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Comomentum Sections and Poisson Maps in Hamiltonian Lie Algebroids

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Abstract. In a Hamiltonian Lie algebroid over a pre-symplectic manifold and over a Poisson manifold, we introduce a map corresponding to a comomentum map, called a comomentum section. We show that the comomentum section gives a Lie algebroid morphism among Lie algebroids. Moreover, we prove that a momentum section on a Hamiltonian Lie algebroid is a Poisson map between proper Poisson manifolds, which is a generalization that a momentum map is a Poisson map between the symplectic manifold to dual of the Lie algebra. Finally, a momentum section is reinterpreted as a Dirac morphism on Dirac structures.

Key words: Poisson geometry; momentum maps; Poisson maps; Dirac structures

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1 Introduction

A momentum map is a fundamental object in symplectic geometry defined on a symplectic manifold with a Lie group action. Then the action is a Hamiltonian action and the total space is called a Hamiltonian G-space.

Analyses of physical models suggest that Lie group actions in momentum maps should be generalized to 'Lie groupoid actions' to realize symmetries and conserved quantities in physical theories [3, 6, 12]. A 'groupoid' generalization of a Lie algebra is a Lie algebroid. A Lie algebroid is an infinitesimal object of a Lie groupoid analogous to the way that a Lie algebra is an infinitesimal object of a Lie group.

Recently, a generalization of a momentum map and a Hamiltonian G-space over a pre-symplectic manifold has been proposed in a Lie algebroid (Lie groupoid) setting [8]. It is inspired by the analysis of the Hamiltonian formalism of general relativity [6] and compatibility of physical models with Lie algebroid structures [25].

Mathematically, the idea is natural in the following sense. A momentum map μ is a map from a smooth manifold M to dual of a Lie algebra $\mathfrak{g}^*$, $\mu\colon M \to \mathfrak{g}^*$. It is also regarded as a section of a trivial vector bundle $M \times \mathfrak{g}^*$. A momentum map on the trivial bundle has been generalized to a section on the dual of a vector bundle A, $\mu \in \Gamma(A^*)$, satisfying certain consistency conditions. The generalization $\mu \in \Gamma(A^*)$ is called a momentum section, and the Hamiltonian G-space is generalized to a Hamiltonian Lie algebroid [8].

Further generalizations have been analyzed. A momentum section and a Hamiltonian Lie algebroid over a Poisson manifold has also been proposed in [7]. A momentum section has been generalized to a momentum section on a Courant algebroid [23], over a pre-multisymplectic manifold [18], over bundle-valued (multi)symplectic structures [19] and over a Dirac structure [20]. In our paper, based on two papers [7, 8], we consider both momentum sections over a (pre-)symplectic manifold and over a Poisson manifold.

Momentum maps have several important and elegant properties. If there exists a momentum map on a symplectic or a Poisson manifold, we have symplectic [29, 31] or Poisson reductions [28]. One essential property to make reductions consistent is that the momentum map is a Poisson map from M to $\mathfrak{g}^*$. In other words, comomentum map $\mu^*\colon \mathfrak{g} \to C^\infty(M)$ is a Lie algebra morphism. However, a momentum section is not necessarily a Poisson map from M to A^* [7]. Our motivation is to improve this problem to consider reductions for the Hamiltonian Lie algebroid setting. One idea to make a momentum section a Poisson map is to impose a condition compatible with a Poisson structure and a Lie algebroid A [19]. We propose another idea to construct a Poisson map in a Hamiltonian Lie algebroid.

In this paper, we generalize two properties of momentum maps to momentum sections. One is a comomentum map and another is the momentum map as a Poisson map.

For a momentum map μ, we can define a *comomentum map* $\mu^*\colon \mathfrak{g} \to C^\infty(M)$ as a dual map, which has properties induced from the momentum map. In particular, a comomentum map has the following property.

Proposition 1.1. *The comomentum map is a Lie algebra morphism from $\mathfrak{g}$ to $C^\infty(M)$, where a Lie algebra structure on $C^\infty(M)$ is defined by the Poisson bracket.*

Here, for two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a Lie algebra morphism is a map $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfying $\phi([e_1, e_2]_1) = [\phi(e_1), \phi(e_2)]_2$ for every $e_1, e_2 \in \mathfrak{g}_1$. A Lie algebra structure on $C^\infty(M)$ is given by the Poisson bracket.

We have the following natural question.

Question 1.2. Construct a comomentum section such that it is a Lie algebroid morphism between suitable two Lie algebroids.

In this paper, we define and analyze a bracket-compatible *comomentum section* $\mu^*\colon \Gamma(A) \to C^\infty(M)$ corresponding to a bracket-compatible momentum section μ. It is proved that the comomentum section μ^* is a *Lie algebroid morphism* from A to a proper space including $C^\infty(M)$.

Definition 1.3. Assume that $(A_1, [-,-]_1, \rho_{A_1})$ and $(A_2, [-,-]_2, \rho_{A_2})$ are two Lie algebroids over M. A *Lie algebroid morphism* between two Lie algebroids A_1 and A_2 is a vector bundle morphism $\phi\colon A_1 \to A_2$ such that

$$\begin{aligned} &\phi([e_1, e_2]_1) = [\phi(e_1), \phi(e_2)]_2, \\ &\rho_{A_2} \circ \phi = \rho_{A_1} \end{aligned}$$

for $e_1, e_2 \in \Gamma(A_1)$.

The idea on a pre-symplectic manifold is that we consider a pair given by the anchor map ρ of the Lie algebroid A and comomentum section μ^*. Since ρ is a map from A to TM, $\rho + \mu^*$ is regarded as a map from A to $TM \oplus \mathbb{R}$. On a Poisson manifold, ρ is replaced to $(\nabla\mu)^*\colon A \to T^*M$. Then $(\nabla\mu)^* + \mu^*$ is a Lie algebroid morphism from A to $T^*M \oplus \mathbb{R}$ if we define a Lie algebroid structure on $T^*M \oplus \mathbb{R}$.

Another important property of a momentum map is that it is a Poisson map between two Poisson manifolds M and $\mathfrak{g}^*$.

The dual of a Lie algebra $\mathfrak{g}^*$ has the so called Kirillov–Kostant–Souriau (KKS) Poisson structure $\pi_{\mathrm{KKS}} \in \wedge^2(T\mathfrak{g}^*)$ [24]. A momentum map μ is a map between two Poisson manifolds (M, π) and $(\mathfrak{g}^*, \pi_{\mathrm{KKS}})$ satisfying the following property.

Proposition 1.4. *A momentum map $\mu\colon M \to \mathfrak{g}^*$ is a Poisson map.*

The purpose of this paper is to generalize this proposition to a momentum section. Here a Poisson map is defined as follows.

Definition 1.5. Let (M_1, π_1) and (M_2, π_2) be Poisson manifolds. A smooth map ψ is a *Poisson map* if $\psi^*\colon C^\infty(M_2) \to C^\infty(M_1)$ satisfies $\psi^*\{f,g\}_2 = \{\psi^* f, \psi^* g\}_1$, for $f, g \in C^\infty(M_2)$, where $\{-,-\}_1$ and $\{-,-\}_2$ are Poisson brackets on M_1 and M_2.

A momentum section $\mu \in \Gamma(A^*)$ is regarded as a map $\mu\colon M \to A^*$. A generalization of Proposition 1.4 for a bracket-compatible momentum section has been analyzed in [7]. On the dual of the Lie algebroid A^*, a bivector field $\widetilde{\pi}_{A^*} = \widetilde{\pi} + \pi_{A^*}$ is defined, where $\widetilde{\pi}$ is the lift of the Poisson bivector field π on M to A^* and π_{A^*} is a Poisson bivector field induced from a Lie algebroid structure on A. In the Hamiltonian Lie algebroid over a Poisson manifold M, $\mu\colon M \to A^*$ is a bivector map if the basic curvature vanishes, ${}^A S = 0$. Here a bivector map $\psi\colon M \to A^*$ is a bilinear map such that $\{\psi^* a, \psi^* b\}_M = \psi^*\{a, b\}_{A^*}$, for every $a, b \in C^\infty(A^*)$, where a bilinear bracket $\{-,-\}_M$ is the Poisson bracket induced from π_M and $\{-,-\}_{A^*}$ is a bilinear bracket induced from $\widetilde{\pi}_{A^*}$. In this result, $\widetilde{\pi}_{A^*}$ is not necessarily a Poisson bivector field on A^*, i.e., $\{-,-\}_{A^*}$ does not satisfy the Jacobi identity, which means that μ is not necessarily a Poisson map.

Question 1.6. Is a momentum section μ regarded as a Poisson map of two proper Poisson manifolds?

We prove that a bracket-compatible momentum section is a Poisson map from $T^*M \oplus \mathbb{R}$ to A^* under Poisson structures induced from the Poisson structure π_M on M and the Poisson structure π_{A^*} induced from the Lie algebroid structure on A.

This paper is organized as follows. Section 2 is the preparation. In Section 2, after a Lie algebroid, connections and related notion are introduced, a Hamiltonian Lie algebroid and a momentum section over a pre-symplectic manifold and a Poisson manifold are explained. Moreover, a Courant algebroid and a Dirac structure are introduced. In Section 3, a relation of a momentum section with the basic curvature is discussed and some formulas are given. In Section 4, comomentum sections are defined for a pre-symplectic case and a Poisson case, and Lie algebroid morphisms induced from momentum sections are constructed. In Section 5, a Poisson map induced from a momentum section is constructed. In Section 6, a momentum section is reinterpreted as a Dirac morphism.

2 Preliminary

In this section, we summarize definitions and previous results, including a Lie algebroid and connections, momentum sections and Hamiltonian Lie algebroids over a pre-symplectic manifold and over a Poisson manifold, as well as a Courant algebroid and a Dirac structure.

2.1 Lie algebroids

Definition 2.1. Let A be a vector bundle over a smooth manifold M. A Lie algebroid $(A, [-,-], \rho = \rho_A)$ is a vector bundle A with a bundle map $\rho = \rho_A\colon A \to TM$ called the anchor map, and a Lie bracket $[-,-] = [-,-]_A\colon \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ satisfying the Leibniz rule, $[e_1, f e_2] = f[e_1, e_2] + \rho_A(e_1) f \cdot e_2$, where $e_1, e_2 \in \Gamma(A)$ and $f \in C^\infty(M)$.

A Lie algebroid is a generalization of a Lie algebra and the space of vector fields on a smooth manifold.

Example 2.2 (Lie algebras)**.** Let a manifold M be one point $M = \{\mathrm{pt}\}$. Then a Lie algebroid is a Lie algebra $\mathfrak{g}$.

Example 2.3 (tangent Lie algebroids)**.** Let a vector bundle A be a tangent bundle TM, $\rho_A = \mathrm{id}$ and a bracket $[-,-]$ is a normal Lie bracket on the space of vector fields $\mathfrak{X}(M)$. TM together with ρ and $[-,-]$ is a Lie algebroid. It is called a *tangent Lie algebroid*.

Example 2.4 (action Lie algebroids)**.** Assume a smooth action of a Lie group G to a smooth manifold M, $M \times G \to M$. The differential map induces an infinitesimal action on the manifold M of the Lie algebra $\mathfrak{g}$ of G. Since $\mathfrak{g}$ acts as a differential operator on M, the differential map is a bundle map $\rho\colon M \times \mathfrak{g} \to TM$. Consistency of a Lie bracket requires that ρ is a Lie algebra morphism such that

$$[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2]), \tag{2.1}$$

where the bracket on the left-hand side of (2.1) is the Lie bracket of vector fields. These data give a Lie algebroid $(A = M \times \mathfrak{g}, [-,-], \rho)$. This Lie algebroid is called an *action Lie algebroid*.

Example 2.5 (Poisson Lie algebroids)**.** A bivector field $\pi \in \Gamma\left(\wedge^2 TM\right)$ is called a Poisson bivector field if $[\pi,\pi]_S = 0$, where $[-,-]_S$ is the Schouten bracket on the space of multivector fields, $\Gamma(\wedge^\bullet TM)$. A smooth manifold M with a Poisson bivector field π is called a Poisson manifold and denoted by (M,π).

Let (M,π) be a Poisson manifold. Then a Lie algebroid structure is induced on T^*M. A bundle map is defined as $\pi^\sharp\colon T^*M \to TM$ by $\left\langle \pi^\sharp(\alpha), \beta\right\rangle = \pi(\alpha,\beta)$ for all $\beta \in \Omega^1(M)$. $\rho = -\pi^\sharp$ is the anchor map, and a Lie bracket on $\Omega^1(M)$ is defined by the Koszul bracket

$$[\alpha,\beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha,\beta)),$$

where $\alpha, \beta \in \Omega^1(M)$. $\left(T^*M, [-,-]_\pi, -\pi^\sharp\right)$ is a Lie algebroid.

One can refer to reviews and textbooks for basic properties of Lie algebroids, see, for instance, [27].

For a Lie algebroid A, sections of the exterior algebra of A^* are called *A-differential forms*. A differential ${}^A\mathrm{d}\colon \Gamma(\wedge^m A^*) \to \Gamma\left(\wedge^{m+1}A^*\right)$ on the spaces of A-differential forms, $\Gamma(\wedge^\bullet A^*)$, called a *Lie algebroid differential*, or an *A-differential*, is defined as follows.

Definition 2.6. For an A-differential form $\eta \in \Gamma(\wedge^m A^*)$, a Lie algebroid differential ${}^A\mathrm{d}\colon \Gamma(\wedge^m A^*) \to \Gamma\left(\wedge^{m+1}A^*\right)$ is defined by

$$\begin{aligned} {}^A\mathrm{d}\eta(e_1,\dots,e_{m+1}) = {}& \sum_{i=1}^{m+1}(-1)^{i-1}\rho_A(e_i)\eta(e_1,\dots,\check{e}_i,\dots,e_{m+1}) \\ &+ \sum_{1\le i<j\le m+1}(-1)^{i+j}\eta([e_i,e_j],e_1,\dots,\check{e}_i,\dots,\check{e}_j,\dots,e_{m+1}), \end{aligned}$$

where $e_i \in \Gamma(A)$, and the haćěk symbol signifies omitting that argument.

The A-differential satisfies $\left({}^A\mathrm{d}\right)^2 = 0$. It is a generalization of the de Rham differential on T^*M and the Chevalley–Eilenberg differential on a Lie algebra.

Definition 2.7. Assume that $(A_1, [-,-]_1, \rho_{A1})$ and $(A_2, [-,-]_2, \rho_{A2})$ are two Lie algebroids over M. A *Lie algebroid morphism* between two Lie algebroids A_1 and A_2 is a vector bundle morphism $\phi\colon A_1 \to A_2$ such that

$$\begin{aligned} &\phi([e_1,e_2]_1) = [\phi(e_1),\phi(e_2)]_2, \\ &\rho_{A2}\circ\phi = \rho_{A1} \end{aligned} \tag{2.2}$$

for $e_1, e_2 \in \Gamma(A_1)$.[1]

[1]For Lie algebroids over different base manifolds, we can define more general Lie algebroid morphism [27]. For two Lie algebroids (A_1, M_1) and (A_2, M_2), a morphism $\phi\colon A_1 \to A_2$ is a vector bundle morphism whose graph $\mathrm{Gr}(\phi) \subset A_1 \times A_2$ is a Lie subalgebroid of $A_1 \times A_2$.

2.2 Connections on Lie algebroids

We introduce several connections on a vector bundle E.

Definition 2.8. A connection is an $\mathbb{R}$-linear map, $\nabla\colon \Gamma(E) \to \Gamma(E \otimes T^*M)$, satisfying the Leibniz rule, $\nabla(fs) = f\nabla s + (\mathrm{d}f) \otimes s$, for $s \in \Gamma(E)$ and $f \in C^\infty(M)$. A dual connection on E^* is defined by the equation

$$\mathrm{d}\langle \mu, s\rangle = \langle \nabla\mu,\, s\rangle + \langle \mu,\, \nabla s\rangle$$

for all sections $\mu \in \Gamma(E^*)$ and $s \in \Gamma(E)$, where $\langle -,\, -\rangle$ is the pairing between E and E^*.

We use the same notation ∇ for the dual connection.

On a Lie algebroid, another derivation called an A-connection is defined.

Definition 2.9. Let A be a Lie algebroid over a smooth manifold M and E be a vector bundle over the same base manifold M. An *A-connection* on a vector bundle E with respect to the Lie algebroid A is a $\mathbb{R}$-linear map, ${}^A\nabla\colon \Gamma(E) \to \Gamma(E \otimes A^*)$, satisfying

$${}^A\nabla_e(fs) = f\,{}^A\nabla_e s + (\rho_A(e)f)s$$

for $e \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

The ordinary connection is regarded as an A-connection for $A = TM$, $\nabla = {}^{TM}\nabla$.

If an ordinary connection ∇ on A as a vector bundle is given, an A-connection on A is simply given by ${}^A\nabla_e e' := \nabla_{\rho_A(e')} e$, for $e, e' \in \Gamma(A)$.

Another A-connection called the *basic A-connection* on the tangent bundle $E = TM$, ${}^A\nabla^{\mathrm{bas}}\colon \Gamma(TM) \to \Gamma(TM \otimes A^*)$ is defined.

Definition 2.10. The *basic A-connection* on TM, ${}^A\nabla^{\mathrm{bas}}\colon \Gamma(TM) \to \Gamma(TM \otimes A^*)$ is defined by

$${}^A\nabla^{\mathrm{bas}}_e v := \mathcal{L}_{\rho_A(e)} v + \rho_A(\nabla_v e) = [\rho_A(e), v] + \rho_A(\nabla_v e), \tag{2.3}$$

where $e \in \Gamma(A)$ and $v \in \mathfrak{X}(M)$.

For a 1-form $\alpha \in \Omega^1(M)$, the basic A-connection is given by

$${}^A\nabla^{\mathrm{bas}}_e \alpha := \mathcal{L}_{\rho_A(e)} \alpha + \langle \rho_A(\nabla e),\, \alpha\rangle. \tag{2.4}$$

Throughout this paper, A-connections ${}^A\nabla$ on TM and T^*M are always the basic A-connection ${}^A\nabla = {}^A\nabla^{\mathrm{bas}}$, (2.3) and (2.4).

Given a connection ∇, a covariant derivative is generalized to the derivation on the space of differential forms taking a value on $\wedge^m A^*$, $\Omega^k(M, \wedge^m A^*)$ called an *exterior covariant derivative*. Similarly, an A-connection ${}^A\nabla$ can be generalized to the derivation satisfying the Leibniz rule for sections on $\wedge^m A^*$. It is called *the A-exterior covariant derivative* ${}^A\nabla$. They are denoted by the same notation ∇ and ${}^A\nabla$.

Let $\Omega^k(M, \wedge^m A^*)$ be the space of k-forms taking values on $\wedge^m A^*$.

Definition 2.11. For $\Omega^k(M, \wedge^m A) = \Gamma\big(\wedge^m A^* \otimes \wedge^k T^*M\big)$, the *$A$-exterior covariant derivative* ${}^A\nabla\colon \Omega^k(M, \wedge^m A^*) \to \Omega^k\big(M, \wedge^{m+1} A^*\big)$ is defined by

$$\begin{aligned}\big({}^A\nabla\alpha\big)(e_1, \ldots, e_{m+1}) := {} & \sum_{i=1}^{m+1} (-1)^{i-1}\,{}^A\nabla_{e_i}(\alpha(e_1, \ldots, \check{e}_i, \ldots, e_{m+1})) \\ & + \sum_{1 \le i < j \le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1})\end{aligned}$$

for $\alpha \in \Omega^k(M, \wedge^m A^*)$ and $e_i \in \Gamma(A)$.

Note that the A-exterior covariant derivative increases the order of $\wedge^m A^*$.

The A-exterior covariant derivative for $E = TM$ is given by taking the pairing,

$$ {}^A\mathrm{d}\langle \phi,\, \alpha\rangle = \big\langle \big({}^A\nabla\phi\big), \alpha\big\rangle + \big\langle \phi, \big({}^A\nabla\alpha\big)\big\rangle $$

for $\phi \in \mathfrak{X}^k(M, \wedge^m A^*)$ and $\alpha_i \in \Omega^k(M)$.

One can refer to [1, 15, 17] about a theory of general and basic A-connections on a Lie algebroid.

For two connections ∇ and ${}^A\nabla$, various torsions and curvatures are introduced. Additional to the normal curvature $R \in \Omega^2(M, A \otimes A^*)$ and the torsion for a vector bundle connection ∇, similar quantities for the A-connections are introduced. An A-*curvature* ${}^AR \in \Gamma\big(\wedge^2 A^* \otimes A \otimes A^*\big)$ is defined by

$$ {}^AR(e, e') := \big[{}^A\nabla_e, {}^A\nabla_{e'}\big] - {}^A\nabla_{[e,e']} $$

for $e, e' \in \Gamma(A)$. It does not appear explicitly in our paper. Important geometric quantities are an ordinary curvature, an A-torsion and a basic curvature [5].

Definition 2.12. An ordinary *curvature* $R \in \Omega^2(M, A \otimes A^*)$, an A-*torsion*, ${}^AT \in \Gamma\big(A \otimes \wedge^2 A^*\big)$, and a *basic curvature*, ${}^AS \in \Omega^1\big(M, \wedge^2 A^* \otimes A\big)$, are defined by

$$ \begin{aligned} R(v, v') &:= [\nabla_v, \nabla_{v'}] - \nabla_{[v,v']}, \\ {}^AT(e, e') &:= \nabla_{\rho(e)}e' - \nabla_{\rho(e')}e - [e, e'], \\ {}^AS(e, e') &:= [e, \nabla e'] - [e', \nabla e] - \nabla[e, e'] - \nabla_{\rho_A(\nabla e)}e' + \nabla_{\rho_A(\nabla e')}e \\ &= (\nabla^A T + 2\mathrm{Alt}\, \iota_\rho R)(e, e') \end{aligned} \tag{2.5} $$

for $v, v' \in \mathfrak{X}(M)$ and $e, e' \in \Gamma(A)$. Note that $\iota_\rho R \in \Omega^1(M, A^* \otimes A^* \otimes A)$ since $\rho \in \Gamma(A^* \otimes TM)$ and ι_ρ gives the contraction between TM and T^*M. Notation Alt means skew symmetrization on $A^* \otimes A^*$, which gives an element in $\Omega^1\big(M, \wedge^2 A^* \otimes A\big)$.

2.3 Momentum sections and Hamiltonian Lie algebroids

In this section, a bracket-compatible momentum section and a Hamiltonian Lie algebroid, which are a generalization of a momentum map on a symplectic manifold, are reviewed [7, 8].

A closed 2-form $\omega \in \Omega^2(M)$ on M is called a pre-symplectic form. A pair (M, ω) of a manifold M and a pre-symplectic form ω is called a pre-symplectic manifold. If ω is nondegenerate, (M, ω) is a symplectic manifold.

On a pre-symplectic manifold M, the following three conditions are introduced.

Definition 2.13 (momentum sections over pre-symplectic manifolds)**.** Suppose that a base manifold (M, ω) is a pre-symplectic manifold, and take a Lie algebroid $(A, [-,-], \rho_A)$ over M.

(S1) A Lie algebroid A is called *pre-symplectically anchored* if ω satisfies

$$ {}^A\nabla\omega = 0. \tag{2.6} $$

(S2) A section $\mu \in \Gamma(A^*)$ is a ∇-*momentum section* if it satisfies[2]

$$ (\nabla\mu)(e) = -\iota_{\rho_A(e)}\omega \tag{2.7} $$

for $e \in \Gamma(A)$.

[2]Notation of the momentum section such as $\mu(e)$, $(\nabla\mu)(e)$, etc. are in fact pairings of A^* and A, $\langle\mu,\, e\rangle$, $\langle\nabla\mu,\, e\rangle$, etc.

(S3) μ is *bracket-compatible* if it satisfies

$$\left({}^A\mathrm{d}\mu\right)(e_1, e_2) = \omega(\rho_A(e_1), \rho_A(e_2)), \tag{2.8}$$

where $e_1, e_2 \in \Gamma(A)$.

Note that the above definition depends on choice of a connection ∇.

Definition 2.14. A Lie algebroid A over a pre-symplectic manifold with a connection ∇ and a section $\mu \in \Gamma(A^*)$ is called *Hamiltonian*[3] if equations (2.6), (2.7) and (2.8) are satisfied.

On a trivial bundle, a momentum section is equivalent to a momentum map. Suppose that M has an action of a Lie group G and ω is a symplectic form. For a Lie algebra $\mathfrak{g}$ of G, a trivial bundle $A = M \times \mathfrak{g}$ has an action Lie algebroid structure in Example 2.4. A section $e \in \Gamma(M \times \mathfrak{g})$ is restricted to the constant section, which is identified to an element of $\mathfrak{g}$. We can take a trivial connection $\nabla = \mathrm{d}$ on the trivial bundle $M \times \mathfrak{g}$. Then conditions of Definition 2.13 reduce to the following conditions. Equation (2.7) is

$$(\mathrm{d}\mu)(e) = -\iota_{\rho_A(e)}\omega, \tag{2.9}$$

where e is a constant section. Equation (2.9) means that $\mu(e)$ is the Hamiltonian function for the Lie algebra action $\rho_A(e)$. Equation (2.6) is ${}^A\nabla_e\omega = \mathcal{L}_{\rho_A(e)}\omega = 0$ from the definition of the A-connection. This equation is trivially satisfied from $\mathrm{d}\omega = 0$ and equation (2.9). Equation (2.8) is equivalent to

$$\rho_A(e_1)\mu(e_2) = \mu([e_1, e_2])$$

under (2.7) and (2.6), which means that μ is infinitesimally equivariant. Since the section $\mu \in \Gamma(M \times \mathfrak{g}^*)$ is a map $\mu\colon M \to \mathfrak{g}^*$, therefore, μ is a momentum map on the pre-symplectic manifold M.

A Hamiltonian Lie algebroid over a Poisson manifold is defined as follows. [7]

Definition 2.15 (momentum sections over Poisson manifolds)**.** Let (M, π) be a Poisson manifold with a Poisson bivector field $\pi \in \Gamma\left(\wedge^2 TM\right)$ and $(A, [-,-], \rho_A)$ be a Lie algebroid over M.

(P1) A is called *Poisson anchored* if π satisfies

$${}^A\nabla\pi = 0. \tag{2.10}$$

(P2) A section $\mu \in \Gamma(A^*)$ is a ∇-*momentum section* if it satisfies

$$\rho_A(e) = -\pi^\sharp((\nabla\mu)(e)) \tag{2.11}$$

for $e \in \Gamma(A)$.

(P3) μ is called *bracket-compatible* if it satisfies

$$\left({}^A\mathrm{d}\mu\right)(e_1, e_2) = -\pi((\nabla\mu)(e_1), (\nabla\mu)(e_2)) \tag{2.12}$$

for $e_1, e_2 \in \Gamma(A)$.

In Definition 2.15, conditions depend on choice of a connection ∇.

Definition 2.16. A Lie algebroid A over a Poisson manifold with a connection ∇ and a section $\mu \in \Gamma(A^*)$ is called *Hamiltonian* if equations (2.10), (2.11) and (2.12) are satisfied.

If π is nondegenerate, M is a symplectic manifold with $\omega = \pi^{-1}$. A Hamiltonian Lie algebroid over a nondegenerate Poisson manifold is a Hamiltonian Lie algebroid over a symplectic manifold as per Definition 2.14.

[3] If the condition is satisfied on a neighborhood of every point in M, it is called *locally Hamiltonian* [8]. All the analysis in this paper are applicable in the locally Hamiltonian case.

2.4 Courant algebroids and Dirac structures

In this subsection, a Courant algebroid and a Dirac structure [14, 26] are introduced as preparations for the following sections.

Definition 2.17. A Courant algebroid is a vector bundle E over M, which has a nondegenerate symmetric bilinear form $\langle -, -\rangle$, a bilinear operation $[-,-]_D$ on $\Gamma(E)$, and a bundle map called an anchor map, $\rho_E\colon E \longrightarrow TM$, satisfying the following properties:

(1) $[e_1,[e_2,e_3]_D]_D = [[e_1,e_2]_D,e_3]_D + [e_2,[e_1,e_3]_D]_D$,

(2) $\rho_E([e_1,e_2]_D) = [\rho_E(e_1),\rho_E(e_2)]$,

(3) $[e_1,fe_2]_D = f[e_1,e_2]_D + (\rho_E(e_1)f)e_2$,

(4) $[e,e]_D = \frac{1}{2}\mathcal{D}\langle e,\, e\rangle$,

(5) $\rho_E(e_1)\langle e_2,\, e_3\rangle = \langle [e_1,e_2]_D,\, e_3\rangle + \langle e_2,\, [e_1,e_3]_D\rangle$,

where $e, e_1, e_2, e_3 \in \Gamma(E)$, $f \in C^\infty(M)$ and $\mathcal{D}$ is a map from $C^\infty(M)$ to $\Gamma(E)$, defined as $\langle \mathcal{D}f,\, e\rangle = \rho_E(e)f$ [26].

The bilinear bracket $[-,-]_D$ is called the Dorfman bracket. A Courant algebroid is encoded in the quadruple $(E, \langle -,\, -\rangle, [-,-]_D, \rho_E)$.

Example 2.18. The *standard Courant algebroid* is the Courant algebroid as defined below on the vector bundle $E = TM \oplus T^*M$.

The three operations of the standard Courant algebroid are defined as follows:

$$\langle u+\alpha,\, v+\beta\rangle = \iota_u\beta + \iota_v\alpha, \qquad \rho_{T\oplus T^*}(u+\alpha) = u,$$
$$[u+\alpha, v+\beta]_D = [u,v] + \mathcal{L}_u\beta - \iota_v \mathrm{d}\alpha + \iota_u\iota_v H$$

for $u+\alpha, v+\beta \in \Gamma(TM\oplus T^*M)$, where u, v are vector fields, α, β are 1-forms, and $H \in \Omega^3(M)$ is a closed 3-form on M.

Definition 2.19. A *Dirac structure* L is a maximally isotropic subbundle of a Courant algebroid E, whose sections are closed under the Dorfman bracket, i.e., L is a subbundle of a Courant algebroid satisfying $\langle e_1,\, e_2\rangle = 0$ (isotropic), $[e_1,e_2]_D \in \Gamma(L)$ (closed) for every $e_1, e_2 \in \Gamma(L)$.

The following proposition is a basic fact for a Dirac structure.

Proposition 2.20 ([26])**.** *A Dirac structure L is a Lie algebroid.*

Example 2.21. Let (M,ω) be a pre-symplectic manifold and

$$(TM\oplus T^*M, \langle -,\, -\rangle, [-,-]_D, \rho_{T\oplus T^*})$$

be a standard Courant algebroid with $H=0$ in Example 2.18. A bundle map $\omega^\flat\colon TM \to T^*M$ is defined by $\omega^\flat(u)(v) = \omega(u,v)$ for every $v \in \mathfrak{X}(M)$. A subbundle $L_\omega \subset TM\oplus T^*M$ given by

$$L_\omega = \mathrm{Gr}(\omega) := \left\{u + \omega^\flat(u) \mid u \in \mathfrak{X}(M)\right\}$$

is a Dirac structure. In fact, $\left\langle u+\omega^\flat(u), v+\omega^\flat(v)\right\rangle = 0$ since $\omega(u,v) = -\omega(v,u)$, and the concrete calculation gives $\left[u+\omega^\flat(u), v+\omega^\flat(v)\right] = [u,v] + \omega^\flat([u,v])$ if $\mathrm{d}\omega = 0$, which means that L_ω is involutive.

Example 2.22. Let (M,π) be a Poisson manifold and $(TM\oplus T^*M, \langle -,\, -\rangle, [-,-]_D, \rho_{T\oplus T^*})$ be a standard Courant algebroid with $H=0$. For a bundle map $\pi^\sharp\colon T^*M \to TM$, a subbundle $L_\pi \subset TM\oplus T^*M$ given by

$$L_\pi = \mathrm{Gr}(\pi) := \left\{-\pi^\sharp(\alpha) + \alpha \mid \alpha \in \Omega^1(M)\right\}$$

is a Dirac structure. In fact, $\left\langle -\pi^\sharp(\alpha)+\alpha, -\pi^\sharp(\beta)+\beta\right\rangle = 0$ since $\pi(\alpha,\beta) = -\pi(\beta,\alpha)$, and the concrete calculation gives $\left[-\pi^\sharp(\alpha)+\alpha, -\pi^\sharp(\beta)+\beta\right] = -\pi^\sharp([\alpha,\beta]_\pi) + [\alpha,\beta]_\pi$ if π is a Poisson bivector field, which means that L_π is involutive.

3 Basic curvatures and momentum sections

Let (A, π, ∇, μ) be a Hamiltonian Lie algebroid over a Poisson manifold. The anchor map ρ_A is a Lie algebroid morphism $\rho_A \colon A \to TM$, i.e., ρ_A satisfies

$$[\rho_A(e_1), \rho_A(e_2)] = \rho_A([e_1, e_2]) \tag{3.1}$$

for $e_1, e_2 \in \Gamma(A)$. The covariant forms of equation (3.1) is

$$(\nabla_{\rho_A(e_1)}\rho_A)(e_2) - (\nabla_{\rho_A(e_2)}\rho_A)(e_1) + \langle \rho_A, {}^A T(e_1, e_2)\rangle = 0. \tag{3.2}$$

In fact, equation (3.2) is proved as follows. Using the Leibniz rule of the connection and equation (2.5), we obtain

$$\begin{aligned}
\langle \rho_A, {}^A T(e_1, e_2)\rangle &= \rho_A(\nabla_{\rho_A(e_1)}e_2) - \rho_A(\nabla_{\rho_A(e_2)}e_1) - \rho_A([e_1, e_2])\\
&= \nabla_{\rho_A(e_1)}(\rho_A(e_2)) - \nabla_{\rho_A(e_2)}(\rho_A(e_1)) - (\nabla_{\rho_A(e_!)}\rho_A)(e_2) + (\nabla_{\rho_A(e_2)}\rho_A)(e_1)\\
&\quad - [\rho_A(e_1), \rho_A(e_2)]\\
&= \mathcal{L}_{\rho_A(e_1)}(\rho_A(e_2)) - \mathcal{L}_{\rho_A(e_2)}(\rho_A(e_1)) - [\rho_A(e_1), \rho_A(e_2)]\\
&\quad - (\nabla_{\rho_A(e_1)}\rho_A)(e_2) + (\nabla_{\rho_A(e_2)}\rho_A)(e_1)\\
&= - (\nabla_{\rho_A(e_1)}\rho_A)(e_2) + (\nabla_{\rho_A(e_2)}\rho_A)(e_1).
\end{aligned}$$

The Jacobi identity of the Lie bracket $[-, -]$ is equivalent to the equation

$$\nabla_{\rho_A(e_1)}{}^A T(e_2, e_3) - {}^A T\big(e_1, {}^A T(e_2, e_3)\big) - R(\rho_A(e_1), \rho_A(e_2), e_3) + \mathrm{Cycl}(e_1, e_2, e_3) = 0.$$

Before we analyze a comomentum section and a Poisson map, we consider a formula between a momentum section and the basic curvature. From the definition of the Hamiltonian Lie algebroid over a Poisson manifold, we obtain the following formula, which is the key identity in our paper.

Lemma 3.1. *In a Hamiltonian Lie algebroid over a Poisson manifold, $\pi^\sharp\langle {}^A S, \mu\rangle = 0$, where $\langle -, -\rangle = 0$ is the pairing of A and A^*.*

Proof. They are proved by using local coordinate expressions.

For a Lie algebroid $(A, \rho, [-, -])$ over a smooth manifold M, let x^i be a local coordinate on M, $e_a \in \Gamma(A)$ be a basis of the fiber of A and $e^a \in \Gamma(A^*)$ be the dual basis on A^*. The basis of TM is denoted by $\partial_i = \frac{\partial}{\partial x^i}$. i, j, etc. are indices on M and a, b, etc. are indices on the fiber of A and A^*. Upper indices are ones for TM and A, and lower indices are ones for T^*M and A^*.[4]

Local coordinate expressions of the anchor map and the Lie bracket are $\rho(e_a) = \rho_a^i(x)\partial_i$ and $[e_a, e_b] = C_{ab}^c(x)e_c$, where $\rho_a^i(x)$ and $C_{ab}^c(x)$ are local functions. Identities of the Lie algebroid are

$$\rho_a^j\partial_j\rho_b^i - \rho_b^j\partial_j\rho_a^i = C_{ab}^c\rho_c^i, \qquad C_{ad}^e C_{bc}^d + \rho_a^i\partial_i C_{bc}^e + \mathrm{Cycl}(abc) = 0. \tag{3.3}$$

The condition of Poisson bivector field π, $[\pi, \pi]_S = 0$, is

$$\pi^{il}\partial_l\pi^{jk} + \mathrm{Cycl}(ijk) = 0. \tag{3.4}$$

[4] Einstein summation convention is used for sums in local coordinate expressions.

A connection 1-form $\omega_a^b = \omega_{ai}^b \mathrm{d}x^i$ for the connection ∇ is introduced as $\nabla_i e_a := \omega_{ai}^b e_b$ for the basis, e_a. Local coordinate expressions of an A-torsion, a curvature and a basic curvature are given by

$$
\begin{aligned}
&T_{ab}^c = -C_{ab}^c + \rho_a^i \omega_{bi}^c - \rho_b^i \omega_{ai}^c, && (3.5)\\
&R_{ijb}^a = \partial_i \omega_{aj}^b - \partial_j \omega_{ai}^b + \omega_{ai}^c \omega_{cj}^b - \omega_{aj}^c \omega_{ci}^b, &&\\
&S_{iab}^c = \nabla_i T_{ab}^c + \rho_b^j R_{ija}^c - \rho_a^j R_{ijb}^c, &&\\
&\qquad = -\partial_i C_{ab}^c - \omega_{di}^c C_{ab}^d + \omega_{ai}^d C_{db}^c + \omega_{bi}^d C_{ad}^c + \rho_a^j \partial_j \omega_{bi}^c - \rho_b^j \partial_j \omega_{ai}^c &&\\
&\qquad\quad + \partial_i \rho_a^j \omega_{bj}^c - \partial_i \rho_b^j \omega_{aj}^c + \omega_{ai}^d \rho_d^j \omega_{bj}^c - \omega_{bi}^d \rho_d^j \omega_{aj}^c, && (3.6)
\end{aligned}
$$

where the covariant derivative $\nabla_i T_{ab}^c$ is given by

$$
\nabla_i T_{ab}^c = \partial_i T_{ab}^c + \omega_{di}^c T_{ab}^d - \omega_{ai}^d T_{db}^c - \omega_{bi}^d T_{ad}^c.
$$

Equations (P1), (P2) and (P3) in the definition of a Hamiltonian Lie algebroid are

$$
\begin{aligned}
&\text{(P1)} \quad {}^A\nabla_a \pi^{ij} = \rho_a^k \partial_k \pi^{ij} - \partial_k \rho_a^i \pi^{kj} + \rho_b^i \omega_{ak}^b \pi^{kj} - \partial_k \rho_a^j \pi^{ik} + \rho_b^j \omega_{ak}^b \pi^{ik} = 0,\\
&\text{(P2)} \quad \rho_a^i = \pi^{ij} \nabla_j \mu_a \big(= \pi^{ij}\big(\partial_j \mu_a - \omega_{aj}^b \mu_b\big)\big), && (3.7)\\
&\text{(P3)} \quad \rho_{[a}^i \partial_i \mu_{b]} - C_{ab}^c \mu_c = -\pi^{ij} \nabla_i \mu_a \nabla_j \mu_b \big(= \pi^{ij}\big(\partial_i \mu_a - \omega_{ai}^c \mu_c\big)\big(\partial_j \mu_b - \omega_{bj}^d \mu_d\big)\big). && (3.8)
\end{aligned}
$$

Substituting (3.7) into the identity (3.3), $\rho_a^j \partial_j \rho_b^i - \rho_b^j \partial_j \rho_a^i = C_{ab}^c \rho_c^i$, we obtain the following equation:

$$
\pi^{ij}\big(-\pi^{kl} \partial_k \nabla_j \mu_a \nabla_l \mu_b - \pi^{kl} \nabla_k \mu_a \partial_l \nabla_j \mu_b - \partial_j \pi^{kl} \nabla_k \mu_a \nabla_l \mu_b - C_{ab}^c \nabla_j \mu_c\big) = 0. \tag{3.9}
$$

Here, we use π is the Poisson bivector field, i.e., equation (3.4). Moreover, equation (3.9) is equivalent to

$$
\pi^{ij}\big(\pi^{kl} \nabla_k \nabla_j \mu_a \nabla_l \mu_b + \pi^{kl} \nabla_k \mu_a \nabla_l \nabla_j \mu_b + \partial_j \pi^{kl} \nabla_k \mu_a \nabla_l \mu_b - T_{ab}^c \nabla_j \mu_c\big) = 0, \tag{3.10}
$$

which is the covariant expression of equation (3.9).

We consider another identity as follows. Substituting (3.7) to (3.8) and using equation (3.4), we obtain

$$
\pi^{ij} \nabla_j \mu_a \partial_i \mu_b - \pi^{ij} \nabla_j \mu_b \partial_i \mu_a + \pi^{ij} \nabla_i \mu_a \nabla_j \mu_b - C_{ab}^c \mu_c = 0.
$$

The covariant expression of this identity is

$$
\pi^{kl} \nabla_k \mu_a \nabla_l \mu_b - T_{ab}^c \mu_c = 0, \tag{3.11}
$$

where equation (3.5) is used. The global form of equation (3.11) is

$$
\pi(\nabla\mu(e_1), \nabla\mu(e_2)) - \big\langle \mu, {}^A T(e_1, e_2)\big\rangle = 0
$$

for $e_1, e_2 \in \Gamma(A)$.

Acting the covariant derivative by ∇_j to equation (3.11), and using the identity (3.6), $S_{iab}^c = \nabla_i T_{ab}^c + \rho_b^j R_{ija}^c - \rho_a^j R_{ijb}^c$, we obtain the equation

$$
\pi^{kl} \nabla_k \nabla_j \mu_a \nabla_l \mu_b + \pi^{kl} \nabla_k \mu_a \nabla_l \nabla_j \mu_b + \nabla_j \pi^{kl} \nabla_k \mu_a \nabla_l \mu_b - T_{ab}^c \nabla_j \mu_c - S_{jab}^c \mu_c = 0. \tag{3.12}
$$

By taking $\pi^{ij} \times$ (3.12), we obtain the identity

$$
\begin{aligned}
&\pi^{ij}\big(\pi^{kl} \nabla_k \nabla_j \mu_a \nabla_l \mu_b + \pi^{kl} \nabla_k \mu_a \nabla_l \nabla_j \mu_b + \nabla_j \pi^{kl} \nabla_k \mu_a \nabla_l \mu_b\\
&\qquad - T_{ab}^c \nabla_j \mu_c - S_{jab}^c \mu_c\big) = 0. && (3.13)
\end{aligned}
$$

Compare equations (3.10) and (3.13), where note that $\nabla_j \pi^{kl} = \partial_j \pi^{kl}$ since we do not introduce a TM-connection on TM.[5] Consistency between equations (3.10) and (3.13) gives the following identity, $\pi^{ij} S^c_{jab} \mu_c = 0$, i.e., $\pi^\sharp \langle {}^A S,\, \mu \rangle = 0$. ■

Lemma 3.1 is equivalent to the following statement.

Lemma 3.2. *In a Hamiltonian Lie algebroid over a Poisson manifold, $\langle {}^A S, \mu \rangle \in \ker(\pi^\sharp)$ for a momentum section μ.*

Now, we assume the equation

$$\langle {}^A S, \mu \rangle (e_1, e_2) = 0 \tag{3.14}$$

for every $e_1, e_2 \in \Gamma(A)$. In the local coordinate, equation (3.14) is

$$S^c_{jab} \mu_c = 0. \tag{3.15}$$

Note that the left-hand side of equation (3.14) is the commutator of the connection and the A-connection on μ,

$$\langle \iota_v {}^A S(e_1, e_2), \mu \rangle = \frac{1}{2}\big(-\big([\nabla_v, {}^A\nabla_{e_1}]\mu\big)(e_2) + \big([\nabla_v, {}^A\nabla_{e_2}]\mu\big)(e_1)\big).$$

Therefore, equation (3.14) is regarded as ∇_v and ${}^A\nabla$ are commutative on the momentum section μ.

The equation (3.14) is satisfied if $\ker(\pi^\sharp) = 0$, or sections $e_1, e_2 \in \Gamma(A)$ are restricted to the subspace of $\Gamma(A)$ satisfying equation (3.14). If π is nondegenerate, $\omega = \pi^{-1}$ is symplectic and $\ker(\pi^\sharp) = 0$ is always satisfied. However, we do not specify geometry of equation (3.14) to symplectic in this paper.

Substituting equation (3.15) into equation (3.13), the following identity is obtained:

$$\pi^{kl} \nabla_k \nabla_j \mu_a \nabla_l \mu_b + \pi^{kl} \nabla_k \mu_a \nabla_l \nabla_j \mu_b + \nabla_j \pi^{kl} \nabla_k \mu_a \nabla_l \mu_b - T^c_{ab} \nabla_j \mu_c = 0. \tag{3.16}$$

We consider a coordinate independent equation of equation (3.16). For it, we consider the Koszul bracket $[-,-]_\pi \colon \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$,

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp \alpha} \beta - \mathcal{L}_{\pi^\sharp \beta} \alpha - \mathrm{d}(\pi(\alpha, \beta)). \tag{3.17}$$

The local coordinate expression of this bracket $[-,-]_\pi$ is

$$([\alpha, \beta]_\pi)_j = \pi^{kl} \partial_k \alpha_j \beta_l + \pi^{kl} \alpha_k \partial_l \beta_j + \partial_j \pi^{kl} \alpha_k \beta_l$$

for 1-form $\alpha = \alpha_i(x) \mathrm{d}x^i$ and $\beta = \beta_i(x) \mathrm{d}x^i$. Using this bracket, equation (3.16) is the following coordinate independent equation:

$$\iota_v([(\nabla\mu)(e_1), (\nabla\mu)(e_2)]_\pi) + (\nabla_v \mu)([e_1, e_2]) = 0$$

or

$$-[(\nabla\mu)(e_1), (\nabla\mu)(e_2)]_\pi = (\nabla\mu)([e_1, e_2]). \tag{3.18}$$

Therefore, if we define the map $(\nabla\mu)^* \colon \Gamma(A) \to \Gamma(T^*M)$ induced from $\nabla\mu$ as $(\nabla\mu)^*(e) := \langle \nabla\mu,\, e \rangle$ for every $e \in \Gamma(A)$, $-(\nabla\mu)^*$ is a Lie algebra morphism from $\Gamma(A)$ to $\Gamma(T^*M)$. Similar to μ^*, since $\nabla\mu$ is a 1-form taking value on A^*, $\nabla\mu \in \Omega^1(M, A^*)$ and $(\nabla\mu)^*$ is defined from the

[5] We have introduced a TM-connection on A, ∇ and an A-connection on TM, ${}^A\nabla = {}^A\nabla^{\mathrm{bas}}$.

pairing of A and A^*, $(\nabla\mu)^*$ maps a vector $a(x) \in A_x$ on the fiber of a point $x \in M$ to a covector $\alpha(x) \in T_x^*M$. Thus $(\nabla\mu)^*$ is regarded as the bundle map, $(\nabla\mu)^*\colon A \to T^*M$.

The condition (P2), equation (2.11), $-\pi^\sharp(\nabla\mu)(e) = \rho_A(e)$ is nothing but the condition for anchor maps in the Lie algebroid morphism $-(\nabla\mu)^*\colon A \to TM$, where the anchor map in A is ρ_A and one in T^*M is $\pi^\sharp$. Therefore, we have proved that $(\nabla\mu)^*$ is a Lie algebroid morphism.

We summarize analysis in this subsection. If M is a symplectic manifold, $\langle {}^A S, \mu\rangle = 0$ is satisfied and we obtain the following proposition.[6]

Proposition 3.3. *Let (M, π, A, μ) be a Hamiltonian Lie algebroid over a symplectic manifold. Then $-(\nabla\mu)^*\colon A \to T^*M$ is a Lie algebroid morphism.*

For a Poisson manifold M, the corresponding proposition is obtained.

Proposition 3.4. *Let (M, π, A, μ) be a Hamiltonian Lie algebroid over a Poisson manifold. Then if $\langle {}^A S, \mu\rangle = 0$, $-(\nabla\mu)^*\colon A \to T^*M$ is a Lie algebroid morphism.*

4 Comomentum sections and Lie algebroid morphisms

We analyze the comomentum description of momentum sections and Hamiltonian Lie algebroids. The comomentum section gives a Lie algebroid morphism from A to a proper vector bundle on M.

4.1 Pre-symplectic case

Let (M, ω) be a pre-symplectic manifold. We define the following bracket on $\mathfrak{X}(M) \oplus C^\infty(M)$,

$$[u + f, v + g]_{TM\oplus\mathbb{R}} = [u, v] + ug - vf - \iota_u\iota_v\omega, \tag{4.1}$$

where $u, v \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. The bracket (4.1) is the $\mathbb{R}$-bilinear Lie bracket if and only if $\mathrm{d}\omega = 0$. Moreover, we define the map, $\rho_{TM\oplus\mathbb{R}}\colon \mathfrak{X}(M) \oplus C^\infty(M) \to \mathfrak{X}(M)$,

$$\rho_{TM\oplus\mathbb{R}}(u + f) = u. \tag{4.2}$$

The space for $\mathfrak{X}(M) \oplus C^\infty(M)$ is denoted by $TM \oplus \mathbb{R}$. An element $u + f \in \mathfrak{X}(M) \oplus C^\infty(M)$ is regarded as a section on the bundle $TM \oplus \mathbb{R}$. Then the map (4.2) is induced from the bundle map $\rho_{TM\oplus\mathbb{R}}\colon TM \oplus \mathbb{R} \to TM$. Two operations (4.1) and (4.2) define a Lie algebroid on $TM \oplus \mathbb{R}$. The Lie algebroid $(TM \oplus \mathbb{R}, [-,-]_{TM\oplus\mathbb{R}}, \rho_{TM\oplus\mathbb{R}})$ is called the Almeida–Molino Lie algebroid [4].

Let $(A, \rho_A, [-,-]_A)$ be a Lie algebroid over M. For a section $\mu \in \Gamma(A^*)$, a map $\mu^*\colon \Gamma(A) \to C^\infty(M)$ is defined by $\langle \mu, e\rangle = \mu^*(e)$. Since μ is a section on A^* and μ^* is defined from the pairing of A and A^*, μ^* maps a vector $a(x) \in A_x$ on the fiber of a point $x \in M$ to a value of a function in $x \in M$, $f(x) \in \mathbb{R}$. Thus, μ^* is regarded as the bundle map $\mu^*\colon A \to M \times \mathbb{R}$ that is the identity map on M. Then $\rho_A + \mu^*$ is a map $\rho_A + \mu^*\colon \Gamma(A) \to \mathfrak{X}(M) \oplus C^\infty(M)$, which is induced from the bundle map $\rho_A + \mu^*\colon A \to TM \oplus \mathbb{R}$. We obtain the following proposition to characterize the condition of a momentum section (S3) as a Lie algebroid morphism.

Proposition 4.1. *$\rho_A + \mu^*$ is a Lie algebroid morphism from A to $TM \oplus \mathbb{R}$ if and only if μ satisfies the condition* (S3)*, i.e., equation* (2.8)*.*

Proof. We prove the equation

$$[(\rho_A + \mu^*)(e_1), (\rho_A + \mu^*)(e_2)]_{TM\oplus\mathbb{R}} = (\rho_A + \mu^*)([e_1, e_2]_A) \tag{4.3}$$

[6] For a momentum section $\mu \in \Gamma(A^*)$, notation $\mu(e)$ in fact means that the pairing of A^* and A, $\langle \mu, e\rangle$. After this section, if μ is regarded as the map $\mu^*\colon \Gamma(A) \to C^\infty(M)$ to consider a morphism, it is denoted by μ^* to emphasize it, which is a *comomentum* defined by $\mu^*(e) = \langle \mu, e\rangle$.

for every $e_1, e_2 \in \Gamma(A)$. The identity $[\rho_A(e_1), \rho_A(e_2)]_{TM} = \rho_A([e_1, e_2]_A)$, is satisfied since ρ_A is a Lie algebroid morphism from A to TM from the definition of a Lie algebroid. It is nothing but the $\rho_A([e_1, e_2]_A)$ part in equation (4.3). Since the Lie algebroid differential ${}^A\mathrm{d}$ is

$$ {}^A\mathrm{d}\mu(e_1, e_2) = \mathcal{L}_{\rho_A(e_1)}(\mu(e_2)) - \mathcal{L}_{\rho_A(e_2)}(\mu(e_1)) - \mu([e_1, e_2]_A), $$

the condition (S3) is equivalent to

$$ \mathcal{L}_{\rho_A(e_1)}(\mu(e_2)) - \mathcal{L}_{\rho_A(e_2)}(\mu(e_1)) + \iota_{\rho_A(e_2)}\iota_{\rho_A(e_1)}\omega = \mu([e_1, e_2]_A), \tag{4.4} $$

which is the $\mu^*([e_1, e_2])$ part in equation (4.3). Therefore, (4.3) is proved, which is the condition (4.4) for a Lie bracket in the definition of a Lie algebroid morphism. The second condition (2.2) for the anchor map,

$$ \rho_A(e) = \rho_{TM\oplus\mathbb{R}} \circ (\rho_A + \mu^*)(e), \tag{4.5} $$

is easily checked. Equations (4.3) and (4.5) show that $\rho_A + \mu^*$ is a Lie algebroid morphism from A to $TM \oplus \mathbb{R}$. ■

Next, we characterize the condition (S2) in terms of a Lie algebroid morphism.

For a section $\mu \in \Gamma(A^*)$, $\nabla\mu \in \Omega^1(M, A^*)$ induces a map $(\nabla_X\mu)^*\colon \Gamma(A) \to \Omega^1(M)$ by defining $\langle\nabla_X\mu, e\rangle = (\nabla_X\mu)^*(e)$ for every $e \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$. Since $(\nabla_X\mu)^*$ is defined by the pairing between A and A^* and $(\nabla_X\mu)^*|_M = \mathrm{id}$, $(\nabla_X\mu)^*$ is considered as a bundle map $(\nabla\mu)^*\colon A \to T^*M$. We consider the map $\rho_A - (\nabla\mu)^*\colon A \to TM \oplus T^*M$.

$TM \oplus T^*M$ naturally has a standard Courant algebroid structure with $H = 0$ in Example 2.18. The pre-symplectic structure on M induces the Dirac structure L_ω. Then the following proposition is obtained.

Proposition 4.2. *The map $\rho_A - (\nabla\mu)^*\colon A \to TM \oplus T^*M$ is a Lie algebroid morphism $\rho_A - (\nabla\mu)^*\colon A \to L_\omega$ if and only if μ satisfies the condition* (S2)*, i.e., equation* (2.7)*.*

Proof. Assume the condition (S2). For $e_1, e_2 \in \Gamma(A)$, using equation (2.7), (S2), we obtain the inner product

$$ \langle(\rho_A - (\nabla\mu)^*)(e_1), (\rho_A - (\nabla\mu)^*)(e_2)\rangle = -\omega(\rho(e_1), \rho(e_2)) - \omega(\rho(e_2), \rho(e_1)) = 0, $$

and the Dorfman bracket

$$ \begin{aligned} &[(\rho_A - (\nabla\mu)^*)(e_1), (\rho_A - (\nabla\mu)^*)(e_2)]_D \\ &\qquad = [\rho_A(e_1), \rho_A(e_2)]_{TM} + \iota_{[\rho_A(e_1), \rho_A(e_2)]_{TM}}\omega - \iota_{\rho_A(e_2)}\iota_{\rho_A(e_1)}\mathrm{d}\omega \\ &\qquad = \rho_A([e_1, e_2]_A) + \iota_{\rho_A([e_1, e_2]_A)}\omega \\ &\qquad = \rho_A([e_1, e_2]_A) - (\nabla\mu)^*([e_1, e_2]_A), \end{aligned} $$

which show $(\rho_A - (\nabla\mu)^*)(e) \in L_\omega$ since $\mathrm{d}\omega = 0$. The condition (2.2) for anchor maps are obvious. Therefore, $\rho - (\nabla\mu)^*$ is a Lie algebroid morphism. If we consider the reverse, the condition (S2) is obtained from the Lie algebroid morphism condition. ■

By combining two Propositions 4.1 and 4.2, we obtain a characterization of the momentum section based on Lie algebroid morphisms.

Theorem 4.3. *Let (M, ω) be a pre-symplectic manifold and $(A, \rho_A, [-, -]_A)$ be a Lie algebroid over M. $\mu \in \Gamma(A^*)$ is a bracket-compatible momentum section if and only if $\rho_A + \mu^*$ is a Lie algebroid morphism from A to $TM \oplus \mathbb{R}$ and $\rho_A - (\nabla\mu)^*$ is a Lie algebroid morphism from A to L_ω.*

The map $\rho_A - (\nabla\mu)^*$ can be considered as a Lie algebroid morphism from A to TM since $L_\omega \simeq TM$.

We define a bracket-compatible comomentum section.

Definition 4.4 (comomentum sections over pre-symplectic manifolds)**.** Let (M,ω) be a pre-symplectic manifold and $(A,[-,-],\rho_A)$ be a Lie algebroid over M. $\mu^*\colon A \to M\times\mathbb{R}$ is called a bracket-compatible if $\rho_A+\mu^*$ is a Lie algebroid morphism from A to $TM\oplus\mathbb{R}$. μ^* is called a bracket-compatible comomentum section if in addition, $\rho_A-(\nabla\mu)^*$ is a Lie algebroid morphism from A to L_ω.

4.2 Poisson case

Let (M,π) be a Poisson manifold. We consider the following bilinear bracket on $\Omega^1(M)\oplus C^\infty(M)$:

$$[\alpha+f,\beta+g]_{T^*M\oplus\mathbb{R}} = [\alpha,\beta]_\pi - \pi^\sharp(\alpha)g + \pi^\sharp(\beta)f - \pi(\alpha,\beta). \tag{4.6}$$

where $\alpha,\beta\in\Omega^1(M)$ and $f,g\in C^\infty(M)$. $[-,-]_\pi$ is the Koszul bracket defined in equation (3.17). The bracket (4.6) is a Lie bracket if and only if π is a Poisson bivector field. If $\alpha=\mathrm{d}h$ and $\beta=\mathrm{d}k$ with $h,k\in C^\infty(M)$, the equation (4.6) reduces to

$$[\mathrm{d}h+f,\mathrm{d}k+g]_{T^*M\oplus\mathbb{R}} = \mathrm{d}(\{h,k\}_M) - \{f,g\}_M,$$

where $\{f,g\}_M = \pi(\mathrm{d}f,\mathrm{d}g)$ is a Poisson bracket on $C^\infty(M)$ defined by π.

We define a bundle map, $\rho_{TM\oplus\mathbb{R}}\colon T^*M\oplus\mathbb{R}\to TM$ as

$$\rho_{TM\oplus\mathbb{R}}(\alpha+f) = \pi^\sharp(\alpha). \tag{4.7}$$

Operations (4.6) and (4.7) give a Lie algebroid structure on $T^*M\oplus\mathbb{R}$. The Lie algebroid $(T^*M\oplus\mathbb{R},\rho_{TM\oplus\mathbb{R}},[-,-]_{T^*M\oplus\mathbb{R}})$ is a Poisson analogue of the Almeida–Molino Lie algebroid in Section 4.1.

Similar to Section 4.1, we consider a map $\mu^*\colon\Gamma(A)\to C^\infty(M)$ on a Lie algebroid A over M. Then $(\nabla\mu)^*$ is a map $(\nabla\mu)^*\colon\Gamma(A)\to\Omega^1(M)$ and $\left({}^A\mathrm{d}\mu\right)^*$ is a map $\left({}^A\mathrm{d}\mu\right)^*\colon\Gamma(A)\times\Gamma(A)\to C^\infty(M)$.

A *comomentum section* on a Poisson manifold is defined from equations (2.11) and (2.12) as follows.

Definition 4.5 (comomentum sections over Poisson manifolds)**.** Let (M,π) be a Poisson manifold with a Poisson bivector field π and $(A,[-,-],\rho_A)$ be a Lie algebroid over M.

A map $\mu^*\colon\Gamma(A)\to C^\infty(M)$ is called a bracket-compatible ∇-comomentum section if it satisfies

$$\text{(PC2)}\quad \rho_A(e) = -\pi^\sharp((\nabla\mu)^*(e)), \tag{4.8}$$

$$\text{(PC3)}\quad \left(\left({}^A\mathrm{d}\mu\right)^*\right)(e_1,e_2) = -\pi((\nabla\mu)^*)(e_1),((\nabla\mu)^*)(e_2)) \tag{4.9}$$

for $e,e_1,e_2\in\Gamma(A)$.

We give characterizations similar to Propositions 4.2 and 4.1 over a pre-symplectic manifold.

Consider the map $-(\nabla\mu)^*+\mu^*\colon\Gamma(A)\to\Omega^1(M)\oplus C^\infty(M)$, which can be regarded as the bundle map, $-(\nabla\mu)^*+\mu^*\colon A\to T^*M\oplus\mathbb{R}$. Then we obtain the following proposition.

Proposition 4.6. *Let μ be a momentum section satisfying equations (4.8) and (4.9). Equivalently, let μ^* be a comomentum section. Assume $\left\langle{}^AS,\mu\right\rangle=0$ in equation (3.14). Then $-(\nabla\mu)^*+\mu^*$ is a Lie algebroid morphism from A to $T^*M\oplus\mathbb{R}$.*

Proof. We prove that the equation

$$[(-(\nabla\mu)^* + \mu^*)(e_1), (-(\nabla\mu)^* + \mu^*)(e_2)]_{T^*M\oplus\mathbb{R}} = (-(\nabla\mu)^* + \mu^*)([e_1, e_2]_A) \tag{4.10}$$

is satisfied under the Lie bracket (4.6) in $T^*M \oplus \mathbb{R}$ for every $e_1, e_2 \in \Gamma(A)$.

If $\langle {}^A S, \mu\rangle = 0$ is imposed, equation (3.18) is satisfied. The left-hand side of the T^*M part of equation (4.10) is $[-(\nabla\mu)^*(e_1), -(\nabla\mu)^*(e_2)]_\pi$. Using equation (3.18), it is equal to $-(\nabla\mu)^*([e_1, e_2])$, which is the T^*M part in the right-hand side of equation (4.10).

Using (2.11), the $\mathbb{R}$ part of equation (4.10) is

$$\mathcal{L}_{\rho_A(e_1)}(\mu(e_2)) - \mathcal{L}_{\rho_A(e_2)}(\mu(e_1)) + \pi((\nabla\mu)^*(e_1), (\nabla\mu)^*(e_2)).$$

It is equal to $\mu^*([e_1, e_2])$ from (P2) and equation (2.12).

The morphism for the anchor maps is

$$\rho_A(e) = \rho_{TM\oplus\mathbb{R}} \circ (-(\nabla\mu)^* + \mu^*)(e),$$

which is equivalent to the condition (PC2). Equations (4.10) and (4.10) show that $-(\nabla\mu)^* + \mu^*$ is the Lie algebroid morphism $A \to T^*M \oplus \mathbb{R}$. ∎

Next, a Lie algebroid morphism corresponding to Proposition 4.2 is introduced.

We consider the map $\rho_A - (\nabla\mu)^*\colon A \to TM \oplus T^*M$. Here $TM \oplus T^*M$ is the standard Courant algebroid with $H = 0$ in Example 2.18.

Proposition 4.7. *Let μ be a comomentum section satisfying* (PC2) *and* (PC3)*. Assume $\langle {}^A S, \mu\rangle = 0$. Then $\rho_A - (\nabla\mu)^*$ is a Lie algebroid morphism from A to L_π.*

Proof. From the condition (4.8), the inner product is

$$\begin{aligned}&\langle(\rho_A - (\nabla\mu)^*)(e_1),\, (\rho_A - (\nabla\mu)^*)(e_2)\rangle\\ &\qquad = -\pi((\nabla\mu)^*(e_1), (\nabla\mu)^*(e_2)) - \pi((\nabla\mu)^*(e_2), (\nabla\mu)^*(e_1)) = 0.\end{aligned} \tag{4.11}$$

Using (PC2) and the π is Poisson, the Dorfman bracket is

$$\begin{aligned}&[(\rho_A - (\nabla\mu)^*)(e_1), (\rho_A - (\nabla\mu)^*)(e_2)]_D\\ &\qquad = \pi^\sharp[(\nabla\mu)^*(e_1), (\nabla\mu)^*(e_2)]_\pi + [(\nabla\mu)^*(e_1), (\nabla\mu)^*(e_2)]_\pi,\end{aligned} \tag{4.12}$$

Using equation (3.18), this equation (4.12) becomes

$$= -\pi^\sharp(\nabla\mu)^*([e_1, e_2]_A) - (\nabla\mu)^*([e_1, e_2]_A) = (\rho - (\nabla\mu)^*)([e_1, e_2]_A), \tag{4.13}$$

which gives

$$[(\rho_A - (\nabla\mu)^*)(e_1), (\rho_A - (\nabla\mu)^*)(e_2)]_D = (\rho - (\nabla\mu)^*)([e_1, e_2]_A). \tag{4.14}$$

Equations (4.11) and (4.13) show that $(\rho_A - (\nabla\mu)^*)(e)$ is an element of the Dirac structure L_π. Equation (4.14) means that $(\rho_A - (\nabla\mu)^*)$ is a Lie algebroid morphism between A and L_π.

The condition (2.2) for anchor maps are obvious. Therefore, $\rho - (\nabla\mu)^*$ is a Lie algebroid morphism from A to L_π. ∎

We combine two Propositions 4.6 and 4.7.

Theorem 4.8. *Let (M, π) be a Poisson manifold and $(A, \rho_A, [-,-]_A)$ be a Lie algebroid over M. Let $\mu^*\colon \Gamma(A) \to C^\infty(M)$ be a bracket-compatible comomentum section. If $\langle {}^A S, \mu\rangle = 0$, $-(\nabla\mu)^* + \mu^*$ is a Lie algebroid morphism from A to $T^*M \oplus \mathbb{R}$ and $\rho_A - (\nabla\mu)^*$ is a Lie algebroid morphism from A to L_π.*

Under the condition $\langle {}^A S, \mu\rangle = 0$, the reverse is proved, i.e., if $-(\nabla\mu)^* + \mu^*$ is a Lie algebroid morphism from A to $T^*M \oplus \mathbb{R}$ and $\rho_A - (\nabla\mu)^*$ is a Lie algebroid morphism from A to L_π, $\mu^*\colon \Gamma(A) \to C^\infty(M)$ be a bracket-compatible comomentum section.

5 Momentum sections as Poisson maps

In this section, we construct a Poisson map by considering a map from $T^*M \oplus \mathbb{R}$ to A^* induced from a momentum section μ. It is an improvement of a "bivector map" in the paper [7] to a Poisson map in some sense. In this section, we concentrate on a Hamiltonian Lie algebroid over a Poisson manifold. Obviously, a Hamiltonian Lie algebroid over a symplectic manifold is a special case.

Definition 5.1 (Poisson map)**.** Let (M_1, π_1) and (M_2, π_2) be Poisson manifolds. A smooth map ψ is a *Poisson map* if $\psi^*\colon C^\infty(M_2) \to C^\infty(M_1)$ satisfies

$$\psi^*\{f, g\}_2 = \{\psi^* f, \psi^* g\}_1$$

for $f, g \in C^\infty(M_2)$, where $\{-,-\}_1$ and $\{-,-\}_2$ are Poisson brackets induced from π_1 and π_2.

The condition of a Poisson map is equivalent to the condition for Poisson bivector fields, $\psi_*\pi_1 = \pi_2$.

As explained in introduction, a momentum map $\mu\colon M \to \mathfrak{g}^*$ is a Poisson map, where a Poisson structure on M is induced from the symplectic structure, or M is a Poisson manifold. A Poisson structure on $\mathfrak{g}^*$ is the KKS Poisson structure. We generalize it to the momentum section.

For construction of a Poisson map, we can use the following fundamental facts. For instance, one can refer to [16, Theorem 13.70] or [30, Proposition 2.13]. The first one is that if A is a Lie algebroid, A^* is a Poisson manifold with a (fiberwise linear) Poisson structure. The second one is as follows.

Proposition 5.2. *Let (A_1, M_1) and (A_2, M_2) be two Lie algebroids. A vector bundle morphism $\phi\colon A_1 \to A_2$ is Lie algebroid morphism if and only if the dual map $\psi\colon A_2^* \to A_1^*$ is a Poisson map.*

Applying Proposition 5.2 to the map $-(\nabla\mu)^*$ in Proposition 3.4, we obtain the following proposition.

Proposition 5.3. *Let (M, π, A, μ) be a Hamiltonian Lie algebroid over a Poisson manifold. Then if $\langle {}^A S, \mu\rangle = 0$, $-\nabla\mu\colon TM \to A^*$ is a Poisson map.*

However, since the map is not based on μ but $\nabla\mu$, this Poisson map is not what we want.

5.1 Dg manifolds for Lie algebroids and graded Poisson manifolds

Let (M, π) be a Poisson manifold. We generalize the Lie bracket (4.6) on $\Gamma(T^*M \oplus \mathbb{R})$ introduced in Section 4.2 to a Poisson bracket. For it, we consider "a space of functions on $T^*M \oplus \mathbb{R}$". Similarly we also consider "a space of functions on A^*" and a Poisson bracket on the space. In other words, one can see $T^*M \oplus \mathbb{R}$ and A^* as graded manifolds. Differential graded manifolds (dg manifolds), which are also called Q-manifolds are graded manifolds with a graded vector field Q such that $Q^2 = 0$. Dg manifolds corresponding to these spaces are introduced and (graded) Poisson brackets are defined on spaces of functions on these graded manifolds. For reviews of graded manifolds, One can refer to [13, 21, 32] and references therein.

At first, we consider a graded manifold for $T^*M \oplus \mathbb{R}$. We introduce the graded manifold $\mathcal{M} = T^*[2](T[1]M \oplus \mathbb{R}[1])$, where [1] means that degree of coordinates is shifted by one. Take local coordinates on $T[1]M \oplus \mathbb{R}[1]$, $\left(x^i, \eta^i, s\right)$ of degree $(0, 1, 1)$, where x^i is a local coordinate on M, η^i is the fiber coordinate on $T[1]M$, and s is one on $\mathbb{R}[1]$. Moreover, take fiber coordinates on $T^*[2](T[1]M \oplus \mathbb{R}[1])$, (ξ_i, y_i, t) of degree $(2, 1, 1)$ corresponds to a 1-form $a_i(x)\mathrm{d}x^i \in \Omega^1(M)$. Here $C^\infty_k(\mathcal{M})$ is a space of degree k functions on a graded manifold $\mathcal{N}$. A section of $T^*M \oplus \mathbb{R}$,

$\alpha + f \in \Omega^1(M) \oplus C^\infty(M)$ is identified to a degree one function $\underline{\alpha} + fs \in C^\infty(T[1]M \oplus \mathbb{R}[1])$. The space of functions $C^\infty(T[1]M \oplus \mathbb{R}[1])$ as an extension of $\Omega^1(M) \oplus C^\infty(M)$, i.e., $\Omega^1(M) \oplus C^\infty(M)$ is a subset of $C^\infty(T[1]M \oplus \mathbb{R}[1])$. Thus,

$$\Omega^1(M) \oplus C^\infty(M) \simeq C_1^\infty(T[1]M \oplus \mathbb{R}[1]) \subset C^\infty(T[1]M \oplus \mathbb{R}[1]).$$

If M is a Poisson manifold, $T^*[1]M \oplus \mathbb{R}[1]$ is a dg manifold (a Q-manifold). Because a homological vector field is constructed from the Poisson structure on M as

$$Q = \pi^{ij}(x) y_j \frac{\partial}{\partial x^i} + \frac{1}{2} \partial_k \pi^{ij}(x) y_i y_j \frac{\partial}{\partial y_k} + \frac{1}{2} \pi^{ij}(x) y_i y_j \frac{\partial}{\partial t}.$$

$Q^2 = 0$ if $\pi = \frac{1}{2}\pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \in \Gamma\big(\wedge^2 TM\big)$ is a Poisson bivector field.

$\mathcal{M}$ has a dg symplectic structure[7] as follows. Since $\mathcal{M}$ is a cotangent bundle, there exist the following canonical graded symplectic form of degree two,

$$\omega_{\mathcal{M}} = \delta x^i \wedge \delta \xi_i + \delta \eta^i \wedge \delta y_i + \delta s \wedge \delta t. \tag{5.1}$$

We consider the following function of degree 3:

$$\begin{aligned} \Theta_{\mathcal{M}} &= \pi(\xi, y) - \iota_y(\mathrm{d}\pi)(y, y) + \pi(y, y) s \\ &= \pi^{ij}(x) \xi_i y_j - \frac{1}{2} \partial_i \pi^{jk}(x) y_j y_k \eta^i + \frac{1}{2} \pi^{jk}(x) y_j y_k s, \end{aligned} \tag{5.2}$$

where $\pi = \frac{1}{2}\pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ is a Poisson bivector field on M. The (graded) Poisson bracket of degree -2, $\{-,-\}_{\mathcal{M}}$, is induced from the symplectic form (5.1) as usual. $\Theta_{\mathcal{M}}$ satisfies $\{\Theta_{\mathcal{M}}, \Theta_{\mathcal{M}}\}_{\mathcal{M}} = 0$ using π is a Poisson bivector field. Thus we obtain a homological vector field $Q_{\mathcal{M}} = \{\Theta_{\mathcal{M}}, -\}$ satisfying $Q_{\mathcal{M}}^2 = 0$. $(\omega_{\mathcal{M}}, \Theta_{\mathcal{M}})$ define a dg symplectic structure on $\mathcal{M}$. The homological function $\Theta_{\mathcal{M}}$ is regarded as "a Poisson bivector field" on $T[1]M \oplus \mathbb{R}[1]$ and define a (graded) Poisson bracket of degree one on $T[1]M \oplus \mathbb{R}[1]$ as follows. In fact, the following derived bracket defines a bracket

$$\{\underline{u}, \underline{v}\}_{T[1]M \oplus \mathbb{R}[1]} := \{\{\underline{u}, \Theta_{\mathcal{M}}\}_{\mathcal{M}}, \underline{v}\}_{\mathcal{M}} \tag{5.3}$$

for $\underline{u}, \underline{v} \in C^\infty(T[1]M \oplus \mathbb{R}[1])$. We can easily prove that $\{-,-\}_{T[1]M \oplus \mathbb{R}[1]}$ is graded skew symmetric, and satisfies the Leibniz rule and the Jacobi identity, thus, a (graded) Poisson bracket on $C^\infty(T[1]M \oplus \mathbb{R}[1])$.

Moreover, if $\underline{u}, \underline{v} \in C_1^\infty(T[1]M \oplus \mathbb{R}[1])$, i.e., $\underline{u} = \underline{\alpha} + fs \simeq \alpha + f$ and $\underline{v} = \underline{\beta} + gs \simeq \beta + g$, the bracket (5.3) is equivalent to the Lie bracket (4.6),

$$[\alpha + f, \beta + g]_{T^*M \oplus \mathbb{R}} = \{\underline{\alpha} + fs, \underline{\beta} + gs\}_{T[1]M \oplus \mathbb{R}[1]}.$$

Thus, the graded manifold $T[1]M \oplus \mathbb{R}[1]$ is a (graded) Poisson manifold with the Poisson bracket (5.3) including the Lie algebroid structure on $T^*M \oplus \mathbb{R}$.

Next, we consider A^*. For $\Gamma(A) \oplus C^\infty(M)$, a bilinear bracket is induced from the Lie algebroid structure on A. The bilinear bracket on A^* is concretely defined by [7]

$$\{f, g\}_{A^*} := 0, \qquad \{a, g\}_{A^*} := \rho_A(a) g, \tag{5.4}$$

$$\{a, b\}_{A^*} := [a, b]_A, \tag{5.5}$$

where $f, g \in C^\infty(M)$ and $a, b \in \Gamma(A)$. Here an element of $a \in \Gamma(A)$ is regarded as a linear function $a\colon \Gamma(A^*) \to \mathbb{R}$ by the pairing of A and A^*. Since the bracket $\{-,-\}_{A^*}$ satisfies

[7] A dg symplectic structure is also called a QP-structure.

the Jacobi identity, this fiberwise bracket is called a 'Poisson bracket' on A^* in literature, by regarding $a \in \Gamma(A)$ is a linear function on A^*.

More generally, this 'Poisson bracket' is induced from a graded Poisson bracket on the space of functions on a dg symplectic manifold. For a vector bundle A over M, $A[1]$ is a graded bundle, of which fiber is shifted by one. For a Lie algebroid A, the graded manifold $A[1]$ is a dg manifold [33]. Let x^i be a local coordinate on the base manifold M, q^a be a fiber coordinate on $A[1]$. A vector field of degree one on the shifted vector bundle $A[1]$ is given by

$$Q = \rho_A{}^i_a(x) q^q \frac{\partial}{\partial x^i} - \frac{1}{2} C^c_{ab}(x) q^a q^b \frac{\partial}{\partial q^c}.$$

Require that $A[1]$ is a dg manifold, i.e., Q is a homological vector field such that $Q^2 = \frac{1}{2}[Q, Q] = 0$. This condition gives a Lie algebroid structure on A. Here the anchor map and the Lie bracket on A are given by $\rho_A(e_a) := \rho_A{}^i_a(x)\partial_i$, $[e_a, e_b] := C^c_{ab} e_c$, for the basis e_a of A.

The corresponding dg symplectic manifold is constructed as follows. We consider the graded cotangent bundle $\mathcal{N} = T^*[1](A[1] \oplus \mathbb{R}[1]) \simeq T^*[1](A^*[1] \oplus \mathbb{R}[1])$. We take the canonical graded symplectic form on the cotangent bundle. Then a Hamiltonian function $\Theta_{\mathcal{N}} \in C^\infty(\mathcal{N})$ for Q is defined by $\delta\Theta_{\mathcal{N}} = -\iota_Q \omega_{\mathcal{N}}$, where δ is the differential on the graded manifold $\mathcal{N}$. Since $Q^2 = 0$, Θ satisfies that

$$\{\Theta_{\mathcal{N}}, \Theta_{\mathcal{N}}\}_{\mathcal{N}} = 0. \tag{5.6}$$

Take local coordinates on $\mathcal{N} = T^*[1](A^*[1] \oplus \mathbb{R}[1])$. (x^i, p_a, s) are local coordinates on $A^*[1] \oplus \mathbb{R}[1]$ of degree $(0, 1, 1)$ and (ξ_i, q^a, t) are the corresponding fiber coordinates of degree $(2, 1, 1)$. The graded symplectic form is $\omega_{can} = \delta x^i \wedge \delta\xi_i + \delta p_a \wedge \delta q^a + \delta s \wedge \delta t$. The local coordinate expression of the homological function $\Theta_{\mathcal{N}}$ is

$$\Theta_{\mathcal{N}} = \rho_A{}^i_a(x) \xi_i q^a + \frac{1}{2} C^c_{ab}(x) q^a q^b p_c. \tag{5.7}$$

Equation (5.6) shows that $\Theta_{\mathcal{N}}$ is regarded as a 'Poisson bivector field' on $A^*[1] \oplus \mathbb{R}[1]$. A Poisson brackets on $A^*[1] \oplus \mathbb{R}[1]$ is defined by the derived bracket

$$\{F, G\}_{A^*[1] \oplus \mathbb{R}[1]} := -\{\{F, \Theta\}_{\mathcal{N}}, G\}_{\mathcal{N}} \tag{5.8}$$

for $F, G \in C^\infty(A^*[1] \oplus \mathbb{R}[1])$.

A degree one function on $A^*[1]$, $\underline{a} = a^a(x) p_a \in C^\infty_1(A^*[1])$ is identified to a section $a = a^a(x) e_a \in \Gamma(A)$. A degree one function on $M \times \mathbb{R}[1]$ is $fs \in C^\infty_1(M \times \mathbb{R}[1])$ is identified to a function $f \in C^\infty(M)$. For degree one functions, $F = \underline{a} + fs \simeq a + f$ and $G = \underline{b} + gs \simeq b + g$, the Poisson bracket (5.8) coincides with equations (5.4)–(5.5), $[a + f, b + g]_A = \{\underline{a} + fs, \underline{b} + gs\}_{A^*[1] \oplus \mathbb{R}[1]}$, i.e., $\{a + f, b + g\}_{A^*} = \{\underline{a} + f, \underline{b} + g\}_{A^*[1] \oplus \mathbb{R}[1]}$. Therefore, the Poisson bracket for linear functions on A^*, equations (5.4)–(5.5), is regarded as the restriction of the Poisson bracket on $C^\infty(A^*[1] \oplus \mathbb{R}[1])$ equation (5.8).

5.2 Poisson maps between dg symplectic manifolds

In this section, we construct a Poisson map from $(T[1]M \oplus \mathbb{R}[1], \{-,-\}_{T[1]M \oplus \mathbb{R}[1]})$ to $(A^*[1], \{-,-\}_{A^*[1] \oplus \mathbb{R}[1]})$ induced from a momentum section μ. Obviously, it gives a Poisson map between two ordinary manifolds, $(T^*M \oplus \mathbb{R}, \{-,-\}_{T^*M \oplus \mathbb{R}})$ to $(A^*, \{-,-\}_{A^*})$.

For $\mu \in \Gamma(A^*)$, the covariant derivative $\nabla\mu \in \Omega^1(M, A^*)$ is a 1-form taking a value on A^*, and regarded as a map $\nabla\mu \colon T[1]M \to A^*[1]$. Thus, we have the map $-\nabla\mu + \mu \colon T[1]M \oplus \mathbb{R}[1] \to A^*[1]$.

Theorem 5.4. *Let (A, π, ∇, μ) be a Hamiltonian Lie algebroid over a Poisson manifold M. If $\langle {}^A S, \mu \rangle = 0$, $-\nabla\mu + \mu \colon T[1]M \oplus \mathbb{R}[1] \to A^*[1]$ is a Poisson map.*

Proof. The statement of theorem is that the following equation is satisfied:

$$\{(-(\nabla\mu)^* + \mu^*)F, (-(\nabla\mu)^* + \mu^*)G\}_{T[1]M\oplus\mathbb{R}[1]} = (-(\nabla\mu)^* + \mu^*)(\{F, G\}_{A^*[1]})$$

for every $F, G \in C^\infty(A^*[1])$.

We can directly prove by calculating this equation using the derived bracket construction of Poisson brackets of two spaces $T^*[1](T[0]M \oplus \mathbb{R}[0])$ and $T^*[2]A^*[1]$ with homological functions, (5.2), (5.3), (5.7) and (5.8). Here $F, G \in C^\infty(A^*[1])$ are arbitrary functions of x^i and p_a, $F = F(x, p)$ and $G = G(x, p)$. $-(\nabla\mu)^* + \mu^*$ corresponds to the degree two element $-(\underline{\nabla\mu})^* + \underline{\mu}^* = (-\nabla_i\mu_a)\eta^i q^a + \mu_a s q^a$ on the dg manifold. By direct calculations, we can prove that if $\overline{\langle {}^A S,\ \mu\rangle} = 0$, the following equation is satisfied in the derived bracket:

$$\{\{\{-\underline{\nabla\mu}^* + \underline{\mu}^*, F\}_{\mathcal{M}}, \Theta_{\mathcal{M}}\}_{\mathcal{M}}, \{-\underline{\nabla\mu}^* + \underline{\mu}^*, G\}_{\mathcal{M}}\}_{\mathcal{M}} = \{-\underline{\nabla\mu}^* + \underline{\mu}^*, \{\{F, \Theta_{\mathcal{N}}\}_{\mathcal{N}}, G\}_{\mathcal{N}}\}_{\mathcal{N}}.$$

Another simpler proof is to use Proposition 5.2 and the result in Section 4.2. $-\nabla\mu + \mu$ is nothing but the dual of $-(\nabla\mu)^* + \mu^*$. In Proposition 4.6, we proved that it $-(\nabla\mu)^* + \mu^*$ is the Lie algebroid morphism between A to $T^*M \oplus \mathbb{R}$. Therefore, Theorem 5.4 is obtained that $-\nabla\mu + \mu$ is a Poisson map from $TM \oplus \mathbb{R}$ to A^*. ■

6 Momentum sections as Dirac morphisms

In previous sections, Lie algebroid morphisms and Poisson maps of momentum sections have been realized in Dirac structures L. This suggests a generalization of a Poisson map to a morphism between Dirac manifolds. Moreover, we propose Hamiltonian Lie algebroids over Dirac structures.

We introduce a Dirac morphism [2] between two Dirac structures. It is a generalization of a Dirac map [9, 10] and is related to the morphism between Dirac structures with other names [11].

Let $\varphi\colon M \to N$ be a smooth map between smooth manifolds M and N. We define a binary relation $\mathbb{T}\varphi$ from $\mathbb{T}M := TM \oplus T^*M$ to $\mathbb{T}N := TN \oplus T^*N$, denoted by $\mathbb{T}\varphi\colon\ \mathbb{T}M \dashrightarrow \mathbb{T}N$, as follows: two elements $u + \sigma \in T_mM \oplus T^*_mM$ and $v + \tau \in T_nN \oplus T^*_nN$ for some $m \in M$, $n \in N$ are said to be in the *relation* $\mathbb{T}\varphi$ if $n = \varphi(m)$, $v = (\mathrm{d}\varphi)_m(u)$ and $\sigma = (\mathrm{d}\varphi)^*_m(\tau)$, hold. Here, the comorphism $(\mathrm{d}\varphi)^*_m$ for each $m \in M$ defines a relation in the opposite direction, from T^*_nN to T^*_mM with $n = \varphi(m)$. We write them as $u + \sigma \mapsto_{\mathbb{T}\varphi} v + \tau$. The relation $\mathbb{T}\varphi$ defines a subset in $\mathbb{T}M \times \mathbb{T}N$. Note that $(\mathrm{d}\varphi)^*$ is a relation, not a map.

We say that sections $u + \alpha \in \Gamma(\mathbb{T}M)$, $v + \beta \in \Gamma(\mathbb{T}N)$ are in the relation $\mathbb{T}\varphi$ which is denote by $u + \alpha \mapsto_{\mathbb{T}\varphi} v + \beta$ if elements $u_m + \alpha_m \in T_mM \oplus T^*_mM$ and $v_n + \beta_n \in T_nN \oplus T^*_nN$ are in the relation $\mathbb{T}\varphi$ at each $m \in M, n \in N$.

Let (M, L_M) and (N, L_N) be two Dirac manifolds. A Dirac morphism is defined as follows.

Definition 6.1. Let $\varphi\colon M \to N$ be a smooth map. A Dirac morphism (or a forward Dirac map) is defined to be a binary relation $\mathbb{T}\varphi\colon (\mathbb{T}M,\ L_M) \dashrightarrow (\mathbb{T}N, L_N)$ satisfying the following property: for any $m \in M$ and $v + \beta \in (L_N)_{\varphi(m)}$, there exists a unique element $u + \alpha \in (L_M)_m$ such that $u + \alpha \mapsto_{\mathbb{T}\varphi} v + \beta$.

There exists the following relation of a Poisson map with a Dirac morphism [30].

Proposition 6.2. *Let (M, π_M) and (N, π_N) be Poisson manifolds. Let $\mathrm{Gr}(\pi_M)$ be the graph of the map $\pi_M\colon T^*M \to TM$. Then $\varphi\colon M \to N$ is a Poisson map if and only if $\mathbb{T}\varphi\colon (TM \oplus T^*M, \mathrm{Gr}(\pi_M)) \dashrightarrow (TN \oplus T^*N, \mathrm{Gr}(\pi_N))$ is a Dirac morphism.*

We apply Proposition 6.2 to our settings.

Let (M, π, A, μ) be a Hamiltonian Lie algebroid over a Poisson manifold. Remember results in Section 5.2. From Theorem 5.4, $-\nabla\mu + \mu\colon TM \oplus \mathbb{R} \to A^*$ is a Poisson map. We can take M and N in Proposition 6.2 as $TM \oplus \mathbb{R}$ and A^*. Then we obtain the following proposition.

Proposition 6.3. *Let (M, π, A, μ) be a Hamiltonian Lie algebroid over a Poisson manifold. Then $\mathbb{T}(-\nabla\mu + \mu)\colon (T(TM \oplus \mathbb{R}) \oplus T^*(TM \oplus \mathbb{R}), \mathrm{Gr}(\pi_{TM\oplus\mathbb{R}})) \dashrightarrow (TA^* \oplus T^*A^*, \mathrm{Gr}(\pi_{A^*}))$ is a Dirac morphism.*

7 Further discussions

Possible applications and future directions are discussed in this section.

One of important applications of momentum maps is symplectic reductions [29, 31] or Poisson reductions [28]. For consistency of reductions, the momentum map must be a Poisson map from M to the dual of the Lie algebra $\mathfrak{g}^*$. A momentum section is not necessarily a Poisson map from M to A^* [7]. In our paper, we have constructed a Poisson map from $T^*[1]M \oplus \mathbb{R}[1]$ to A^* from a momentum section. The map will give a consistent reduction. One idea to make a momentum section a Poisson map is to impose a condition compatible with a Poisson structure and a Lie algebroid A [19]. The idea to construct a Poisson map in this paper is another one.

Hamiltonian Lie algebroids appear in some physical models, constrained mechanics and sigma models [22]. Analysis of Hamiltonian Lie algebroids is directly connected to understanding these physical systems. In particular, quantizations of these physical models will give a kind of 'quantizations' of Hamiltonian Lie algebroids. Precise definition and meaning is not clear. It will give interesting insights to new relations to geometry and quantization.

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