group.

The space of period vectors $\mathcal{A}$ has the following remarkable pairing, called *intersection pairing*,

$$(\alpha|\beta) := \big(I^{(0)}_\alpha(t,\lambda), (\lambda - E\bullet) I^{(0)}_\beta(t,\lambda)\big),$$

where E is the Euler vector field (see Sections 2.1 and 2.2 for more details). Using the differential equations (2.1)–(2.2), it is easy to check that the right-hand side of the above formula is independent of t and λ. In particular, the intersection pairing is monodromy invariant. The local structure of the period vectors near a generic point on the discriminant is described by the following simple lemma.

Lemma 1.4. *Suppose that C is a reference path from (t°,λ°) to a point $(t,\lambda)\in(M\times\mathbb{C})'$ sufficiently close to a generic point (t',u') on the discriminant. Then*

(a) *The subspace of $\beta\in\mathcal{A}$, such that, $I^{(m)}_\beta(t,\lambda)$ extends analytically in a neighborhood of (t',u') is a co-dimension 1 subspace of $\mathcal{A}$.*

(b) *Up to a sign, there exists a unique period vector $\alpha\in\mathcal{A}$, such that, the analytic continuation along a small loop around (t',u') transforms $I^{(m)}_\alpha(t,\lambda)\mapsto -I^{(m)}_\alpha(t,\lambda)$ $(\forall m)$ and $(\alpha|\alpha)=2$.*

In the case $m=0$, this is exactly Lemma 5.3 in [11]. Dubrovin's proof works in general too after some minor modifications. Moreover, the solutions to $\nabla^{(0)}$ constructed by Dubrovin in [11, Lemma 5.3], are in fact periods, i.e., we can include them in a sequence satisfying the conditions of Definition 1.2 — see Section 2.3 where this sequence is constructed in terms of Givental's R-matrix.

Definition 1.5. A period vector $\alpha\in\mathcal{A}$ is said to be a *reflection vector* if there exists a reference path C approaching a generic point on the discriminant, such that, $\pm\alpha$ is the unique vector from part (b) in Lemma 1.4.

It is an easy corollary of Lemma 1.4 that if C is a simple loop in $(M\times\mathbb{C})'$ around a generic point on the discriminant, then the analytic continuation along C defines a linear transformation

$$w_\alpha\colon\ \mathcal{A}\to\mathcal{A}, \qquad x\mapsto x-(x|\alpha)\alpha,$$

where α is the reflection vector corresponding to C, i.e., this is an orthogonal reflection in $\mathcal{A}$ with respect to the hyperplane $\alpha^\perp := \{y \in \mathcal{A} \mid (\alpha|y) = 0\}$. This is our main motivation to call α a reflection vector. Note that if $\pi_1(M, t^\circ) = 1$, then the monodromy group $\pi_1((M \times \mathbb{C})', (t^\circ, \lambda^\circ))$ is generated by simple loops around the discriminant. Therefore, in this case, the stable monodromy group of M is a reflection group.

1.3 Calibrated Frobenius manifolds

We would like to construct an isomorphism $\mathcal{A} \cong H$ where $H = T_{t^\circ} M$ is the tangent space at the reference point, or equivalently the space of flat vector fields. This isomorphism depends on the choice of a fundamental solution $S(t, z) z^\theta z^{-\rho}$ of Dubrovin's connection near $z = \infty$, such that, the operator series satisfies the symplectic condition $S(t, z) S(t, -z)^T = 1$ (see Section 2.2 for more details). In the case of quantum cohomology of a manifold X, there is a canonical choice of such solution: ρ is the operator of classical cup product multiplication by $c_1(TX)$ and $S(t, z)$ is defined in terms of genus-0, 1-point descendent Gromov–Witten invariants of X (see (2.11)). In general, there is an ambiguity in the choice of such a solution (see [11], Lemma 2.7). Following Givental [20], we call a Frobenius manifold equipped with the choice of a fundamental solution *calibrated*. In this case, the operator series S is called *calibration*. Let us point out that in all examples of Frobenius manifolds coming from geometry, we have the following commutation relation: $[\theta, \rho] = -\rho$. Following Givental again, we will say that θ is a *Hodge grading operator* if there exists a calibration for which $[\theta, \rho] = -\rho$.

Suppose now that M is a calibrated Frobenius manifold for which θ is a Hodge grading operator. Then we can construct an $\mathrm{End}(H)$-valued solution $I^{(m)}(t, \lambda)$ of the second structure connection $\nabla^{(m)}$, such that, $\partial_\lambda I^{(m)}(t, \lambda) = I^{(m+1)}(t, \lambda)$ — see formula (2.3). Moreover, if m is sufficiently negative, then the operator $I^{(m)}(t, \lambda)$ is invertible. Therefore, the map

$$H \to \mathcal{A}, \qquad a \mapsto I^{(m)}(t, \lambda) a, \qquad m \in \mathbb{Z} \tag{1.1}$$

is an isomorphism of vector spaces. Under this isomorphism, the intersection pairing on $\mathcal{A}$ can be expressed in terms of the operators θ and ρ — see formulas (2.6) and (2.7).

1.4 Quantum cohomology

Suppose that X is a smooth projective variety with semisimple quantum cohomology. There is no geometric interpretation of the reflection vectors in this case unless the manifold admits a mirror in the sense of Givental. Nevertheless, there is a remarkable conjectural description of the set of reflection vectors partially motivated by the examples from mirror symmetry. Let us give a precise statement. Since the quantum cohomology of X is semisimple, the Dolbeault cohomology groups $H^{p,q}(X) = 0$ for $p \neq q$ (see [24]). In particular, there exists a set of ample line bundles $L_1, \dots, L_r$, such that, the first Chern classes $p_i := c_1(L_i)$ $(1 \le i \le r)$ form a $\mathbb{Z}$-basis of $H^2(X, \mathbb{Z})_{\mathrm{t.f.}}$ (the torsion free part). Let $q_1, \dots, q_r$ be formal variables. Following Iritani, we introduce the following map (see [25]):

$$\Psi_q \colon\ K^0(X) \to H^*(X, \mathbb{C}), \tag{1.2}$$

defined by

$$\Psi_q(E) := (2\pi)^{\frac{1-n}{2}} \widehat{\Gamma}(X) \cup e^{-\sum_{i=1}^r p_i \log q_i} \cup (2\pi \mathbf{i})^{\deg}(\mathrm{ch}(E)),$$

where deg is the complex degree operator, that is, $\deg(\phi) = i\phi$ for $\phi \in H^{2i}(X; \mathbb{C})$, $\mathbf{i} := \sqrt{-1}$, $n = \dim_{\mathbb{C}}(X)$, and $\widehat{\Gamma}(X) = \widehat{\Gamma}(TX)$ is the Γ-class of X. Recall that for a vector bundle E, the Γ-class of E is defined by

$$\widehat{\Gamma}(E) := \prod_{x\colon \text{Chern roots of } E} \Gamma(1 + x).$$

The map Ψ_q is multivalued with respect to q. If q_i is sufficiently close to 1 for all $1 \le i \le r$, then we define $\log q_i$ via the principal branch of the logarithm. In general, one has to fix a reference path in $(\mathbb{C}^*)^r$ between $q = (q_1, \dots, q_r)$ and $(1, \dots, 1)$ and define Ψ_q via analytic continuation along the reference path. Let us introduce the following pairing

$$\langle\ ,\ \rangle\colon\ H^*(X,\mathbb{C}) \otimes H^*(X,\mathbb{C}) \to \mathbb{C}, \qquad \langle a, b\rangle := \frac{1}{2\pi}\int_X a \cup \mathrm{e}^{\pi \mathbf{i}\theta} \circ \mathrm{e}^{\pi\mathbf{i}\rho}(b),$$

where the linear operators θ and ρ are defined respectively by

$$\theta\colon\ H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C}), \qquad \theta(\phi) := \frac{n\phi}{2} - \deg(\phi),$$

and

$$\rho\colon\ H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C}), \qquad \rho(\phi) := c_1(TX) \cup \phi.$$

By using the Hierzebruch–Riemann–Roch formula, we get

$$\langle \Psi_q(E), \Psi_q(F)\rangle = \chi(E^\vee \otimes F),$$

where χ is the holomorphic Euler characteristic, that is, $\chi(E) = \sum_{i=0}^\infty (-1)^i \dim H^i(X, E)$. We will refer to $\langle\ ,\ \rangle$ as the *Euler pairing*. In case the manifold X admits a mirror model in the sense of Givental, the Euler pairing $\langle\ ,\ \rangle$ can be identified with the Seifert form and therefore its symmetrization

$$(a|b) := \langle a, b\rangle + \langle b, a\rangle, \qquad a, b \in H^*(X,\mathbb{C})$$

corresponds to the intersection pairing. For that reason we refer to the symmetrization $(\ |\ )$ of the Euler pairing as the *intersection pairing*.

Let us denote by $D^b(X)$ the derived category of the category of bounded complexes of coherent sheaves on X, that is, the bounded derived category of X (see [18] for some background on derived categories). For $\mathcal{E}, \mathcal{F} \in D^b(X)$ we denote by $\mathcal{E}[i]$ the shifted complex: $(\mathcal{E}[i])^k := \mathcal{E}^{k+i}$ and $\mathrm{Ext}^k(\mathcal{E},\mathcal{F}) := \mathrm{Hom}(\mathcal{E}, \mathcal{F}[k])$ where Hom is computed in the derived category $D^b(X)$. Recall that an object $\mathcal{E} \in D^b(X)$ is called *exceptional* if

$$\mathrm{Ext}^k(\mathcal{E},\mathcal{E}) = \begin{cases} \mathbb{C} & \text{if } k = 0, \\ 0 & \text{otherwise.}\end{cases}$$

A sequence of exceptional objects $(\mathcal{E}_1, \dots, \mathcal{E}_N)$ in $D^b(X)$ is called an *exceptional collection* if $\mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $i > j$ and $k \in \mathbb{Z}$. An exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_N)$ is called *full exceptional collection* if the smallest subcategory of $D^b(X)$ that contains $\mathcal{E}_i$ $(1 \le i \le N)$ and is closed under isomorphisms, shifts, and cones, is $D^b(X)$ itself.

Conjecture 1.6.

(*a*) *If the quantum cohomology of X is convergent and semisimple, then $\Psi_q(\mathcal{E})$ is a reflection vector for every exceptional object $\mathcal{E} \in D^b(X)$.*

(*b*) *If $(\mathcal{E}_1, \dots, \mathcal{E}_N)$ is a full exceptional collection in $D^b(X)$, then the reflection vectors $\alpha_i := \Psi_q(\mathcal{E}_i)$ $(1 \le i \le N)$ generate the set $\mathcal{R}$ of all reflection vectors in the following sense:*

  (*i*) *The reflections $x \mapsto x - (x|\alpha_i)\alpha_i$ $(1 \le i \le N)$ generate the monodromy group W of quantum cohomology.*

  (*ii*) *For every $\alpha \in \mathcal{R}$ there exists $w \in W$, such that, $w(\alpha) \in \{\alpha_1, \dots, \alpha_N\}$.*

The reflection vectors $\mathcal{R}$ in quantum cohomology are defined through the Frobenius manifold structure on the domain $M \subset H^*(X,\mathbb{C})$ of convergence of the big quantum cup product. Using the calibration (2.11) we turn M into a calibrated Frobenius manifold. The space of period vectors of M is identified with $H^*(X,\mathbb{C})$ via the isomorphism (1.1) which give us an embedding of the set of reflection vectors $\mathcal{R}$ in $H^*(X,\mathbb{C})$. More details will be given in Sections 2.2 and 2.5. Conjecture 1.6 follows easily from the work of Iritani (see [25]) in the case when X is a weak compact Fano toric orbifold that admits a full exceptional collection consisting only of line bundles. In general, since the second structure connection is a Laplace transform of Dubrovin's connection, Conjecture 1.6 should be equivalent to the so-called Dubrovin's conjecture (see [10, Conjecture 4.2.2]) or to its improved version proposed by Galkin–Golyshev–Iritani (see [16, Conjecture 4.6.1]). Dubrovin's conjecture was originally stated for Fano manifolds but shortly afterwards Arend Bayer suggested that the Fano condition should be removed (see [3]). Dubrovin already proved that the intersection pairing can be expressed in terms of the Stokes multipliers for the first structure connection (see [11, Lemma 5.4]). We expect that Dubrovin's argument already has the necessary ingredients to prove that Conjecture 1.6 is equivalent to Dubrovin's conjecture. We are planning to return to this problem in the near future.

Let us state the main result in our paper. Let $\mathrm{Bl}(X)$ be the blowup of X at one point, $\pi\colon \mathrm{Bl}(X) \to X$ be the blowup map, and $j\colon \mathbb{P}^{n-1} \to \mathrm{Bl}(X)$ be the closed embedding that identifies $\mathbb{P}^{n-1}$ with the exceptional divisor E.

Theorem 1.7. *If the quantum cohomology of X is convergent and semisimple and the quantum cohomology of $\mathrm{Bl}(X)$ is convergent, then $\Psi_q(\mathcal{O}_E(k))$, where $\mathcal{O}_E(k) := j_*\mathcal{O}_{\mathbb{P}^{n-1}}(k)$, $k \in \mathbb{Z}$, are reflection vectors for the quantum cohomology of $\mathrm{Bl}(X)$.*

Several remarks are in order. It is known by the results of Bayer (see [3]) that the blowup at a point preserves semi-simplicity of the quantum cohomology. We believe that our requirement that the quantum cohomology of $\mathrm{Bl}(X)$ is convergent is redundant, that is, the blowup operation preserves the convergence in quantum cohomology. Let us point out that recently Giordano Cotti (see [7, Theorem 6.6]) was able to prove the convergence of quantum cohomology under the assumption that the *small* quantum cohomology is semisimple and convergent. His result is not quite sufficient for our purposes but it might be interesting to apply his techniques to study convergence of blowups in general. We will return to this problem in the near future. Furthermore, we would like to prove that Conjecture 1.6 is compatible with the blowup operation. Let us recall that by the work of Orlov (see [31]) if $(\mathcal{E}_1,\dots,\mathcal{E}_N)$ is a full exceptional collection of X, then $(\mathcal{O}_E(-n+1),\dots,\mathcal{O}_E(-1),\pi^*\mathcal{E}_1,\dots,\pi^*\mathcal{E}_N)$ is a full exceptional collection of $\mathrm{Bl}(X)$. In order to complete the proof of Conjecture 1.6 for the blowups at finitely many points we still have to prove that $\pi^*\mathcal{E}_i$ are reflection vectors. The methods used in the current paper, after some modification, should be sufficient to do this. Nevertheless, our attempts to modify the arguments were unsuccessful so far, so we left this problem for a separate project too.

The paper is organized as follows. Sections 2 and 3 contain the background which we need to formulate and state our main result, i.e., Theorem 1.7. In Section 4, we investigate the fundamental solution of the second structure connection of a blowup. The goal is to expand the solution in a Laurent series at $Q=0$, where Q is the Novikov variable corresponding to the exceptional divisor, and to compute explicitly the coefficients of the leading order terms. In Sections 5 and 6, we compute the monodromy of the leading order coefficients in the Q-expansion. The monodromy of the leading order coefficients allows us to determine the reflection vectors corresponding to certain class of simple loops which yields our main result. The logic in our proof is the following. Let us look at the fundamental solution $I^{(m)}(t,\lambda)$ of the second structure connection defined in terms of the calibration — see Section 2.2. This fundamental solution is a Laurent series whose coefficients are genus 0, 1-point descendent GW invariants. Since the line bundle corresponding to the exceptional divisor E is not ample, the GW invariants are in general

Laurent series in Q. Our first observation is that if we rescale appropriately the fundamental solution $I^{(m)}(t,\lambda)$, then we will obtain a power series in Q. Moreover, we can extract the leading order terms of the Taylor series expansion at $Q=0$ up to order Q^n where $n=\dim(X)$. This is done in Section 4 (see Propositions 4.3, 4.4 and 4.5) by using a generalization of Gathman's vanishing result. The latter is proved in Section 3, Proposition 3.4. The next step is to analyze the singularities of the second structure connection, i.e., the dependence of the canonical coordinates u_j on Q. Again using Gathman's vanishing result we prove (see Proposition 5.2) that the canonical coordinates split into two groups such that Qu_j is either sufficiently close to 0 (there are $N=\dim H^*(X)$ such coordinates) or sufficiently close to $-(n-1)v_k$ $(1\leq k\leq n-1)$. Suppose now that we have a simple loop γ_k around $-(n-1)v_k$ that contains the corresponding canonical coordinate and let $\alpha =: Q^{-(n-1)e}\beta$ be the corresponding reflection vector, where the dependence of α on Q follows from the divisor equation (see Section 5.1). Let us decompose $\beta=\beta_b+\beta_e$ where $\beta_b\in H^*(X)$ and $\beta_e\in \widetilde{H}^*(E)$ where E is the exceptional divisor. In Section 5, by analyzing the monodromy of the leading order terms in the expansion at $Q=0$ we prove that $\beta_b=0$. There is a slight complication in proving the vanishing of the top degree part of β_b because one of the coefficients in the Q-expansion (see Proposition 4.3, the term involving $Q^n\phi_N$) is an infinite series so its monodromy is not straightforward to compute. In Section 6, we prove that this problematic coefficient is a Mellin–Barnes integral and we compute its monodromy by standard techniques based on deforming the contour. Finally, in order to compute β_e, we look again at the leading order term of the Q-expansion and we see that the corresponding coefficient is a fundamental solution for the second structure connection in quantum cohomology of $\mathbb{P}^{n-2}$ (see Sections 5.3 and 5.4). We get that β_e must be a reflection vector in the quantum cohomology of $\mathbb{P}^{n-2}$ but the latter were computed in our previous work [28] (see also [25]).

2 Frobenius manifolds

Following Dubrovin [9], we recall the notion of a Frobenius manifold. Then we proceed by defining the so-called *second structure connection* and *reflection vectors* of a semisimple Frobenius manifold. Finally, we would like to recall the construction of a Frobenius manifold in the settings of Gromov–Witten theory.

2.1 First and second structure connections

Suppose that M is a complex manifold and $\mathcal{T}_M$ is the sheaf of holomorphic vector fields on M. The manifold M is equipped with the following structures:

(F1) A non-degenerate symmetric bilinear pairing

$$(\cdot,\cdot)\colon\ \mathcal{T}_M\otimes\mathcal{T}_M\to\mathcal{O}_M.$$

(F2) A Frobenius multiplication: commutative associative multiplication

$$\cdot\bullet\cdot\colon\ \mathcal{T}_M\otimes\mathcal{T}_M\to\mathcal{T}_M,$$

such that $(v_1\bullet w,v_2)=(v_1,w\bullet v_2)\ \forall v_1,v_2,w\in\mathcal{T}_M$.

(F3) A unit vector field: global vector field $\mathbf{1}\in\mathcal{T}_M(M)$, such that,

$$\mathbf{1}\bullet v=v,\qquad \nabla^{\mathrm{L.C.}}_v\mathbf{1}=0,\qquad \forall v\in\mathcal{T}_M,$$

where $\nabla^{\mathrm{L.C.}}$ is the Levi-Civita connection of the pairing $(\cdot,\cdot)$.

(F4) An Euler vector field: global vector field $E \in \mathcal{T}_M(M)$, such that, there exists a constant $n \in \mathbb{C}$, called *conformal dimension*, and

$$E(v_1, v_2) - ([E, v_1], v_2) - (v_1, [E, v_2]) = (2 - n)(v_1, v_2)$$

for all $v_1, v_2 \in \mathcal{T}_M$.

Note that the complex manifold $TM \times \mathbb{C}^*$ has a structure of a holomorphic vector bundle with base $M \times \mathbb{C}^*$: the fiber over $(t, z) \in M \times \mathbb{C}^*$ is $T_tM \times \{z\} \cong T_tM$ which has a natural structure of a vector space. Given the data (F1)–(F4), we define the so called *Dubrovin's connection* on the vector bundle $TM \times \mathbb{C}^*$

$$\nabla_v := \nabla_v^{\mathrm{L.C.}} - z^{-1} v\bullet, \qquad v \in \mathcal{T}_M, \qquad \nabla_{\partial/\partial z} := \frac{\partial}{\partial z} - z^{-1}\theta + z^{-2} E\bullet,$$

where z is the standard coordinate on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, where $v\bullet$ is an endomorphism of $\mathcal{T}_M$ defined by the Frobenius multiplication by the vector field v, and where $\theta\colon \mathcal{T}_M \to \mathcal{T}_M$ is an $\mathcal{O}_M$-modules morphism defined by

$$\theta(v) := \nabla_v^{\mathrm{L.C.}}(E) - \left(1 - \frac{n}{2}\right) v.$$

Definition 2.1. The data $((\cdot,\cdot), \bullet, \mathbf{1}, E)$, satisfying the properties $(F1)$–$(F4)$, is said to be a *Frobenius structure* of conformal dimension n if the corresponding Dubrovin connection is flat, that is, if $(t_1, \dots, t_N)$ are holomorphic local coordinates on M, then the set of $N+1$ differential operators $\nabla_{\partial/\partial t_i}$ $(1 \le i \le N)$, $\nabla_{\partial/\partial z}$ pairwise commute.

Let us proceed with recalling the notion of second structure connection and reflection vectors. We follow the exposition from [28]. We are going to work only with Frobenius manifolds satisfying the following 4 additional conditions:

(i) The tangent bundle TM is trivial and it admits a trivialization given by a frame of global flat vector fields.

(ii) Recall that the operator

$$\mathrm{ad}_E\colon\ \mathcal{T}_M \to \mathcal{T}_M, \qquad v \mapsto [E, v]$$

preserves the space of flat vector fields. We require that the restriction of ad_E to the space of flat vector fields is a diagonalizable operator with eigenvalues rational numbers ≤ 1.

(iii) The Frobenius manifold has a *calibration* for which the grading operator is a Hodge grading operator (see Sections 1.3 and 2.2).

(iv) The Frobenius manifold has a direct product decomposition $M = \mathbb{C} \times B$ such that if we denote by $t_1\colon M \to \mathbb{C}$ the projection along B, then $\mathrm{d}t_1$ is a flat 1-form and $\langle \mathrm{d}t_1, \mathbf{1}\rangle = 1$.

Conditions (i)–(iv) are satisfied for all Frobenius manifolds constructed by quantum cohomology or by the primitive forms in singularity theory.

Let us fix a base point $t^\circ \in M$ and a basis $\{\phi_i\}_{i=1}^N$ of the reference tangent space $H := T_{t^\circ}M$. Furthermore, let $(t_1, \dots, t_N)$ be a local flat coordinate system on an open neighborhood of t° such that $\partial/\partial t_i|_{t^\circ} = \phi_i$ in H. The flat vector fields $\partial/\partial t_i$ $(1 \le i \le N)$ extend to global flat vector fields on M and provide a trivialization of the tangent bundle $TM \cong M \times H$. This allows us to identify the Frobenius multiplication $\bullet$ with a family of associative commutative multiplications $\bullet_t\colon H \otimes H \to H$ depending analytically on $t \in M$. Modifying our choice of $\{\phi_i\}_{i=1}^N$ and $\{t_i\}_{i=1}^N$ if necessary we may arrange that

$$E = \sum_{i=1}^N ((1 - d_i)t_i + r_i)\partial/\partial t_i,$$

where $\partial/\partial t_1$ coincides with the unit vector field $\mathbf{1}$ and the numbers

$$0 = d_1 \le d_2 \le \cdots \le d_N = n$$

are symmetric with respect to the middle of the interval $[0, n]$. The number n is known as the *conformal dimension* of M. The operator $\theta\colon \mathcal{T}_M \to \mathcal{T}_M$ defined above preserves the subspace of flat vector fields. It induces a linear operator on H, known to be skew symmetric with respect to the Frobenius pairing $(\ ,\ )$. Following Givental, we refer to θ as the *Hodge grading operator*.

There are two flat connections that one can associate with the Frobenius structure. The first one is the *Dubrovin connection* — defined above. The Dubrovin connection in flat coordinates takes the following form:

$$\nabla_{\partial/\partial t_i} = \frac{\partial}{\partial t_i} - z^{-1}\phi_i \bullet, \qquad \nabla_{\partial/\partial z} = \frac{\partial}{\partial z} + z^{-1}\theta - z^{-2}E\bullet,$$

where z is the standard coordinate on $\mathbb{C}^* = \mathbb{C} - \{0\}$ and for $v \in \Gamma(M, \mathcal{T}_M)$ we denote by $v\bullet\colon H \to H$ the linear operator of Frobenius multiplication by v.

Our main interest is in the *second structure connection*

$$\nabla^{(m)}_{\partial/\partial t_i} = \frac{\partial}{\partial t_i} + (\lambda - E\bullet_t)^{-1}(\phi_i \bullet_t)(\theta - m - 1/2), \tag{2.1}$$

$$\nabla^{(m)}_{\partial/\partial \lambda} = \frac{\partial}{\partial \lambda} - (\lambda - E\bullet_t)^{-1}(\theta - m - 1/2), \tag{2.2}$$

where $m \in \mathbb{C}$ is a complex parameter. This is a connection on the trivial bundle

$$(M \times \mathbb{C})' \times H \to (M \times \mathbb{C})',$$

where

$$(M \times \mathbb{C})' = \{(t, \lambda) \mid \det(\lambda - E\bullet_t) \neq 0\}.$$

The hypersurface $\det(\lambda - E\bullet_t) = 0$ in $M \times \mathbb{C}$ is called the *discriminant*.

2.2 Reflection vectors for calibrated Frobenius manifolds

We would like to construct a fundamental solution to the second structure connection $\nabla^{(m)}$ for m sufficiently negative. As we already explained in Section 1.3, this would allow us to embed the reflection vectors (see Definition 1.5) of the Frobenius manifold M in H.

Suppose that M is a calibrated Frobenius manifold with calibration $S(t, z)$ for which the grading operator is a Hodge grading operator. By definition (see [20]), the calibration is an operator series $S(t, z) = 1 + \sum_{k=1}^{\infty} S_k(t) z^{-k}$, $S_k(t) \in \operatorname{End}(H)$ depending holomorphically on t and z for t sufficiently close to the base point t° and $z \in \mathbb{C}^*$, such that, the Dubrovin's connection has a fundamental solution near $z = \infty$ of the form

$$S(t, z) z^{\theta} z^{-\rho},$$

where $\rho \in \operatorname{End}(H)$ is a nilpotent operator and the following symplectic condition holds

$$S(t, z) S(t, -z)^T = 1,$$

where ^{T} denotes transposition with respect to the Frobenius pairing. We say that θ is a Hodge grading operator if $[\theta, \rho] = -\rho$.

Let us fix a reference point $(t^\circ, \lambda^\circ) \in (M \times \mathbb{C})'$ such that λ° is a sufficiently large positive real number. It is easy to check that the following function is a solution to the second structure connection $\nabla^{(m)}$

$$I^{(m)}(t, \lambda) = \sum_{k=0}^{\infty} (-1)^k S_k(t) \widetilde{I}^{(m+k)}(\lambda), \tag{2.3}$$

where

$$\widetilde{I}^{(m)}(\lambda) = \mathrm{e}^{-\rho \partial_\lambda \partial_m} \left(\frac{\lambda^{\theta - m - \frac{1}{2}}}{\Gamma\big(\theta - m + \frac{1}{2}\big)} \right).$$

Note that both $I^{(m)}(t, \lambda)$ and $\widetilde{I}^{(m)}(\lambda)$ take values in $\mathrm{End}(H)$. From now on we restrict $m \in \mathbb{Z}$. The second structure connection has a Fuchsian singularity at infinity, therefore the series $I^{(m)}(t, \lambda)$ is convergent for all (t, λ) sufficiently close to (t°, λ°). Using the differential equations (2.1)–(2.2), we extend $I^{(m)}$ to a multi-valued analytic function on $(M \times \mathbb{C})'$ taking values in $\mathrm{End}(H)$. We define the following multi-valued functions taking values in H:

$$I_a^{(m)}(t, \lambda) := I^{(m)}(t, \lambda) a, \qquad a \in H, \quad m \in \mathbb{Z}. \tag{2.4}$$

Clearly, for each fixed $a \in H$, the sequence $I_a^{(m)}(t, \lambda)$ $(m \in \mathbb{Z})$ is a period vector in the sense of Definition 1.2. Moreover, if $m \in \mathbb{Z}$ is sufficiently negative, then $I^{(m)}(t, \lambda)$ is an invertible operator. Therefore, all period vectors of M have the form (2.4). Using analytic continuation we get a representation

$$\pi_1((M \times \mathbb{C})', (t^\circ, \lambda^\circ)) \to \mathrm{GL}(H)$$

called the *monodromy representation* of the Frobenius manifold. The image W of the monodromy representation is called the *monodromy group* or *stable monodromy group* (see Remark 1.3).

Under the semi-simplicity assumption, we may choose a generic reference point t° on M, such that the Frobenius multiplication $\bullet_{t^\circ}$ is semisimple and the operator $E \bullet_{t^\circ}$ has N pairwise different eigenvalues u_i° $(1 \le i \le N)$. The fundamental group $\pi_1((M \times \mathbb{C})', (t^\circ, \lambda^\circ))$ fits into the following exact sequence

$$\pi_1(F^\circ, \lambda^\circ) \xrightarrow{i_*} \pi_1((M \times \mathbb{C})', (t^\circ, \lambda^\circ)) \xrightarrow{p_*} \pi_1(M, t^\circ) \longrightarrow 1, \tag{2.5}$$

where $p\colon (M \times \mathbb{C})' \to M$ is the projection on M, $F^\circ = p^{-1}(t^\circ) = \mathbb{C} \setminus \{u_1^\circ, \dots, u_N^\circ\}$ is the fiber over t°, and $i\colon F^\circ \to (M \times \mathbb{C})'$ is the natural inclusion. For a proof we refer to [35, Proposition 5.6.4] or [30, Lemma 1.5 (C)]. Using the exact sequence (2.5), we get that the monodromy group W is generated by the monodromy transformations representing the lifts of the generators of $\pi_1(M, t^\circ)$ in $\pi_1((M \times \mathbb{C})', (t^\circ, \lambda^\circ))$ and the generators of $\pi_1(F^\circ, \lambda^\circ)$.

The image of $\pi_1(F^\circ, \lambda^\circ)$ under the monodromy representation is a reflection group that can be described as follows. Let us introduce the bi-linear pairing

$$\langle a, b \rangle = \frac{1}{2\pi} \big(a, \mathrm{e}^{\pi \mathbf{i} \theta} \mathrm{e}^{\pi \mathbf{i} \rho} b\big), \qquad a, b \in H. \tag{2.6}$$

Motivated by the applications to mirror symmetry, we will refer to $\langle\ ,\ \rangle$ as the *Euler pairing*. Its symmetrization

$$(a|b) := \langle a, b \rangle + \langle b, a \rangle, \qquad a, b \in H, \tag{2.7}$$

also plays an important role in mirror symmetry and we will refer to it as the *intersection pairing*. It can be checked that the intersection pairing can be expressed in terms of the period vectors as follows:

$$(a|b) := \big(I^{(0)}_a(t,\lambda), (\lambda - E\bullet) I^{(0)}_b(t,\lambda)\big).$$

Using the differential equations of the second structure connection, it is easy to prove that the right-hand side of the above identity is independent of t and λ. However, the fact that the constant must be $(a|b)$ requires some additional work (see [29]).

Suppose now that γ is a simple loop in F°, i.e., a loop that starts at λ°, approaches one of the punctures u_i° along a path γ' that ends at a point sufficiently close to u_i°, goes around u_i°, and finally returns back to λ° along γ'. By analyzing the second structure connection near $\lambda = u_i$ it is easy to see that up to a sign there exists a unique $a \in H$ such that $(a|a) = 2$ and the monodromy transformation of a along γ is $-a$. The monodromy transformation representing $\gamma \in \pi_1(F^\circ, \lambda^\circ)$ is the reflection defined by the following formula:

$$w_a(x) = x - (a|x)a.$$

Let us denote by $\mathcal{R}$ the set of all $a \in H$ as above determined by all possible choices of simple loops in F°. Under the isomorphism (1.1), the set $\mathcal{R}$ coincides with the set of reflection vectors of M.

2.3 The anti-invariant solution

We would like to construct the unique solution to the second structure connection appearing in part (b) of Lemma 1.4. We refer to it as the anti-invariant solution because the analytic continuation around the discriminant changes its sign. Our construction is very similar to (2.3), except that now instead of the singularity of $\nabla^{(m)}$ at $\lambda = \infty$, we will consider the singularity at $\lambda = u_i(t)$ and instead of fundamental solution of Dubrovin's connection near $z = \infty$ we will make use of the formal asymptotic solution to Dubrovin's connection near $z = 0$.

Let us recall Givental's R-matrix (see [20])

$$R(t,z) = 1 + R_1(t)z + R_2(t)z^2 + \cdots, \qquad R_k(t) \in \mathrm{End}(H)$$

defined for all semisimple $t \in M$ as the unique solution to the following system of differential equations:

$$\frac{\partial R}{\partial t_a}(t,z) = -R(t,z)\frac{\partial \Psi}{\partial t_a}\Psi^{-1} + z^{-1}[\phi_a\bullet, R(t,z)],$$
$$\frac{\partial R}{\partial z}(t,z) = -z^{-1}\theta R(t,z) - z^{-2}[E\bullet, R(t,z)],$$

where $\phi_a\bullet$ and $E\bullet$ are the operators of Frobenius multiplication respectively by the flat vector field $\partial/\partial t_a$ and by the Euler vector field E and Ψ is the $(N\times N)$-matrix with entries

$$\Psi_{ai} := \sqrt{\Delta_i}\frac{\partial t_a}{\partial u_i}, \qquad 1 \le a, i \le N,$$

where $u_1, \dots, u_N$ are the canonical coordinates in a neighborhood of the base point t°, that is, a local coordinate system, such that,

$$\frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_j}, \qquad \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \frac{\delta_{ij}}{\Delta_i},$$

where δ_{ij} is the Kronecker delta symbol and $\Delta_i \in \mathcal{O}_{M,t^\circ}$ is a holomorphic function that has no zeroes in a neighborhood of t°. It is known that the canonical coordinates coincide with the eigenvalues of the operator $E\bullet_t$. Here $\mathrm{End}(H)$ is identified with the space of $(N\times N)$-matrices via the basis $\phi_1, \dots, \phi_N$, that is, the entries A_{ab} of $A \in \mathrm{End}(H)$ are defined by $A(\phi_b) =: \sum_a \phi_a A_{ab}$.

Remark 2.2. The matrix Ψ up to the normalization factors Δ_i is the Jacobian matrix of the change from canonical to flat coordinates. The above definition of the R-matrix differed from the original definition in [20] by conjugation by Ψ, that is, $\Psi^{-1}R(t,z)\Psi$ is the R-matrix of Givental.

Suppose that $\alpha \in H$ is a reflection vector. Let us fix a generic semisimple point $t \in M$, such that, the canonical coordinates $u_1(t), \dots, u_N(t)$ are pairwise distinct. Let us fix a reference path from (t°, λ°) to a neighborhood of a point on the discriminant $(t, u_i(t))$ for some i, such that, the period vector $I^{(-m)}_\alpha(t,\lambda)$ transforms into $-I^{(-m)}_\alpha(t,\lambda)$ under the analytic continuation in λ along a closed loop around $u_i(t)$. We claim that the period vector has the following expansion at $\lambda = u_i(t)$:

$$I^{(-m)}_\alpha(t,\lambda) = \sqrt{2\pi}\sum_{k=0}^{\infty}(-1)^k\frac{(\lambda-u_i)^{k+m-1/2}}{\Gamma(k+m+1/2)}R_k(t)\Psi(t)e_i, \tag{2.8}$$

where e_i is the vector column with 1 on the ith position and 0 elsewhere, that is, Ψe_i is the column representing the vector field $\sum_{a=1}^N \sqrt{\Delta_i}\frac{\partial t_a}{\partial u_i}\phi_a = \sqrt{\Delta_i}\partial/\partial u_i$. Let us prove this claim. Using the differential equations for $R(t,z)$, it is easy to check that the right-hand side of the above formula is a solution to the second structure connection. Therefore, the right-hand side of (2.8) and the reference path determine a vector $\alpha \in H$ for which formula (2.8) holds. Moreover,

$$\big(I^{(0)}_\alpha(t,\lambda), (\lambda - E\bullet)I^{(0)}_\alpha(t,\lambda)\big) = \frac{2\pi}{\Gamma(1/2)^2} + O(\lambda-u_i) = 2 + O(\lambda-u_i).$$

Since the left-hand side is independent of λ and u_i, the higher order terms $O(\lambda-u_i)$ in the above formula must vanish. This proves that $(\alpha|\alpha) = 2$. Finally, since the analytic continuation around $\lambda = u_i$ of the right-hand side of (2.8) changes the sign of the right-hand side, we conclude that α must be a reflection vector and that (2.8) is the expansion of the corresponding period vector near the discriminant.

2.4 Gromov–Witten theory

Let us recall some basics on Gromov–Witten (GW) theory. For further details we refer to [26]. Let $\mathrm{Eff}(X) \subset H_2(X,\mathbb{Z})_{\mathrm{t.f.}}$ be the monoid of all homology classes that can be represented in the form $\sum_i k_i[C_i]$, where k_i is a non-negative integer and $[C_i]$ is the fundamental class of a holomorphic curve $C_i \subset X$. The main object in GW theory is the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X,\beta)$, where g, k are non-negative integers and $\beta \in \mathrm{Eff}(X)$. By definition, a stable map consists of the following data $(\Sigma, z_1, \dots, z_k, f)$:

(1) Σ is a Riemann surface with at most nodal singular points.

(2) $z_1, \dots, z_k$ are *marked points*, that is, smooth pairwise-distinct points on Σ.

(3) $f\colon \Sigma \to X$ is a holomorphic map, such that, $f_*[\Sigma] = \beta$.

(4) The map is stable, i.e., the automorphism group of $(\Sigma, z_1, \dots, z_k, f)$ is finite.

Two stable maps $(\Sigma, z_1, \dots, z_k, f)$ and $(\Sigma', z_1', \dots, z_k', f')$ are called equivalent if there exists a biholomorphism $\phi\colon \Sigma \to \Sigma'$, such that, $\phi(z_i) = z_i'$ and $f' \circ \phi = f$. The moduli space of equivalence classes of stable maps is known to be a proper Delign–Mumford stack with respect to the étale topology on the category of schemes (see [6]). The corresponding coarse moduli space $\overline{M}_{g,k}(X,\beta)$ has a structure of a projective variety, which however could be very singular.

We have the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,k+1}(X,\beta) \arrow[r, "\mathrm{ev}_{k+1}"] \arrow[d, "\pi"'] & X \\
\overline{\mathcal{M}}_{g,k}(X,\beta) \arrow[r, "\mathrm{ev}_i"] \arrow[d, "\mathrm{ft}"'] & X \qquad 1 \leq i \leq k, \\
\overline{\mathcal{M}}_{g,k},
\end{tikzcd}$$

where $\mathrm{ev}_i(\Sigma, z_1, \ldots, z_k, f) := f(z_i)$, π is the map forgetting the last marked point an contracting all unstable components, and ft is the map forgetting the holomorphic map f and contracting all unstable components. The moduli space has natural orbifold line bundles L_i $(1 \leq i \leq k)$ whose fiber at a point $(\Sigma, z_1, \ldots, z_k, f)$ is the cotangent line $T^*_{z_i}\Sigma$ equipped with the action of the automorphism group of $(\Sigma, z_1, \ldots, z_k, f)$. Let $\psi_i = c_1(L_i)$ be the first Chern class. The most involved construction in GW theory is the construction of the so called *virtual fundamental cycle*. The construction has as an input the complex $(R\pi_*\mathrm{ev}^*_{k+1}TX)^\vee$ which gives rise to a perfect obstruction theory on $\overline{\mathcal{M}}_{g,k}(X,\beta)$ relative to $\overline{\mathcal{M}}_{g,k}$ (see [4, 5]) and yields a homology cycle in $\overline{M}_{g,k}(X,\beta)$ of complex dimension

$$3g - 3 + k + n(1-g) + \langle c_1(TX), \beta \rangle,$$

known as the virtual fundamental cycle. Gromov–Witten invariants are by definition the following correlators:

$$\left\langle a_1\psi^{l_1}, \ldots, a_k\psi^{l_k} \right\rangle_{g,k,\beta} = \int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{virt}}} \mathrm{ev}_1^*(a_1)\cdots \mathrm{ev}_k^*(a_k)\psi_1^{l_1}\cdots\psi_k^{l_k},$$

where $a_1, \ldots, a_k \in H^*(X;\mathbb{C})$ and $l_1, \ldots, l_k$ are non-negative integers.

Let us recall the so-called *string* and *divisor* equations. Suppose that either $\beta \neq 0$ or $2g - 2 + k > 0$, then

$$\left\langle 1, a_1\psi^{l_1}, \ldots, a_k\psi^{l_k} \right\rangle_{g,k+1,\beta} = \sum_{i=1}^{k} \left\langle a_1\psi^{l_1}, \ldots, a_i\psi^{l_i - 1}, \ldots, a_k\psi^{l_k} \right\rangle_{g,k,\beta},$$

and if $p \in H^2(X,\mathbb{C})$ is a divisor class, then

$$\begin{aligned}\left\langle p, a_1\psi^{l_1}, \ldots, a_k\psi^{l_k} \right\rangle_{g,k+1,\beta} = & \left(\int_\beta p\right) \left\langle a_1\psi^{l_1}, \ldots, a_k\psi^{l_k} \right\rangle_{g,k,\beta} \\ & + \sum_{i=1}^{k} \left\langle a_1\psi^{l_1}, \ldots, p \cup a_i\psi^{l_i-1}, \ldots, a_k\psi^{l_k} \right\rangle_{g,k,\beta},\end{aligned}$$

where if $l_i = 0$, then we define $\psi_i^{l_i - 1} := 0$. We will need also the *genus*-0 *topological recursion relations*, that is, if $k \geq 2$, then the following relation holds:

$$\begin{aligned}&\left\langle a\psi^{l+1}, b_1\psi^{m_1}, \ldots, b_k\psi^{m_k} \right\rangle_{0,k+1,\beta} \\ &\quad = \sum_{i,I,\beta'} \left\langle a\psi^l, \phi_i, b_{i_1}\psi^{m_{i_1}}, \ldots, b_{i_r}\psi^{m_{i_r}} \right\rangle_{0,2+r,\beta'} \left\langle \phi^i, b_{j_1}\psi^{m_{j_1}}, \ldots, b_{j_s}\psi^{m_{j_s}} \right\rangle_{0,1+s,\beta''},\end{aligned}$$

where the sum is over all $1 \leq i \leq N$, all subsequences $I = (i_1, \ldots, i_r)$ of the sequence $(1, 2, \ldots, k)$ including the empty one, and all homology classes $\beta' \in \mathrm{Eff}(X)$, such that, $\beta'' := \beta - \beta' \in \mathrm{Eff}(X)$. The sequence $(j_1, \ldots, j_s)$ is obtained from $(1, 2, \ldots, k)$ by removing the subsequence I. In particular, $r + s = k$.

2.5 Quantum cohomology of X

Let us recall the notation L_i, $p_i := c_1(L_i)$, and q_i $(1 \leq i \leq r)$ from the introduction. If $\beta \in \mathrm{Eff}(X)$, then we put $q^\beta = q_1^{\langle p_1,\beta\rangle} \cdots q_r^{\langle p_r,\beta\rangle}$. The group ring $\mathbb{C}[\mathrm{Eff}(X)]$ is called the *Novikov ring* of X and the variables q_i are called *Novikov variables*. Note that the Novikov variables determine an embedding of the Novikov ring into the ring of formal power series $\mathbb{C}[\![q_1,\dots,q_r]\!]$. Let us fix a homogeneous basis ϕ_i $(1 \leq i \leq N)$ of $H^*(X;\mathbb{C})$, such that, $\phi_1 = 1$ and $\phi_{i+1} = p_i$ for all $1 \leq i \leq r$. Let $t = (t_1,\dots,t_N)$ be the corresponding linear coordinates. The quantum cup product $\bullet_{t,q}$ of X is a deformation of the classical cup product defined by

$$(\phi_a \bullet_{t,q} \phi_b, \phi_c) := \langle \phi_a, \phi_b, \phi_c\rangle_{0,3}(t) = \sum_{m=0}^{\infty} \sum_{\beta\in\mathrm{Eff}(X)} \frac{q^\beta}{m!} \langle \phi_a, \phi_b, \phi_c, t, \dots, t\rangle_{0,3+m,\beta}.$$

Using string and divisor equation, we get that the structure constants of the quantum cup product, i.e., the 3-point genus-0 correlators in the above formula are independent of t_1 and are formal power series in the following variables:

$$q_1 e^{t_2}, \dots, q_r e^{t_r}, t_{r+1}, \dots, t_N.$$

We are going to consider only manifolds X, such that, the quantum cup product is analytic. More precisely, let us allow for the Novikov variables to take values $0 < |q_i| < 1$ $(1 \leq i \leq r)$. Then we will assume that there exists an $\epsilon > 0$, such that, the structure constants of the quantum cup product are convergent power series for all t satisfying

$$\mathrm{Re}(t_i) < \log\epsilon, \quad 2 \leq i \leq r+1, \qquad |t_j| < \epsilon, \quad r+1 < j \leq N. \tag{2.9}$$

The inequalities (2.9) define an open subset $M \subset H^*(X;\mathbb{C})$. The main fact about genus-0 GW invariants is that M has a Frobenius structure, such that, the Frobenius pairing is the Poincaré pairing, the Frobenius multiplication is the quantum cup product, the unit $\mathbf{1} = \phi_1$, and the Euler vector field is

$$E = \sum_{i=1}^{N} (1-d_i) t_i \frac{\partial}{\partial t_i} + \sum_{j=2}^{r+1} \big(c_1(TX), \phi^j\big) \frac{\partial}{\partial t_j},$$

where d_i is the complex degree of ϕ_i, that is, $\phi_i \in H^{2d_i}(X;\mathbb{C})$ and ϕ^j $(1 \leq j \leq N)$ is the basis of $H^*(X;\mathbb{C})$ dual to ϕ_i $(1 \leq i \leq N)$ with respect to the Poincaré pairing. Let us point out that in case the quantum cup product is semisimple we have $H^{\mathrm{odd}}(X;\mathbb{C}) = 0$. Otherwise, in general M has to be given the structure of a super-manifold (see [26]). The conformal dimension of M is $n = \dim_{\mathbb{C}}(X)$ and the Hodge grading operator takes the form

$$\theta(\phi_i) = \left(\frac{n}{2} - d_i\right) \phi_i, \qquad 1 \leq i \leq N. \tag{2.10}$$

Finally, there is a standard choice for a calibration $S(t,q,z) = 1 + \sum_{k=1}^{\infty} S_k(t,q) z^{-k}$, where $S_k(t,q) \in \mathrm{End}(H^*(X;\mathbb{C}))$ is defined by

$$(S_k(t,q)\phi_i, \phi_j) = \sum_{m=0}^{\infty} \sum_{\beta\in\mathrm{Eff}(X)} \frac{q^\beta}{m!} \big\langle \phi_i \psi^{k-1}, \phi_j, t, \dots, t\big\rangle_{0,2+m,\beta}. \tag{2.11}$$

Suppose that the Frobenius manifold M corresponding to quantum cohomology is semisimple. Recalling the construction from Section 2.2, we get the notion of a reflection vector.

3 The geometry of blowups

Let $\mathrm{Bl}(X)$ be the blowup of X at a point $\mathrm{pt} \in X$, $\pi\colon \mathrm{Bl}(X) \to X$ be the corresponding blowup map, and $E := \pi^{-1}(\mathrm{pt})$ the exceptional divisor. Put $e = c_1(\mathcal{O}(E)) = \mathrm{P.D.}(E)$. We would like to recall some well known facts about $\mathrm{Bl}(X)$ which will be used later on.

3.1 Cohomology of the blowup

Using a Mayer–Vietories sequence argument, it is easy to prove the following two facts:

(1) The pullback map $\pi^*\colon\ H^*(X;\mathbb{C}) \longrightarrow H^*(\mathrm{Bl}(X);\mathbb{C})$ is injective, so we can view the cohomology $H^*(X;\mathbb{C})$ as a subvector space of $H^*(\mathrm{Bl}(X);\mathbb{C})$.

(2) We have a direct sum decomposition

$$H^*(\mathrm{Bl}(X);\mathbb{C}) = H^*(X;\mathbb{C}) \bigoplus \widetilde{H}^*(E),$$

where $\widetilde{H}^*(E) = \bigoplus_{i=1}^{n-1} \mathbb{C}e^i$ is the reduced cohomology of E.

The Poincaré pairing of $\mathrm{Bl}(X)$ can be computed as follows. Let us choose a basis ϕ_i $(1 \le i \le N)$ of $H^*(X;\mathbb{C})$, such that,

(i) $\phi_1 = 1$ and $\phi_N = \mathrm{P.D.}(\mathrm{pt})$,

(ii) $\phi_{i+1} = p_i = c_1(L_i)$ $(1 \le i \le r)$, where L_i $(1 \le i \le r)$ is a set of ample line bundles on X, such that, p_i $(1 \le i \le r)$ form a $\mathbb{Z}$-basis of $H^2(X,\mathbb{Z})_{\mathrm{t.f.}}$.

Lemma 3.1. *Let* $(\ ,\ )^{\mathrm{Bl}(X)}$ *and* $(\ ,\ )^X$ *be the Poincaré pairings on respectively* $\mathrm{Bl}(X)$ *and* X. *Then we have*

(a) $(\phi_i,\phi_j)^{\mathrm{Bl}(X)} = (\phi_i,\phi_j)^X$ *for all* $1 \le i,j \le N$.

(b) $(\phi_i, e^k)^{\mathrm{Bl}(X)} = 0$ *for* $1 \le i \le N$ *and* $1 \le k \le n-1$.

(c) $e^n = (-1)^{n-1}\phi_N$ *and* $\left(e^k, e^{n-k}\right)^{\mathrm{Bl}(X)} = (-1)^{n-1}$.

Proof. Parts (a) and (b) follow easily by the projection formula and Poincaré duality. The second part of (c) is a consequence of the first part, so we need only to prove that $e^n = (-1)^{n-1}\phi_N$. We have $e^n = c\phi_N$ for dimension reasons. Note that $E \cong \mathbb{P}^{n-1}$ and $\mathcal{O}(E)|_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Therefore, $e|_E = c_1(O(E)|_E) = -p$, where $p = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ is the standard hyperplane class of $\mathbb{P}^{n-1}$. We get

$$c = \int_{[\mathrm{Bl}(X)]} e^n = \int_{[E]} e^{n-1} = \int_{[\mathbb{P}^{n-1}]} (-p)^{n-1} = (-1)^{n-1}. \qquad \blacksquare$$

The ring structure of $H^*(\mathrm{Bl}(X);\mathbb{C})$ with respect to the cup product is also easy to compute. We have

(1) $H^*(X;\mathbb{C})$ is a subring of $H^*(\mathrm{Bl}(X);\mathbb{C})$.

(2) $\phi_1 \cup e^k = e^k$ and $\phi_i \cup e^k = 0$, $2 \le i \le N$, $1 \le k \le n-1$.

(3)

$$e^k \cup e^l = \begin{cases} e^{k+l} & \text{if } k+l < n, \\ (-1)^{n-1}\phi_N & \text{if } k+l = n, \\ 0 & \text{if } k+l > n. \end{cases}$$

Property (1) follows from the fact that pullback in cohomology is a ring homomorphism. The formulas in (3) follow from Lemma 3.1 (c). Finally, (2) follows from (1), (3) and Lemma 3.1 (b).

3.2 K-ring of the blowup

For some background on topological K-theory we refer to [13, Chapter 6]. Let us compute the topological K-ring of $\mathrm{Bl}(X)$. We will be interested only in manifolds X, such that, the corresponding quantum cohomology is semisimple. Such X are known to have cohomology classes of Hodge type (p,p) only. In particular, $K^1(X)\otimes\mathbb{Q}=0$. To simplify the exposition, let us assume that $K^1(X)=0$. In our arguments below we will have to work with non-compact manifolds. However, in all cases the non-compact manifolds are homotopy equivalent to finite CW-complexes. We define the corresponding K-groups by taking the K-groups of the corresponding finite CW-complexes.

Proposition 3.2.

(a) *The K-theoretic pullback $\pi^*\colon K^0(X)\to K^0(\mathrm{Bl}(X))$ is injective.*

(b) *We have*

$$K^0(\mathrm{Bl}(X)) = K^0(X)\oplus\bigoplus_{j=1}^{n-1}\mathbb{Z}\mathcal{O}_E^j,$$

where $K^0(X)$ is viewed as a subring of $K^0(\mathrm{Bl}(X))$ via the K-theoretic pullback π^* and $\mathcal{O}_E:=\mathcal{O}-\mathcal{O}(-E)$ is the structure sheaf of the exceptional divisor.

Proof. Let $U\subset X$ be a small open neighborhood of the center of the blowup pt and $V:=X\setminus\{\mathrm{pt}\}$. Note that $\{U,V\}$ is a covering of X. Put $\widetilde{U}=\pi^{-1}(U)$ and $\widetilde{V}:=\pi^{-1}(V)$, then $\{\widetilde{U},\widetilde{V}\}$ is a covering of $\mathrm{Bl}(X)$. Let us compare the reduced K-theoretic Mayer–Vietories sequences of these two coverings. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{K}^{-1}(X) \arrow[r]\arrow[d] & \widetilde{K}^{-1}(V)\oplus\widetilde{K}^{-1}(U) \arrow[r]\arrow[d] & \widetilde{K}^{-1}(U\setminus\mathrm{pt}) \arrow[r]\arrow[d, "\cong"'] & \widetilde{K}^0(X) \arrow[r]\arrow[d] & \widetilde{K}^0(V)\oplus\widetilde{K}^0(U) \arrow[d]\\
\widetilde{K}^{-1}(\mathrm{Bl}(X)) \arrow[r] & \widetilde{K}^{-1}(\widetilde{V})\oplus\widetilde{K}^{-1}(\widetilde{U}) \arrow[r] & \widetilde{K}^{-1}(\widetilde{U}\setminus E) \arrow[r] & \widetilde{K}^0(\mathrm{Bl}(X)) \arrow[r] & \widetilde{K}^0(\widetilde{V})\oplus\widetilde{K}^0(\widetilde{U}),
\end{tikzcd}$$

where the vertical arrows in the above diagram are induced by the K-theoretic pullback π^*. Note that $\widetilde{K}^{\mathrm{ev}}(U\setminus\mathrm{pt})=\widetilde{K}^0(\widetilde{U}\setminus E)=0$ because $\widetilde{U}\setminus E\cong U\setminus\mathrm{pt}$ is homotopic to $\mathbb{S}^{2n-1}$ – the $(2n-1)$-dimensional sphere. Therefore, the horizontal arrows in the first and the last square of the above diagram are respectively injections and surjections. Furthermore, $\widetilde{K}^{-1}(U)=\widetilde{K}^0(U)=0$ because U is contractible and $\widetilde{K}^{-1}(\widetilde{U})=0$ because $\widetilde{U}$ is homotopy equivalent to $E\cong\mathbb{P}^{n-1}$. We get that the second vertical arrow is an isomorphism $(V\cong\widetilde{V})$ and hence, recalling the 5-lemma or by simple diagram chasing, we get $\widetilde{K}^{-1}(\mathrm{Bl}(X))\cong\widetilde{K}^{-1}(X)$. By assumption $\widetilde{K}^{-1}(X)=0$, so $\widetilde{K}^{-1}(\mathrm{Bl}(X))=0$. A straightforward diagram chasing shows that the 4th vertical arrow is injective, i.e., we proved (a).

Note that the above diagram yields the following short exact sequence:

$$0\longrightarrow\widetilde{K}^0(X)\xrightarrow{\pi^*}\widetilde{K}^0(\mathrm{Bl}(X))\xrightarrow{|_E}\widetilde{K}^0(\mathbb{P}^{n-1})\longrightarrow 0,\tag{3.1}$$

where the map $|_E$ is the restriction to the exceptional divisor $E\cong\mathbb{P}^{n-1}$. The above exact sequence splits because $\widetilde{K}^0(\mathbb{P}^{n-1})\cong\mathbb{Z}^{n-1}$ is a free module. Note that $\mathcal{O}_E|_E=\mathcal{O}_{\mathbb{P}^{n-1}}-\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is the generator of $\widetilde{K}^0(\mathbb{P}^{n-1})$, so part (b) follows from the exactness of (3.1). ∎

Let us compute the K-theoretic product of the torsion free part $K^0(\mathrm{Bl}(X))_{\mathrm{t.f.}}$. Note that $\pi_*(\mathcal{O}_{\mathrm{Bl}(X)})=\mathcal{O}_X$. Therefore, $\pi_*\pi^*(F)=F$ for every $F\in K^0(X)$. Let us compute $\mathcal{O}_E\otimes\pi^*F$

for $F \in \widetilde{K}^0(X)$. The restriction of $\mathcal{O}_E \otimes \pi^* F$ to E is 0. Recalling the exact sequence (3.1), we get $\mathcal{O}_E \otimes \pi^* F = \pi^* G$ for some $G \in \widetilde{K}^0(X)$. Taking pushforward, we get

$$G = \pi_*(\mathcal{O}_E \otimes \pi^* F) = \pi_*(\mathcal{O}_E) \otimes F = \mathbb{C}_{\mathrm{pt}} \otimes F = \mathrm{rk}(F)\mathbb{C}_{\mathrm{pt}} = 0,$$

where $\mathbb{C}_{\mathrm{pt}}$ is the skyscraper sheaf on X and in the 3rd equality we used the exact sequence

$$0 \longrightarrow \mathcal{O}(-E) \longrightarrow \mathcal{O} \longrightarrow j_*(\mathcal{O}_{\mathbb{P}^{n-1}}) \longrightarrow 0,$$

where $j\colon \mathbb{P}^{n-1} \to \mathrm{Bl}(X)$ is the embedding whose image is the exceptional divisor. This sequence implies $\mathcal{O}_E = j_* \mathcal{O}_{\mathbb{P}^{n-1}}$ and hence $\pi_* \mathcal{O}_E = (\pi \circ j)_* \mathcal{O}_{\mathbb{P}^{n-1}} = \mathbb{C}_{\mathrm{pt}}$. We proved that

$$\mathcal{O}_E \otimes \pi^* F = 0, \qquad \forall F \in \widetilde{K}^0(X).$$

It remains only to compute $\mathcal{O}_E^n$. The restriction of $\mathcal{O}_E^n$ to E is $(1-\mathcal{O}_{\mathbb{P}^{n-1}}(-1))^n = 0$. Therefore, $\mathcal{O}_E^n = \pi^* F$. The Chern character $\mathrm{ch}(\mathcal{O}_E^n) = (1 - \exp(-c_1(\mathcal{O}(E))))^n = e^n = (-1)^{n-1}\phi_N$, where we used Lemma 3.1 (c). On the other hand, the Chern character of the skyscraper sheaf can be computed easily with the Grothendieck–Riemann–Roch formula. Namely, we have

$$\mathrm{ch}(\iota^\circ_*(\mathbb{C})) \cup \mathrm{td}(X) = \iota^\circ_*(\mathrm{ch}(\mathbb{C}) \cup \mathrm{td}(\mathrm{pt})) = \iota^\circ_*(1) = \mathrm{P.D.}(\mathrm{pt}) = \phi_N,$$

where $\iota^\circ\colon \mathrm{pt} \to X$ is the natural inclusion of the point pt. Thus, $\mathrm{ch}(\mathbb{C}_{\mathrm{pt}}) = \phi_N$. Comparing with the formula for $\mathrm{ch}(\mathcal{O}_E^n)$, we get

$$\mathcal{O}_E^n = (-1)^{n-1}\mathbb{C}_{\mathrm{pt}} \qquad \mathrm{mod}\ \ker(\mathrm{ch}).$$

Finally, let us finish this section by quoting the formula for the K-theoretic class of the tangent bundle (see [15, Lemma 15.4]):

$$T\,\mathrm{Bl}(X) = TX - n - 1 + n\mathcal{O}(-E) + \mathcal{O}(E).$$

3.3 Quantum cohomology of the blowup

Let us first compare the effective curve cones $\mathrm{Eff}(X)$ and $\mathrm{Eff}(\mathrm{Bl}(X))$. We have an exact sequence

$$0 \longrightarrow H_2\big(\mathbb{P}^{n-1};\mathbb{Z}\big) \xrightarrow{\ j_*\ } H_2(\mathrm{Bl}(X);\mathbb{Z}) \xrightarrow{\ \pi_*\ } H_2(X;\mathbb{Z}) \longrightarrow 0,$$

where $j\colon \mathbb{P}^{n-1} \to \mathrm{Bl}(X)$ is the natural closed embedding of the exceptional divisor. The proof of the exactness is similar to the proof of (3.1). In particular, since the torsion free part of the above sequence splits, we get

$$H_2(\mathrm{Bl}(X);\mathbb{Z})_{\mathrm{t.f.}} = H_2(X;\mathbb{Z})_{\mathrm{t.f.}} \oplus \mathbb{Z}\ell,$$

where $\ell \in H_2(E;\mathbb{Z})$ is the class of a line in the exceptional divisor. The cone of effective curve classes $\mathrm{Eff}(\mathrm{Bl}(X)) \subset \mathrm{Eff}(X) \oplus \mathbb{Z}\ell$. The Novikov variables of the blowup will be fixed to be the Novikov variables of X and an extra variable corresponding to the line bundle $\mathcal{O}(E)$. In other words, for $\widetilde{\beta} = \beta + d\ell \in \mathrm{Eff}(\mathrm{Bl}(X))$, put

$$q^{\widetilde{\beta}} = q^\beta q_{r+1}^{\langle c_1(O(E)), \widetilde{\beta}\rangle} = q_1^{\langle \phi_2, \beta\rangle} \cdots q_r^{\langle \phi_{r+1}, \beta\rangle} q_{r+1}^{-d}.$$

Note that $\mathcal{O}(E)$ is not an ample line bundle: for example, $\ell \cdot E = -1 < 0$. Our choice of q_{r+1} makes the structure constants formal Laurent (not power) series in q_{r+1}. Following Bayer (see [3]), we write $q_{r+1} = Q^{n-1}$ for some formal variable Q. Let us recall the basis ϕ_i $(1 \le i \le N)$

of $H^*(X;\mathbb{C})$. Put $\phi_{N+k} = e^k$ $(1 \le k \le n-1)$. Then ϕ_i $\big(1 \le i \le \widetilde{N} := N+n-1\big)$ is a basis of $H^*(\mathrm{Bl}(X);\mathbb{C})$. Let $t = (t_1, \dots, t_{\widetilde{N}})$ be the corresponding linear coordinate system on $H^*(\mathrm{Bl}(X);\mathbb{C})$. The structure constants of the quantum cohomology of $\mathrm{Bl}(X)$ take the form

$$(\phi_a \bullet_{t,q} \phi_b, \phi_c) := \langle \phi_a, \phi_b, \phi_c \rangle_{0,3}(t) = \sum_{m=0}^{\infty} \sum_{\widetilde{\beta}=(\beta,d)} \frac{q^\beta Q^{-d(n-1)}}{m!} \langle \phi_a, \phi_b, \phi_c, t, \dots, t \rangle_{0,3+m,\widetilde{\beta}}.$$

Remark 3.3. The quantum cup product of $H^*(X)$ depends only on $(t_2, \dots, t_N)$. Suppose that these $N-1$ parameters are generic such that the quantum cup product of $H^*(X)$ is semisimple. Then, according to Bayer [3] (see also Proposition 4.6), even if we restrict the remaining parameters to 0, that is, set $t_1 = t_{N+1} = \cdots = t_{\widetilde{N}} = 0$, then the quantum cup product of the blowup is still semisimple. Therefore, for our purposes, it is sufficient to work with $t \in H^*(\mathrm{Bl}(X))$, such that, $t_1 = t_{N+1} = \cdots = t_{\widetilde{N}} = 0$.

3.4 Twisted GW invariants of $\mathbb{P}^{n-1}$

It turns out that genus-0 GW invariants of $\mathrm{Bl}(X)$ whose degree $\widetilde{\beta} = d\ell$ with $d \ne 0$ can be identified with certain twisted GW invariants of $\mathbb{P}^{n-1}$. Suppose that $(C, z_1, \dots, z_k, f)$ is a stable map representing a point in $\overline{\mathcal{M}}_{0,k}(\mathrm{Bl}(X), d\ell)$. Let $\pi\colon \mathrm{Bl}(X) \to X$ be the blowup map. Since $\pi_* \circ f_*[C] = 0$ and π induces a biholomorphism between $\mathrm{Bl}(X) \setminus E$ and $X \setminus \{\mathrm{pt}\}$, we get that $f(C)$ is contained in E. Therefore, we have a canonical identification

$$\overline{\mathcal{M}}_{0,k}(\mathrm{Bl}(X), d\ell) = \overline{\mathcal{M}}_{0,k}(E, d),$$

where $E \cong \mathbb{P}^{n-1}$ is the exceptional divisor. Let us compare the virtual tangent spaces of the two moduli spaces at $(C, z_1, \dots, z_k, f)$. For the left-hand side, we have

$$\begin{aligned}\mathcal{T}_{0,k,d\ell} = {} & H^1(C, \mathcal{T}_C(-z_1 - \cdots - z_k)) - H^0(C, \mathcal{T}_C(-z_1 - \cdots - z_k)) \\ & + H^0(C, f^*T_{\mathrm{Bl}(X)}) - H^1(C, f^*T_{\mathrm{Bl}(X)}),\end{aligned}$$

while for the right-hand side we have

$$\begin{aligned}\mathcal{T}_{0,k,d} = {} & H^1(C, \mathcal{T}_C(-z_1 - \cdots - z_k)) - H^0(C, \mathcal{T}_C(-z_1 - \cdots - z_k)) \\ & + H^0(C, f^*T_E) - H^1(C, f^*T_E),\end{aligned}$$

where $\mathcal{T}_C$ is the tangent sheaf of C and $\mathcal{T}_C(-z_1 - \cdots - z_k)$ is the sub sheaf of $\mathcal{T}_C$ consisting of sections vanishing at $z_1, \dots, z_k$. On the other hand, we have an exact sequence

$$0 \longrightarrow T_E \longrightarrow T_{\mathrm{Bl}(X)}|_E \longrightarrow \mathcal{O}_E(-1) \longrightarrow 0,$$

where we used that $\mathcal{O}_E(-1)$ is the normal bundle to the exceptional divisor in $\mathrm{Bl}(X)$. Pulling back the exact sequence to C via the stable map and taking the long exact sequence in cohomology, we get

$$\begin{aligned}& 0 \longrightarrow H^0(C, f^*T_E) \longrightarrow H^0(C, f^*T_{\mathrm{Bl}(X)}) \longrightarrow H^0(C, f^*\mathcal{O}_E(-1)) \longrightarrow \\ & \longrightarrow H^1(C, f^*T_E) \longrightarrow H^1(C, f^*T_{\mathrm{Bl}(X)}) \longrightarrow H^1(C, f^*\mathcal{O}_E(-1)) \longrightarrow 0.\end{aligned}$$

Note that $H^0(C, f^*\mathcal{O}_E(-1)) = 0$ because C is a rational curve. Indeed, if C' is an irreducible component of C and $d' = f_*[C']$ is its contribution to the degree of f, then $C' \cong \mathbb{P}^1$ and $f^*\mathcal{O}_E(-1)|_{C'} = \mathcal{O}_{\mathbb{P}^1}(-d')$. Therefore, $H^0(C', f^*\mathcal{O}_E(-1)) = 0$ and we get that the restrictions of the sections of $f^*\mathcal{O}_E(-1)$ to the irreducible components of C are 0 which implies that there are no non-zero global sections. Let us recall the Riemann–Roch formula for nodal curves (easily proved by induction on the number of nodes)

$$\dim H^0(C,\mathcal{L}) - \dim H^1(C,\mathcal{L}) = 1-g+\int_{[C]} c_1(\mathcal{L}),$$

where $\mathcal{L}$ is a holomorphic line bundle on C and g is the genus of C. Applying the Riemann–Roch formula to $f^*\mathcal{O}_E(-1)$, we get that

$$\dim H^1(C, f^*\mathcal{O}_E(-1)) = -1 - \int_{f_*[C]} c_1(\mathcal{O}_E(-1)) = d-1.$$

The cohomology group $H^1(C, f^*\mathcal{O}_E(-1))$ is the fiber of a holomorphic vector bundle $\mathbb{N}_{0,k,d}$ on $\overline{\mathcal{M}}_{0,k}(E,d)$ of rank $d-1$. The virtual tangent bundles are related by $\mathcal{T}_{0,k,d\ell} = \mathcal{T}_{0,k,d} - \mathbb{N}_{0,k,d}$. Recalling the construction of the virtual fundamental cycle [5], we get

$$\left[\overline{\mathcal{M}}_{0,k}(\mathrm{Bl}(X), d\ell)\right]^{\mathrm{virt}} = \left[\overline{\mathcal{M}}_{0,k}(E,d)\right]^{\mathrm{virt}} \cap e(\mathbb{N}_{0,k,d}).$$

The above formula for the virtual fundamental class yields the following formula:

$$\langle \alpha_1\psi^{m_1},\dots,\alpha_k\psi^{m_k}\rangle_{0,k,d\ell} = \int_{[\overline{\mathcal{M}}_{0,k}(E,d)]^{\mathrm{virt}}} \prod_{i=1}^k \mathrm{ev}_i^*(\alpha_i|_E)\psi_i^{m_i} \cup e(\mathbb{N}_{0,k,d}).$$

Later on we will need the 3-point GW invariants with $d=1$. Let us compute them. If $d=1$, then $e(\mathbb{N}_{0,k,d})=1$ and the above formula implies that the GW invariants of the blowup coincide with the GW invariants of the exceptional divisor, that is,

$$\langle \alpha_1\psi^{m_1},\dots,\alpha_k\psi^{m_k}\rangle_{0,k,\ell}^{\mathrm{Bl}(X)} = \langle \alpha_1|_E\psi^{m_1},\dots,\alpha_k|_E\psi^{m_k}\rangle_{0,k,1}^{E},$$

where we used the superscripts $\mathrm{Bl}(X)$ and E in order to specify that the correlators are GW invariants of respectively $\mathrm{Bl}(X)$ and E. Note that if $p=c_1\mathcal{O}_E(1)$ is the hyperplane class, then $e|_E=-p$. The quantum cohomology of $\mathbb{P}^{n-1}$ is well known to be $\mathbb{C}[p]/(p^n-Q)$. In particular, the 3-point correlators

$$\left\langle p^i,p^j,p^k\right\rangle_{0,3,1} = \begin{cases} 1 & \text{if } i+j+k=2n-1,\\ 0 & \text{otherwise,}\end{cases} \qquad \forall 0\le i,j,k\le n-1.$$

Therefore,

$$\left\langle e^i,e^j,e^k\right\rangle_{0,3,\ell} = \begin{cases} -1 & \text{if } i+j+k=2n-1,\\ \ \ 0 & \text{otherwise.}\end{cases}$$

Let us specialize $k=1$. Using the divisor equation (recall that $\int_\ell e=-1$), we get

$$\left\langle e^i,e^j\right\rangle_{0,2,\ell} = \begin{cases} 1 & \text{if } i=j=n-1,\\ 0 & \text{otherwise.}\end{cases}$$

3.5 The vanishing theorem of Gathmann

Gathmann discovered a very interesting vanishing criteria for the GW invariants of the blowup (see [17]). We need a slight generalization of his result which can be stated as follows. Following Gathmann, we assign a weight to each basis vector

$$\mathrm{wt}(\phi_a) = \begin{cases} 0 & \text{if } 1\le a\le N,\\ a-N-1 & \text{if } N<a\le N+n-1.\end{cases}$$

In other words, the exceptional class e^k has weight $k-1$ for all $1\le k\le n-1$ and in all other cases the weight is 0.

Proposition 3.4. *Suppose that we have a GW invariant*

$$\big\langle \phi_a\psi^k, \phi_{b_1}, \dots, \phi_{b_m}, e^{l_1}, \dots, e^{l_s}\big\rangle_{0,\beta+d\ell,1+m+s}, \tag{3.2}$$

where $1 \le a \le \widetilde{N}$, $1 \le b_1, \dots, b_m \le N$, *and* $2 \le l_1, \dots, l_s \le n-1$, *satisfying the following* 3 *conditions:*

(i) $\beta \neq 0$.

(ii) $\mathrm{wt}(\phi_a) + \sum_{i=1}^s (l_i - 1) > 0$ *or* $d > 0$.

(iii) $\mathrm{wt}(\phi_a) + \sum_{i=1}^s (l_i - 1) < (d+1)(n-1) - k$.

Then the GW invariant (3.2) *must be* 0.

Proof. The proof is done by induction on k. Gathmann's result is the case when $k = 0$. The inductive step uses the genus-0 topological recursion relations (see Section 2.4). Suppose that the proposition is proved for k and let us prove it for $k+1$. Using the TRRs, we write the correlator (3.2) with k replaced by $k+1$ in the following form:

$$\sum_{c=1}^{N+n-1} \sum \big\langle \phi_a\psi^k, \phi_c, \phi_{B'}, e^{L'}\big\rangle_{\beta'+d'\ell} \big\langle \phi^c, \phi_{B''}, e^{L''}\big\rangle_{\beta''+d''\ell},$$

where the second sum is over all possible splittings $B' \sqcup B'' = \{b_1, \dots, b_m\}$, $L' \sqcup L'' = \{l_1, \dots, l_s\}$, $\beta' + \beta'' = \beta$ and $d' + d'' = d$. The notation is as follows. We dropped the genus and the number of marked points from the correlator notation because the genus is always 0 and the number of marked points is the same as the number of insertions. The insertion of all $\phi_{b'}$ with $b' \in B'$ is denoted by $\phi_{B'}$ and the insertions of all $e^{l'}$ with $l' \in L'$ is denoted by $e^{L'}$. Similar conventions apply for $\phi_{B''}$ and $e^{L''}$ in the second correlator. The first correlator has $2+m'+s'$ insertions while the second one $1 + m'' + s''$, where m', m'', s', and s'' are respectively the number of elements of respectively B', B'', L', and L''. We have to prove that if the 3 conditions in the proposition are satisfied where k should be replaced by $k+1$, then the above sum is 0. We will refer to the correlator involving B' and L' as the first correlator and to the correlator involving B'' and L'' as the second correlator. We will prove that for each term in the above sum either the first or the second correlator vanishes. The proof will be divided into 4 cases.

Case 1*:* if $\beta' = 0$ and the second correlator does not satisfy condition (ii), that is, $\mathrm{wt}(\phi^c) + \sum_{l'' \in L''}(l'' - 1) \le 0$ and $d'' \le 0$. Note that since $\beta'' = \beta \neq 0$, the second correlator satisfies condition (i). Since $\beta' = 0$ we need to consider only c, such that, $\phi_c|_E \neq 0$ and hence $\phi_c \in \{1, e, \dots, e^{n-1}\}$. Moreover, the weight of ϕ^c is 0 so $\phi_c \in \{1, e^{n-1}\}$ and $\phi^c \in \{\phi_N, e\}$. Since $l'' \ge 2$ for all $l'' \in L''$ the set L'' must be empty. The corresponding term in the sum in this case takes the form

$$\big\langle \phi_a\psi^k, \phi_c, 1, \dots, 1, e^{l_1}, \dots, e^{l_s}\big\rangle_{d'\ell} \langle \phi^c, \phi_{B''}\rangle_{\beta+d''\ell},$$

where the insertions from $\phi_{B'}$ all must be 1 otherwise $\phi_b|_E = 0$ and the correlator vanishes. Using the dimension formula, we get

$$\deg(\phi_a) + k + \deg(\phi_c) + \sum_{i=1}^s l_i = (d'+1)(n-1) + s + m'.$$

Note that ϕ_a must satisfy $\phi_a|_E \neq 0$, otherwise the correlator is 0. Therefore, $\phi_a \in \{1, e, \dots, e^{n-1}\}$ which implies that $\deg(\phi_a) \le \mathrm{wt}(\phi_a) + 1$ with inequality only if $\phi_a = 1$. We get

$$\begin{aligned}(d'+1)(n-1) + m' &= \deg(\phi_a) + k + \deg(\phi_c) + \sum_{i=1}^s (l_i - 1) \\ &\le \mathrm{wt}(\phi_a) + 1 + k + \sum_{i=1}^s (l_i - 1) + \deg(\phi_c).\end{aligned}$$

On the other hand, let us recall that the correlator (3.2) (with $k+1$ instead of k) satisfies condition (iii), that is,

$$\mathrm{wt}(\phi_a) + \sum_{i=1}^{s} (l_i - 1) < (d+1)(n-1) - k - 1.$$

We get $(d'+1)(n-1) + m' < \deg(\phi_c) + (d+1)(n-1)$. Recall that there are two possibilities for ϕ_c: $\phi_c = 1$ or $\phi_c = e^{n-1}$. In the first case, we get $0 \leq m' < d''(n-1)$ and hence $d'' > 0$ contradicting our assumption that $d'' \leq 0$. In the second case, we get $0 \leq m' < (d''+1)(n-1)$. This implies that $d'' > -1$ which together with $d'' \leq 0$ implies that $d'' = 0$. However, since $\phi^c = e$, we get that the second correlator vanishes by the divisor equation. This completes the proof of our claim in Case 1.

Case 2*:* $\beta' = 0$ and the second correlator satisfies condition (ii). Since $\beta'' = \beta \neq 0$, the second correlator satisfies condition (i) too, so it will vanish unless condition (iii) fails, that is,

$$\mathrm{wt}(\phi^c) + \sum_{l'' \in L''} (l'' - 1) \geq (d''+1)(n-1).$$

On the other hand, similarly to Case 1, we must have $\phi_{b'} = 1$ for all $b' \in B'$, so the dimension formula applied to the first correlator yields

$$\deg(\phi_a) + k + \deg(\phi_c) + \sum_{l' \in L'} (l' - 1) = (d'+1)(n-1) + m'.$$

Adding up the above inequality and identity, we get

$$\deg(\phi_a) + k + \deg(\phi_c) + \mathrm{wt}(\phi^c) + \sum_{i=1}^{s} (l_i - 1) \geq (d+1)(n-1) + n - 1 + m'.$$

Again $\deg(\phi_a) \leq \mathrm{wt}(\phi_a) + 1$, so

$$m' + n - 1 - \deg(\phi_c) - \mathrm{wt}(\phi^c) \leq \mathrm{wt}(\phi_a) + 1 + k + \sum_{i=1}^{s} (l_i - 1) - (d+1)(n-1).$$

Recalling again condition (iii), we get that the right-hand side of the above inequality is < 0, and hence $m' + n - 1 < \deg(\phi_c) + \mathrm{wt}(\phi^c)$. Similarly to Case 1, we may assume that $\phi_c|_E \neq 0$, that is, $\phi_c \in \left\{1, e, \ldots, e^{n-1}\right\}$ which implies that $\deg(\phi_c) + \mathrm{wt}(\phi^c) \leq n - 1$. This is a contradiction with $m' + n - 1 < \deg(\phi_c) + \mathrm{wt}(\phi^c)$.

Case 3*:* if $\beta' \neq 0$ and the first correlator does not satisfy condition (ii), that is, $\mathrm{wt}(\phi_a) + \mathrm{wt}(\phi_c) + \sum_{l' \in L'} (l' - 1) = 0$ and $d' \leq 0$. Note that we must have $L' = \varnothing$ and either $d'' > 0$ or $\sum_{i=1}^{s} (l_i - 1) > 0$. Therefore, the second correlator satisfies condition (ii).

Suppose that $\beta'' = 0$ ($\Leftrightarrow$ condition (i) fails). We must have $\phi_{b''}|_E \neq 0$ for all $b'' \in B''$ $\Rightarrow$ $\phi_{b''} = 1$ for all $b'' \in B''$. Recalling the dimension formula for the second correlator, we get

$$\deg(\phi^c) + \sum_{i=1}^{s} (l_i - 1) = (d''+1)(n-1) + m'' - 1.$$

On the other hand, since $\mathrm{wt}(\phi_a) = 0$ for the case under consideration, condition (iii) implies that $\sum_{i=1}^{s} (l_i - 1) < (d+1)(n-1) - k - 1$. Combining this estimate with the above equality, we get $m'' + k < \deg(\phi^c) + d'(n-1)$. If $m'' > 0$, then the second correlator has at least one insertion by 1 ($\because$ $B'' \neq \varnothing$). Since the second correlator does not have descendants it will vanish unless $d'' = 0$. However, if $\beta'' = d'' = 0$ the second correlator is non-zero only if the number of insertion is 3 because the moduli space is $\overline{\mathcal{M}}_{0,1+m''+s} \times \mathrm{Bl}(X)$, that is, $m'' = s = 1$. Moreover, $\phi^c \cup e^{l_1}$ up to a constant must be ϕ_N hence $\phi^c = e^{n-l_1}$ and $\phi_c = e^{l_1}$. However, $l_1 \geq 2$ by definition, so $\mathrm{wt}(\phi_c) = l_1 - 1 > 0$ — contradicting the assumption that the first correlator does not satisfy condition (ii). We get $m'' = 0$ and the estimate that we did above yields $k < \deg(\phi^c) + d'(n-1)$. Note that $\deg(\phi^c) \leq n-1$. Therefore, $d' > -1$. Recall that we are assuming that $d' \leq 0$, so $d' = 0$. Recalling the divisor equation we get $\phi_c \neq e$. Since $\beta'' = 0$ the restriction $\phi^c|_E \neq 0$ hence $\phi^c \in \{1, e, \dots, e^{n-1}\}$. Moreover, $\phi^c \neq 1$ thanks to the string equation. We get $\phi_c = e^l$ for some $l \geq 2$ contradicting the assumption that $\mathrm{wt}(\phi_c) = 0$.

Suppose now that $\beta'' \neq 0$. Then the second correlator satisfies both conditions (i) and (ii). Therefore, condition (iii) must fail, that is,

$$\mathrm{wt}(\phi^c) + \sum_{i=1}^{s}(l_i - 1) \geq (d''+1)(n-1). \tag{3.3}$$

Using that $\mathrm{wt}(\phi^c) \leq n-2$ and $\sum_i (l_i - 1) < (d+1)(n-1) - k - 1$, we get

$$(d''+1)(n-1) < n - 2 + (d+1)(n-1) - k - 1,$$

which implies that $k + 1 < d'(n-1) + n - 2$. In particular, $d' > -1$ and since by assumption $d' \leq 0$ we get $d' = 0$. If $\phi_c = e$, then using the divisor equation we get

$$\langle \phi_a \psi^k, \phi_c, \phi_{B'} \rangle_{\beta'} = \langle e \cup \phi_a \psi^{k-1}, \phi_{B'} \rangle_{\beta'}.$$

Since $\mathrm{wt}(\phi_a) = 0$ the cup product $e \cup \phi_a \neq 0$ only if $\phi_a = e$. This however implies that $e \cup \phi_a = e^2$ has positive weight and hence the correlator on the right-hand side of the above identity satisfies both conditions (i) and (ii). Condition (iii) must fail, so $1 \geq n - 1 - (k-1) = n - k$, that is, $k \geq n-1$. On the other hand, recall that we already have the estimate $k+1 < d'(n-1)+n-2 = n-2$ which contradicts the inequality in the previous sentence. We get $\phi_c \neq e$ which together with $\mathrm{wt}(\phi_c) = 0$ implies that $\phi^c \notin \{e^2, \dots, e^{n-1}\}$ and hence $\mathrm{wt}(\phi^c) = 0$. Recalling (3.3), we get

$$(d''+1)(n-1) \leq \sum_{i=1}^{s}(l_i - 1) < (d+1)(n-1) - k - 1.$$

Since $d' = 0$, we get $0 < -k - 1$ which is clearly a contradiction. This completes the proof of the vanishing claim in Case 3.

Case 4*:* if $\beta' \neq 0$ and the first correlator satisfies condition (ii). Then condition (iii) for the first correlator must fail, that is,

$$\mathrm{wt}(\phi_a) + \mathrm{wt}(\phi_c) + \sum_{l' \in L'}(l' - 1) \geq (d'+1)(n-1) - k. \tag{3.4}$$

We claim that the second correlator also satisfies conditions (i) and (ii). Indeed, suppose that (i) is not satisfied, that is, $\beta'' = 0$. All insertions in $\phi_{B''}$ must be 1. Recalling the dimension formula, we get

$$\deg(\phi^c) + \sum_{l'' \in L''}(l'' - 1) = (d''+1)(n-1) - 1 + m''.$$

Adding up the above identity and the inequality (3.4), we get

$$\begin{aligned}\mathrm{wt}(\phi_a) + \deg(\phi^c) + \mathrm{wt}(\phi_c) + \sum_{i=1}^{s}(l_i - 1) &\geq (d+2)(n-1) - k - 1 + m'' \\ &= (d+1)(n-1) - k - 1 + n - 1 + m'',\end{aligned}$$

which is equivalent to

$$n-1+m''-\deg(\phi^c)-\mathrm{wt}(\phi_c)\leq \mathrm{wt}(\phi_a)+\sum_{i=1}^{s}(l_i-1)+k+1-(d+1)(n-1)<0,$$

where for the last inequality we used that the correlator whose vanishing we want to prove satisfies condition (iii). We get $m''+n\leq \deg(\phi^c)+\mathrm{wt}(\phi_c)$. On the other hand, since $\phi^c|_E\neq 0$, we have $\phi^c\in\{1,e,\dots,e^{n-1}\}$ which implies that $\deg(\phi^c)+\mathrm{wt}(\phi_c)\leq n-1$ – contradiction. This proves that $\beta''\neq 0$.

Suppose that the second correlator does not satisfy condition (ii). Then, $d''\leq 0$, $\mathrm{wt}(\phi^c)=0$, and $L''=\varnothing$. Since $L''=\varnothing$, the inequality (3.4) takes the form

$$\mathrm{wt}(\phi_a)+\mathrm{wt}(\phi_c)+\sum_{i=1}^{s}(l_i-1)\geq (d'+1)(n-1)-k.$$

On the other hand, recalling again condition (iii) for the correlator whose vanishing we wish to prove, we get

$$\mathrm{wt}(\phi_a)+\sum_{i=1}^{s}(l_i-1)<(d+1)(n-1)-k-1.$$

Combining with the above estimate, we get

$$(d'+1)(n-1)-k<\mathrm{wt}(\phi_c)+(d+1)(n-1)-k-1,$$

which becomes $(-d'')(n-1)<\mathrm{wt}(\phi_c)-1$. Since $\mathrm{wt}(\phi^c)=0$ and $d''\leq 0$ the above inequality is possible only if $\phi^c=e$. Then we get $d''\neq 0$ thanks to the divisor equation, that is, $d''\leq -1$ and hence $\mathrm{wt}(\phi_c)-1>n-1$. This is a contradiction because the maximal possible value of $\mathrm{wt}(\phi_c)$ is $n-2$. This completes the proof of our claim that the second correlator satisfies conditions (i) and (ii).

Finally, in order for the second correlator to be non-zero, condition (iii) must fail. We get

$$\mathrm{wt}(\phi^c)+\sum_{l''\in L''}(l''-1)\geq (d''+1)(n-1).$$

Adding up the above inequality and (3.4), we get

$$\mathrm{wt}(\phi_a)+\mathrm{wt}(\phi_c)+\mathrm{wt}(\phi^c)+\sum_{i=1}^{s}(l_i-1)\geq (d+2)(n-1)-k=(d+1)(n-1)-k-1+n.$$

On the other hand, recalling the inequality

$$\mathrm{wt}(\phi_a)+\sum_{i=1}^{s}(l_i-1)<(d+1)(n-1)-k-1,$$

we get

$$n-\mathrm{wt}(\phi_c)-\mathrm{wt}(\phi^c)<0.$$

The inequality clearly does not hold if one of the weights is 0. If both weights are non-zero, then we will have $\mathrm{wt}(\phi_c)+\mathrm{wt}(\phi^c)=n-2$ which again contradicts the above inequality. The conclusion is that either the first or the second correlator satisfies condition (iii) and hence one of the two correlators must vanish according to the inductive assumption. ■

4 Second structure connection and blowups

Let us recall the notation already fixed in Sections 3.1 and 3.3. From now on, for a complex variety Y, we denote by $H(Y) := H^*(Y, \mathbb{C})$ and $\widetilde{H}(Y)$ respectively the cohomology and the reduced cohomology of Y with complex coefficients. Using the direct sum decomposition $H(\mathrm{Bl}(X)) = H(X) \oplus \widetilde{H}(E)$, we define the $H(X)$-component (resp. $\widetilde{H}(E)$-component) of a vector $v \in H(\mathrm{Bl}(X))$ to be the projection of v on $H(X)$ (resp. $\widetilde{H}(E)$).

We will view quantum cohomology of $\mathrm{Bl}(X)$ as a family of Frobenius manifolds parametrized by the Novikov variables $q = (q_1, \dots, q_{r+1}) \in (\mathbb{C}^*)^{r+1}$ defined in Section 3.3. Recall that $q_{r+1} = Q^{n-1}$. We will be interested in the Laurent series expansion of the second structure connection of $\mathrm{Bl}(X)$ with respect to Q at $Q = 0$, while the remaining parameters $q_1, \dots, q_r$ remain fixed. The main goal in this section is to determine the leading order terms of this expansion.

4.1 Period vectors for the blowup

Let us denote by $\widetilde{\rho}$ and ρ the operators of classical cup product multiplications by respectively $c_1(T\,\mathrm{Bl}(X))$ and $c_1(TX)$. Let $\widetilde{\theta}$ and θ be the grading operators of the Frobenius structures underlying the quantum cohomologies of respectively $\mathrm{Bl}(X)$ and X (see (2.10)). In the lemma below, we will need the following notation. Suppose that $A\colon \mathbb{C} \to \mathrm{End}(H)$ is an $\mathrm{End}(H)$-valued smooth function. Let m be the standard coordinate on $\mathbb{C}$. We define *right* differential operator on $\mathbb{C}$, to be a formal expression of the form

$$L\big(m, \overleftarrow{\partial}_m\big) = \sum_{k=0}^{r} B_k(m) \overleftarrow{\partial}_m^{\,k}, \tag{4.1}$$

where the coefficients $B_k(m) \in \mathrm{End}(H)$ depend smoothly on m. We define the action of L on A by

$$A(m) L\big(m, \overleftarrow{\partial}_m\big) := \sum_{k=0}^{r} \partial_m^k \big(A(m) \circ B_k(m)\big),$$

where $\circ$ is the composition operation in $\mathrm{End}(V)$. Given two right differential operators L_1 and L_2, there exists a unique right differential operator $L_1 \circ L_2$, such that,

$$\big(A(m) L_1\big(m, \overleftarrow{\partial}_m\big)\big) L_2\big(m, \overleftarrow{\partial}_m\big) =: A(m)(L_1 \circ L_2)\big(m, \overleftarrow{\partial}_m\big).$$

We say that $L_1 \circ L_2$ is the composition of L_1 and L_2. One can check that this operation is associative and therefore, the set of all right differential operators of the form (4.1) has a structure of an associative algebra acting from the right on the space of smooth $\mathrm{End}(H)$-valued functions on $\mathbb{C}$.

Lemma 4.1.

(a) *The following formula holds:*

$$\widetilde{I}^{(-m)}(\lambda) := e^{\widetilde{\rho} \partial_\lambda \partial_m} \left(\frac{\lambda^{\widetilde{\theta} + m - 1/2}}{\Gamma\big(\widetilde{\theta} + m + 1/2\big)} \right) = \left(\frac{\lambda^{\widetilde{\theta} + m - 1/2}}{\Gamma\big(\widetilde{\theta} + m + 1/2\big)} \right) e^{\widetilde{\rho} \overleftarrow{\partial}_m},$$

where the first identity is just a definition and $\overleftarrow{\partial}_m$ denotes the right action by a derivation with respect to m.

(*b*) *The following identity holds:*

$$\widetilde{I}^{(-m)}(Q^{-1}\lambda) = \left(\frac{\lambda^{\widetilde{\theta}+m-1/2}}{\Gamma\big(\widetilde{\theta}+m+1/2\big)}\right) e^{Q\widetilde{\rho}\overleftarrow{\partial}_m} Q^{-(\widetilde{\theta}+m-1/2)} Q^{-\widetilde{\rho}}.$$

Proof. (a) By definition

$$\begin{aligned} e^{\widetilde{\rho}\partial_\lambda\partial_m}\left(\frac{\lambda^{\widetilde{\theta}+m-1/2}}{\Gamma\big(\widetilde{\theta}+m+1/2\big)}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{\rho}^k\partial_m^k\left(\frac{\lambda^{\widetilde{\theta}+m-k-1/2}}{\Gamma\big(\widetilde{\theta}+m-k+1/2\big)}\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\partial_m^k\left(\frac{\lambda^{\widetilde{\theta}+m-1/2}}{\Gamma\big(\widetilde{\theta}+m+1/2\big)}\right)\widetilde{\rho}^k, \end{aligned}$$

where we used that $\widetilde{\rho}\widetilde{\theta} = \big(\widetilde{\theta}+1\big)\widetilde{\rho}$. The above expression is by definition the right action of $e^{\widetilde{\rho}\overleftarrow{\partial}_m}$ on $\lambda^{\widetilde{\theta}+m-1/2}/\Gamma\big(\widetilde{\theta}+m+1/2\big)$.

(b) Using the formula from part (a), we get that the identity that we have to prove is equivalent to the following conjugation formulas:

$$Q^{\widetilde{\theta}+m}Q^{-Q\widetilde{\rho}}Q^{-(\widetilde{\theta}+m)} = Q^{-\widetilde{\rho}},$$

and

$$Q^{-(\widetilde{\theta}+m)}e^{\widetilde{\rho}\overleftarrow{\partial}_m}Q^{\widetilde{\theta}+m} = Q^{-Q\widetilde{\rho}}e^{Q\widetilde{\rho}\overleftarrow{\partial}_m}. \tag{4.2}$$

The first identity follows easily from $\big[\widetilde{\theta},\widetilde{\rho}\big] = -\widetilde{\rho}$. Let us prove (4.2). To begin with, note that this is an identity between operators acting from the right. We will use the following fact. Suppose that we have an associative algebra $\mathcal{A}$ acting on a vector space V from the right, that is, $v\cdot(AB) = (v\cdot A)\cdot B$ for all $A, B\in\mathcal{A}$ and $v\in V$. Then the following formula holds:

$$v\cdot\big(e^{A}Be^{-A}\big) = v\cdot\big(e^{\mathrm{ad}_A}(B)\big), \tag{4.3}$$

where $\mathrm{ad}_A(X) = AX - XA$. In our case, $\mathcal{A}$ is the algebra of differential operators in m with coefficients in $\mathrm{End}(H^*(\mathrm{Bl}(X)))$, that is, as a vector space $\mathcal{A}$ consists of elements of the form

$$\sum_{k=0}^{k_0} c_k(m)\overleftarrow{\partial}_m^{\,k}, \qquad c_k(m)\in\mathrm{End}(H^*(\mathrm{Bl}(X))),$$

and the product in $\mathcal{A}$ is determined by the natural composition of endomorphisms of $H^*(\mathrm{Bl}(X))$ and the commutation relation $\big[m,\overleftarrow{\partial}_m\big] = 1$. Let us apply the conjugation formula (4.3) to (4.2). The main difficulty is to prove that

$$\mathrm{ad}^k_{\widetilde{\theta}+m}\big(\widetilde{\rho}\overleftarrow{\partial}_m\big) = (-1)^k\widetilde{\rho}\overleftarrow{\partial}_m + (-1)^{k-1}k\widetilde{\rho}, \tag{4.4}$$

for all $k\geq 0$. We argue by induction on k. For $k=0$, the identity is true. Suppose that it is true for k. Then we get

$$\begin{aligned} \big[\widetilde{\theta}+m,(-1)^k\widetilde{\rho}\overleftarrow{\partial}_m+(-1)^{k-1}k\widetilde{\rho}\big] &= (-1)^k\big(-\widetilde{\rho}\overleftarrow{\partial}_m+\widetilde{\rho}\big)+(-1)^k k\widetilde{\rho} \\ &= (-1)^{k+1}\widetilde{\rho}\overleftarrow{\partial}_m+(-1)^k(k+1)\widetilde{\rho}, \end{aligned}$$

where we used that $\big[\widetilde{\theta},\widetilde{\rho}\big] = -\widetilde{\rho}$ and $\big[m,\overleftarrow{\partial}_m\big] = 1$. Using the conjugation formula (4.3) and formula (4.4), we get that the left-hand side of (4.2) is equal to

$$\exp\left(\sum_{k=0}^{\infty}\frac{1}{k!}(-\log Q)^k\big((-1)^k\widetilde{\rho}\overleftarrow{\partial}_m+(-1)^{k-1}k\widetilde{\rho}\big)\right) = e^{Q\widetilde{\rho}\overleftarrow{\partial}_m}e^{-(\log Q)Q\widetilde{\rho}},$$

which is the same as the right-hand side of (4.2). ∎

Put $q=(q_1,\dots,q_r)$ and let $S(t,q,Q,z)$ be the calibration of the blowup $\mathrm{Bl}(X)$. Let us recall the fundamental solution of the second structure connection for the quantum cohomology of $\mathrm{Bl}(X)$

$$I^{(-m)}(t,q,Q,\lambda)=\sum_{k=0}^{\infty}(-1)^k S_k(t,q,Q)\partial_\lambda^k \widetilde{I}^{(-m)}(\lambda).$$

Recalling Lemma 4.1, we get

$$\begin{aligned} & I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big)Q^{\widetilde{\rho}}Q^{\widetilde{\theta}+m-1/2} \\ & \quad = \sum_{i,l=0}^{\infty}(-1)^l Q^l S_l(t,q,Q)\frac{\partial_m^i}{i!}\left(\frac{\lambda^{\widetilde{\theta}+m-l-1/2}}{\Gamma\big(\widetilde{\theta}+m-l+1/2\big)}(Q\widetilde{\rho})^i\right). \end{aligned} \tag{4.5}$$

Let us extend the Hodge grading operator θ of X to $H^*(\mathrm{Bl}(X))$ in such a way that $\theta\big(e^k\big)=\frac{n}{2}e^k$ for all $1\le k\le n-1$. Let $\Delta:=\widetilde{\theta}-\theta$, then in the basis

$$\phi_i,\quad 1\le i\le N,\qquad e^k,\quad 1\le k\le n-1 \tag{4.6}$$

of $H^*(\mathrm{Bl}(X))$, the operator Δ takes the form

$$\Delta(\phi_i)=0,\quad 1\le i\le N,\qquad \Delta\big(e^k\big)=-ke^k,\quad 1\le k\le n-1.$$

Let us point out that the basis of $H^*(\mathrm{Bl}(X))$ dual to the basis (4.6) with respect to the Poincaré pairing is given by ϕ^i $(1\le i\le N)$, e_k $(1\le k\le n-1)$, where ϕ^i $(1\le i\le N)$ is a basis of $H^*(X)$ dual to ϕ_i $(1\le i\le N)$ with respect to the Poincaré pairing (on X) and $e_k:=(-1)^{n-1}e^{n-k}$.

Lemma 4.2. *Suppose that $t=\sum_{b=2}^N t_b\phi_b\in H^*(X)$ and that $l\ge 1$. Then*

$$\begin{aligned} Q^{\Delta}Q^{k+l}\big(S_l(t,q,Q)e^k\big) = {} & \sum_{k''=1}^{n-1}\sum_{d=0}^{\infty}\big\langle\psi^{l-1}e^k,e^{k''}\big\rangle_{0,2,d\ell}e_{k''}+\big\langle\psi^{l-1}e^k,1\big\rangle_{0,2,d\ell}Q^n(-1)^{n-1}\phi_N \\ & + O\big(Q\widetilde{H}(E)+Q^{n+1}H(X)\big), \end{aligned}$$

where the O-term denotes a power series in Q with values in $H(\mathrm{Bl}(X))$ whose $\widetilde{H}(E)$-component involves only positive powers of Q and its $H(X)$-component involves only powers of Q of degree $\ge n+1$.

Proof. Recall that every $\widetilde{\beta}\in\mathrm{Eff}(\mathrm{Bl}(X))$ has the form $\widetilde{\beta}=\beta+d\ell$ for some $\beta\in\mathrm{Eff}(X)$ and $d\in\mathbb{Z}$. Recalling the definition of the calibration, we get

$$\begin{aligned} Q^{\Delta}Q^{k+l}\big(S_l(t,q,Q)e^k\big) = Q^{\Delta}Q^{k+l} & \sum_{\widetilde{\beta}\in\mathrm{Eff}(\mathrm{Bl}(X))}\left(\sum_{b=1}^{N}\big\langle e^k\psi^{l-1},\phi_b\big\rangle_{0,2,\beta+d\ell}(t)\phi^b\right. \\ & \left. +\sum_{k''=1}^{n-1}\big\langle e^k\psi^{l-1},e^{k''}\big\rangle_{0,2,\beta+d\ell}(t)e_{k''}\right)q^{\beta}Q^{-d(n-1)}. \end{aligned}$$

Let us examine first the correlator

$$\big\langle e^k\psi^{l-1},\phi_b\big\rangle_{0,2,\beta+d\ell}(t)=\sum_{r=0}^{\infty}\sum_{b_1,\dots,b_r=2}^{N}\big\langle e^k\psi^{l-1},\phi_b,\phi_{b_1},\dots,\phi_{b_r}\big\rangle_{0,2+r,\beta+d\ell}t_{b_1}\cdots t_{b_r}.$$

There are 3 cases.

Case 1*:* if $\beta = 0$. Then the correlator is a twisted GW invariant of E and since $\phi_b|_E = 0$ for $2 \leq b \leq N$, we may assume that $b = 1$, that is, $\phi_b = \phi_1 = 1$. For similar reasons, we may assume that $r = 0$. The dimension of the virtual fundamental cycle of $\overline{\mathcal{M}}_{0,2}(\mathrm{Bl}(X), d\ell)$ is $(d+1)(n-1)$. Therefore, $k + l - 1 = (d+1)(n-1)$ or equivalently $k + l - d(n-1) = n$, that is, in this case the correlator coincides with the term of order Q^n on the right-hand side of the formula that we want to prove.

Case 2*:* if $\beta \neq 0$ and the correlator does not satisfy condition (ii) in Gathmann's vanishing theorem. Then $d \leq 0$ and $k - 1 \leq 0$, that is, $k = 1$. If $d \leq -1$, then $k + l - d(n-1) \geq k + l + n - 1 \geq n + 1$, so the correlator contributes to the terms of order $O\big(Q^{n+1}H(X)\big)$. It remains to consider the case when $k = 1$ and $d = 0$. We will prove that if the correlator $\big\langle e\psi^{l-1}, \phi_b\big\rangle_{0,2,\beta}(t)$ is non-zero, then $l \geq n$.

First, by using the divisor equation we may reduce to the cases when $2 \leq b \leq N$. Indeed, if $b = 1$ then by using the string equation $\big\langle e\psi^{l-1}, 1\big\rangle_{0,2,\beta}(t) = \big\langle e\psi^{l-2}\big\rangle_{0,1,\beta}(t)$. Since $\beta \neq 0$, there exists a divisor class $p \in H^2(X)$, such that, $\int_\beta p \neq 0$. Recalling the divisor equation, we get $\big\langle e\psi^{l-2}, p\big\rangle_{0,2,\beta} = \int_\beta p \big\langle e\psi^{l-2}\big\rangle_{0,1,\beta}$, that is, the correlator for $b = 1$ can be expressed in terms of correlators involving only $2 \leq b \leq N$.

Suppose now that $2 \leq b \leq N$. Recalling the divisor equation, we get

$$\big\langle e\psi^{l-1}, \phi_b\big\rangle_{0,2,\beta}(t) = \big\langle e, \psi^l, \phi_b\big\rangle_{0,3,\beta}(t),$$

which according to the topological recursion relations (TRR) can be written as

$$\sum_{\beta'+\beta''=\beta} \sum_{d\in\mathbb{Z}} \left(\sum_{j=1}^{n-1} \big\langle \psi^{l-1}, e^j\big\rangle_{0,2,\beta'+d\ell}(t) \langle e_j, e, \phi_b\rangle_{0,3,\beta''-d\ell}(t) \right.$$
$$\left. + \sum_{a=1}^{N} \big\langle \psi^{l-1}, \phi_a\big\rangle_{0,2,\beta'+d\ell}(t) \langle \phi^a, e, \phi_b\rangle_{0,3,\beta''-d\ell}(t) \right).$$

Let us consider the two correlators that involve β'. If $\beta' = 0$, then in both correlators $d > 0$ and in the second correlator $\phi_a = 1$. Since $t|_E = 0$ we may assume that $t = 0$. By the dimension formula, we get $l - 1 + j = (d+1)(n-1)$ for the 1st correlator and $l - 1 = (d+1)(n-1)$ for the second correlator. In both cases, since $d \geq 1$ and $n \geq 2$, we have $l \geq n$. Suppose now that $\beta' \neq 0$ and that the 1st (resp. second) correlator does not satisfy condition (ii) in Gathmann's vanishing theorem, that is $j = 1$ and $d \leq 0$. Note that for the correlators involving β'' we must have $\beta'' \neq 0$ because $\phi_b|_E = 0$ and $d \neq 0$ due to the divisor equation for the divisor class e. We get $d < 0$. Therefore, the correlator involving β'' satisfies both conditions (i) and (ii) in Gathmann's vanishing theorem. Therefore, condition (iii) must fail, that is, $n - j - 1 \geq (-d+1)(n-1)$ and $0 \geq (-d+1)(n-1)$. Since $-d \geq 1$, both inequalities lead to a contradiction. It remains only the possibility that $\beta' \neq 0$ and that the correlators involving β' satisfy condition (ii) in Gathmann's vanishing theorem. Then condition (iii) must fail, so $j - 1 + l - 1 \geq (d+1)(n-1)$ and $l - 1 \geq (d+1)(n-1)$. If $d \geq 1$, then these inequalities will imply that $l \geq n + 1$. If $d = 0$, then $\beta'' = 0$, otherwise the correlator involving β'' will be 0 by the divisor equation. But then the correlator becomes $\int_{\mathrm{Bl}(X)} e_j \cup e \cup \phi_b$ which is 0 because $\phi_b \cup e = 0$. Finally, if $d \leq -1$, then, since $\phi_b|_E = 0$ we must have $\beta'' \neq 0$, so the correlator involving β'' satisfies conditions (i) and (ii) in Gathmann's vanishing theorem. The 3rd condition must fail, that is, $n - j - 1 \geq (-d+1)(n-1)$ and $0 \geq (-d+1)(n-1)$. Both inequalities are impossible and this completes the analysis in the second case.

Case 3*:* if $\beta \neq 0$ and the correlator does satisfy condition (ii). Then condition (iii) in Gathmann's vanishing theorem does not hold, that is,

$$k - 1 \geq (d+1)(n-1) - l + 1 = d(n-1) + n - l.$$

This inequality is equivalent to $k+l-d(n-1)\geq n+1$. We get that the correlators in this case contribute to the terms of order $O\big(Q^{n+1}\big)$.

Similarly, let us examine the correlators

$$\big\langle e^k\psi^{l-1}, e^{k''}\big\rangle_{0,2,\beta+d\ell}(t) = \sum_{r=0}^{\infty}\sum_{b_1,\dots,b_r=2}^{N}\big\langle e^k\psi^{l-1}, e^{k''}, \phi_{b_1},\dots,\phi_{b_r}\big\rangle_{0,2+r,\beta+d\ell} t_{b_1}\cdots t_{b_r}.$$

Since $\Delta(e_{k''}) = -(n-k'')e_{k''}$, we get that we have to prove that if the above correlator is non-zero, then $k+l-d(n-1)-(n-k'')\geq 0$ and that if the equality holds, then $r=0$ and $\beta=0$. Again we will consider 3 cases.

Case 1: if $\beta=0$. Just like above, $r=0$ because we can identify the correlator with a twisted GW invariant of the exceptional divisor and the restriction of ϕ_{b_i} to E is 0. Using the dimension formula for the virtual fundamental cycle, we get

$$k+l-1+k'' = \dim\big[\overline{\mathcal{M}}_{0,2}(\mathrm{Bl}(X), d\ell)\big]^{\mathrm{virt}} = n-1+(n-1)d.$$

We get

$$k+l-d(n-1)-(n-k'') = k+l+k''-n-d(n-1) = 0.$$

Cases 2*:* if $\beta\neq 0$ and condition (ii) does not hold. Then $d\leq 0$ and $k-1+k''-1\leq 0$, that is, $k=k''=1$. If $d\leq -1$, then

$$k+l-d(n-1)-(n-k'')\geq k+l+n-1-n+k'' = 1+l\geq 2.$$

The correlator contributes to the terms of order $O\big(Q^2\big)$. Suppose that $d=0$. The insertion $e^{k''}=e$ can be removed via the divisor equation, that is, the correlator in front of $t_{b_1}\cdots t_{b_r}$ takes the form

$$\big\langle e^k\psi^{l-1}, e^{k''}, \phi_{b_1},\dots,\phi_{b_r}\big\rangle_{0,2+r,\beta} = \big\langle e^2\psi^{l-2}, \phi_{b_1},\dots,\phi_{b_r}\big\rangle_{0,1+r,\beta}.$$

The above correlator does satisfy condition (ii) of Proposition 3.4. Therefore, in order to have a non-trivial contribution, condition (iii) in Gathmann's vanishing theorem must fail, that is, $1\geq n-1-l+2$ or equivalently $l\geq n$. We get

$$k+l-d(n-1)-(n-k'') = 1+l-(n-1) = 2+l-n\geq 2>0,$$

so the equality that we need to prove holds.

Case 3*:* if $\beta\neq 0$ and condition (ii) holds. In other words, conditions (i) and (ii) in Gathmann's vanishing theorem (see Proposition 3.4) hold for the correlators

$$\big\langle e^k\psi^{l-1}, e^{k''}, \phi_{b_1},\dots,\phi_{b_r}\big\rangle_{0,2+r,\beta+d\ell}.$$

Again, in order to have a non-trivial contribution, condition (iii) must fail, so

$$k-1+k''-1\geq (d+1)(n-1)-l+1 = d(n-1)+n-l,$$

or equivalently $k+l+k''\geq 2+n+d(n-1)$. We get

$$k+l-d(n-1)-(n-k'') = k+l+k''-n-d(n-1)\geq 2>0.$$

This completes the proof of the lemma. ■

Note that

$$\widetilde{\rho}^i e^k = (c_1(TX) - (n-1)e)^i e^k = (-n+1)^i e^{k+i},$$

and that $\widetilde{\theta}\big(e^{k+i}\big) = \big(\frac{n}{2} - k - i\big)e^{k+i}$. Therefore, using formula (4.5) and Lemma 4.2, we get the following proposition.

Proposition 4.3. *The following formula holds:*

$$\begin{aligned}
&\big(Q^{\Delta} I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big) Q^{\widetilde{\rho}} Q^{\widetilde{\theta}+m-1/2} Q^{-\Delta}\big) e^k \\
&= \sum_{l,d=0}^{\infty} \sum_{i=0}^{n-1-k} \sum_{k''=1}^{n-1} \langle e^{k+i}\psi^{l-1}, e^{k''}\rangle_{0,2,d\ell} e_{k''} (-\partial_\lambda)^l \frac{(-(n-1)\partial_m)^i}{i!} \left(\frac{\lambda^{\frac{n-1}{2}+m-k-i}}{\Gamma\big(\frac{n-1}{2}+m-k-i+1\big)} \right) \\
&\quad + \sum_{l,d=0}^{\infty} \sum_{i=0}^{n-1-k} \langle e^{k+i}\psi^{l-1}, 1\rangle_{0,2,d\ell} (-1)^{n-1} Q^n \phi_N (-\partial_\lambda)^l \frac{(-(n-1)\partial_m)^i}{i!} \\
&\quad\quad \times \left(\frac{\lambda^{\frac{n-1}{2}+m-k-i}}{\Gamma\big(\frac{n-1}{2}+m-k-i+1\big)} \right) + O\big(Q\widetilde{H}(E) + Q^{n+1}H(X)\big),
\end{aligned}$$

where $1 \le k \le n-1$ and the notation involving O is the same as in Lemma 4.2.

Proposition 4.4. *If $2 \le a \le N$, then the following formula holds:*

$$\big(Q^{\Delta} I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big) Q^{\widetilde{\rho}} Q^{\widetilde{\theta}+m-1/2} Q^{-\Delta}\big) \phi_a = \frac{\lambda^{\theta+m-1/2}}{\Gamma(\theta+m+1/2)} \phi_a + O(Q).$$

Moreover, in the above expansion, the $H(X)$-component of the coefficient in front of Q^m for $0 \le m \le n-1$ is a Laurent polynomial in $\lambda^{1/2}$ (with coefficients in $H(X)$).

Proof. Using formula (4.5), we get that the left-hand side of the formula that we want to prove is equal to

$$Q^{\Delta} \sum_{i,l=0}^{\infty} (-1)^l Q^l S_l(t,q,Q) \frac{\partial_m^i}{i!} \left(\frac{\lambda^{\widetilde{\theta}+m-l-1/2}}{\Gamma(\widetilde{\theta}+m-l+1/2)} (Q\widetilde{\rho})^i \right) \phi_a.$$

Since $\widetilde{\rho}\phi_a = \rho\phi_a$, the above formula takes the form

$$\sum_{l=0}^{\infty} \sum_{i=0}^{\infty} Q^{\Delta+l+i} S_l(t,q,Q) \rho^i \phi_a (-\partial_\lambda)^l \frac{\partial_m^i}{i!} \left(\frac{\lambda^{(n-1)/2+m-i-\deg(\phi_a)}}{\Gamma((n+1)/2+m-i-\deg(\phi_a))} \right). \tag{4.7}$$

Note that the term in the above double sum corresponding to $l = i = 0$ coincides with the leading order term on the right-hand side of the formula that we have to prove. Therefore, recalling the definition of the calibration, we get that we have to prove the following two statements. First, if $l + i > 0$, then the following expression

$$\sum_{b=1}^{N} \langle \rho^i \phi_a \psi^{l-1}, \phi_b \rangle_{0,2,\beta+d\ell}(t) \phi^b Q^{l+i-d(n-1)} + \sum_{k=1}^{n-1} \langle \rho^i \phi_a \psi^{l-1}, e^k \rangle_{0,2,\beta+d\ell}(t) e_k Q^{k+l+i-d(n-1)-n}$$

has order at least $O(Q)$ for all $\beta + d\ell \in \operatorname{Eff}(\operatorname{Bl}(X))$. Second, there are only finitely many d and l, such that, in the first sum the coefficient in front of Q^m for $0 \le m \le n-1$ is non-zero. Let us consider the correlators in the first sum.

Case 1: if $\beta = 0$. Then the correlator is a twisted GW invariant of the exceptional divisor E and since $\phi_a|_E = 0$, we get that the correlator must be 0.

Case 2: if $\beta \neq 0$ and condition (ii) in Gathmann's vanishing theorem does not hold. Then $d \leq 0$ and we get $l+i-d(n-1) \geq l+i > 0$. In order for the correlator to contribute to the coefficient in front of Q^m for some $0 \leq m \leq n-1$, we must have $-1 \leq d \leq 0$ and $0 \leq l < n$. Clearly, there are only finitely many d and l satisfying these inequalities.

Case 3: if $\beta \neq 0$ and condition (ii) holds. Then condition (iii) in Gathmann's vanishing theorem must fail, that is, $0 \geq (d+1)(n-1)-l+1$ or equivalently $l-d(n-1) \geq n$. The power of Q is $l+i-d(n-1) \geq n+i$. Therefore, the correlators satisfying the conditions of this case contribute only to the coefficients in front of Q^m with $m \geq n$.

The argument for the correlators in the second sum is similar.

Case 1: if $\beta = 0$. Then the correlator is a twisted GW invariant of the exceptional divisor E and since $\phi_a|_E = 0$, we get that the correlator must be 0.

Case 2: if $\beta \neq 0$ and condition (ii) in Gathmann's vanishing theorem does not hold. Then $d \leq 0$ and $k = 1$. The divisor equation implies that if $d = 0$, then the correlator vanishes. Therefore, $d \leq -1$. We get $k+l+i-d(n-1)-n = l+i-(d+1)(n-1) \geq l+i > 0$.

Case 3: if $\beta \neq 0$ and condition (ii) holds. Then condition (iii) in Gathmann's vanishing theorem must fail, that is, $k-1 \geq (d+1)(n-1)-l+1$ or equivalently $k+l-d(n-1) \geq n+1$. The power of Q is $k+l+i-d(n-1)-n \geq i+1 \geq 1$. ■

Proposition 4.5. *The following formula holds:*

$$\big(Q^{\Delta} I^{(-m)}(t,q,Q,Q^{-1}\lambda) Q^{\widetilde{\rho}} Q^{\widetilde{\theta}+m-1/2} Q^{-\Delta}\big)\phi_1 = \frac{\lambda^{\theta+m-1/2}}{\Gamma(\theta+m+1/2)}\phi_1$$
$$+ \sum_{d,l \geq 0} \sum_{i=0}^{n-1} \sum_{k=1}^{n-1} \big\langle (-(n-1)e)^i \psi^{l-1}, e^k \big\rangle_{0,2,d\ell} e_k (-\partial_\lambda)^l \frac{\partial_m^i}{i!} \left(\frac{\lambda^{(n-1)/2+m-i}}{\Gamma((n+1)/2+m-i)} \right) \tag{4.8}$$
$$+ \sum_{l=1}^{\infty} \sum_{\beta \in \mathrm{Eff}(X)} \big\langle \psi^{l-1}, e \big\rangle_{0,2,\beta}(t) e_1 q^{\beta} Q^{l-n+1} (-\partial_\lambda)^l \left(\frac{\lambda^{(n-1)/2+m}}{\Gamma((n+1)/2+m)} \right) + O(Q), \tag{4.9}$$

where $e_k = (-1)^{n-1} e^{n-k}$ *and the correlator*

$$\big\langle (-(n-1)e)^i \psi^{l-1}, e^k \big\rangle_{0,2,d\ell} = \big\langle (-(n-1)e)^i \psi^{l}, e^k, 1 \big\rangle_{0,3,d\ell}$$

can be defined also for $l = 0$.

Proof. Note that if $i > 0$,then $\widetilde{\rho}^i = \rho^i + (-(n-1)e)^i$. Just like in the proof of Proposition 4.4, we get that the left-hand side of the identity that we would like to prove is equal to the sum of (4.7) with $a = 1$ and

$$\sum_{l=0}^{\infty} \sum_{i=1}^{\infty} Q^{\Delta+l+i} S_l(t,q,Q)(-(n-1)e)^i (-\partial_\lambda)^l \frac{\partial_m^i}{i!} \left(\frac{\lambda^{(n-1)/2+m-i}}{\Gamma((n+1)/2+m-i)} \right). \tag{4.10}$$

Let us discuss first the contribution of (4.7). The same argument as in the proof of Proposition 4.4 yields that if $i > 0$, then the corresponding terms in the sum have order at least $O(Q)$. If $i = 0$ and $l = 0$, then the corresponding term in the sum becomes $\lambda^{\theta+m-1/2}/\Gamma(\theta+m+1/2)$ which is precisely the first term on the right-hand side of the formula that we would like to prove. Finally, we are left with the case $i = 0$ and $l \geq 1$. By definition, $Q^{\Delta+l} S_l(t,q,Q)\phi_1$ is

$$\sum_{\beta+d\ell} \sum_{b=1}^{N} \big\langle \psi^{l-1}, \phi_b \big\rangle_{0,2,\beta+d\ell}(t) \phi^b Q^{l-d(n-1)} + \sum_{k=1}^{n-1} \big\langle \psi^{l-1}, e^k \big\rangle_{0,2,\beta+d\ell}(t) e_k q^{\beta} Q^{k+l-d(n-1)-n}, \tag{4.11}$$

where the first sum is over all effective curve classes $\beta + d\ell \in \mathrm{Eff}(\mathrm{Bl}(X))$. Let us consider the following 3 cases for the correlators in (4.11).

Case 1*:* if $\beta = 0$. we may assume that $t = 0$ because $t|_E = 0$. The sum over b is non-zero only if $\phi_b = 1$, that is, $b = 1$. Recalling the formula for the dimension of the moduli space, we get

$$l - 1 = -1 + n + d(n-1) \quad \Rightarrow \quad l - d(n-1) = n.$$

Therefore, in this case the contribution has order $O(Q^n)$. The sum over k in (4.11) (when $\beta = 0$) is independent of Q because by matching the degree of the correlator insertion with the dimension of the virtual fundamental cycle we get $k+l-1 = -1+n+d(n-1)$. Therefore, the contribution to the sum (4.7) with $a = 1$ of the terms with $i = 0$, $l \geq 1$, and degree $\beta = 0$ is

$$\sum_{l=1}^{\infty} \sum_{d=0}^{\infty} \langle \psi^{l-1}, e^k \rangle_{0,2,d\ell} e_k (-\partial_\lambda)^l \left(\frac{\lambda^{(n-1)/2+m}}{\Gamma((n+1)/2+m)} \right).$$

Note that the above sum coincides with the $i = 0$ component of (4.8).

Case 2*:* if $\beta \neq 0$ and condition (ii) in Gathmann's vanishing theorem does not hold. Since $d \leq 0$ and $l \geq 1$ the sum over b has order at least $O(Q)$. For the sum over k, only for $k = 1$ condition (ii) does not hold and if $d \leq -1$ then the term has order at least $O(Q)$. Therefore, only the terms with $k = 1$ and $d = 0$ satisfy the conditions of this case and do not have order $O(Q)$. The corresponding contribution to the sum (4.7) with $a = 1$ becomes

$$\sum_{l \geq 1} \langle \psi^{l-1}, e \rangle_{0,2,\beta}(t) e_1 q^\beta Q^{1+l-n} (-\partial_\lambda)^l \left(\frac{\lambda^{(n-1)/2+m}}{\Gamma((n+1)/2+m)} \right).$$

Note that the above sum coincides with the sum in (4.9).

Case 3*:* if $\beta \neq 0$ and condition (ii) holds. Then condition (iii) does not hold. For the correlators in the sum over b we get $0 \geq (d+1)(n-1) - l + 1 = d(n-1) + n - l$. This inequality implies that the sum over b has order $O(Q^n)$. Similarly, for the correlators in the sum over k, we get

$$k - 1 \geq (d+1)(n-1) - l + 1 = d(n-1) + n - l \quad \Rightarrow \quad k + l - d(n-1) - n \geq 1.$$

In other words, the sum over k has order at least $O(Q)$.

This completes the analysis of the contributions from the sum (4.7) with $a = 1$. It remains to analyze the contributions from the sum (4.10). This is done in a similar way. To begin with, note that the sum of the terms with $l = 0$ is equal to

$$\sum_{i=1}^{n} Q^{\Delta+i} (-(n-1)e)^i (-\partial_\lambda)^l \frac{\partial_m^i}{i!} \left(\frac{\lambda^{(n-1)/2+m-i}}{\Gamma((n+1)/2+m-i)} \right).$$

Note that only the term with $i = n$ depends on Q, that is, it has order $O(Q^n)$. Therefore, up to terms of order $O(Q)$ the above sum coincides with the sum of the terms in (4.8) with $l = 0$ and $i \geq 1$. Suppose that $l > 0$. By definition, $Q^{\Delta+l+i} S_l(t,q,Q) e^i$ is equal to

$$\sum_{\beta+d\ell} \sum_{b=1}^{N} \langle e^i \psi^{l-1}, \phi_b \rangle_{0,2,\beta+d\ell}(t) \phi^b Q^{l+i-d(n-1)}$$
$$+ \sum_{k=1}^{n-1} \langle e^i \psi^{l-1}, e^k \rangle_{0,2,\beta+d\ell}(t) e_k q^\beta Q^{k+l+i-d(n-1)-n}.$$

Let us consider the following 3 cases for the correlators in the above sum.

Case 1*:* if $\beta = 0$. Again since $t|_E = 0$, we may assume that $t = 0$. Let us consider first the correlators in the sum over b. Since $\phi_b|_E = 0$ for $b > 1$, the only non-trivial contribution will come from the term with $b = 1$. The dimension constraint now yields $i+l-1 = -1+n+d(n-1)$ and hence $l+i-d(n-1) = n$. Therefore, the contribution to (4.10) has order $O(Q^n)$. Let us consider now the correlators in the sum over k. The dimension constraint takes the form $i+l-1+k = -1+n+d(n-1)$ and hence $k+l+i-d(n-1)-n = 0$. Therefore, the contribution of these terms to the sum (4.10) is

$$\sum_{l=1}^{\infty}\sum_{i,k=1}^{n-1}\sum_{d=0}^{\infty}\langle(-(n-1)e)^i\psi^{l-1}, e^k\rangle_{0,2,d\ell}e_k(-\partial_\lambda)^l\frac{\partial_m^i}{i!}\left(\frac{\lambda^{(n-1)/2+m-i}}{\Gamma((n+1)/2+m-i)}\right).$$

The above sum coincides with the sum of the terms in (4.8) with $l \geq 1$ and $i \geq 1$.

Note that at this point all terms in the formula that we would like to prove are already matched with contributions from (4.7) and (4.10). It remains only to check that in the remaining two cases the contributions have order $O(Q)$.

Case 2*:* if $\beta \neq 0$ and condition (ii) does not hold. For the sum over b, since $d \leq 0$ and $l \geq 1$, the powers of Q are positive. For the sum over k, in addition to $d \leq 0$, we also have $i-1+k-1 = 0$, that is, $i = k = 1$. If $d \leq -1$, then the power of Q is positive. Suppose that $d = 0$. Using the divisor equation we get $\langle e\psi^{l-1}, e\rangle_{0,2,\beta}(t) = \langle e^2\psi^{l-2}\rangle_{0,1,\beta}(t)$. The latter satisfies both conditions (i) and (ii) of Gathmann's vanishing theorem. In order for the correlator to be non-zero, condition (iii) must fail $1 \geq (d+1)(n-1)-l+2 = d(n-1)+n-l+1$. The power of Q becomes

$$k+l+i-d(n-1)-n = 2+l-d(n-1)-n \geq 2.$$

Case 3*:* if $\beta \neq 0$ and condition (ii) holds. Then condition (iii) must fail. For the correlators in the sum over b, we get $i-1 \geq (d+1)(n-1)-l+1 = d(n-1)+n-l$. Therefore, the power of Q is $l+i-d(n-1) \geq n+1$. For the correlators in the sum over k, we get $i-1+k-1 \geq d(n-1)+n-l$, that is, the power of Q is $k+l+i-d(n-1)-n \geq 2$. ■

4.2 Quantum cohomology of the blowup

Let us recall the result of Bayer [3]. Suppose that $t \in \widetilde{H}^*(X) \subset H^*(\mathrm{Bl}(X))$, that is, $t_1 = t_{N+1} = \cdots = t_{N+n-1} = 0$. Let us denote by $\widetilde{\Omega}_i(t,q,Q)$ $(1 \leq i \leq N+n-1)$ the linear operator in $H^*(\mathrm{Bl}(X))$ defined by quantum multiplication $\phi_i\bullet_{t,q,Q}$ for $1 \leq i \leq N$ and by quantum multiplication by $e^k\bullet_{t,q,Q}$ for $i = N+k$, $1 \leq k \leq n-1$. Slightly abusing the notation let us denote by the same letters $\widetilde{\Omega}_i$ the matrices of the corresponding linear operators with respect to the basis ϕ_i $(1 \leq i \leq N+n-1)$, where recall that $\phi_{N+k} := e^k$ $(1 \leq k \leq n-1)$. Note that the matrix of Δ is diagonal with diagonal entries $0,\dots,0,-1,-2,\dots,-n+1$ (0 appears N times). The main observation of Bayer (see [3, Section 3.4]) can be stated as follows.

Proposition 4.6. *The matrices of the linear operators $\widetilde{\Omega}_i$ $(1 \leq i \leq N+n-1)$ with respect to the basis $Q^{-\Delta}\phi_i$ $(1 \leq i \leq N+n-1)$ have the following Laurent series expansions at $Q = 0$:*

$$Q^{\Delta}\widetilde{\Omega}_i(t,q,Q)Q^{-\Delta} = \begin{bmatrix}\Omega_i(t,q)+O\big(Q^{n-1}\big) & O(Q^n)\\ O(1) & \delta_{i,1}\,\mathrm{Id}_{n-1}+O(Q)\end{bmatrix}, \qquad 1 \leq i \leq N, \tag{4.12}$$

where Id_{n-1} is the identity matrix of size $(n-1)\times(n-1)$ and

$$Q^{\Delta}\widetilde{\Omega}_{N+a}(t,q,Q)Q^{-\Delta} = Q^{-a}\begin{bmatrix}O(Q^n) & O(Q^n)\\ O(1) & \epsilon^a+O\big(Q^2\big)\end{bmatrix}, \qquad 1 \leq a \leq n-1, \tag{4.13}$$

where $\Omega_i(t,q)$ *is the matrix of the linear operator in* $H^*(X)$ *defined by quantum multiplication by* $\phi_i \bullet_{t,q}$ *with respect to the basis* ϕ_i $(1 \le i \le N)$ *and* ϵ *is the following* $(n-1)\times(n-1)$*-matrix:*

$$\epsilon = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^n \\ 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Proof. The proof is based on Gathmann's vanishing theorem and it is very similar to the proof of Lemma 4.2. Since the proofs of (4.12) and (4.13) are similar, let us prove only (4.13). We have

$$Q^\Delta\big(e^a \bullet \big(Q^{-\Delta} e^k\big)\big) = \sum_{\widetilde{\beta}=\beta+d\ell} \left(\sum_{j=1}^N \big\langle e^a, e^k, \phi^j \big\rangle_{0,3,\beta+d\ell}(t)\phi_j \right.$$
$$\left. + \sum_{l=1}^{n-1} \big\langle e^a, e^k, e_l \big\rangle_{0,3,\beta+d\ell}(t) e^l Q^{-l} \right) q^\beta Q^{k-d(n-1)}.$$

Let us examine the correlators in the sum over j, that is,

$$\big\langle e^a, e^k, \phi^j \big\rangle_{0,3,\beta+d\ell}(t) Q^{k-d(n-1)}.$$

There are 3 cases.

Case 1*:* if $\beta = 0$. The correlator is a twisted GW invariant of the exceptional divisor E. The restriction $\phi^j|_E$ is non-zero only if $\phi^j = 1$, that is, $j = N$. Recalling the string equation, we get that the correlator is non-zero only if $\phi^j = 1$ and $d = 0$. Therefore, the contribution takes the form

$$\int_{\mathrm{Bl}(X)} e^a \cup e^k \cup 1 \; \phi_N Q^k = (-1)^{n-1} Q^{n-a} \delta_{a+k,n} \; \phi_N.$$

Case 2*:* if $\beta \neq 0$ and condition (ii) in Gathmann's vanishing theorem does not hold. Here we have in mind the correlator $\big\langle e^a, e^k, \phi^j \big\rangle_{0,3,\beta+d\ell}(t)$. Note that the weight of this correlator is $a-1+k-1$. If condition (ii) does not hold, then $a-1+k-1 \le 0$ and $d \le 0$. Since $a, k \ge 1$, this case is possible only if $a = k = 1$. Moreover, if $d = 0$, then the correlator vanishes by the divisor equation. Therefore, we may assume that $d \le -1$. The power of Q becomes $k - d(n-1) \ge 1 + n - 1 = n$, that is, the contribution in this case has order $O(Q^n) = O\big(Q^{n+1-a}\big)$.

Case 3*:* if $\beta \neq 0$ and condition (ii) holds. According to Gathmann's vanishing theorem, condition (iii) does not hold, that is,

$$a-1+k-1 \ge (d+1)(n-1) = d(n-1) + n - 1 \quad \Rightarrow \quad k - d(n-1) \ge n+1-a.$$

We get that the contribution in this case has order $O\big(Q^{n+1-a}\big)$.

Combining the results of the 3 cases, we get that the sum over j has the form

$$Q^{-a}\big((-1)^{n-1}\delta_{a+k,n} Q^n \phi_N + O\big(Q^{n+1}\big)\big).$$

Let us examine the correlators in the sum over l. Just like above, there are 3 cases.

Case 1*:* if $\beta = 0$. The correlator $\big\langle e^a, e^k, e_l \big\rangle_{0,3,d\ell}(t)$ can be computed explicitly. Indeed, such a correlator is a twisted GW invariant of the exceptional divisor, so it is independent of $t \in H^*(X)$, that is, we may substitute $t = 0$. Moreover, since $d\ell$ must be an effective curve class in E, we have $d \ge 0$. Recall that $e_l = (-1)^{n-1} e^{n-l}$ and note that the dimension of the virtual fundamental cycle of $\overline{\mathcal{M}}_{0,3}(\mathrm{Bl}(X), d\ell)$ is $d(n-1)+n$. Therefore, $a+k-l=d(n-1)$. We conclude that $d=0$ or $d=1$, that is,

$$\langle e^a, e^k, e_l\rangle_{0,3,d\ell}(t) = \begin{cases} (-1)^n & \text{if } d=1 \text{ and } l=a+k-n+1, \\ 1 & \text{if } d=0 \text{ and } l=a+k, \\ 0 & \text{otherwise.} \end{cases}$$

The contribution to the sum over l becomes

$$\begin{cases} (-1)^n e^{a+k-n+1} Q^{-a} & \text{if } a+k>n-1, \\ e^{a+k} Q^{-a} & \text{if } a+k \le n-1. \end{cases} \tag{4.14}$$

Note that the matrix ϵ^a has entries

$$\epsilon^a_{ij} = \begin{cases} (-1)^n & \text{if } j=i+n-1-a, \\ 1 & \text{if } i=j+a, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with formula (4.14), we get that the contribution in this case to formula (4.13) coincides with the matrix $Q^{-a}\epsilon^a$.

Case 2*:* if $\beta \neq 0$ and condition (ii) does not hold. Then $d \le 0$ and $a-1+k-1+n-l-1 \le 0$. Since $a, k, n-l \ge 1$, this case is possible only if $a=k=1$ and $l=n-1$. Since $\beta \neq 0$, the divisor equation implies that $d \neq 0$, that is, $d \le -1$. In other words, if condition (ii) in Gathmann's vanishing theorem does not hold, then the power of Q, must be $k-l-d(n-1) \ge 1-(n-1)+n-1 = 1 = -a+2$.

Case 3*:* if $\beta \neq 0$ and condition (ii) holds. Then condition (iii) must fail, that is, $a-1+k-1+n-l-1 \ge (d+1)(n-1) = d(n-1)+n-1$, or equivalently $k-l-d(n-1) \ge -a+2$.

Combining the results of these 3 cases, we get that the contribution of the sum over l matches the (2,2)-block of the matrix on the right-hand side in formula (4.13) with the factor Q^{-a} inserted.

In order to complete the argument, we have to repeat the above discussion by replacing e^k with ϕ_i ($1 \le i \le N$), that is, we have to determine the contribution to the right-hand side of (4.13) of the following expression:

$$\begin{aligned} &Q^{\Delta}\big(e^a \bullet \big(Q^{-\Delta}\phi_i\big)\big) \\ &\quad = \sum_{\widetilde{\beta}=\beta+d\ell} \left(\sum_{j=1}^{N} \langle e^a, \phi_i, \phi^j\rangle_{0,3,\beta+d\ell}(t)\phi_j + \sum_{l=1}^{n-1} \langle e^a, \phi_i, e_l\rangle_{0,3,\beta+d\ell}(t) e^l Q^{-l} \right) q^{\beta} Q^{-d(n-1)}. \end{aligned}$$

First, let us determine the contribution of the correlators in the sum over j.

Case 1*:* if $\beta = 0$. The correlator could be non-zero only if $\phi^j = 1$ and $d=0$. In the latter case, since $\int_{\mathrm{Bl}(X)} e^a \cup \phi_i \cup 1 = 0$, we get that the correlator still vanishes. There is no contribution in this case.

Case 2*:* if $\beta \neq 0$ and condition (ii) does not hold. Then $d \le 0$ and the weight $a-1 \le 0$, that is, $a=1$. Due to divisor equation, $d \neq 0$, so $d \le -1$ and $-d(n-1) \ge n-1 = n-a$. We get that the contribution in this case has order $O\big(Q^{n-1}\big)$.

Case 3*:* if $\beta \neq 0$ and condition (ii) does hold. Then condition (iii) does not hold, so $a-1 \ge (d+1)(n-1)$ and $-d(n-1) \ge n-a$. We get that the contribution in this case is still of order $O(Q^{n-a})$.

Combining the results of the 3 cases, we get that the order of the elements in the (1,1)-block of the matrix $Q^{\Delta}\widetilde{\Omega}_{N+a}Q^{-\Delta}$ is $O(Q^{n-a})$, that is, the same as in formula (4.13).

Finally, it remains to determine the contribution of the correlators in the sum over l.

Case 1*:* if $\beta = 0$. The correlator could be non-zero only if $\phi_i = 1$, that is, $i = 1$ and $d = 0$. We get that the contribution in this case is $\delta_{i,1} e^a Q^{-a}$.

Case 2*:* if $\beta \neq 0$ and condition (ii) does not hold. Then $d \leq 0$ and the weight $a - 1 + n - l - 1 \leq 0$, that is, $a = n - l = 1$. Due to divisor equation, $d \neq 0$, so $d \leq -1$ and $-l - d(n-1) \geq -l + n - 1 = 0 = -a + 1$. The contribution in this case has order $O\big(Q^{-a+1}\big)$.

Case 3*:* if $\beta \neq 0$ and condition (ii) does hold. Then condition (iii) must fails. We get

$$a - 1 + n - l - 1 \geq (d+1)(n-1) \quad \Rightarrow \quad -l - d(n-1) \geq -a + 1.$$

The contribution in this case also has order $O\big(Q^{-a+1}\big)$.

Combining the results in the 3 cases we get that the elements in the (2,1)-block of the matrix $Q^{\Delta} \widetilde{\Omega}_{N+a} Q^{-\Delta}$ have the form $Q^{-a}(E_{a,1} + O(Q))$, where $E_{a,1}$ denotes the matrix whose $(a,1)$-entry is 1 and the remaining entries are 0. ■

5 The exceptional component of a reflection vector

Suppose that $\alpha \in H^*(\mathrm{Bl}(X))$ is a reflection vector. Let us decompose $\alpha = \alpha_e + \alpha_b$, where $\alpha_e \in \widetilde{H}^*(E)$ and $\alpha_b \in H^*(X)$. We will refer to α_e and α_b as respectively the *exceptional* and the *base* components of α. Using Proposition 4.3, we would like to classify the exceptional components of the reflection vectors.

5.1 Dependence on the Novikov variables

Since the quantum cohomology is a Frobenius manifold depending on the parameters $q := (q_1, \ldots, q_r)$ and $q_{r+1} := Q^{n-1}$, the reflection vectors depend on q_i too. We claim that if α is a reflection vector, then

$$\alpha = q_1^{-p_1} \cdots q_r^{-p_r} q_{r+1}^{-e} \beta, \tag{5.1}$$

where $\beta \in H^*(\mathrm{Bl}(X))$ is independent of q_i $(1 \leq i \leq r+1)$. To proof this fact, we will make use of the divisor equation. Suppose that the basis of divisor classes is part of the basis $\{\phi_i\}_{1 \leq i \leq N+n-1}$, such that, $p_i = \phi_{i+1}$ for $1 \leq i \leq r$ and $p_{r+1} = e = \phi_{N+1}$. Let τ_i $(1 \leq i \leq r+1)$ be the linear coordinates corresponding to the divisor classes p_i, that is, $\tau_i := t_{i+1}$ for $1 \leq i \leq r$ and $\tau_{r+1} = t_{N+1}$. Using the divisor equation we get that the calibration satisfies the following differential equations:

$$\begin{aligned} &z \frac{\partial}{\partial \tau_i} S(t,q,Q,z) = p_i \bullet S(t,q,Q,z), \\ &z q_i \frac{\partial}{\partial q_i} S(t,q,Q,z) = z \frac{\partial}{\partial \tau_i} S(t,q,Q,z) - S(t,q,Q,z) p_i \cup . \end{aligned}$$

Therefore,

$$S(t,q,Q,z) = T(t,q,Q,z) e^{\sum_{i=1}^{r+1} \tau_i p_i \cup / z},$$

where for fixed z the operator series $T(t,q,Q,z)$ is a function on the variables

$$t_1, q_1 e^{t_2}, \ldots, q_r e^{t_{r+1}}, t_{r+2}, \ldots, t_N, q_{r+1} e^{t_{N+1}}, t_{N+2}, \ldots, t_{N+n-1}. \tag{5.2}$$

As we already pointed out before (see Section 2.5), due to the divisor equation, the operators of quantum multiplication $\phi_i \bullet_{t,q,Q}$ are represented by matrices whose entries are functions in the variables (2.5) too. Since the canonical coordinates $u_i(t,q,Q)$ are eigenvalues of $E \bullet_{t,q,Q}$, it follows that they have the same property. Moreover, using the chain rule, we get that the partial derivatives $\frac{\partial u_j}{\partial t_a}$ are also functions in (5.2). On the other hand, if α is a reflection vector, then the Laurent series expansion of $I^{(-m)}_\alpha(t,q,Q,\lambda)$ at a point $\lambda = u_i(t,q,Q)$ has coefficients that are rational functions in the canonical coordinates $u_j(t,q,Q)$ and their partial derivatives $\frac{\partial u_j}{\partial t_a}$ (see Section 2.3). Therefore,

$$\left(\frac{\partial}{\partial \tau_i} - q_i \frac{\partial}{\partial q_i}\right) I^{(-m)}_\alpha(t,q,Q,\lambda) = 0. \tag{5.3}$$

By definition,

$$\begin{aligned} I^{(-m)}(t,q,Q,\lambda) &= S\big(t,q,Q,-\partial_\lambda^{-1}\big)\widetilde{I}^{-m}(\lambda) = T\big(t,q,Q,-\partial_\lambda^{-1}\big)e^{-\sum_{i=1}^{r+1}\tau_i p_i \cup \partial_\lambda}\widetilde{I}^{(-m)}(\lambda) \\ &= T\big(t,q,Q,-\partial_\lambda^{-1}\big)\widetilde{I}^{-m}(\lambda)e^{-\sum_{i=1}^{r+1}\tau_i p_i}, \end{aligned}$$

where for the last equality we used the following relation (see also the proof of Lemma 4.1 (a)):

$$-p \cup \partial_\lambda \frac{\lambda^{\theta+\alpha-1}}{\Gamma(\theta+\alpha)} = \frac{\lambda^{\theta+\alpha-1}}{\Gamma(\theta+\alpha)}(-p).$$

Since $I^{(-m)}_\alpha(t,q,Q,\lambda) = I^{(-m)}(t,q,Q,\lambda)\alpha$, from equation (5.3) we get

$$q_i \frac{\partial \alpha}{\partial q_i} + p_i \cup \alpha = 0, \qquad \forall 1 \le i \le r+1.$$

Our claim that the reflection vector has the form (5.1) follows.

5.2 Canonical coordinates

We would like to determine the dependence of the canonical coordinates $u_i(t,q,Q)$ $(1 \le i \le N+n-1)$ on Q, where the parameter $t \in \widetilde{H}(X)$, that is, $t_1 = t_{N+1} = \cdots = t_{N+n-1} = 0$. Using the identity $u_i = \widetilde{E}(u_i)$, we get

$$u_i(t,q,Q) = \sum_{a=2}^{N}(1-\deg\phi_a)t_a\frac{\partial u_i}{\partial t_a}(t,q,Q) + \sum_{j=1}^{r}\rho_j\frac{\partial u_i}{\partial \tau_j}(t,q,Q) - (n-1)\frac{\partial u_i}{\partial t_{N+1}}(t,q,Q),$$

where ρ_j are the coefficients in the decomposition $c_1(TX) = \sum_{j=1}^N \rho_j p_j$ and $\tau_j = t_{j+1}$. The above formula allows us to reduce the problem to investigating the dependence on Q of the partial derivatives $\frac{\partial u_i}{\partial t_j}$ $(1 \le i \le N+n-1,\ 1 \le j \le N+1)$. The advantage now is that the eigenvalues of the operator $\widetilde{\Omega}_j(t,q,Q) = \phi_j \bullet_{t,q,Q}$ of quantum multiplication by ϕ_j are precisely $\frac{\partial u_i}{\partial t_j}$ $(1 \le i \le N+n-1)$.

Lemma 5.1. *Suppose that $U(Q)$ is a square matrix of size $k \times k$ whose entries are functions holomorphic at $Q = 0$.*

- (*a*) *There exists an integer $b > 0$, such that, every eigenvalue of $U(Q)$ has an expansion of the form $\lambda_0 + \sum_{i=1}^\infty \lambda_i Q^{i/b}$.*
- (*b*) *If λ_0 is an eigenvalue of $U(0)$ of multiplicity 1, then $U(Q)$ has a unique eigenvalue of multiplicity one of the form $\lambda_0 + \sum_{i=1}^\infty \lambda_i Q^i$.*

Proof. The eigenvalues are roots of the characteristic polynomial $\det(\lambda - U(Q))$. This is a monic polynomial in λ of degree k with coefficients in $C\{Q\}$ — the ring of convergent power series in Q. Therefore, in order to prove (a), it is sufficient to prove the following statement.

Let $f(Q,\lambda)\in C\{Q\}[\lambda]$ be a monic polynomial. Then the roots of $f(Q,\lambda)$ have the expansion stated in the lemma. Let us decompose

$$f(0,\lambda)=(\lambda-w_1)^{b_1}\cdots(\lambda-w_s)^{b_s},$$

where $w_i\neq w_j$ for $i\neq j$. Recalling Hensel's lemma (see [22, Chapter 2, Section 2]), we get that $f(Q,\lambda)=f_1(Q,\lambda)\cdots f_r(Q,\lambda)$, where $f_i(Q,\lambda)\in C\{Q\}[\lambda]$ is a monic polynomials of degree b_i, such that, $f_i(0,\lambda)=(\lambda-w_i)^{b_i}$. Note that if $b_i=1$ for some i, then the unique zero of $f_i(Q,\lambda)=0$ is a holomorphic at $Q=0$ and its value at $Q=0$ is w_i. Therefore, part (b) is an elementary consequence of Hensel's lemma. If $s>1$, then the lemma follows from the inductive assumption. Suppose that $s=1$, that is,

$$f(Q,\lambda)=\lambda^k+a_1(Q)\lambda^{k-1}+\cdots+a_k(Q),$$

where $a_i(0)=0$. We may assume that the sub-leading coefficient $a_1(Q)=0$. Indeed, using the substitution $\lambda\mapsto\lambda-a_1(Q)/k$ we can transform the polynomial to one for which the sub-leading coefficient is 0. The roots of the two polynomials are related by a shift of $a_1(Q)/k$, so it is sufficient to prove our claim for one of them. Let $\operatorname{ord}(a_i)$ be the order of vanishing of a_i at $Q=0$. If $a_i(Q)=0$, then we define the order of vanishing to be $+\infty$. Put $\nu:=\min_{1\le i\le k}\frac{\operatorname{ord}(a_i)}{i}$. Substituting $\lambda=Q^\nu\mu$ in the equation $f(Q,\lambda)=0$ and dividing by $Q^{\nu k}$, we get

$$\mu^k+\sum_{i=2}^k a_i(Q)Q^{-\nu i}\mu^{k-i}=0.$$

Since $\operatorname{ord}(a_i)\ge\nu i$ with equality for at leats one i, we get that the left-hand side of the above equation is a monic polynomial $g\big(Q^{1/b},\mu\big)$ in $\mathbb{C}\big\{Q^{1/b}\big\}[\mu]$ for some integer $b>0$. Note that $g(0,\mu)$ has at least two different zeroes because its sub-leading coefficient is 0. Therefore, just like above, we can use Hensel's lemma to reduce the proof to a case in which the inductive assumption can be applied. This completes the proof. ■

Proposition 5.2. *Let $u_j(t,q,Q)$ $(1\le j\le N+n-1)$ be the canonical coordinates of the quantum cohomology of* $\mathrm{Bl}(X)$*, where the parameter $t\in\widetilde{H}(X)$. After renumbering, the canonical coordinates split into two groups*

$$u_j(t,q,Q)\in\mathbb{C}\{Q\},\qquad 1\le j\le N,$$

and

$$u_j(t,q,Q)=-(n-1)v_kQ^{-1}+O(1),\qquad j=N+k,\qquad 1\le k\le n-1,$$

where v_k $(1\le k\le n-1)$ are the solutions of the equation $\lambda^{n-1}=(-1)^n$.

Proof. Let us apply the above lemma to the matrix of the linear operator

$$\sum_{a=2}^N(1-\deg\phi_a)t_a\widetilde{\Omega}_a(t,q,Q)+\sum_{j=1}^r\rho_j\widetilde{\Omega}_{j+1}(t,q,Q)+Q\widetilde{\Omega}_{N+1}(t,q,Q)\tag{5.4}$$

with respect to the basis $Q^{-\Delta}\phi_i$ $(1\le i\le N+n-1)$. Recalling Proposition 4.6, we get that the entries of the matrix of the operator (5.4) are holomorphic at $Q=0$ and that its specialization to $Q=0$ has the form

$$\begin{bmatrix}E\bullet_{t,q} & 0\\ * & \epsilon\end{bmatrix}.$$

The eigenvalues of the above matrix are the canonical coordinates $u_i^X(t,q)$ $(1 \le i \le N)$ of the quantum cohomology of X and the solutions v_k $(1 \le k \le n-1)$ of the equation $\lambda^{n-1} = (-1)^n$. Note that for a generic choice of t the eigenvalues are pairwise distinct. On the other hand, the canonical vector fields $\frac{\partial}{\partial u_j}$ $(1 \le j \le N+n-1)$ form an eigenbasis for the operator (5.4). Let us enumerate the canonical coordinates in such a way that the eigenvalues corresponding to $\frac{\partial}{\partial u_j}$ for $1 \le j \le N$ and $j = N+k$ with $1 \le k \le n-1$ are respectively $u_j^X(t,q) + O(Q)$ and $v_k + O(Q)$. Recall that the eigenvalues of the operators $\widetilde{\Omega}_a(t,q,Q)$ are $\frac{\partial u_j}{\partial t_a}(t,q,Q)$ $(1 \le j \le N+n-1)$. Recalling Lemma 5.1 (b), we get that the functions

$$E(u_j) + Q\frac{\partial u_j}{\partial t_{N+1}}, \qquad 1 \le j \le N+n-1$$

are holomorphic at $Q = 0$, where $E := \sum_{a=2}^N (1-\deg(\phi_a))t_a\partial/\partial t_a + \sum_{j=1}^r \rho_j \partial/\partial t_{j+1}$. Moreover, the restriction to $Q = 0$ satisfies

$$\left(E(u_j) + Q\frac{\partial u_j}{\partial t_{N+1}}\right)\Big|_{Q=0} = \begin{cases} u_j^X(t,q) & \text{if } 1 \le j \le N, \\ v_k & \text{if } j = N+k. \end{cases}$$

On the other hand, note that $E(u_j)$ are the eigenvalues of the matrix

$$\sum_{a=2}^N (1-\deg\phi_a)t_a\widetilde{\Omega}_a(t,q,Q) + \sum_{j=1}^r \rho_j\widetilde{\Omega}_{j+1}(t,q,Q)$$

and that the restriction of the above matrix at $Q = 0$ is

$$\begin{bmatrix} E\bullet_{t,q} & 0 \\ * & 0 \end{bmatrix}.$$

Recalling Lemma 5.1, we get that N of the eigenvalues $E(u_j)$ $(1 \le j \le N+n-1)$ are holomorphic at $Q = 0$ and have the form $u_i^X(t,q) + O(Q)$ $(1 \le i \le N)$, while the remaining $n-1$ ones have order $O(Q^\alpha)$ for some rational number $\alpha > 0$. Similarly, by applying Lemma 5.1 to the matrix $Q\widetilde{\Omega}_{N+1}$, we get that its eigenvalues $Q\frac{\partial u_j}{\partial t_{N+1}}$ split into two groups. The first group consist of $n-1$ functions holomorphic at $Q = 0$ with an expansion of the form $v_k + O(Q)$, while the second group consist of N functions that have an expansion in possibly fractional powers of Q of order $O\big(Q^\beta\big)$ for some $\beta > 0$. Let (t,q) be generic, such that, the canonical coordinates $u_i^X(t,q)$ $(1 \le i \le N)$ are pairwise distinct and non-zero. Then for every $1 \le j \le N+n-1$, the sum $E(u_j) + Q\frac{\partial u_j}{\partial t_{N+1}} \ne 0$. Therefore, the two numbers $E(u_j)$ and $Q\frac{\partial u_j}{\partial t_{N+1}}$ can not be vanishing at $Q = 0$, that is, either $E(u_j)$ is holomorphic at $Q = 0$ of the form $u_i^X(t,q) + O(Q)$ or $Q\frac{\partial u_j}{\partial t_{N+1}}$ is holomorphic at $Q = 0$ of the form $v_k + O(Q)$. In the first case, since $E(u_j)$ is holomorphic at $Q = 0$ and the sum $E(u_j) + Q\frac{\partial u_j}{\partial t_{N+1}}$ is also holomorphic at $Q = 0$, we get that $Q\frac{\partial u_j}{\partial t_{N+1}}$ is holomorphic at $Q = 0$. Similarly, the holomorphicity of $Q\frac{\partial u_j}{\partial t_{N+1}}$ implies that $E(u_j)$ is holomorphic. Therefore, $E(u_j)$ and $Q\frac{\partial u_j}{\partial t_{N+1}}$ are holomorphic at $Q = 0$ for all j. In particular, the numbers α and β must be integral. Note that since $E(u_j)$ and $Q\frac{\partial u_j}{\partial t_{N+1}}$ can not vanish simultaneously at $Q = 0$, we get that for every $1 \le j \le N+n-1$ either

$$E(u_j) = u_i^X(t,q) + O(Q), \qquad Q\frac{\partial u_j}{\partial t_{N+1}} = O(Q)$$

for some i or

$$E(u_j) = O(Q), \qquad Q\frac{\partial u_j}{\partial t_{N+1}} = v_k + O(Q)$$

for some k. In the first case, we will get that

$$u_j(t,q,Q) = E(u_j) - (n-1)\frac{\partial u_j}{\partial t_{N+1}} \in \mathbb{C}\{Q\},$$

while in the second case

$$u_j(t,q,Q) = E(u_j) - (n-1)\frac{\partial u_j}{\partial t_{N+1}} = -(n-1)v_k Q^{-1} + O(1). \qquad \blacksquare$$

5.3 Twisted periods of $\mathbb{P}^{n-1}$

Let us recall the reduced cohomology $\widetilde{H}(E)$ of the exceptional divisor. It has a basis given by e^i ($1 \le i \le n-1$). The Poincaré pairing on $H(\mathrm{Bl}(X))$ induces a non-degenerate pairing on $\widetilde{H}(E)$:

$$\big(e^i, e^j\big) = (-1)^{n-1}\delta_{i+j,n}, \qquad 1 \le i,j \le n-1.$$

The *twisted periods* will be multi-valued analytic functions with values in $\widetilde{H}(E)$. Let us define the following linear operators on $\widetilde{H}(E)$:

$$^{tw}\theta\big(e^i\big) := \Big(\frac{n}{2} - i\Big)\, e^i, \qquad {}^{tw}\rho\big(e^i\big) := \begin{cases} -(n-1)e^{i+1} & \text{if } 1 \le i < n-1, \\ 0 & \text{if } i = n-1. \end{cases}$$

Let us define first the calibrated twisted periods:

$$^{tw}\widetilde{I}^{(-m)}_{\beta}(\lambda) = e^{{}^{tw}\rho\partial_\lambda\partial_m}\left(\frac{\lambda^{{}^{tw}\theta + m - 1/2}}{\Gamma\big({}^{tw}\theta + m + 1/2\big)}\right)\beta, \qquad \beta \in \widetilde{H}(E),$$

and the twisted calibration

$$^{tw}S(Q,z) = \sum_{k=0}^{\infty} {}^{tw}S_k(Q) z^{-k} \in \mathrm{End}\,\big(\widetilde{H}(E)\big)\big[\big[z^{-1}\big]\big],$$

where ${}^{tw}S_0(Q) = 1$ and

$$\big({}^{tw}S_k(Q)e^i, e^j\big) = \sum_{d=0}^{\infty} \big\langle e^i\psi^{k-1}, e^j\big\rangle_{0,2,d\ell} Q^{-d(n-1)}, \qquad 1 \le i,j \le n-1.$$

Note that in the above sum only one value of d contributes, because the degree of the cohomology class in the correlator, that is, $k-1+i+j$ must be equal to the dimension of the virtual fundamental cycle of $\overline{\mathcal{M}}_{0,2}(\mathrm{Bl}(X), d\ell)$ which is $(n-1)(d+1)$ and we get $d(n-1) = k+i+j-n$. The twisted periods are defined by

$$^{tw}I^{(-m)}_{\beta}(Q,\lambda) := \sum_{l=0}^{\infty} {}^{tw}S_l(Q)(-\partial_\lambda)^l\ {}^{tw}\widetilde{I}^{(-m)}_{\beta}(\lambda), \qquad \beta \in \widetilde{H}(E).$$

The twisted periods satisfy a system of ODEs with respect to Q and λ. Let us derive these differential equations.

Lemma 5.3. *We have*

$$\big(\lambda - {}^{tw}\rho\big)\partial_\lambda\, {}^{tw}\widetilde{I}^{(-m)}_{\beta}(\lambda) = \Big({}^{tw}\theta + m - \frac{1}{2}\Big)\, {}^{tw}\widetilde{I}^{(-m)}_{\beta}(\lambda).$$

The proof is straightforward and it is left as an exercise.

Lemma 5.4. *We have*

$$Q\partial_Q {}^{tw}S_l + {}^{tw}\theta {}^{tw}S_l - {}^{tw}S_l {}^{tw}\theta = -l {}^{tw}S_l.$$

Proof. Let us apply the operator on the left-hand side to e^i and compute the pairing with e^j for an arbitrary $1 \le i, j \le n-1$. We get

$$Q\partial_Q \big({}^{tw}S_l e^i, e^j\big) + \big({}^{tw}\theta {}^{tw}S_l e^i, e^j\big) - \big({}^{tw}S_l {}^{tw}\theta e^i, e^j\big).$$

Since ${}^{tw}\theta e^i = \left(\frac{n}{2} - i\right) e^i$ and ${}^{tw}\theta$ is skew-symmetric with respect to the pairing $(\ , \ )$, the above expression becomes

$$Q\partial_Q \big({}^{tw}S_l e^i, e^j\big) - (n-i-j)\big({}^{tw}S_l e^i, e^j\big).$$

We saw above that the expression $\big({}^{tw}S_l e^i, e^j\big)$ is proportional to $Q^{-d(n-1)}$ where $d(n-1) = l + i + j - n$. Therefore, the above expression becomes $-l\big({}^{tw}S_l e^i, e^j\big)$. ■

In order to state the next result we need to introduce the linear operator

$$e\bullet_{tw}\colon\ \widetilde{H}(E) \to \widetilde{H}(E), \qquad e^i \mapsto e \bullet_{tw} e^i,$$

where the quantum product is defined by

$$\big(e \bullet_{tw} e^i, e^j\big) = \sum_{d=0}^{\infty} \big\langle e, e^i, e^j \big\rangle_{0,3,d\ell} Q^{-d(n-1)}.$$

For dimensional reasons, that is, $1 + i + j = n + d(n-1)$, we get that the contributions to the quantum product could be non-trivial only in degree $d = 0$ and $d = 1$. Recalling our computations from Section 3.4, we get the following formulas:

$$e \bullet_{tw} e^i = \begin{cases} e^{i+1} & \text{if } 1 \le i \le n-2, \\ (-1)^n Q^{-(n-1)} e & \text{if } i = n-1. \end{cases}$$

In other words, the matrix of $e\bullet_{tw}$ with respect to the basis $e, e^2, \dots, e^{n-1}$ is

$$e\bullet_{tw} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^n Q^{-(n-1)} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Lemma 5.5. *We have*

$$Q\partial_Q {}^{tw}S_l = (n-1) e \bullet_{tw} {}^{tw}S_{l-1} + {}^{tw}S_{l-1}\, {}^{tw}\rho, \qquad \forall l \ge 1.$$

Proof. The lemma is an easy consequence of the divisor equation and the topological recursion relations for the GW invariants of the blowup $\mathrm{Bl}(X)$. We have, by the divisor equation,

$$\big\langle e, e^i\psi^l, e^j \big\rangle_{0,3,d\ell} = -d\big\langle e^i\psi^l, e^j\big\rangle_{0,2,d\ell} + \big\langle e \cup e^i\psi^{l-1}, e^j\big\rangle_{0,3,d\ell}.$$

The left-hand side, according to the topological recursion relations is equal to

$$\sum_{d'+d''=d} \sum_{k=1}^{n-1} \big\langle e^i\psi^{l-1}, e_k\big\rangle_{0,2,d'\ell} \big\langle e^k, e, e^j\big\rangle_{0,3,d''\ell}.$$

Multiplying the above identity by $(n-1)Q^{-d(n-1)}$ and summing over all $d \ge 0$, we get

$$\big(S_l e^i, (n-1) e \bullet_{tw} e^j\big) = Q\partial_Q \big(S_{l+1} e^i, e^j\big) + \big(S_l (n-1) e \cup e^i, e^j\big).$$

Note that the above expression is 0 for $i = n-1$ because $e^n = (-1)^{n-1}\phi_N$ is a cohomology class on $\mathrm{Bl}(X)$ whose restriction to the exceptional divisor E is 0. Therefore, we may replace $(n-1)e \cup e^i$ with $-{}^{tw}\rho(e^i)$. The lemma follows. ■

Using Lemmas 5.3, 5.4 and 5.5, we get that the twisted periods satisfy the following system of differential equations

$$(\lambda + (n-1)e\bullet_{tw})\partial_\lambda\, {}^{tw}I^{(-m)}_\alpha(Q,\lambda) = \left({}^{tw}\theta + m - \frac{1}{2}\right) {}^{tw}I^{(-m)}_\alpha(Q,\lambda), \tag{5.5}$$

$$Q\partial_Q\, {}^{tw}I^{(-m)}_\alpha(Q,\lambda) = -(n-1)e \bullet_{tw} \partial_\lambda\, {}^{tw}I^{(-m)}_\alpha(Q,\lambda), \tag{5.6}$$

where $\alpha = Q^{{}^{tw}\rho}\beta$ with $\beta \in \widetilde{H}(E)$ independent of Q and λ. Note that the determinant

$$\det(\lambda + (n-1)e\bullet_{tw}) = \lambda^{n-1} + \big((n-1)Q^{-1}\big)^{n-1}.$$

We get that the twisted periods are multivalued analytic functions on the complement of the hypersurface in $\mathbb{C}^* \times \mathbb{C}$ defined by the equation $(Q\lambda)^{n-1} + (n-1)^{n-1} = 0$.

5.4 Periods of $\mathbb{P}^{n-2}$

We would like to compute the monodromy of the system of differential equations (5.5)–(5.6). We will do this by identifying the twisted periods with the periods of $\mathbb{P}^{n-2}$. To begin with, let us recall the definition of the periods of $\mathbb{P}^{n-2}$. We have $H^*\big(\mathbb{P}^{n-2}\big) = \mathbb{C}[p]/p^{n-1}$, where $p = c_1(\mathcal{O}(1))$ is the hyperplane class. We have an isomorphism of vector spaces

$$\widetilde{H}(E) \cong H\big(\mathbb{P}^{n-2}\big), \qquad e^i \mapsto p^{i-1}.$$

Note that under this isomorphism ${}^{tw}\theta$ coincides with the grading operator $\theta_{\mathbb{P}^{n-2}}$ and ${}^{tw}\rho$ coincides with $-c_1\big(T\mathbb{P}^{n-2}\big)\cup$. Therefore, the calibrated periods in the twisted GW theory of $\mathbb{P}^{n-1}$ and the GW theory of $\mathbb{P}^{n-2}$ are related by $e^{{}^{tw}\theta\pi\mathbf{i}}\, {}^{tw}\widetilde{I}^{(-m)}_\beta(\lambda) = \widetilde{I}^{(-m)}_{\sigma(\beta)}(\lambda)$, where $\sigma(\beta) := e^{\pi\mathbf{i}\theta}\beta$, where θ is the grading operator of $\mathbb{P}^{n-2}$.

Let us compare the S-matrices. In the GW theory of $\mathbb{P}^{n-2}$, we have

$$S(q,z)^{-1}1 = 1 + \sum_{d=1}^{\infty} \frac{q^d}{\prod_{m=1}^{d}(p-mz)^{n-1}},$$

where q is the Novikov variable corresponding to $\mathcal{O}(1)$. Using the divisor equation $(-zq\partial_q + p\cup)S(q,z)^{-1} = S(q,z)^{-1}p\bullet$, where $p\bullet$ is the operator of quantum multiplication by p, we get

$$S(q,z)^{-1}p^i = p^i + \sum_{d=1}^{\infty} \frac{q^d(p-dz)^i}{\prod_{m=1}^{d}(p-mz)^{n-1}}, \qquad 0 \le i \le n-2. \tag{5.7}$$

On the other hand, the twisted S-matrix ${}^{tw}S(Q,z)$ can be computed from the S-matrix of the blowup $\mathrm{Bl}(\mathbb{P}^n)$ of $\mathbb{P}^n$ at one point which is known explicitly. Namely, let us recall that $\mathrm{Bl}(\mathbb{P}^n)$ is the submanifold of $\mathbb{P}^{n-1}\times\mathbb{P}^n$ defined by the quadratic equations $x_iy_j = x_jy_i$ $(0 \le i,j \le n-1)$, where $x = [x_0,\dots,x_{n-1}]$ and $y = [y_0,\dots,y_n]$ are the homogeneous coordinate systems on respectively $\mathbb{P}^{n-1}$ and $\mathbb{P}^n$. We have two projection maps $\pi_1\colon \mathrm{Bl}(\mathbb{P}^n) \to \mathbb{P}^{n-1}$ and $\pi_2\colon \mathrm{Bl}(\mathbb{P}^n) \to \mathbb{P}^n$. Note that π_2 is the projection of the blowup — the exceptional divisor E is the fiber over $[0,0,\dots,0,1] \in \mathbb{P}^n$. Let L_1 and L_2 be the pullbacks of the hyperplane bundles $\mathcal{O}(1)$ on respectively $\mathbb{P}^{n-1}$ and $\mathbb{P}^n$. Let us denote by ${}^{bl}S(q_1,q_2,z)$ the S-matrix in the GW theory of $\mathrm{Bl}(\mathbb{P}^n)$, where q_1 and q_2 are the Novikov variables corresponding to the line bundles L_1 and L_2. Then we have

$${}^{bl}S(q_1,q_2,z)^{-1}1 = \sum_{d_1,d_2\ge 0} \frac{q_1^{d_1}q_2^{d_2}\prod_{m=-\infty}^{0}(p_2-p_1-mz)}{\prod_{m=1}^{d_1}(p_1-mz)^n\prod_{m=1}^{d_2}(p_2-mz)\prod_{m=-\infty}^{d_2-d_1}(p_2-p_1-mz)},$$

where q_1 and q_2 are the Novikov variables. The degree class in $\mathrm{Bl}(\mathbb{P}^n)$ corresponding to a pair (d_1, d_2) is $d_1e_1 + d_2e_2$, where e_1 is the class of a line in E and $e_2 = \pi_2^{-1}$(line in $\mathbb{P}^n$ avoiding $[0, 0, \ldots, 0, 1]$). It can be checked that the cohomology ring of the blowup is

$$H(\mathrm{Bl}(\mathbb{P}^n)) = \mathbb{C}[p_1, p_2]/\langle p_2(p_2 - p_1) = 0, p_1^n = 0\rangle$$

and that $\mathcal{O}(E) = L_2L_1^{-1}$, that is, the Poincaré dual of the exceptional divisor E is $e = p_2 - p_1$. In order to compute the twisted S-matrix ${}^{tw}S$, we have to restrict ${}^{bl}S$ to $q_2 = 0$ and substitute $q_1 = Q^{-(n-1)}$. We get

$$ {}^{bl}S\big(Q^{-(n-1)}, 0, z\big)^{-1}1 = 1 + \sum_{d=1}^{\infty} \frac{Q^{-d(n-1)} \prod_{m=-d+1}^{0}(p_2 - p_1 - mz)}{\prod_{m=1}^{d}(p_1 - mz)^n}. \tag{5.8}$$

Note that the numerator is proportional to $p_2 - p_1$. Using the relation $p_2(p_2 - p_1) = 0$, we get that $p_1 - mz$ can be replaced by $p_1 - p_2 - mz = -e - mz$. The above formula takes the form

$$ {}^{bl}S\big(Q^{-(n-1)}, 0, z\big)^{-1}1 = 1 + \sum_{d=1}^{\infty} \frac{(-1)^{dn}Q^{-d(n-1)}e}{(e + dz)^n \prod_{m=1}^{d-1}(e + mz)^{n-1}}. \tag{5.9}$$

Using the above formula and the divisor equation

$$\left(-\frac{1}{n-1} zQ\partial_Q + e\cup\right) {}^{bl}S\big(Q^{-(n-1)}, 0, z\big)^{-1} = {}^{bl}S\big(Q^{-(n-1)}, 0, z\big)^{-1} e\bullet,$$

whose proof is the same as the proof of Lemma 5.5, we get

$$ {}^{tw}S(Q, z)^{-1}e^i = e^i + \sum_{d=1}^{\infty} \frac{(-1)^{dn}Q^{-d(n-1)}e}{(e + dz)^{n-i} \prod_{m=1}^{d-1}(e + mz)^{n-1}}, \qquad 1 \le i \le n - 1, \tag{5.10}$$

where the right-hand side should be expanded into a power series in z^{-1} and e should be identified with the linear operator

$$e\cup_{tw}\colon\ \widetilde{H}(E) \to \widetilde{H}(E), \qquad e \cup_{tw} e^i := \begin{cases} e^{i+1} & \text{if } 1 \le i \le n-2, \\ 0 & \text{if } i = n-1. \end{cases}$$

Comparing formulas (5.7) and (5.10), we get that if we put $q = (-1)^n Q^{-(n-1)}$, then the matrices of $S(q, z)$ and ${}^{tw}S(Q, -z)$ with respect to respectively the bases $1, p, \ldots, p^{n-2}$ and $e, e^2, \ldots, e^{n-1}$ coincide. Now we are in position to prove the following key formula.

Proposition 5.6. *Under the isomorphism $\widetilde{H}(E) \cong H\big(\mathbb{P}^{n-2}\big)$ the following identity holds:*

$$ {}^{tw}I^{(-m)}_{\beta}(Q, \lambda) = e^{-\pi \mathbf{i}\theta}\, I^{(-m)}_{\sigma(\beta)}\big(-Q^{-(n-1)}, \lambda\big),$$

where $\sigma = e^{\pi\mathbf{i}\theta}$ and θ is the grading operator of $\mathbb{P}^{n-2}$.

Proof. By definition,

$$ {}^{tw}I^{(-m)}_{\beta}(Q, \lambda) = \sum_{l\in\mathbb{Z}} \sum_{i=1}^{n-1} \mathrm{Res}\, dz z^{l-1}(-\partial_\lambda)^l \big({}^{tw}\widetilde{I}^{(-m)}_{\beta}(\lambda), {}^{tw}S(Q, -z)^{-1}e^i\big)e_i,$$

where the residue is defined formally as the coefficient in front of dz/z. Under the isomorphism $\widetilde{H}(E) \cong H(\mathbb{P}^{n-2})$ the period

$$ {}^{tw}\widetilde{I}^{(-m)}_{\beta}(\lambda) = e^{-\pi\mathbf{i}\theta}\widetilde{I}^{(-m)}_{\sigma(\beta)}(\lambda), \qquad {}^{tw}S(Q, -z)^{-1}e^i = S\big((-1)^n Q^{-(n-1)}, z\big)^{-1}p^{i-1},$$

and $e_i = (-1)^{n-1}p^{n-1-i}$. Note that the Poincaré pairing on $H\big(\mathbb{P}^{n-2}\big)$ differs from the pairing on $\widetilde{H}(E)$ by the sign $(-1)^{n-1}$. The above formula for the period takes the form

$$
\begin{aligned}
{}^{tw}I^{(-m)}_{\beta}(Q,\lambda) = &\sum_{l\in\mathbb{Z}}\sum_{i=1}^{n-1}\operatorname{Res} dz z^{l-1}(-\partial_\lambda)^l \\
&\times \big(e^{-\pi\mathbf{i}\theta}\widetilde{I}^{(-m)}_{\sigma(\beta)}(\lambda), S\big((-1)^nQ^{-(n-1)},z\big)^{-1}p^{i-1}\big)p^{n-1-i}.
\end{aligned}
$$

Since $e^{\pi\mathbf{i}\theta}p = -pe^{\pi\mathbf{i}\theta}$, using formula (5.7), we get

$$
e^{\pi\mathbf{i}\theta}S(q,z)^{-1}p^{i-1} = e^{\pi\mathbf{i}(\frac{n}{2}-i)}S\big((-1)^{n-1}q,-z\big)^{-1}p^{i-1}.
$$

The formula for the period takes the form

$$
\begin{aligned}
{}^{tw}I^{(-m)}_{\beta}(Q,\lambda) = &\sum_{l\in\mathbb{Z}}\sum_{i=1}^{n-1}\operatorname{Res} dz z^{l-1}(-\partial_\lambda)^l \\
&\times \big(\widetilde{I}^{(-m)}_{\sigma(\beta)}(\lambda), S\big(-Q^{-(n-1)},-z\big)^{-1}p^{i-1}\big)\sigma^{-1}\big(p^{n-1-i}\big),
\end{aligned}
$$

where we used that $\sigma^{-1}\big(p^{n-1-i}\big) = e^{\pi\mathbf{i}(\frac{n}{2}-i)}p^{n-1-i}$. Clearly, the right-hand side of the above identity coincides with $\sigma^{-1}\big(I^{(-m)}_{\sigma(\beta)}\big(-Q^{-(n-1)},\lambda\big)\big)$. The lemma follows. ■

5.5 Monodromy of the twisted periods of $\mathbb{P}^{n-1}$

Let us describe the monodromy group of the system of differential equations (5.5)–(5.6), that is, the monodromy of the twisted periods of $\mathbb{P}^{n-1}$. According to Proposition 5.6, it is sufficient to recall the monodromy group for the periods of $\mathbb{P}^{n-2}$. Let us first fix $q=1$ and $\lambda^\circ \in \mathbb{R}_{>0}$ sufficiently large — any $\lambda^\circ > n-1$ works. The value of the period $I^{(-m)}(q,\lambda)$ depends on the choice of a path from $(1,\lambda^\circ)$ to (q,λ) avoiding the discriminant

$$
\{(q,\lambda) \mid \det(\lambda - (n-1)p\bullet) = 0\}.
$$

For fixed q, the equation of the discriminant has $n-1$ solutions

$$
u_k(q) := (n-1)\eta^{-2k}q^{1/(n-1)}, \qquad 0\le k\le n-2,
$$

where $\eta = e^{\pi\mathbf{i}/(n-1)}$. Let us focus first on the monodromy of the twisted periods for $q=1$. The fundamental group

$$
\pi_1(\mathbb{C}\setminus\{u_0(1),\dots,u_{n-2}(1)\},\lambda^\circ)
$$

is a free group generated by the simple loops γ_k° corresponding to the paths C_k° from λ° to $u_k(1)$ defined as follows. C_k° consists of two pieces. First, an arc on the circle with center 0 and radius λ° starting at λ° and rotating clockwise on angle $2\pi\mathbf{i}k/(n-1)$. The second piece is the straight line segment from $\lambda^\circ\eta^{-2k}$ to $u_k(1) = (n-1)\eta^{-2k}$. It turns out that the reflection vector corresponding to the simple loop γ_k° is precisely $\Psi(\mathcal{O}(k))$, where Ψ is the Iritani's map for the integral structure of the quantum cohomology of $\mathbb{P}^{n-2}$ (see formula (1.2)), that is,

$$
\Psi(\mathcal{O}(k)) = (2\pi)^{\frac{3-n}{2}}\Gamma(1+p)^{n-1}e^{2\pi\mathbf{i}kp}.
$$

If $q\in\mathbb{C}^* := \mathbb{C}\setminus\{0\}$ is arbitrary, then we construct the path $\big(q,\lambda^\circ q^{1/(n-1)}\big)$ by letting q vary continuously along some reference path. This path allows us to determine the value of $I^{(-m)}_\alpha(q,\lambda)$ at $\lambda = \lambda^\circ q^{1/(n-1)}$ which we declare to be the base point of $\mathbb{C}\setminus\{u_0(q),\dots,u_{n-2}(q)\}$. Let $\gamma_k(q)$

be the simple loop obtained from γ_k° by rescaling $\lambda \in \gamma_k^\circ \mapsto \lambda q^{1/(n-1)}$. The reflection vectors corresponding to $\gamma_k(q)$ are precisely

$$\Psi_q(\mathcal{O}(k)) = (2\pi)^{\frac{3-n}{2}}\Gamma(1+p)^{n-1}q^{-p}e^{2\pi\mathbf{i}kp}.$$

The proof of the above facts in the case of $\mathbb{P}^2$ (that is $n=4$) can be found in [28]. In general, the argument is straightforward to generalize. Now let us apply the above construction and Proposition 5.6 in order to describe the monodromy of the twisted periods of $\mathbb{P}^{n-1}$. We have $q = -Q^{-(n-1)}$. Let us assume that $Q \in \mathbb{R}_{>0}$ is a real number. We pick a reference path from 1 to q consisting of the interval $\left[1, Q^{-(n-1)}\right]$ and the arc in the upper half-plane from $Q^{-(n-1)}$ to $q = -Q^{-(n-1)}$. Note that with such a choice of the reference path $q^{1/(n-1)} = \eta Q^{-1}$. Therefore, $\gamma_k(q)$ becomes a simple loop around $u_k(q) = (n-1)\eta^{-2k+1}Q^{-1}$ which are precisely the singularities of the differential equation (5.5). We get the following corollary.

Corollary 5.7. *If $\beta \in \widetilde{H}(E)$ is such that the analytic continuation of ${}^{tw}I_\beta^{(-m)}(Q,\lambda)$ along $\gamma_k(q)$ is ${}^{tw}I_{-\beta}^{(-m)}(Q,\lambda)$, then β must be proportional to*

$$\Psi(\mathcal{O}_E(-k+1)) = (2\pi)^{\frac{1-n}{2}}\Gamma(\mathrm{Bl}(X))Q^{-e(n-1)}(2\pi\mathbf{i})^{\deg}\mathrm{ch}(\mathcal{O}_E(-k+1)),$$

where $\mathcal{O}_E(-k+1) := \mathcal{O}(-(k-1)E) - \mathcal{O}(-kE)$.

Proof. According to the above discussion and Proposition 5.6, under the isomorphism $\widetilde{H}(E) \cong H\left(\mathbb{P}^{n-2}\right)$, $\sigma(\beta)$ must be proportional to $\Psi_q(\mathcal{O}(k))$, that is, β is proportional to

$$e^{-\pi\mathbf{i}\theta}\ (2\pi)^{\frac{3-n}{2}}\Gamma(1+p)^{n-1}q^{-p}e^{2\pi kp} = (2\pi)^{(3-n)/2}\mathbf{i}^{2-n}\Gamma(1-p)^{n-1}q^pe^{-2\pi\mathbf{i}kp}.$$

Note that $q^p = e^{\pi\mathbf{i}p}Q^{-p(n-1)}$. Therefore, under the isomorphism $\widetilde{H}(E) \cong H\left(\mathbb{P}^{n-2}\right)$, the above expression becomes

$$(2\pi)^{(3-n)/2}\mathbf{i}^{2-n}\Gamma(1-e)^{n-1}Q^{-e(n-1)}e^{(-2k+1)\pi\mathbf{i}e}e.$$

We have to check that the above expression is proportional to the image of the Iritani map for $\mathrm{Bl}(X)$ of the exceptional object $\mathcal{O}((-k+1)E) - \mathcal{O}(-kE)$. We have

$$\Gamma(\mathrm{Bl}(X)) = \Gamma(X)\Gamma(1-e)^n\Gamma(1+e),$$
$$\Gamma(1-e)\Gamma(1+e) = \frac{2\pi\mathbf{i}e}{e^{\pi\mathbf{i}e} - e^{-\pi\mathbf{i}e}} = \frac{2\pi\mathbf{i}e}{e^{2\pi\mathbf{i}e} - 1}e^{\pi\mathbf{i}e},$$

and

$$(2\pi\mathbf{i})^{\deg}\mathrm{ch}(\mathcal{O}((-k+1)E) - \mathcal{O}(-kE)) = e^{-2\pi\mathbf{i}ke}\left(e^{2\pi\mathbf{i}e} - 1\right).$$

Since $\Gamma(X) \cup e = e$, the image of the Iritani map becomes

$$(2\pi)^{(3-n)/2}\mathbf{i}\Gamma(1-e)^{n-1}Q^{-e(n-1)}e^{(-2k+1)\pi\mathbf{i}e}e.$$

The claim of the lemma follows. ■

5.6 Isomonodromic analytic continuation

Let $D(u,r)$ be the open disk in $\mathbb{C}$ with center u and radius r. Put $\mathbb{D}_r := D(0,r)$. Let $\epsilon > 0$ be a real number, $V \subset \mathbb{C}$ an open subset, and $u_i \colon \mathbb{D}_\epsilon \to V$ $(1 \le i \le m)$ be m holomorphic functions, such that, there exists a positive real number $\delta > 0$ satisfying

(i) The m disks $D(u_i(0),\delta)$ $(1 \le i \le m)$ are pairwise disjoint and contained in V.

(ii) We have $u_i(Q) \in D(u_i(0),\delta)$ for all $Q \in \mathbb{D}_\epsilon$.

Suppose that I is a multi-valued analytic function on $\mathbb{D}_\epsilon \times V \setminus \Sigma$ with values in a finite dimensional vector space H, where

$$\Sigma := \{(Q,\lambda) \in \mathbb{D}_\epsilon \times V \mid \lambda = u_i(Q) \text{ for some } i\}.$$

Let us fix $\lambda^\circ \in V$, such that, $\mathbb{D}_\epsilon \times \{\lambda^\circ\}$ is disjoint from Σ. Then I is analytic at $(Q,\lambda) = (0,\lambda^\circ)$ and I extends analytically along any path in $\mathbb{D}_\epsilon \times V \setminus \Sigma$ starting at $(0,\lambda^\circ)$. In particular, we can extend uniquely $I(Q,\lambda)$ for all $Q \in \mathbb{D}_\epsilon$ and λ sufficiently close to λ°. Let us expand $I(Q,\lambda) = \sum_{d=0}^\infty I_d(\lambda) Q^d$, where each coefficient I_d is an H-valued analytic function at $\lambda = \lambda^\circ$. Clearly, $I_d(\lambda)$ extends analytically along any path in $V \setminus \{u_1(0), \dots, u_m(0)\}$.

Lemma 5.8. *Suppose that γ is a closed loop based at λ° in*

$$V \setminus D(u_1(0),\delta) \sqcup \cdots \sqcup D(u_m(0),\delta),$$

such that, for every fixed $Q \neq 0$, the analytic extension of $I(Q,\lambda)$ along the path $\{Q\} \times \gamma$ transforms $I(Q,\lambda)$ into $A(I(Q,\lambda))$, where λ is sufficiently close to λ° and $A \in \mathrm{GL}(H)$ is a linear operator. If the operator A is independent of Q, then the analytic continuation along γ transforms the coefficient $I_d(\lambda)$ into $A(I_d(\lambda))$.

Proof. Since $I_d(\lambda) = \frac{1}{d!}\frac{\partial^d I}{\partial Q^d}(0,\lambda)$ by replacing the function $I(Q,\lambda)$ with its partial derivative $\frac{1}{d!}\frac{\partial^d I}{\partial Q^d}(Q,\lambda)$, we can reduce the general case to the case when $d = 0$.

Let us cover the path γ with small *closed* disks D_j $(1 \le j \le N)$, such that,

(i) D_j is disjoint from $D(u_i(0),\delta)$ for all i.

(ii) $D_j \cap D_{j+1} \neq \varnothing$.

(iii) $D_N = D_1$.

In other words, the union of the disks D_j give a fattening of the path γ. Let $I(Q,\lambda_j)$ $\forall \lambda_j \in D_j$ be the analytic extension of $I(Q,\lambda)$ along γ. Let us fix an arbitrary $\epsilon' > 0$. There exists a small $\rho_j > 0$, such that, $I(Q,\lambda_j)$ is a uniformly continuous function in $(Q,\lambda) \in \mathbb{D}_{\rho_j} \times D_j$. Therefore, there exists $0 < \delta_j' < \rho_j$, such that,

$$\|I(Q,\lambda_j) - I(0,\lambda_j)\| < \epsilon', \qquad \forall (Q,\lambda_j) \in \mathbb{D}_{\delta_j'} \times D_j,$$

where $\| \ \|$ is any norm on H — for example fix an isomorphism $H \cong \mathbb{R}^{\dim(H)}$ and choose the standard Euclidean metric. Since there are only finitely many disks D_j, we can choose ρ and δ' that work for all j simultaneously, that is, $\rho_j = \rho$ and $\delta_j' = \delta'$. Using the triangle inequality, we get

$$\begin{aligned}\|I(0,\lambda_N) - AI(0,\lambda_1)\| \le \|I(0,\lambda_N) - I(Q,\lambda_N)\| &+ \|I(Q,\lambda_N) - AI(Q,\lambda_1)\| \\ &+ \|A\|\|I(Q,\lambda_1) - I(0,\lambda_1)\|.\end{aligned}$$

Note that the middle term on the right-hand side of the inequality is 0 by definition. Choosing $|Q| < \delta'$, we get that the right-hand side of the above inequality is bounded by $\epsilon'(1 + \|A\|)$. Since ϵ' can be chosen arbitrary small, we get $I(0,\lambda_N) = A(I(0,\lambda_1))$ which is exactly what we had to prove. ∎

We will need a result which is slightly more general then Lemma 5.8. Namely, suppose that I is a multi-valued analytic function on $\mathbb{D}_\epsilon^* \times V \setminus \Sigma$, where $\mathbb{D}_\epsilon^* := \mathbb{D}_\epsilon \setminus \{0\}$ is the punctured disk.

Definition 5.9. The singularity of $I(Q,\lambda)$ at $Q=0$ is said to be *at most logarithmic* if the following two conditions hold:

(i) The function has an expansion of the form

$$I(Q,\lambda) = \sum_{s=0}^{n} \sum_{d=0}^{\infty} I_{s,d}(\lambda) Q^d (\log Q)^s,$$

where λ is sufficiently close to λ° and $Q \in \mathbb{D}_\epsilon \setminus (-\epsilon, 0]$.

(ii) The functions $I_s(Q,\lambda) := \sum_{d=0}^{\infty} I_{s,d}(\lambda) Q^d$ $(0 \le s \le n)$ extend analytically along any path in $\mathbb{D}_\epsilon \times V \setminus \Sigma$.

Remark 5.10. Condition (ii) might be redundant but we could not prove it in this generality. For our purposes, both conditions are easy to verify, because $I(Q,\lambda)$ will be a solution to an ODE in Q that has a Fuchsian singularity at $Q=0$.

Proposition 5.11. *Suppose that I is a multi-valued analytic function on $\mathbb{D}_\epsilon^* \times V \setminus \Sigma$ and that it has at most a logarithmic singularity at $Q=0$. Furthermore, suppose that γ is a closed loop based at λ° in*

$$V \setminus D(u_1(0),\delta) \sqcup \cdots \sqcup D(u_m(0),\delta),$$

such that, for every fixed $Q \in \mathbb{D}_\epsilon \setminus (-\epsilon, 0]$, the analytic extension of $I(Q,\lambda)$ along the path $\{Q\} \times \gamma$ transforms $I(Q,\lambda)$ into $A(I(Q,\lambda))$, where λ is sufficiently close to λ° and $A \in \mathrm{GL}(H)$ is a linear operator. If the operator A is independent of Q, then the analytic continuation along γ transforms the coefficient $I_{s,d}(\lambda)$ into $A(I_{s,d}(\lambda))$.

Proof. Let $I_s(Q,\lambda)$ be as in condition (ii) in Definition 5.9. We have $I(Q,\lambda) = \sum_{s=0}^{n} I_s(Q,\lambda) \times (\log Q)^s$. The analytic continuation along γ yields $A(I(Q,\lambda)) = \sum_{s=0}^{n} \widetilde{I}_s(Q,\lambda)(\log Q)^s$, where $\widetilde{I}_s(Q,\lambda)$ is the analytic extension of $I_s(Q,\lambda)$ along γ. It is easy to prove by letting $Q \to 0$ that such an identity is possible only if the coefficients in front of the powers of $\log Q$ are equal, that is, $A(I_s(Q,\lambda)) = \widetilde{I}_s(Q,\lambda)$. It remains only to recall Lemma 5.8. ■

5.7 Vanishing of the base component

Let $\alpha = Q^{-(n-1)e}\beta$, where $\beta \in H^*(\mathrm{Bl}(X))$ is a vector independent of Q and t. Let $\beta = \beta_e + \beta_b$. We would like to extract the leading order terms in the power series expansion at $Q=0$ of

$$Q^{\Delta+m+(n-1)/2} I^{(-m)}\left(t,q,Q,Q^{-1}\lambda\right)\alpha, \tag{5.11}$$

where $m > 0$ is a sufficiently large integer, that is, we choose m so big that the operator $\widetilde{\theta} + m + 1/2$ has only positive eigenvalues. Moreover, we would like to determine the structure of the following terms in the expansion up to order Q^n. Note that

$$Q^\Delta Q^{-\widetilde{\theta}-m+\frac{1}{2}} Q^{-\widetilde{\rho}}\alpha = Q^{-m-(n-1)/2}\big(\beta_e + Q^{\deg}(Q^{-\rho}\beta_b)\big).$$

Therefore, we have

$$\begin{aligned} Q^{\Delta+m+(n-1)/2} I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big)\alpha = {} & \big(Q^\Delta I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big) Q^{\widetilde{\rho}} Q^{\widetilde{\theta}+m-\frac{1}{2}} Q^{-\Delta}\big) \\ & \times \big(\beta_e + Q^{\deg}(Q^{-\rho}\beta_b)\big). \end{aligned}$$

Let us look at the contribution of β_e to (5.11), that is, the expression

$$\big(Q^{\Delta} I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big)Q^{\widetilde{\rho}}Q^{\widetilde{\theta}+m-\frac{1}{2}}Q^{-\Delta}\big)\beta_e. \tag{5.12}$$

According to Proposition 4.3, the leading order term of the $\widetilde{H}(E)$-component of (5.12) is at degree 0 and it is precisely ${}^{tw}I^{(-m)}_{\beta_e}(1,\lambda)$, that is, the twisted period of $\mathbb{P}^{n-1}$ at $Q=1$. The leading order term of the $H(X)$-component of (5.12) is at degree n and the corresponding coefficient in front of Q^n is given by

$$\sum_{l,d=0}^{\infty}(-\partial_\lambda)^l\big\langle\psi^{l-1}\widetilde{I}^{(-m)}_{\beta_e}(\lambda),1\big\rangle_{0,2,d\ell}\,(-1)^{n-1}\phi_N, \tag{5.13}$$

where

$$\widetilde{I}^{(-m)}_{\beta_e}(\lambda)=e^{-(n-1)e\partial_\lambda\partial_m}\left(\frac{\lambda^{\widetilde{\theta}+m-1/2}}{\Gamma(\widetilde{\theta}+m+1/2)}\right)\beta_e$$

is the calibrated period of $\mathrm{Bl}(X)$ and for $l=0$ the correlator should be understood via the string equation as $\big\langle\psi^l\widetilde{I}^{(-m)}_{\beta_e}(\lambda),1,1\big\rangle_{0,3,d\ell}$.

Let us look at the contribution to (5.11) corresponding to β_b, that is, the expression

$$\big(Q^{\Delta} I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big)Q^{\widetilde{\rho}}Q^{\widetilde{\theta}+m-\frac{1}{2}}Q^{-\Delta}\big)Q^{\deg}\big(Q^{-\rho}\beta_b\big). \tag{5.14}$$

Let us decompose $\beta_b=\sum_{a=1}^N\beta_{b,a}\phi_a$. The $H(X)$-component of (5.14) is a power series in Q whose coefficients are polynomials in $\log Q$ whose coefficients are in $H(X)$. According to Propositions (4.5) and (4.4), the coefficient in front of $Q^M(\log Q)^0$ with $0\le M\le n$ has the form

$$\sum_{a\colon\ \deg(\phi_a)=M}\frac{\lambda^{\theta+m-1/2}}{\Gamma(\theta+m+1/2)}\beta_{b,a}\phi_a+\sum_{a'\colon\ \deg(\phi_{a'})<M}\beta_{b,a'}f_{M,a'}(\lambda), \tag{5.15}$$

where $f_{M,a'}(\lambda)$ is the $H(X)$-component of the coefficient in front of $Q^{M-\deg(\phi_{a'})}$ in the expansion at $Q=0$ of

$$\big(Q^{\Delta} I^{(-m)}\left(t,q,Q,Q^{-1}\lambda\right)Q^{\widetilde{\rho}}Q^{\widetilde{\theta}+m-\frac{1}{2}}Q^{-\Delta}\big)\phi_{a'}.$$

Let us summarize our analysis.

Proposition 5.12. *Let $\alpha=Q^{-(n-1)e}\beta$, where $\beta\in H(\mathrm{Bl}(X))$ is independent of t and Q. Then*

- (*a*) *The $H(X)$-component of (5.11) expands as a power series in Q whose coefficients are polynomials in $\log Q$. The coefficient in front of $(\log Q)^0Q^M$ for $0\le M\le n-1$ is given by (5.15), while for $M=n$ it is given by the sum of (5.15) (with $M=n$) and (5.13).*
- (*b*) *If $\beta_{b,1}=0$, then the $\widetilde{H}(E)$-component of (5.11) expands as a power series in Q. The corresponding leading order term is ${}^{tw}I^{(-m)}_{\beta_e}(1,\lambda)$.*

Let us discuss now the analytic properties of the series (5.13). It is convenient to introduce the following series:

$$\Phi_\beta(Q,\lambda):=\sum_{l,d=0}^{\infty}(-\partial_\lambda)^l\big\langle\psi^{l-1}\widetilde{I}^{(-m)}(\lambda)Q^{-(n-1)e}\beta,1\big\rangle_{0,2,d\ell}Q^{-d(n-1)},\qquad \beta\in\widetilde{H}(E).$$

Note that (5.13) coincides with $\Phi_{\beta_e}(1,\lambda)(-1)^{n-1}\phi_N$. Recalling the definition of the period vector $I^{(-m)}_\alpha(t,q,Q,\lambda)$, we get

$$\Phi_\beta(Q,\lambda)=\big(I^{(-m)}_{Q^{-(n-1)e}\beta}(t,0,Q,\lambda),1\big),\qquad \forall\beta\in\widetilde{H}(E).$$

Proposition 5.13. *Let Q be a positive real number and $\gamma_k(q)$ with $q = -Q^{-(n-1)}$ be the same simple loop as in Corollary 5.7. If*

$$\beta = \Psi(\mathcal{O}_E(-k+1)) = (2\pi)^{(1-n)/2}\Gamma(1-e)^{n-1}Q^{-(n-1)e}e^{(-2k+1)\pi\mathbf{i}e}2\pi\mathbf{i}e,$$

then the analytic continuation of $\Phi_\beta(Q,\lambda)$ along $\gamma_k(q)$ is $-\Phi_\beta(Q,\lambda)$.

The proof of Proposition 5.13 will be given in Section 6.

5.8 Proof of Theorem 1.7

Now we are in position to prove the main result of this paper. Let us fix $t \in \widetilde{H}(X)$ and the Novikov variables $q = (q_1, \dots, q_r)$ of X to be generic, such that, the quantum cohomology of X is semisimple and the conclusions of Proposition 5.2 hold. Let us pick a real number $R > 0$, such that, $u_j(t,q,0) \in \mathbb{D}_R$ for all $1 \le j \le N$, where recall that $\mathbb{D}_R$ denotes the circle with center 0 and radius R. Let us choose a real number $\epsilon > 0$ so small that the quantum cup product of the blowup $\mathrm{Bl}(X)$ at (t,q,Q) is convergent for all $|Q| < \epsilon$, $u_j(t,q,Q) \in \mathbb{D}_R$ for all $|Q| < \epsilon$ and $1 \le j \le N$, and $R < (n-1)\epsilon^{-1}$. We would like to use the results from Section (5.6) in the following settings: the domain $V := \{\lambda \mid |\lambda| > R\epsilon\}$, $m := n-1$, and the $(n-1)$ holomorphic functions (denoted by u_i in Section 5.6) will be given by $Qu_{N+k}(t,q,Q)$, $1 \le k \le n-1$. Here we are using Proposition 5.2 to conclude that $Qu_{N+k}(t,q,Q) = -(n-1)v_k + O(Q)$ is analytic at $Q = 0$. Let us choose $\delta > 0$, such that, the disks $D(-(n-1)v_k, 2\delta)$ $(1 \le k \le n-1)$ are pairwise disjoint. If necessary, we decrease ϵ even further so that condition (ii) given in the beginning of Section 5.6 is satisfied. Note that condition (i) is satisfied according to our choice of δ. Before we continue further let us fix the solutions v_k of $\lambda^{n-1} = (-1)^n$ to be given by $v_k = -\eta^{-2k+1}$, where $\eta := e^{\pi\mathbf{i}/(n-1)}$. Then $-(n-1)v_k = (n-1)\eta^{-2k+1}$. Finally, for a reference point $\lambda^\circ \in V$ we pick any positive real $\lambda^\circ > (n-1) > R\epsilon$.

Let us define the loop γ_k in $V \setminus D\big((n-1)\eta^{-1},\delta\big) \sqcup \cdots \sqcup D\big((n-1)\eta^{-2n+3},\delta\big)$ to be the simple loop around $(n-1)\eta^{-2k+1}$ based at λ° corresponding to the path from λ° to $(n-1)\eta^{-2k+1}$ consisting of the following two pieces: an arc along the circle $|\lambda| = \lambda^\circ$ obtained by rotating from λ° clock-wise on angle $(2k-1)\pi/(n-1)$ and the second piece is the straight segment from $\lambda^\circ\eta^{-2k+1}$ to $(n-1)\eta^{-2k+1}$.

Suppose now that $Q \in \mathbb{D}_\epsilon$ is a positive real number. Note that by re-scaling the path γ_k, we obtain a path $\gamma_k \cdot Q^{-1}$ which is a simple loop around $(n-1)\eta^{-2k+1}Q^{-1}$. The simple loop γ_k goes around $(n-1)\eta^{-2k+1}$ along a circle with center $(n-1)\eta^{-2k+1}$ and radius r, where $\delta < r < 2\delta$. We claim that by decreasing ϵ if necessary, we can arrange that the circle with center $(n-1)\eta^{-2k+1}Q^{-1}$ and radius rQ^{-1} contains the canonical coordinate $u_{N+k}(t,q,Q)$. Indeed, we have

$$\big|u_{N+k}(t,q,Q) - (n-1)\eta^{-2k+1}Q^{-1}\big| = \big|Qu_{N+k}(t,q,Q) - (n-1)\eta^{-2k+1}\big|Q^{-1}$$

and since $\big|Qu_{N+k}(t,q,Q) - (n-1)\eta^{-2k+1}\big|$ has order $O(Q)$, by choosing ϵ small enough we can arrange that $\big|Qu_{N+k}(t,q,Q) - (n-1)\eta^{-2k+1}\big| < r$ for all $|Q| < \epsilon$. In other words, the re-scaled loop $\gamma_k \cdot Q^{-1}$ is a simple loop around the canonical coordinate $u_{N+k}(t,q,Q)$. Let us denote by $\alpha \in H(\mathrm{Bl}(X))$ the reflection vector corresponding to the simple loop $\gamma_k \cdot Q^{-1}$. Let us recall Proposition 5.11 for the series (5.11), that is,

$$I(Q,\lambda) := Q^{\Delta+m+(n-1)/2}I^{(-m)}\big(t,q,Q,Q^{-1}\lambda\big)\alpha.$$

The singularities of $I(Q,\lambda)$ are precisely at $Q^{-1}\lambda = u_j(t,q,Q)$ for $1 \le j \le N+k$, that is, $\lambda = Qu_j(t,q,Q)$. Note that by definition of R, the first N singularities $Qu_j(t,q,Q)$ $(1 \le j \le N)$ are in $\mathbb{D}_{R\epsilon}$. Therefore, $I(Q,\lambda)$ is a multi-valued analytic function in $(Q,\lambda) \in \mathbb{D}_\epsilon \times V \setminus \Sigma$. Although we are not going to give a complete proof, let us outline how to prove that $I(Q,\lambda)$ has at most logarithmic singularity at $Q=0$ (see Definition 5.9). Recall the divisor equation (5.3) with $i=r+1$. Note that $q_{r+1}\partial_{q_{r+1}} = \frac{1}{n-1}Q\partial_Q$. Combining the divisor equation and the differential equation of the second structure connection with respect to $\tau_{r+1}=t_{N+1}$, it is easy to prove that for every $\lambda\in V\setminus D\big((n-1)\eta^{-1},\delta\big)\sqcup\cdots\sqcup D\big((n-1)\eta^{-2n+3},\delta\big)$ the function $I(Q,\lambda)$ is a solution to a differential equation that has a Fuchsian singularity at $Q=0$. Now the conclusion follows from the theory of Fuchsian singularities.

The analytic continuation of $I(Q,\lambda)$ along γ_k transforms $I(Q,\lambda)$ into $-I(Q,\lambda)$ because when λ changes along γ_k, $Q^{-1}\lambda$ changes along $\gamma_k\cdot Q^{-1}$ which is the simple loop used to define the reflection vector α. Let us look at the expansion of $I(Q,\lambda)$ at $Q=0$ in the powers of Q and $\log Q$. To begin with, we know that $\alpha = Q^{-(n-1)e}\beta$ where $\beta\in H(\mathrm{Bl}(X))$ is independent of t and Q (it could depend on q). Let us decompose $\beta=\beta_e+\beta_b$, where $\beta_e\in\widetilde{H}(E)$ and $\beta_b\in H(X)$. Put $\beta_b =: \sum_{i=1}^N \beta_{b,i}\phi_i$. We claim that $\beta_b=0$. According to Proposition 5.12 (a), the coefficient in front of $Q^0(\log Q)^0$ in the expansion of the $H(X)$-component of $I(Q,\lambda)$ is

$$\frac{\lambda^{\theta+m-1/2}}{\Gamma(\theta+m+1/2)}\beta_{b,1}\phi_1 = \frac{\lambda^{m+(n-1)/2}}{\Gamma(m+(n+1)/2)}\beta_{b,1}\phi_1.$$

According to Proposition 5.11, the analytic continuation along γ_k of the above expression should change the sign. However, the function $\lambda^{m+(n-1)/2}$ is invariant under the analytic continuation along γ_k. Therefore, the only possibility is that $\beta_{b,1}=0$. Suppose that M is the smallest number, such that, $\beta_{b,a}\neq 0$ for some ϕ_a of degree M. If $M\leq n-1$, then since $\beta_{b,a'}=0$ for all a', such that, $\deg(\phi_{a'})<M$, Proposition 5.12 (a) yields that the coefficient in front of $Q^M(\log Q)^0$ in the expansion of the $H(X)$-component of $I(Q,\lambda)$ is

$$\sum_{a:\deg(\phi_a)=M}\frac{\lambda^{\theta+m-1/2}}{\Gamma(\theta+m+1/2)}\beta_{b,a}\phi_a.$$

Just like before, the above expression is invariant under the analytic continuation along γ_k, while Proposition 5.11 implies that the analytic continuation must change the sign. The conclusion is again that $\beta_{b,a}=0$ for all a for which ϕ_a has degree M. We get that all $\beta_{b,a}=0$ except possibly for $\beta_{b,N}$. Let us postpone the analysis of $\beta_{b,N}$ and consider β_e first. Recalling Proposition 5.12 (b), we get that the coefficient in front of Q^0 in the expansion of the $\widetilde{H}(E)$-component of $I(Q,\lambda)$ is the twisted period ${}^{tw}I^{(-m)}_{\beta_e}(1,\lambda)$. Therefore, the analytic continuation of ${}^{tw}I^{(-m)}_{\beta_e}(1,\lambda)$ along γ_k must be ${}^{tw}I^{(-m)}_{-\beta_e}(1,\lambda)$. Recalling Corollary 5.7, we get that β_e must be proportional to

$$\Psi(\mathcal{O}_E(-k+1)) = (2\pi)^{\frac{1-n}{2}}\Gamma(\mathrm{Bl}(X))(2\pi\mathbf{i})^{\deg}\operatorname{ch}(\mathcal{O}_E(-k+1)).$$

Let us prove that $\beta_{b,N}=0$. According to Proposition 5.12, the coefficient in front of $Q^n(\log Q)^0$ in the expansion of the $H(X)$-component of $I(Q,\lambda)$ is

$$\left(\frac{\lambda^{m-(n+1)/2}}{\Gamma(m+(1-n)/2)}\beta_{b,N}+\Phi_{\beta_e}(1,\lambda)(-1)^{n-1}\right)\phi_N.$$

Let us analytically continue the above expression along γ_k. Just like above, the analytic continuation should change the sign. However, recalling Proposition 5.13, we get

$$\left(\frac{\lambda^{m-(n+1)/2}}{\Gamma(m+(1-n)/2)}\beta_{b,N}-\Phi_{\beta_e}(1,\lambda)(-1)^{n-1}\right)\phi_N.$$

Therefore, $\beta_{b,N}=0$ and this completes the proof of our claim that $\beta_b=0$. Moreover, we proved that $\beta=\beta_e$ is proportional to $\Psi(\mathcal{O}_E(-k+1))$. In order to conclude that the proportionality coefficient is ±1, we need only to check that the Euler pairing $\langle\Psi(\mathcal{O}_E(-k+1)),\Psi(\mathcal{O}_E(-k+1))\rangle = 1$.

For simplicity, let us consider only the case when $k = 1$. In fact, the general case follows easily by using analytic continuation with respect to q around $q = 0$: the clock-wise analytic continuation transforms $\Psi_q(\mathcal{O}_E(-k+1))$ to $\Psi_q(\mathcal{O}_E(-k+1) \otimes \mathcal{O}(-E)) = \Psi_q(\mathcal{O}_E(-k))$. We have

$$\Psi_q(\mathcal{O}_E) = (2\pi)^{(1-n)/2}\Gamma(1-e)^{n-1}q^e(2\pi\mathbf{i}e),$$

where $q = -Q^{-(n-1)}$ and the branch of $\log q$ is fixed in such a way that $q^e = e^{\pi\mathbf{i}e}Q^{-(n-1)e}$. Recalling formula (2.6), after a straightforward computation, we get $\langle \Psi_q(\mathcal{O}_E), \Psi_q(\mathcal{O}_E)\rangle = 1$.

6 Mirror model for the twisted periods

The goal in this section is to prove Proposition 5.13. The idea is to prove that the Laplace transform of $\Phi_\beta(Q,\lambda)$ with respect to λ can be identified with an appropriate oscillatory integral whose integration cycle is swept out by a family of vanishing cycles. Once this is done, the statement of the proposition follows easily by an elementary local computation. It is more convenient to construct an oscillatory integral when $q := -Q^{-(n-1)}$ is a positive real number. Therefore, let us reformulate the statement of Proposition 5.13 by analytically continuing $\Phi_\beta(Q,\lambda)$ with respect to Q along an arc in the counter-clockwise direction connecting the rays $\mathbb{R}_{>0}$ and $\eta\mathbb{R}_{>0}$, where $\eta := e^{\pi\mathbf{i}/(n-1)}$. Note that the value of $\log Q$ will change to $\log|Q| + \frac{\pi\mathbf{i}}{n-1}$. In other words, we will assume that $Q = \eta q^{-1/(n-1)}$ where $q \in \mathbb{R}_{>0}$ is a positive real number. Note that $Q^{-(n-1)e} = e^{-\pi\mathbf{i}e}q^e$ and that the formula for $\Phi_\beta(Q,\lambda)$ takes the form

$$\Phi_\beta(q,\lambda) = \sum_{l,d=0}^{\infty}(-\partial_\lambda)^l\big\langle\psi^{l-1}\widetilde{I}^{(-m)}(\lambda)q^e e^{-\pi\mathbf{i}e}\beta, 1\big\rangle_{0,2,d\ell}(-q)^d. \tag{6.1}$$

Furthermore, it is sufficient to prove Proposition 5.13 only in the case when $k = 0$, because the general case will follow from that one by taking an appropriate analytic continuation with respect to q around $q = 0$. Let us assume $k = 0$, so that

$$e^{-\pi\mathbf{i}e}\beta = (2\pi)^{(1-n)/2}\Gamma(1-e)^{n-1}(2\pi\mathbf{i}e).$$

Note that $\gamma_0(q)$ is a simple loop approaching $u_0(q) = (n-1)q^{1/(n-1)} \in \mathbb{R}_{>0}$ along the positive real axis. From now on we assume the above settings and denote $\Phi_\beta(Q,\lambda)$ and $u_0(q)$ simply by respectively $\Phi(q,\lambda)$ and $u(q)$. We have to prove that the analytic continuation of $\Phi(q,\lambda)$ along $\gamma_0(q)$ is $-\Phi(q,\lambda)$. Finally, let us point out that in the previous sections we denoted by q the sequence of Novikov variables $(q_1,\dots,q_r)$ of X, while in this section we denote by q just a positive real number. We will never have to deal with X, so there will be no confusion in doing so.

6.1 Contour integral

The Gromov–Witten invariants involved in the definition of the series $\Phi(q,\lambda)$ can be extracted from formula (5.8). Indeed, we have

$$\begin{aligned}\Phi(q,\lambda) &= \sum_{l=0}^{\infty}(-\partial_\lambda)^l\big({}^{bl}S_l(-q,0)\widetilde{I}^{(-m)}(\lambda)q^e e^{-\pi\mathbf{i}e}\beta, 1\big)\\ &= \sum_{l=0}^{\infty}(-\partial_\lambda)^l\big(\widetilde{I}^{(-m)}(\lambda)q^e e^{-\pi\mathbf{i}e}\beta, {}^{bl}S_l(-q,0)^T 1\big).\end{aligned}$$

Recall that ${}^{bl}S(-q,0,-\partial_\lambda)^T = {}^{bl}S(-q,0,\partial_\lambda)^{-1}$. Using formula (5.9), we get

$$\Phi(q,\lambda) = \int_{\mathrm{Bl}(X)} \sum_{d=0}^{\infty} \frac{(-1)^{dn}(-q)^d e\partial_\lambda}{(e\partial_\lambda + d)^n \prod_{i=1}^{d-1}(e\partial_\lambda + i)^{n-1}} \; \partial_\lambda^{d(n-1)} \; \widetilde{I}^{(-m)}(\lambda) q^e e^{-\pi \mathbf{i} e}\beta. \tag{6.2}$$

Using that $\theta e = e(\theta - 1)$, we get the relation $\widetilde{I}^{(-m)}(\lambda)e = e\partial_\lambda \widetilde{I}^{(-m)}(\lambda)$ (see Lemma 4.1 (a)), where slightly abusing the notation we denoted by e the operator of classical cup product multiplication by e. Therefore,

$$\begin{aligned}\widetilde{I}^{(-m)}(\lambda)q^e e^{-\pi\mathbf{i}e}\beta &= q^{e\partial_\lambda}(2\pi)^{(1-n)/2}\Gamma(1-e\partial_\lambda)^{n-1}(2\pi\mathbf{i}e\partial_\lambda)\widetilde{I}^{(-m)}(\lambda)1 \\ &= q^{e\partial_\lambda}(2\pi)^{(1-n)/2}\Gamma(1-e\partial_\lambda)^{n-1}(2\pi\mathbf{i}e\partial_\lambda)e^{-(n-1)e\partial_\lambda\partial_m}\left(\frac{\lambda^{\frac{n}{2}+m-\frac{1}{2}}}{\Gamma\big(\frac{n}{2}+m+\frac{1}{2}\big)}\right).\end{aligned}$$

Let us substitute the above formula for $\widetilde{I}^{(-m)}(\lambda)q^e e^{-\pi\mathbf{i}e}\beta$ in (6.2). Note that everywhere the operator e comes together with the differentiation operator ∂_λ. On the other hand, since in the entire expression only the coefficient in front of e^n contributes, we may remove ∂_λ from $e\partial_\lambda$ and apply to the entire expression the differential operator ∂_λ^n, that is, change $\partial_\lambda^{d(n-1)}$ to $\partial_\lambda^{d(n-1)+n}$. We get the following formula for $\Phi(q,\lambda)$

$$\begin{aligned}&(2\pi)^{(1-n)/2}2\pi\mathbf{i}\sum_{d=0}^{\infty}\int_{\mathrm{Bl}(X)}\frac{(-1)^{dn+d}q^{d+e}e^2}{(e+d)^n\prod_{i=1}^{d-1}(e+i)^{n-1}}\Gamma(1-e)^{n-1} \\ &\quad\times\partial_\lambda^{d(n-1)+n}e^{-(n-1)e\partial_m}\left(\frac{\lambda^{\frac{n}{2}+m-\frac{1}{2}}}{\Gamma\big(\frac{n}{2}+m+\frac{1}{2}\big)}\right).\end{aligned}$$

Note that

$$\partial_\lambda^{d(n-1)+n}\, e^{-(n-1)e\partial_m}\left(\frac{\lambda^{\frac{n}{2}+m-\frac{1}{2}}}{\Gamma\big(\frac{n}{2}+m+\frac{1}{2}\big)}\right) = \frac{\lambda^{-\frac{n}{2}-(n-1)(d+e)+m-\frac{1}{2}}}{\Gamma\big(-\frac{n}{2}-(n-1)(d+e)+m+\frac{1}{2}\big)}$$

and

$$\Gamma(1-e) = (-e)(-e-1)\cdots(-e-d)\Gamma(-e-d) = (-1)^{d+1}e(e+1)\cdots(e+d)\Gamma(-e-d).$$

Since $\int_{\mathrm{Bl}(X)} e^n = (-1)^{n-1}$, we can replace $\int_{\mathrm{Bl}(X)}$ with $(-1)^{n-1}\operatorname{Res}_{e=0}\frac{de}{e^{n+1}}$. Note that $dn+d+(d+1)(n-1)+n-1 = 2dn+2n-2$ is an even number, so that the signs that appear in our formula cancel out exactly. We get

$$\begin{aligned}\Phi(q,\lambda) = (2\pi)^{(1-n)/2}2\pi\mathbf{i}&\sum_{d=0}^{\infty}\operatorname{Res}_{e=0}\frac{de}{d+e}q^{d+e}\Gamma(-d-e)^{n-1} \\ &\times\frac{\lambda^{-\frac{n}{2}-(n-1)(d+e)+m-\frac{1}{2}}}{\Gamma(-\frac{n}{2}-(n-1)(d+e)+m+\frac{1}{2})}.\end{aligned}$$

Let us substitute $x := -e-d$, then the above formula becomes

$$\Phi(q,\lambda) = (2\pi)^{(1-n)/2}2\pi\mathbf{i}\sum_{d=0}^{\infty}\operatorname{Res}_{x=-d}\frac{\mathrm{d}x}{x}q^{-x}\Gamma(x)^{n-1}\frac{\lambda^{-\frac{n}{2}+(n-1)x+m-\frac{1}{2}}}{\Gamma\big(-\frac{n}{2}+(n-1)x+m+\frac{1}{2}\big)}.$$

The sum of infinitely many residues can be replaced with an integral of the form $\int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty}\mathrm{d}x$, where $\epsilon > 0$ is a positive real number. Let us sketch the proof of this claim. Let us fix a real number $\delta \in \big(\frac{1}{2},1\big)$, such that, $\mu := (n-1)\big(\delta - 1/2\big) \in \big(\frac{1}{2},1\big)$. Suppose that $K \geq 1$ is an integer.

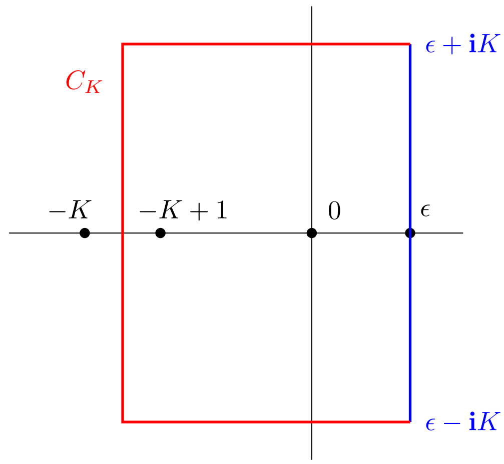

Figure 1. Integration contours.

Let us consider the rectangular contour given by the boundary of the rectangle with vertices $\epsilon - \mathbf{i}K, \delta - K - \mathbf{i}K, \delta - K + \mathbf{i}K$, and $\epsilon + \mathbf{i}K$. The contour is divided into two parts: the straight line segment $[\epsilon - \mathbf{i}K, \epsilon + \mathbf{i}K]$ and its complement which we denote by C_K — see Figure 1 where these two pieces are colored respectively with blue and red. By the Cauchy residue formula, the integral along this contour coincides with the partial sum $2\pi\mathbf{i}\sum_{d=0}^{K-1} \operatorname{Res}_{x=-d}$. On the other hand, using the standard asymptotic estimates for the Γ-function (see Appendix A), one can prove that if $\lambda > (n-1)q^{1/(n-1)}$ is a real number then the integral along C_K tends to 0 when $K \to \infty$. We get

$$\Phi(q, \lambda) = (2\pi)^{(1-n)/2} \int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} q^{-x}\Gamma(x)^{n-1} \frac{\lambda^{-\frac{n}{2}+(n-1)x+m-\frac{1}{2}}}{\Gamma\big(-\frac{n}{2}+(n-1)x+m+\frac{1}{2}\big)} \frac{\mathrm{d}x}{x}. \tag{6.3}$$

Let us denote by $G(q, \lambda)$ the right-hand side of (6.3). Note that $G(q, \lambda)$, after replacing $1/x$ with $\Gamma(x)/\Gamma(x+1)$, becomes a *Mellin–Barnes* integral. The analytic properties of such integrals are well known (see [2, 32]). Using the standard asymptotic estimates for the Γ-function it is easy to prove that the integral is convergent for all positive real λ and that it is divergent for $\operatorname{Im}(\lambda) \neq 0$. Since the series (6.1), viewed as a Laurent series in λ^{-1}, is convergent for $|\lambda| > u(q) = (n-1)q^{1/(n-1)}$, we get that $\Phi(q, \lambda)$ is the analytic continuation of the restriction of $G(q, \lambda)$ to the interval $\big[(n-1)q^{1/(n-1)}, +\infty\big)$.

Lemma 6.1. *The Mellin–Barnes integral* $G(q, \lambda)$ *is* 0 *for all* $0 < \lambda \leq (n-1)q^{1/(n-1)}$.

Proof. Let $R > 0$ be a sufficiently big positive number. Let us fix $\delta \in (0, 1)$. We would like to deform the contour $\epsilon + \mathbf{i}\mathbb{R}$ into the contour consisting of the 3 linear pieces $-\mathbf{i} - s$ $(-\infty < s \leq -\epsilon)$, $s\mathbf{i} + \epsilon$ $(-1 \leq s \leq 1)$, and $\mathbf{i} + s$ $(\epsilon \leq s < +\infty)$. The integral $G(q, \lambda)$ is a limit as $R \to \infty$ of the integral over $s\mathbf{i} + \epsilon$ $(-T \leq s \leq T)$ where $T := \sqrt{R^2 - \epsilon^2}$, while the integral over the deformed contour is a limit as $R \to \infty$ of the integral over the contour consisting of the 3 linear pieces $-\mathbf{i} - s$ $(-\sqrt{R^2-1} < s \leq -\epsilon)$, $s\mathbf{i} + \epsilon$ $(-1 \leq s \leq 1)$, and $\mathbf{i} + s$ $\big(\epsilon \leq s < \sqrt{R^2-1}\big)$. The difference between the two integrals is an integral over the two arcs $C_R\colon Re^{\mathbf{i}\theta}$ $(\arcsin(1/R) \leq \theta \leq \arcsin T/R)$ and $\overline{C}_R\colon Re^{\mathbf{i}\theta}$ $(-\arcsin(T/R) \leq \theta \leq -\arcsin 1/R)$. One has to prove that

$$\lim_{R \to +\infty} \int_{C_R \text{ or } \overline{C}_R} q^{-x}\Gamma(x)^{n-1} \frac{\lambda^{-\frac{n}{2}+(n-1)x+m-\frac{1}{2}}}{\Gamma\big(-\frac{n}{2}+(n-1)x+m+\frac{1}{2}\big)} \frac{\mathrm{d}x}{x} = 0.$$

This is proved in the same way as in [8, Section 5]. Namely, divide C_R into two pieces $C'_R\colon Re^{\mathbf{i}\theta}$ $(\arcsin(1/R) \leq \theta \leq \delta)$ and $C''_R\colon Re^{\mathbf{i}\theta}$ $(\delta \leq \theta \leq \arcsin T/R)$ and then use the standard asymptotic estimates for the Γ-function and the assumption $|\lambda| \leq (n-1)q^{1/(n-1)}$.

Finally, to complete the proof. Note that the integral is independent of $\epsilon > 0$, because the Γ-functions in $G(q,\lambda)$ do not have poles on the positive real axis. Letting $\epsilon \to +\infty$ and using again the standard asymptotic estimates for the Γ-function, we get that $G(q,\lambda) = 0$ for $\lambda \le (n-1)q^{1/(n-1)}$. ■

Using the above lemma, we get

$$\int_{u(q)}^{\infty} e^{-\lambda s} G(q,\lambda)\mathrm{d}\lambda = \int_0^{\infty} e^{-\lambda s} G(q,\lambda)\mathrm{d}\lambda.$$

Substituting $G(q,\lambda)$ with the corresponding Mellin–Barnes integral, exchanging the order of integration and using that

$$\int_0^{\infty} e^{-\lambda s} \frac{\lambda^{-\frac{n}{2}+(n-1)x+m-\frac{1}{2}}}{\Gamma\big(-\frac{n}{2}+(n-1)x+m+\frac{1}{2}\big)}\mathrm{d}\lambda = s^{\frac{n}{2}-(n-1)x-m-\frac{1}{2}},$$

we get

$$\int_{u(q)}^{\infty} e^{-\lambda s} G(q,\lambda)\mathrm{d}\lambda = (2\pi)^{(1-n)/2} \int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} q^{-x}\Gamma(x)^{n-1} s^{\frac{n}{2}-(n-1)x-m-\frac{1}{2}} \frac{\mathrm{d}x}{x}. \tag{6.4}$$

6.2 Oscillatory integral

Let us consider the following family of functions

$$f(x,q) = x_1 + \cdots + x_{n-2} + \frac{q}{x_1\cdots x_{n-2}}\big(1 + x_{n-1}^2 + x_n^2\big),$$

where q is a positive real number and

$$x = (x_1,\dots,x_n) \in V := \mathbb{C}^n \setminus \big\{x_1\cdots x_{n-2}\big(1+x_{n-1}^2+x_n^2\big) = 0\big\}.$$

Let $\Gamma := \mathbb{R}_{>0}^{n-2} \times \mathbb{R}^2 \subset V$, that is, Γ is the real n-dimensional cycle in V consisting of points $x = (x_1,\dots,x_n)$, such that, the first $n-2$ coordinates are positive real numbers and the last two ones are arbitrary real numbers. Note that the cycle Γ belongs to the following group of semi-infinite homology cycles:

$$\varprojlim H_n(V, \mathrm{Re}(f(x,q)) > M, \mathbb{Z}) \cong \mathbb{Z}^{n-1},$$

where the inverse limit is taken over all $M \in \mathbb{R}$.

Proposition 6.2. *Under the above notation the following identity holds:*

$$2\mathbf{i}\int_{\Gamma} e^{-f(x,q)} \frac{\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n}{x_1\cdots x_{n-2}(1+x_{n-1}^2+x_n^2)} = \int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} q^{-x}\Gamma(x)^{n-1}\frac{\mathrm{d}x}{x},$$

where the orientation of Γ *is induced from the standard orientation on* $\mathbb{R}^n$.

Proof. Let us integrate out x_{n-1} and x_n. Using polar coordinates $x_{n-1} = r\cos\theta$ and $x_n = r\sin\theta$, since $\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n = r\mathrm{d}r\wedge\theta$, we get

$$\int_{\mathbb{R}^2} e^{-K(1+x_{n-1}^2+x_n^2)}\frac{\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n}{1+x_{n-1}^2+x_n^2} = \int_0^{\infty} e^{-K(1+r^2)}\int_0^{2\pi}\frac{r\mathrm{d}r\wedge\mathrm{d}\theta}{1+r^2} = \pi\int_1^{\infty} e^{-Ku}\frac{\mathrm{d}u}{u},$$

where K is a positive real number and for the second equality we used the substitution $u = 1+r^2$. Applying the above formula to our oscillatory integral, we get

$$\int_\Gamma e^{-f(x,q)} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{x_1 \cdots x_{n-2}\big(1 + x_{n-1}^2 + x_n^2\big)} = \pi \int_{\mathbb{R}_{>0}^{n-2}} \int_1^\infty e^{-(x_1+\cdots+x_{n-2}+\frac{qu}{x_1\cdots x_{n-2}})} \frac{\mathrm{d}u}{u} \frac{\mathrm{d}x_1\cdots \mathrm{d}x_{n-2}}{x_1\cdots x_{n-2}}, \tag{6.5}$$

where $\mathrm{d}x_1 \cdots \mathrm{d}x_{n-2}$ is the standard Lebesgue measure on $\mathbb{R}_{>0}^{n-2}$. On the other hand, let us recall the oscillatory integral

$$J(q) := \int_{\mathbb{R}_{>0}^{n-2}} \exp\left(-\left(x_1 + \cdots + x_{n-2} + \frac{q}{x_1\cdots x_{n-2}}\right)\right) \frac{\mathrm{d}x_1\cdots \mathrm{d}x_{n-2}}{x_1\cdots x_{n-2}}.$$

Note that the Mellin transform of $J(q)$ is

$$\{\mathcal{M}J\}(x) = \int_0^\infty q^{x-1} J(q)\mathrm{d}q = \Gamma(x)^{n-1}.$$

Recalling the Mellin inversion theorem, we get

$$J(q) = \frac{1}{2\pi \mathbf{i}} \int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} q^{-x}\Gamma(x)^{n-1}\mathrm{d}x,$$

where $\epsilon > 0$ is a positive real number. Let us apply the above formula to (6.5). Namely, on the right-hand side of (6.5), after exchanging the order of the integration, we get

$$\pi \int_1^\infty J(qu)\frac{\mathrm{d}u}{u} = \frac{1}{2\mathbf{i}} \int_1^\infty \int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} (qu)^{-x}\Gamma(x)^{n-1}\mathrm{d}x\frac{\mathrm{d}u}{u}.$$

Exchanging again the order of integration and using that

$$\int_1^\infty u^{-x}\frac{\mathrm{d}u}{u} = \frac{u^{-x}}{-x}\bigg|_{u=1}^{u=\infty} = \frac{1}{x},$$

we get the formula stated in the proposition. ■

6.3 Laplace transform

The function $f(x,q)$ has a minimum over $x \in \Gamma$ achieved at the critical point $x_1 = \cdots = x_{n-2} = q^{1/(n-1)}$, $x_{n-1} = x_{n-2} = 0$. Note that the corresponding critical value is $u(q) = (n-1)q^{1/(n-1)}$. Let us consider the map $\Gamma \to [u(q), +\infty)$, $x \mapsto f(x,q)$. The fiber over $\lambda \in (u(q), +\infty)$ is the real algebraic hypersurface $\Gamma_\lambda \subset \Gamma$ defined by

$$x_1 + \cdots + x_{n-2} + \frac{q}{x_1\cdots x_{n-2}}\big(1 + x_{n-1}^2 + x_n^2\big) = \lambda.$$

It is easy to see that Γ_λ is compact and it has the homotopy type of a sphere. Indeed, the map

$$\Gamma \setminus \{u(q)\} \to (u(q), +\infty), \qquad x \mapsto f(x,q)$$

is proper and regular. Therefore, according to the Ehresmann's fibration theorem, it must be a locally trivial fibration and hence a trivial fibration, because $(u(q), +\infty)$ is a contractible manifold. If λ is sufficiently close to $u(q)$, then Γ_λ is contained in a Morse coordinate neighborhood of the critical point $\big(q^{1/(n-1)},\dots,q^{1/(n-1)},0,0\big)$. Switching to Morse coordinates for f, we get that the fiber Γ_λ is diffeomorphic to the $(n-1)$-dimensional sphere.

Let use denote by $\Gamma_{\leq\lambda}$ the subset of Γ defined by the inequality

$$x_1+\dots+x_{n-2}+\frac{q}{x_1\cdots x_{n-2}}\big(1+x_{n-1}^2+x_n^2\big)\leq\lambda.$$

Note that $\Gamma_{\leq\lambda}$ is a manifold with boundary and its boundary is precisely $\partial\Gamma_{\leq\lambda}=\Gamma_\lambda$. Put

$$\mathcal{I}(q,\lambda):=\int_{\Gamma_{\leq\lambda}}\frac{(\lambda-f(x,q))^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\big(m-\frac{n}{2}+\frac{1}{2}\big)}\omega,$$

where

$$\omega:=\frac{\mathrm{d}x_1\wedge\dots\wedge\mathrm{d}x_n}{x_1\cdots x_{n-2}\big(1+x_{n-1}^2+x_n^2\big)}.$$

Lemma 6.3. *The following formula holds:*

$$\int_\Gamma e^{-f(x,q)s}\omega=s^{m-\frac{n}{2}+\frac{1}{2}}\int_{u(q)}^\infty e^{-\lambda s}\mathcal{I}(q,\lambda)\mathrm{d}\lambda.$$

Proof. Using Fubini's theorem, we transform

$$\mathcal{I}(q,\lambda)=\int_{u(q)}^\lambda\frac{(\lambda-\mu)^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\big(m-\frac{n}{2}+\frac{1}{2}\big)}\int_{\Gamma_\mu}\frac{\omega}{\mathrm{d}f}\mathrm{d}\mu.$$

Therefore,

$$\int_{u(q)}^\infty e^{-\lambda s}\mathcal{I}(q,\lambda)\mathrm{d}\lambda=\int_{u(q)}^\infty\int_{u(q)}^\lambda e^{-\lambda s}\frac{(\lambda-\mu)^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\big(m-\frac{n}{2}+\frac{1}{2}\big)}\int_{\Gamma_\mu}\frac{\omega}{\mathrm{d}f}\mathrm{d}\mu\mathrm{d}\lambda.$$

Exchanging the order of the integration, we get

$$\int_{u(q)}^\infty\left(\int_\mu^\infty e^{-\lambda s}\frac{(\lambda-\mu)^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\big(m-\frac{n}{2}+\frac{1}{2}\big)}\mathrm{d}\lambda\right)\int_{\Gamma_\mu}\frac{\omega}{\mathrm{d}f}\mathrm{d}\mu=s^{-m+\frac{n}{2}-\frac{1}{2}}\int_{u(q)}^\infty e^{-\mu s}\int_{\Gamma_\mu}\frac{\omega}{\mathrm{d}f}\mathrm{d}\mu.$$

Recalling again Fubini's theorem we get that the above iterated integral coincides with

$$\int_\Gamma e^{-f(x,q)s}\omega.$$

The formula stated in the lemma follows. ■

According to the above lemma, the Laplace transform of the integral $\mathcal{I}(q,\lambda)$ is given by the following formula:

$$\int_{u(q)}^\infty e^{-\lambda s}\mathcal{I}(q,\lambda)=s^{-m+\frac{n}{2}-\frac{1}{2}}\int_\Gamma e^{-f(x,q)s}\omega=:F(s).$$

Let us recall Proposition 6.2 and note that $f(x,q)$ has the following rescaling symmetry:

$$f\big(s\cdot x,s^{n-1}q\big)=f(x,q)s,$$

where

$$s\cdot(x_1,\dots,x_n)=(sx_1,\dots,sx_{n-2},x_{n-1},x_n).$$

Note that if $s > 0$ is a positive real number, then the integration cycle and the holomorphic form ω are invariant under the rescaling action by s. Therefore, the formula from Proposition 6.2 yields the following formula:

$$\int_\Gamma e^{-f(x,q)s}\omega = \frac{1}{2\mathbf{i}}\int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} q^{-x}\Gamma(x)^{n-1}s^{-(n-1)x}\frac{\mathrm{d}x}{x}.$$

Therefore, the function

$$F(s) = \frac{1}{2\mathbf{i}}\int_{\epsilon-\mathbf{i}\infty}^{\epsilon+\mathbf{i}\infty} q^{-x}\Gamma(x)^{n-1}s^{-(n-1)x-m+\frac{n}{2}-\frac{1}{2}}\frac{\mathrm{d}x}{x}.$$

Comparing the above formula with (6.4) and using that the Laplace transformation is injective on smooth functions, we get that $G(q,\lambda) = 2\mathbf{i}(2\pi)^{(n-1)/2}\mathcal{I}(q,\lambda)$. Finally, in order to complete the proof of Proposition 5.13, we need only to check that the analytic continuation of $\mathcal{I}(q,\lambda)$ around $\lambda = u(q) = (n-1)q^{1/(n-1)}$ transforms $\mathcal{I}(q,\lambda)$ into $-\mathcal{I}(q,\lambda)$. This however is a local computation. Indeed, if λ is sufficiently close to $(n-1)q^{1/(n-1)}$, then the integration cycle defining $\mathcal{I}(q,\lambda)$ is sufficiently close to the critical point $\left(q^{1/(n-1)},\dots,q^{1/(n-1)},0,0\right)$. By switching to Morse coordinates, we get

$$\int_{\Gamma_\mu}\omega/df = (\mu-u(q))^{\frac{n}{2}-1}P(q,\mu),$$

where $P(q,\mu)$ is holomorphic at $\mu = u(q)$ (see [1, Section 12.1, Lemma 2]). Therefore,

$$\mathcal{I}(q,\lambda) = \int_{u(q)}^{\lambda}\frac{(\lambda-\mu)^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\left(m-\frac{n}{2}+\frac{1}{2}\right)}(\mu-u(q))^{\frac{n}{2}-1}P(q,\mu)\mathrm{d}\mu. \tag{6.6}$$

Changing the variables $\mu-u(q) = t(\lambda-u(q))$, we get $\lambda-\mu = (1-t)(\lambda-u(q))$ and $\mathrm{d}\mu = (\lambda-u(q))\mathrm{d}t$, we get

$$\begin{aligned}\int_{u(q)}^{\lambda}\frac{(\lambda-\mu)^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\left(m-\frac{n}{2}+\frac{1}{2}\right)}(\mu-u(q))^{i+\frac{n}{2}-1}\mathrm{d}\mu &= \int_0^1\frac{(1-t)^{m-\frac{n}{2}-\frac{1}{2}}}{\Gamma\left(m-\frac{n}{2}+\frac{1}{2}\right)}t^{i+\frac{n}{2}-1}\mathrm{d}t\,(\lambda-u(q))^{i+m-1/2}\\ &= \Gamma(i+n/2)\frac{(\lambda-u(q))^{i+m-1/2}}{\Gamma(i+m+1/2)}.\end{aligned}$$

Substituting the Taylor series expansion of $P(q,\mu) = \sum_{i=0}^\infty P_i(q)(\mu-u(q))^i$ at $\mu = u(q)$ in (6.6) and using the above formula, we get

$$\mathcal{I}(q,\lambda) = (\lambda-u(q))^{m-1/2}\sum_{i=0}^\infty\frac{\Gamma(i+n/2)}{\Gamma(i+m+1/2)}P_i(q)(\lambda-u(q))^i.$$

The above expansion is clearly anti-invariant under the analytic continuation around $\lambda = u(q)$.

A Bending the contour

For the sake of completeness we would like to prove that if λ is a positive real number, such that, $\lambda > (n-1)q^{1/(n-1)}$, then

$$\lim_{K\to+\infty}\int_{C_K} q^{-x}\lambda^{(n-1)x}\frac{\Gamma(x)^{n-1}}{\Gamma\left(-\frac{n}{2}+(n-1)x+m+\frac{1}{2}\right)}\frac{\mathrm{d}x}{x} = 0,$$

where C_K is the contour defined in Section 6.1 (see Figure 1). The integrand of the above integral differs from the integrand in (6.3) by the constant factor $\lambda^{-\frac{n}{2}+m-\frac{1}{2}}$. Therefore, the vanishing result needed in the derivation of (6.3) follows from the above statement.

Let us consider first the upper horizontal part of C_K, that is, $x = a+\mathbf{i}K$, $\delta - K \leq a \leq \epsilon$. The estimate in this case is a direct consequence of the Stirling's formula for the gamma function. Namely, recall that if $x = a + \mathbf{i}b \notin (-\infty, 0]$, then

$$|\Gamma(x)| = \sqrt{2\pi} e^{-a-|b||\mathrm{Arg}(x)|}|x|^{a-1/2}(1+o(1)),$$

where $-\pi < \mathrm{Arg}(x) < \pi$ and $o(1) \to 0$ uniformly when $|x| \to \infty$ in any proper subsector $-\pi < \alpha \leq \mathrm{Arg}(x) \leq \beta < \pi$. Put $c := -\frac{n}{2} + m + \frac{1}{2}$. Using Stirling's formula, we get

$$\begin{aligned}|\Gamma((n-1)x+c)| = \sqrt{2\pi} & e^{-(n-1)a-c-(n-1)|b||\mathrm{Arg}(x+c/(n-1))|} \\ & \times |(n-1)x+c|^{(n-1)a+c-1/2}(1+o(1)).\end{aligned}$$

Note that $|\mathrm{Arg}(x + c/(n-1))| \leq |\mathrm{Arg}(x)|$ because we may choose m so big that $c > 0$ while

$$|(n-1)x+c|^{(n-1)a+c-1/2} = (n-1)^{(n-1)a}|x|^{(n-1)a+c-1/2}O(1).$$

Moreover, both $(n-1)x + c$ and x belong to the sector $-\frac{3\pi}{4} \leq \mathrm{Arg}(x) \leq \frac{3\pi}{4}$ for all x in the horizontal integration contour. Therefore, we have an estimate of the form

$$|\Gamma((n-1)x+c)|^{-1} \leq \mathrm{const}(n-1)^{-(n-1)a}|x|^{-(n-1)a-c+1/2}e^{(n-1)a+(n-1)|b||\mathrm{Arg}(x+c/(n-1))|},$$

for all x in the upper horizontal part of C_K, where the constant is independent of K. Note that $|q^{-x}\lambda^{(n-1)x}| = q^{-a}\lambda^{(n-1)a}$. Combining all these estimates together, we get that the absolute value of the integrand along the upper horizontal contour can be bounded from above by

$$\mathrm{const}((n-1)q^{1/(n-1)}/\lambda)^{(n-1)a}|x|^{-m-1/2}|\mathrm{d}a| \leq \mathrm{const}\, K^{-m-1/2}|\mathrm{d}a|,$$

where we used that $\lambda > (n-1)q^{1/(n-1)}$ and $|x|^2 \leq K^2 + (K+\epsilon-\delta)^2 \leq (1+|\epsilon-\delta|)^2K^2$ for all $x = a+\mathbf{i}K$ $(\delta - K \leq a \leq \epsilon)$. Therefore, up to a constant independent of K the integral is bounded by $K^{-m+1/2}$ which proves that the integral vanishes in the limit $K \to \infty$.

The estimate for the lower horizontal part of C_K, that is, $x = a - \mathbf{i}K$, $\delta - K \leq a \leq \epsilon$ is the same as above. Let us consider the vertical part $x = \delta - K + \mathbf{i}b$, $-K \leq b \leq K$. In order to apply Stirling's formula, let us first recall the reflection formula for the gamma function

$$\Gamma(x) = \Gamma(1-x)^{-1}\frac{2\pi\mathbf{i}}{e^{2\pi\mathbf{i}x}-1}e^{\pi\mathbf{i}x}.$$

If x is on the vertical part of the integration contour, then $-x$ belongs to a proper subsector of $-\pi < \mathrm{Arg}(x) < \pi$ in which the Stirling's formula for $\Gamma(1-x) = (-x)\Gamma(-x)$ can be applied, that is,

$$|\Gamma(x)| = \sqrt{2\pi}\frac{e^{-\pi b}}{\left|e^{2\pi\mathbf{i}a}e^{-2\pi b}-1\right|}|x|^{a-1/2}e^{-a+|b||\mathrm{Arg}(-x)|}(1+o(1)),$$

where $x = a + \mathbf{i}b$. Similarly,

$$\begin{aligned}|\Gamma((n-1)x+c)| = \sqrt{2\pi} & \frac{e^{-\pi(n-1)b}}{\left|e^{2\pi\mathbf{i}((n-1)a+c)}e^{-2\pi(n-1)b}-1\right|}|(n-1)x+c|^{(n-1)a+c-1/2} \\ & \times e^{-(n-1)a-c+(n-1)|b||\mathrm{Arg}(-x-c/(n-1))|}(1+o(1)).\end{aligned}$$

Note that if $x = a+\mathbf{i}b$ is on the integration contour, then $a = \delta - K$ and $(n-1)a+c = \mu+m-K$, where $\mu = (n-1)(\delta - 1/2)$. Therefore, $e^{2\pi\mathbf{i}a} = e^{2\pi\mathbf{i}\delta}$ and $e^{2\pi\mathbf{i}((n-1)a+c)} = e^{2\pi\mathbf{i}\mu}$ are constants independent of K. Moreover, we chose both μ and δ to be non-integers, so $e^{2\pi\mathbf{i}\delta}-1$ and $e^{2\pi\mathbf{i}\mu}-1$ are non-zero. We get

$$\frac{|\Gamma(x)|^{n-1}}{|\Gamma((n-1)x+c)x|} \leq \text{const}\, \frac{\left|e^{2\pi\mathbf{i}\mu}e^{-2\pi(n-1)b}-1\right|}{\left|e^{2\pi\mathbf{i}\delta}e^{-2\pi b}-1\right|^{n-1}} \frac{|x|^{(n-1)(a-1/2)-1}}{|(n-1)x+c|^{(n-1)a+c-1/2}} \times e^{(n-1)|b|(|\mathrm{Arg}(-x))|-|\mathrm{Arg}(-x-c/(n-1))|}(1+o(1)).$$

The first fraction is clearly a bounded function in $b \in \mathbb{R}$. For the second one, we have

$$\frac{|x|^{(n-1)(a-1/2)-1}}{|(n-1)x+c|^{(n-1)a+c-1/2}} \leq \text{const}\, \frac{|x|^{-m-1/2}}{(n-1)^{(n-1)a}}.$$

Finally, for the exponential term, let us look at the triangle formed by vectors $-x$ and $-x - c/(n-1)$. The area of this triangle is $\frac{|b|c}{2(n-1)}$. On the other hand, the difference $\theta := |\mathrm{Arg}(-x))| - |\mathrm{Arg}(-x-c/(n-1))|$ as $K \to \infty$ tends to 0 uniformly in $x = \delta - K + \mathbf{i}b$ for $|b| \leq K$. Therefore, up to a constant independent of K we can bound θ from above by $\sin\theta$. Using that the area of the triangle is also $\frac{1}{2}|x||x+c/(n-1)|\sin\theta$, we get

$$\begin{aligned}&(n-1)|b|(|\mathrm{Arg}(-x))| - |\mathrm{Arg}(-x-c/(n-1))| \\ &\quad = (n-1)|b|\theta \leq \text{const}\,|b|\sin\theta \leq \text{const}\, \frac{b^2c}{|x||(n-1)x+c|}.\end{aligned}$$

The above expression is bounded by a constant independent of K. We get the following estimate:

$$\frac{|\Gamma(x)|^{n-1}}{|\Gamma((n-1)x+c)x|} \leq \text{const}\, K^{-m-1/2}(n-1)^{(n-1)K}$$

for all $x = \delta - K + \mathbf{i}b$, $-K \leq b \leq K$, where the constant is independent of K. Finally, since $\left|q^{-x}\lambda^{(n-1)x}\right| = q^{-a}\lambda^{(n-1)a}$, we get the following estimate:

$$\left|q^{-x}\lambda^{(n-1)x}\frac{\Gamma(x)^{n-1}}{\Gamma((n-1)x+c)x}\right| \leq \text{const}\left((n-1)q^{\frac{1}{n-1}}/\lambda\right)^{(n-1)K}K^{-m-1/2}.$$

Since $\lambda > (n-1)q^{\frac{1}{n-1}}$ the integral along the vertical segment of C_K, up to a constant, is bounded by $K^{-m+1/2}$. Therefore, the integral vanishes in the limit $K \to \infty$.

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