Positive Weighted Partitions Generated by Double Series

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Abstract. We investigate some weighted integer partitions whose generating functions are double-series. We will establish closed formulas for these q-double series and deduce that their coefficients are non-negative. This leads to inequalities among integer partitions.

Key words: partitions; q-series; positive q-series

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In honor of Steve Milne's 75th birthday

1 Introduction

Throughout, let q denote a complex number satisfying $|q| < 1$ and let m and n denote nonnegative integers. We adopt the following standard notation from the theory of q-series [4, 15]

$$(a;q)_0 = 1, \qquad v(a;q)_n = \prod_{j=0}^{n-1}\bigl(1-aq^j\bigr), \qquad (a;q)_\infty = \prod_{j=0}^{\infty}\bigl(1-aq^j\bigr),$$

$$(a_1,\dots,a_k;q)_n = \prod_{j=1}^{k}(a_j;q)_n, \qquad \text{and} \qquad (a_1,\dots,a_k;q)_\infty = \prod_{j=1}^{k}(a_j;q)_\infty.$$

We shall need the following basic facts

$$(a;q)_{n+m} = (a;q)_m(aq^m;q)_n \qquad \text{and} \qquad (a;q)_\infty = (a;q)_n(aq^n;q)_\infty. \tag{1.1}$$

In this paper, we consider certain q-double series in one single variable which turn out to be natural generating functions for weighted integer partitions.

Weighted integer partitions have been extensively studied in the past. A first systematic investigation of identities for weighted partitions is due to Alladi [1, 2]. For other references on weighted partitions and their applications, see, for instance, [14, 20].

A power series $\sum_{n\geq 0} a_n q^n$ is called positive, written $\sum_{n\geq 0} a_n q^n \succeq 0$, if $a_n \geq 0$ for any nonnegative integer n. Accordingly, we will write $\sum_{n\geq 0} a_n q^n \succeq \sum_{n\geq 0} b_n q^n$ to mean that $\sum_{n\geq 0} a_n q^n - \sum_{n\geq 0} b_n q^n \succeq 0$. Positivity results for q-series have been intensively studied in the past to some extent in connection with Borwein's famous positivity conjecture [3]. For

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more on this, see, for instance, [11, 12, 19, 21]. Positivity results for alternating sums have also received much attention in recent years, see for example [6, 7, 13, 16].

An important application of positivity results is the fact that positive series which are generating functions for weighted partitions give rise to inequalities of integer partitions. About the interplay between positive q-series and inequalities of integer partitions, we refer the reader to [5, 8, 9, 10, 18].

Our main goal in this work is to prove that certain q-double series are positive. As these series turn out to be generating functions for weighted partitions, our results yield inequalities of integer partitions.

The paper is organized as follows. In Section 2, we introduce our series through the partitions they generate and we state our main results. In Section 3, we collect the lemmas needed to prove the main theorems. Sections 4–6 are devoted to the proofs of the main results and their corollaries.

2 Main results

Definition 2.1. For any positive integer N, let $F_1(N)$ be the number of partitions of N, where if the partition has n ones then the largest part is $2n+2k+1$ for some k and all parts >1 are in the interval $[2n+2, 2n+2k+1]$, no even parts are repeated, and the partition is counted with weight $(-1)^j$, where j is the number of even parts. Then we have

$$\sum_{n\geq 0} F_1(n) q^n = \sum_{k,n\geq 0} \frac{\left(q^{2n+2};q^2\right)_k}{\left(q^{2n+3};q^2\right)_k} q^{2k+3n+1}.$$

We now state our first main result.

Theorem 2.2. *We have*

$$\sum_{n\geq 0} F_1(n) q^n = \frac{1}{\left(1-q^2\right)} \frac{\left(q^2;q^2\right)_\infty^2}{\left(q;q^2\right)_\infty^2} - \frac{\left(1+q^2\right)}{\left(1-q\right)\left(1-q^3\right)}.$$

With the help of Theorem 2.2, we will derive the following positivity result.

Corollary 2.3. *There holds* $\sum_{n\geq 0} F_1(n) q^n \succeq 0$.

We now introduce our second integer partitions.

Definition 2.4. For any positive integer N, let $F_2(N)$ be the number of partitions of N such that for each $j=0,1,2$ satisfying $3 \mid 2k+j$ for some k, there are $n+j-2$ ones and $(2k+j)/3$ threes, the remaining parts lie in the set $\{2n+2\} \cup (2n+3, 2n+2k+3]$, no even parts are repeated, and the partition is counted with weight $(-1)^j$, where j is the number of even parts. Then we have

$$\sum_{n\geq 0} F_2(n) q^n = \sum_{k,n\geq 0} \frac{\left(q^{2n+2};q^2\right)_k}{\left(q^{2n+5};q^2\right)_k} q^{2k+n+2}.$$

Theorem 2.5. *We have*

$$\sum_{n\geq 0} F_2(n) q^n = \frac{q\left(1-q^3\right)}{\left(1-q\right)\left(1-q^2\right)} \frac{\left(q^2;q^2\right)_\infty^2}{\left(q;q^2\right)_\infty^2} - \frac{q\left(1+q^2\right)}{\left(1-q\right)^2}.$$

Corollary 2.6. *There holds* $\sum_{n\geq 0} F_2(n) q^n \succeq 0$.

We now deal with our third example of integer partitions.

Definition 2.7. For any positive integer N, let $G(N)$ be the number of partitions of N, where if the partition has $3n$ ones then the largest part is $2n+2k+2$ for some k and all parts are in the interval $[2n+1, 2n+2k+2]$, no even parts are repeated, and the partition is counted with weight $(-1)^j$, where j is the number of even parts. Then we have

$$\sum_{n\geq 0} G(n)q^n = \sum_{k,n\geq 0} \frac{\left(q^{2n+2};q^2\right)_k}{\left(q^{2n+1};q^2\right)_k} q^{2k+5n+2}.$$

Theorem 2.8. *We have*

$$\sum_{n\geq 0} G(n)q^n = \frac{q^3}{(1+q)\left(1-q^3\right)} \frac{\left(q^2;q^2\right)_\infty^2}{\left(q;q^2\right)_\infty^2} - \frac{q^2(1-q)\left(-1+q^3+q^4+q^5\right)}{\left(1-q^3\right)^2\left(1-q^5\right)}.$$

Corollary 2.9. *There holds* $\sum_{n\geq 0} G(n)q^n \succeq 0$.

Our proofs for Corollaries 2.3, 2.6 and 2.9 on positive weighted partitions are all analytic. Obviously, each of these three corollaries is equivalent to an inequality of integer partitions. So, it is natural to ask for injective proofs for these inequalities.

3 Preliminary lemmas

In this section, we collect several lemmas which we will need to prove our main results. To simplify the presentation, we introduce the following sequences.

Definition 3.1. For any positive integers m and n, let

$$\sum_{n\geq 0} A(m,n)q^n = \sum_{n\geq 0} \frac{q^{mn}\left(q^{2n+2};q^2\right)_\infty\left(q^{2n+4};q^2\right)_\infty}{\left(q^{2n-1};q^2\right)_\infty\left(q^{2n+1};q^2\right)_\infty},$$

$$\sum_{n\geq 0} A'(m,n)q^n = \sum_{n\geq 0} \frac{q^{m(n+1)}\left(q^{2n+2};q^2\right)_\infty\left(q^{2n+6};q^2\right)_\infty}{\left(q^{2n+1};q^2\right)_\infty\left(q^{2n+3};q^2\right)_\infty},$$

$$\sum_{n\geq 0} B(m,n)q^n = \sum_{n\geq 0} \frac{q^{mn}\left(q^{2n+2};q^2\right)_\infty\left(q^{2n+4};q^2\right)_\infty}{\left(q^{2n-3};q^2\right)_\infty\left(q^{2n+3};q^2\right)_\infty},$$

$$\sum_{n\geq 0} B'(m,n)q^n = \sum_{n\geq 0} \frac{q^{m(n+1)}\left(q^{2n+2};q^2\right)_\infty\left(q^{2n+6};q^2\right)_\infty}{\left(q^{2n-1};q^2\right)_\infty\left(q^{2n+5};q^2\right)_\infty}.$$

Lemma 3.2. *We have*

$$\sum_{n\geq 0} A(2,n)q^n = \frac{-q\left(1+q^2\right)}{(1-q)^2\left(1-q^3\right)} \tag{3.1}$$

and for any positive integer $m\geq 2$,

$$\sum_{n\geq 0} A(2m,n)q^n = \frac{-1}{(1-q)^2} \sum_{n\geq 0} \frac{q^{n+1}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+3};q^2\right)_{m-2}}. \tag{3.2}$$

Proof. Throughout we will use Heine's transformations [15, Appendix III, equations (III.1)–(III.2)]

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q,\ z\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, & z \\ & az \end{matrix}; q,\ b\right], \tag{3.3}
$$

$$
= \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left[\begin{matrix} abz/c, & b \\ & bz \end{matrix}; q,\ c/b\right], \tag{3.4}
$$

where

$$
{}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q,\ z\right] = \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n.
$$

We will also use the q-binomial theorem [4, equation (2.2.1)],

$$
\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{3.5}
$$

and Ramanujan's ${}_1\psi_1$ summation formula [15, equation (5.2.1)]

$$
\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(q, b/a, az, q/(az); q)_\infty}{(b, q/a, z, b/(az); q)_\infty}. \tag{3.6}
$$

We start with (3.1). By (1.1) and (3.4), we obtain

$$
\begin{aligned}
\sum_{n\geq 0} A(2,n) q^{2n} &= \sum_{n\geq 0} \frac{q^{2n}\left(q^{2n+2}, q^{2n+4}; q^2\right)_\infty}{\left(q^{2n-1}, q^{2n+1}; q^2\right)_\infty} = \frac{\left(q^2, q^4; q^2\right)_\infty}{\left(q^{-1}, q; q^2\right)_\infty} \sum_{n\geq 0} \frac{\left(q^{-1}, q; q^2\right)_n q^{2n}}{\left(q^2, q^4; q^2\right)_n} \\
&= \frac{\left(q^2, q^4; q^2\right)_\infty}{\left(q^{-1}, q; q^2\right)_\infty} {}_2\phi_1\left[\begin{matrix} q^{-1}, & q \\ & q^4 \end{matrix}; q^2,\ q^2\right] \\
&= \frac{\left(q^2, q^4; q^2\right)_\infty}{\left(q^{-1}, q; q^2\right)_\infty} \frac{\left(q^3, q^3; q^2\right)_\infty}{\left(q^2, q^4; q^2\right)_\infty} {}_2\phi_1\left[\begin{matrix} q^{-2}, & q \\ & q^3 \end{matrix}; q^2,\ q^3\right] \\
&= \frac{-q}{(1-q)^3}\left(1 - \frac{q(1-q)\left(1-q^2\right)}{\left(1-q^2\right)\left(1-q^3\right)}\right) = \frac{-q\left(1+q^2\right)}{(1-q)^2\left(1-q^3\right)}.
\end{aligned}
$$

Now assume that m is a positive integer such that $m \geq 2$. We get by using (1.1) and (3.3)

$$
\begin{aligned}
\sum_{n\geq 0} A(2m,n) q^n &= \sum_{n\geq 0} \frac{q^{2mn}\left(q^{2n+2}, q^{2n+4}; q^2\right)_\infty}{\left(q^{2n-1}, q^{2n+1}; q^2\right)_\infty} = \frac{\left(q^2, q^4; q^2\right)_\infty}{\left(q^{-1}, q; q^2\right)_\infty} \sum_{n\geq 0} \frac{\left(q^{-1}, q; q^2\right)_n q^{2mn}}{\left(q^2, q^4; q^2\right)_n} \\
&= \frac{\left(q^2, q^4; q^2\right)_\infty}{\left(q^{-1}, q; q^2\right)_\infty} {}_2\phi_1\left[\begin{matrix} q^{-1}, & q \\ & q^4 \end{matrix}; q^2,\ q^{2m}\right] \\
&= \frac{\left(q^2, q^4; q^2\right)_\infty}{\left(q^{-1}, q; q^2\right)_\infty} \frac{\left(q, q^{2m-1}; q^2\right)_\infty}{\left(q^4, q^{2m}; q^2\right)_\infty} {}_2\phi_1\left[\begin{matrix} q^3, & q^{2m} \\ & q^{2m-1} \end{matrix}; q^2,\ q\right] \\
&= \frac{\left(q^2; q^2\right)_\infty}{\left(q^{-1}; q^2\right)_\infty} \frac{\left(q^{2m-1}; q^2\right)_\infty}{\left(q^{2m}; q^2\right)_\infty} \frac{\left(q^3, q^{2m}; q^2\right)_\infty}{\left(q^2, q^{2m-1}; q^2\right)_\infty} \sum_{n\geq 0} \frac{q^n\left(q^{2n+2}, q^{2n+2m-1}; q^2\right)_\infty}{\left(q^{2n+3}, q^{2n+2m}; q^2\right)_\infty} \\
&= \frac{-q}{(1-q)^2} \sum_{n\geq 0} \frac{q^n\left(q^{2n+2}; q^2\right)_{m-1}}{\left(q^{2n+3}; q^2\right)_{m-2}} = \frac{-1}{(1-q)^2} \sum_{n\geq 1} \frac{q^n\left(q^{2n}; q^2\right)_{m-1}}{\left(q^{2n+1}; q^2\right)_{m-2}},
\end{aligned}
$$

which confirms (3.2). ∎

Lemma 3.3. *For any positive integer, we have*

$$\sum_{n\geq 0} A'(2m,n)q^n = \frac{1}{1-q^3}\sum_{n\geq 0}\frac{q^{n+2m}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+5};q^2\right)_{m-1}} \tag{3.7}$$

$$= \frac{1}{1-q}\sum_{n\geq 0}\frac{q^{3n+2m}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+3};q^2\right)_{m-1}}. \tag{3.8}$$

Proof. We start with (3.7). We have by (1.1)

$$\begin{aligned}
\sum_{n\geq 0} A'(2m,n)q^n &= \sum_{n\geq 0}\frac{q^{2m(n+1)}\left(q^{2n+2};q^2\right)_\infty\left(q^{2n+6};q^2\right)_\infty}{\left(q^{2n+1};q^2\right)_\infty\left(q^{2n+3};q^2\right)_\infty} \\
&= q^{2m}\frac{\left(q^2,q^6;q^2\right)_\infty}{\left(q,q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{q^{2m}\left(q,q^3;q^2\right)_n}{\left(q^2,q^6;q^2\right)_n} \\
&= q^{2m}\frac{\left(q^2,q^6;q^2\right)_\infty}{\left(q,q^3;q^2\right)_\infty}{}_2\phi_1\left[\begin{matrix}q,\ q^3\\ q^6\end{matrix};q^2,\ q^{2m}\right] \\
&= q^{2m}\frac{\left(q^2,q^6;q^2\right)_\infty}{\left(q,q^3;q^2\right)_\infty}\frac{\left(q,q^{2m+3};q^2\right)_\infty}{\left(q^6,q^{2m};q^2\right)_\infty}{}_2\phi_1\left[\begin{matrix}q^5,\ q^{2m}\\ q^{2m+3}\end{matrix};q^2,\ q\right] \\
&= \frac{q^{2m}}{1-q^3}\sum_{n\geq 0}\frac{q^n\left(q^{2n+2},q^{2m+2n+3};q^2\right)_\infty}{\left(q^{2n+5},q^{2m+2n};q^2\right)_\infty} \\
&= \frac{1}{1-q^3}\sum_{n\geq 0}\frac{q^{2m+n}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+5};q^2\right)_{m-1}},
\end{aligned}$$

where in the fourth step we applied (3.3) with $(a,b,c,z) = \left(q^3,q,q^6,q^{2m}\right)$. This proves the desired result.

As for (3.8), we omit the details as the proof follows exactly the same steps as in the proof of (3.7) with the exception that (3.3) is employed with $(a,b,c,z) = \left(q,q^3,q^6,q^{2m}\right)$ rather than $(a,b,c,z) = \left(q^3,q,q^6,q^{2m}\right)$. ∎

Lemma 3.4. *There holds* $\sum_{n\geq 0} A'(m,n)q^n = \sum_{n\geq 0}(A(m,n)-A(m+2,n))q^n$.

Proof. We need the following contiguous relation which can be found in [17, equation (2.1)]:

$${}_2\phi_1\left[\begin{matrix}a,\ b\\ c\end{matrix};q,\ z\right] - {}_2\phi_1\left[\begin{matrix}a,\ b\\ c\end{matrix};q,\ qz\right] = z\frac{(1-a)(1-b)}{1-c}{}_2\phi_1\left[\begin{matrix}qa,\ qb\\ qc\end{matrix};q,\ z\right]. \tag{3.9}$$

Then by (3.9) applied with $q\to q^2$ and $(a,b,c,z) = \left(q^{-1},q,q^4,q^m\right)$, we get

$$\begin{aligned}
&\sum_{n\geq 0}(A(m,n)-A(m+2,n))q^n \\
&\quad= \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-1},q;q^2\right)_\infty}\left({}_2\phi_1\left[\begin{matrix}q^{-1},\ q\\ q^4\end{matrix};q^2,\ q^m\right] - {}_2\phi_1\left[\begin{matrix}q^{-1},\ q\\ q^4\end{matrix};q^2,\ q^{m+2}\right]\right) \\
&\quad= \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-1},q;q^2\right)_\infty}q^m\frac{\left(1-q^{-1}\right)(1-q)}{1-q^4}{}_2\phi_1\left[\begin{matrix}q,\ q^3\\ q^6\end{matrix};q^2,\ q^m\right] \\
&\quad= \frac{\left(q^2,q^6;q^2\right)_\infty}{\left(q,q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{q^{m(n+1)}\left(q,q^3;q^2\right)_n}{\left(q^2,q^6;q^2\right)_n} = \sum_{n\geq 0}\frac{q^{m(n+1)}\left(q^{2n+2};q^2\right)_\infty\left(q^{2n+6};q^2\right)_\infty}{\left(q^{2n+1};q^2\right)_\infty\left(q^{2n+3};q^2\right)_\infty}.
\end{aligned}$$

This proves the lemma. ∎

Lemma 3.5. *We have*

$$\sum_{n\geq 0} B(2,n)q^n = \frac{-q^3\left(1-q^3-q^4-q^5\right)}{\left(1-q^3\right)^2\left(1-q^5\right)} \tag{3.10}$$

and for any positive integer $m\geq 2$,

$$\sum_{n\geq 0} B(2m,n)q^n = \frac{1}{(1-q)\left(1-q^3\right)}\sum_{n\geq 0}\frac{q^{3n+4}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+1};q^2\right)_{m-2}}. \tag{3.11}$$

By (1.1) and (3.4), we obtain

$$\begin{aligned}
\sum_{n\geq 0} B(2,n)q^{2n} &= \sum_{n\geq 0}\frac{q^{2n}\left(q^{2n+2},q^{2n+4};q^2\right)_\infty}{\left(q^{2n-3},q^{2n+3};q^2\right)_\infty} = \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-3},q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(q^{-3},q^3;q^2\right)_n q^{2n}}{\left(q^2,q^4;q^2\right)_n} \\
&= \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-3},q^3;q^2\right)_\infty}\, {}_2\phi_1\left[\begin{matrix} q^{-3},\ q^3 \\ q^4\end{matrix};q^2,\ q^2\right] \\
&= \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-1},q;q^2\right)_\infty}\frac{\left(q,q^5;q^2\right)_\infty}{\left(q^2,q^4;q^2\right)_\infty}\, {}_2\phi_1\left[\begin{matrix} q^{-2},\ q^3 \\ q^5\end{matrix};q^2,\ q\right] \\
&= \frac{q^4}{(1-q)\left(1-q^3\right)^2}\left(1+\frac{q\left(1-q^{-2}\right)\left(1-q^3\right)}{\left(1-q^2\right)\left(1-q^5\right)}\right) \\
&= \frac{q^4}{(1-q)\left(1-q^3\right)^2} - \frac{q^3}{(1-q)\left(1-q^3\right)\left(1-q^5\right)} = \frac{-q^3\left(1-q^3-q^4-q^5\right)}{\left(1-q^3\right)^2\left(1-q^5\right)}.
\end{aligned}$$

This proves (3.10).

Now assume that m is a positive integer such that $m\geq 2$. Then by using (1.1) and (3.3), we have

$$\begin{aligned}
\sum_{n\geq 0} B(2m,n)q^n &= \sum_{n\geq 0}\frac{q^{2mn}\left(q^{2n+2},q^{2n+4};q^2\right)_\infty}{\left(q^{2n-3},q^{2n+3};q^2\right)_\infty} \\
&= \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-3},q^3;q^2\right)_\infty}\, {}_2\phi_1\left[\begin{matrix} q^{-3},\ q^3 \\ q^4\end{matrix};q^2,\ q^{2m}\right] \\
&= \frac{\left(q^2,q^4;q^2\right)_\infty}{\left(q^{-3},q^3;q^2\right)_\infty}\frac{\left(q^3,q^{2m-3};q^2\right)_\infty}{\left(q^4,q^{2m};q^2\right)_\infty}\, {}_2\phi_1\left[\begin{matrix} q,\ q^{2m} \\ q^{2m-3}\end{matrix};q^2,\ q^3\right] \\
&= \frac{\left(q^2,q^{2m-3};q^2\right)_\infty}{\left(q^{-3},q^{2m};q^2\right)_\infty}\frac{\left(q,q^{2m};q^2\right)_\infty}{\left(q^2,q^{2m-3};q^2\right)_\infty}\sum_{n\geq 0}\frac{q^{3n}\left(q^{2n+2},q^{2n+2m-3};q^2\right)_\infty}{\left(q^{2n+1},q^{2n+2m};q^2\right)_\infty} \\
&= \frac{q^4}{(1-q)\left(1-q^3\right)}\sum_{n\geq 0}\frac{q^{3n}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+1};q^2\right)_{m-2}},
\end{aligned}$$

which yields (3.11).

Lemma 3.6. *For any positive integer, we have*

$$\sum_{n\geq 0} B'(2m,n)q^n = \frac{-1}{1-q}\sum_{n\geq 0}\frac{q^{5n+2m+1}\left(q^{2n+2};q^2\right)_{m-1}}{\left(q^{2n+1};q^2\right)_{m-1}}.$$

Proof. By (1.1) and (3.3) applied to $(a,b,c,z)=\big(q^{-1},q^5,q^6,q^{2m}\big)$, we find

$$\begin{aligned}\sum_{n\geq 0}B'(2m,n)q^n&=q^{2m}\frac{\big(q^2,q^6;q^2\big)_\infty}{\big(q^{-1},q^5;q^2\big)_\infty}{}_2\phi_1\left[\begin{matrix}q^{-1},\ q^5\\ q^6\end{matrix};q^2,\ q^{2m}\right]\\
&=q^{2m}\frac{\big(q^2,q^6;q^2\big)_\infty}{\big(q^{-1},q^5;q^2\big)_\infty}\frac{\big(q^5,q^{2m-1};q^2\big)_\infty}{\big(q^6,q^{2m};q^2\big)_\infty}{}_2\phi_1\left[\begin{matrix}q,\ q^{2m}\\ q^{2m-1}\end{matrix};q^2,\ q^5\right]\\
&=q^{2m}\frac{\big(q^2,q^{2m-1};q^2\big)_\infty}{\big(q^{-1},q^{2m};q^2\big)_\infty}\sum_{n\geq 0}\frac{q^{5n}\big(q,q^{2m};q^2\big)_n}{\big(q^2,q^{2m-1};q^2\big)_n}\\
&=\frac{-1}{1-q}\sum_{n\geq 0}\frac{q^{5n+2m+1}\big(q^{2n+2},q^{2n+2m-1};q^2\big)_\infty}{\big(q^{2n+1},q^{2n+2m};q^2\big)_\infty}\\
&=\frac{-1}{1-q}\sum_{n\geq 0}\frac{q^{5n+2m+1}\big(q^{2n+2};q^2\big)_{m-1}}{\big(q^{2n+1};q^2\big)_{m-1}},\end{aligned}$$

which is the desired identity. ■

Lemma 3.7. *There holds* $\sum_{n\geq 0}B'(m,n)q^n=\sum_{n\geq 0}(B(m,n)-B(m+2,n))q^n$.

Proof. By (3.9) applied with $q\to q^2$ and $(a,b,c,z)=\big(q^{-3},q^3,q^4,q^m\big)$, we get

$$\begin{aligned}&\sum_{n\geq 0}(B(m,n)-B(m+2,n))q^n\\
&\quad=\frac{\big(q^2,q^4;q^2\big)_\infty}{\big(q^{-3},q^3;q^2\big)_\infty}\left({}_2\phi_1\left[\begin{matrix}q^{-3},\ q^3\\ q^4\end{matrix};q^2,\ q^m\right]-{}_2\phi_1\left[\begin{matrix}q^{-3},\ {}^3\\ q^4\end{matrix};q^2,\ q^{m+2}\right]\right)\\
&\quad=\frac{\big(q^2,q^4;q^2\big)_\infty}{\big(q^{-3},q^3;q^2\big)_\infty}q^m\frac{\big(1-q^{-3}\big)\big(1-q^3\big)}{1-q^4}{}_2\phi_1\left[\begin{matrix}q^{-1},\ q^5\\ q^6\end{matrix};q^2,\ q^m\right]\\
&\quad=\frac{\big(q^2,q^6;q^2\big)_\infty}{\big(q^{-1},q^5;q^2\big)_\infty}\sum_{n\geq 0}\frac{q^{m(n+1)}\big(q^{-1},q^5;q^2\big)_n}{\big(q^2,q^6;q^2\big)_n}=\sum_{n\geq 0}\frac{q^{m(n+1)}\big(q^{2n+2};q^2\big)_\infty\big(q^{2n+6};q^2\big)_\infty}{\big(q^{2n-1};q^2\big)_\infty\big(q^{2n+5};q^2\big)_\infty},\end{aligned}$$

as desired. ■

4 Proof of Theorem 2.2 and Corollary 2.3

Proof of Theorem 2.2. By virtue of Lemma 3.4, we obtain

$$\begin{aligned}\sum_{n\geq 0}A(2m+2,n)q^n&=\sum_{n\geq 0}A(2m,n)q^n-\sum_{n\geq 0}A'(2m,n)q^n\\
&=\sum_{n\geq 0}A(2m-2,n)q^n-\left(\sum_{n\geq 0}A'(2m-2,n)q^n+\sum_{n\geq 0}A'(2m,n)q^n\right)\\
&\cdots\\
&=\sum_{n\geq 0}A(2,n)q^n-\sum_{k=1}^{m}\sum_{n\geq 0}A'(2k,n)q^n.\end{aligned}\tag{4.1}$$

Now combine (4.1) with Lemma 3.2 and (3.8) to deduce

$$\frac{-1}{(1-q)^2}\sum_{n\geq 0}\frac{q^{n+1}\big(q^{2n+2};q^2\big)_m}{\big(q^{2n+3};q^2\big)_{m-1}}=\frac{-q\big(1+q^2\big)}{(1-q)^2\big(1-q^3\big)}-\frac{1}{1-q}\sum_{n\geq 0}\sum_{k=1}^{m}\frac{q^{2k+3n}\big(q^{2n+2};q^2\big)_{k-1}}{\big(q^{2n+3};q^2\big)_{k-1}},$$

which by letting $m\to\infty$ yields

$$\frac{-q}{(1-q)^2}\sum_{n\geq 0}\frac{q^n\big(q^{2n+2};q^2\big)_\infty}{\big(q^{2n+3};q^2\big)_\infty}=\frac{-q\big(1+q^2\big)}{(1-q)^2\big(1-q^3\big)}-\frac{1}{1-q}\sum_{n,k\geq 0}\frac{q^{2k+5n+2}\big(q^{2n+2};q^2\big)_k}{\big(q^{2n+1};q^2\big)_k}. \tag{4.2}$$

In addition, by (3.5) and simplification we have

$$\begin{aligned}\sum_{n\geq 0}\frac{q^n\big(q^{2n+2};q^2\big)_\infty}{\big(q^{2n+3};q^2\big)_\infty}&=\frac{\big(q^2;q^2\big)_\infty}{\big(q^3;q^2\big)_\infty}\sum_{n\geq 0}\frac{\big(q^3;q^2\big)_n q^n}{\big(q^2;q^2\big)_n}=\frac{\big(q^2;q^2\big)_\infty}{\big(q^3;q^2\big)_\infty}\frac{\big(q^4;q^2\big)_\infty}{\big(q;q^2\big)_\infty}\\&=\frac{1-q}{1-q^2}\frac{\big(q^2;q^2\big)_\infty^2}{\big(q;q^2\big)_\infty^2}.\end{aligned} \tag{4.3}$$

Finally, combine (4.3) with (4.2) and rearrange to obtain

$$\sum_{n,k\geq 0}\frac{q^{2k+3n+2}\big(q^{2n+2};q^2\big)_k}{\big(q^{2n+3};q^2\big)_k}=\frac{1}{1-q^2}\frac{\big(q^2;q^2\big)_\infty^2}{\big(q;q^2\big)_\infty^2}-\frac{1+q^2}{(1-q)\big(1-q^3\big)},$$

which gives the desired formula. ■

Proof of Corollary 2.3. An application of (3.6) with $q\to q^4$ and $(a,b,z)=\big(q,q^5,q\big)$ yields after simplification

$$\sum_{n=-\infty}^{\infty}\frac{(q;q^4)_n q^n}{(q^5;q^4)_n}=(1-q)\frac{\big(q^2;q^2\big)_\infty^2}{\big(q;q^2\big)_\infty^2},$$

that is, $\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}}=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}$, or equivalently,

$$\sum_{n=0}^{\infty}\left(\frac{q^n}{1-q^{4n+1}}-\frac{q^{3n+2}}{1-q^{4n+3}}\right)=\frac{\big(q^2;q^2\big)_\infty^2}{\big(q;q^2\big)_\infty^2}. \tag{4.4}$$

Thus, by using Theorem 2.2 and (4.4) we obtain

$$\begin{aligned}\sum_{n\geq 0}F_1(n)q^n&=\frac{-\big(1+q^2\big)}{(1-q)\big(1-q^3\big)}+\frac{1}{1-q^2}\frac{\big(q^2;q^2\big)_\infty^2}{\big(q;q^2\big)_\infty^2}\\&=\frac{-\big(1+q^2\big)}{(1-q)\big(1-q^3\big)}+\frac{1}{1-q^2}\left(\frac{1}{1-q}-\frac{q^2}{1-q^3}\right)\\&\quad+\sum_{n\geq 1}\left(\frac{q^n}{1-q^{4n+1}}-\frac{q^{3n+2}}{1-q^{4n+3}}\right)\frac{1}{1-q^2}\\&=\frac{-q^2}{(1-q)\big(1-q^3\big)}+\sum_{n\geq 1}\left(\frac{q^n\big(1-q^{2n}\big)}{\big(1-q^{4n+1}\big)\big(1-q^2\big)}+\frac{q^{3n}}{\big(1-q^{4n+1}\big)\big(1-q^{4n+3}\big)}\right)\\&\quad\succeq\frac{-q^2}{(1-q)\big(1-q^3\big)}+\sum_{n\geq 1}\frac{q^n\big(1-q^{2n}\big)}{1-q^2}+\sum_{n\geq 1}q^{3n}\\&=\frac{-q^2}{(1-q)\big(1-q^3\big)}+\frac{q}{(1-q)\big(1-q^2\big)}-\frac{q^3}{\big(1-q^2\big)\big(1-q^3\big)}+\frac{q^3}{1-q^3}\\&=\frac{q+q^3}{1-q^3}\succeq 0.\end{aligned}$$

This completes the proof. ■

5 Proof of Theorem 2.5 and Corollary 2.6

Proof of Theorem 2.5. Combining (4.1) with Lemma 3.2 and (3.7), we derive

$$\frac{1}{(1-q)^2}\sum_{n\geq 0}\frac{q^{n+1}\left(q^{2n+2};q^2\right)_m}{\left(q^{2n+3};q^2\right)_{m-1}}=\frac{q\left(1+q^2\right)}{(1-q)^2\left(1-q^3\right)}+\frac{1}{1-q^3}\sum_{n\geq 0}\sum_{k=1}^{m}\frac{q^{2k+n}\left(q^{2n+2};q^2\right)_{k-1}}{\left(q^{2n+5};q^2\right)_{k-1}},$$

which by letting $m\to\infty$ implies

$$\frac{1}{(1-q)^2}\sum_{n\geq 0}\frac{q^{n+1}\left(q^{2n+2};q^2\right)_\infty}{\left(q^{2n+3};q^2\right)_\infty}=\frac{q\left(1+q^2\right)}{(1-q)^2\left(1-q^3\right)}+\frac{1}{1-q^3}\sum_{n,k\geq 0}\frac{q^{2k+2+n}\left(q^{2n+2};q^2\right)_k}{\left(q^{2n+5};q^2\right)_k},$$

or equivalently,

$$\sum_{n,k\geq 0}\frac{q^{2k+2+n}\left(q^{2n+2};q^2\right)_k}{\left(q^{2n+5};q^2\right)_k}=\frac{1-q^3}{(1-q)^2}\sum_{n\geq 0}\frac{q^{n+1}\left(q^{2n+2};q^2\right)_\infty}{\left(q^{2n+3};q^2\right)_\infty}-\frac{q\left(1+q^2\right)}{(1-q)^2}.$$

Now using (4.3), we derive

$$\sum_{n,k\geq 0}\frac{q^{2k+2+n}\left(q^{2n+2};q^2\right)_k}{\left(q^{2n+5};q^2\right)_k}=\frac{q\left(1-q^3\right)}{(1-q)^2}\frac{1-q}{1-q^2}\frac{\left(q^2;q^2\right)_\infty^2}{\left(q;;q^2\right)_\infty^2}-\frac{q\left(1+q^2\right)}{(1-q)^2},$$

which clearly is equivalent to the desired formula. ■

Proof of Corollary 2.6. From Theorem 2.5 and (4.4), we obtain

$$\begin{aligned}
\sum_{n\geq 0}F_2(n)q^n &= \frac{-q\left(1+q^2\right)}{(1-q)^2}+\frac{q\left(1-q^3\right)}{(1-q)\left(1-q^2\right)}\frac{\left(q^2;q^2\right)_\infty^2}{\left(q;q^2\right)_\infty^2}\\
&= \frac{-q\left(1+q^2\right)}{(1-q)^2}+\frac{q\left(1-q^3\right)}{(1-q)\left(1-q^2\right)}\left(\frac{1-q^2}{(1-q)\left(1-q^3\right)}\right.\\
&\qquad\left.+\sum_{n\geq 1}\left(\frac{q^n}{1-q^{4n+1}}-\frac{q^{3n+2}}{1-q^{4n+3}}\right)\right)\\
&= \frac{-q^3}{(1-q)^2}+\sum_{n\geq 1}\left(\frac{q^n\left(1-q^{2n}\right)}{\left(1-q^{4n+1}\right)\left(1-q^2\right)}+\frac{q^{3n}}{\left(1-q^{4n+1}\right)\left(1-q^{4n+3}\right)}\right)\\
&\succeq \frac{-q^3}{(1-q)^2}+\sum_{n\geq 1}\left(\frac{q^n\left(1-q^{2n}\right)}{1-q^2}+q^{3n}\right)\\
&= \frac{-q^3}{(1-q)^2}+\frac{q}{(1-q)\left(1-q^2\right)}-\frac{q^3}{\left(1-q^2\right)\left(1-q^3\right)}+\frac{q^3}{1-q^3}\\
&= \frac{-q^3}{(1-q)^2}+\frac{q}{(1-q)\left(1-q^2\right)}+\frac{q^3-q^4}{(1-q)\left(1-q^3\right)}\\
&= \frac{2-2q^4+q^5}{(1-q)\left(1-q^3\right)}=\frac{q\left(1-q^3\right)^2+q^5-q^7}{(1-q)\left(1-q^3\right)}\\
&= q\left(1+q+q^2\right)+\frac{q^5(1+q)}{1-q^3}\succeq 0.
\end{aligned}$$

This gives the desired result. ■

6 Proof of Theorem 2.8 and Corollary 2.9

Proof of Theorem 2.8. By Lemma 3.7, we get

$$\begin{aligned}\sum_{n\geq 0} B(2m+2,n)q^n &= \sum_{n\geq 0} B(2m,n)q^n - \sum_{n\geq 0} B'(2m,n)q^n \\ &= \sum_{n\geq 0} B(2m-2,n)q^n - \Big(\sum_{n\geq 0} B'(2m-2,n)q^n + \sum_{n\geq 0} B'(2m,n)q^n\Big) \\ &\dots \\ &= \sum_{n\geq 0} B(2,n)q^n - \sum_{k=1}^{m}\sum_{n\geq 0} B'(2k,n)q^n. \end{aligned} \tag{6.1}$$

Now use (6.1), Lemmas 3.5 and 3.6 to deduce

$$\begin{aligned}&\frac{1}{(1-q)(1-q^3)}\sum_{n\geq 0}\frac{q^{3n+4}(q^{2n+2};q^2)_m}{(q^{2n+1};q^2)_{m-1}} \\ &\quad = \frac{q^3(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)} + \frac{q}{1-q}\sum_{n\geq 0}\sum_{k=1}^{m}\frac{q^{2k+5n}(q^{2n+2};q^2)_{k-1}}{(q^{2n+1};q^2)_{k-1}},\end{aligned}$$

which by letting $m\to\infty$ and using (3.5) gives

$$\begin{aligned}\sum_{n,k\geq 0}\frac{q^{2k+3n+2}(q^{2n+2};q^2)_k}{(q^{2n+3};q^2)_k} &= \frac{q^3}{1-q^3}\sum_{n\geq 0}\frac{q^{3n}(q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} - \frac{q^2(1-q)(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)} \\ &= \frac{q^3}{1-q^3}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{(q^4;q^2)_\infty}{(q^3;q^2)_\infty} - \frac{q^2(1-q)(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)} \\ &= \frac{q^3(1-q)}{(1-q^2)(1-q^3)}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} - \frac{q^2(1-q)(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)} \\ &= \frac{q^3}{(1+q)(1-q^3)}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} - \frac{q^2(1-q)(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)},\end{aligned}$$

which is the desired identity. ■

Proof of Corollary 2.9. By virtue of Theorem 2.8 and (4.4), we have

$$\begin{aligned}\sum_{n\geq 0} G(n)q^n &= \frac{q^3}{(1+q)(1-q^3)}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} - \frac{q^2(1-q)(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)} \\ &= \frac{q^3}{(1+q)(1-q^3)}\left(\frac{1}{1-q} - \frac{q^2}{1-q^3}\right) \\ &\quad + \frac{q^3}{(1+q)(1-q^3)}\sum_{n\geq 1}\left(\frac{q^n}{1-q^{4n+1}} - \frac{q^{3n+2}}{1-q^{4n+3}}\right) \\ &\quad - \frac{q^2(1-q)(-1+q^3+q^4+q^5)}{(1-q^3)^2(1-q^5)} \\ &= \frac{q^2}{(1-q^3)(1-q^5)} + \sum_{n\geq 1}\frac{q^{n+3}-q^{3n+5}-q^{5n+6}+q^{7n+6}}{(1+q)(1-q^3)(1-q^{4n+1})(1-q^{4n+3})}\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^2}{(1-q^3)(1-q^5)} + \sum_{n\geq 1} \frac{q^{n+3}(1-q^{2n+2}) - q^{5n+6}(1-q^{2n})}{(1+q)(1-q^3)(1-q^{4n+1})(1-q^{4n+3})} \\
&\succeq \frac{q^2}{(1-q^3)(1-q^5)} + \sum_{n\geq 1} \frac{q^{n+3}(1-q^{2n+2})}{(1+q)(1-q^3)} - \sum_{n\geq 1} \frac{q^{5n+6}(1-q^{2n})}{(1+q)(1-q^3)} \\
&= \frac{q^2}{(1-q^3)(1-q^5)} + \frac{q^3}{(1+q)(1-q^3)} \left(\frac{1}{1-q} - \frac{q^2}{1-q^3} \right) \\
&\quad + \frac{q^6}{(1+q)(1-q^3)} \left(\frac{-1}{1-q^5} + \frac{1}{1-q^7} \right) \\
&= \frac{q^2}{(1-q^3)(1-q^5)} + \frac{q^3}{(1-q^3)^2} - \frac{q^{11}(1-q)}{(1-q^3)(1-q^5)(1-q^7)} \\
&\succeq \frac{q^2}{(1-q^3)(1-q^5)} - \frac{q^{11}(1-q)}{(1-q^3)(1-q^5)(1-q^7)} \\
&= \frac{q^2 - q^9 - q^{11} + q^{12}}{(1-q^3)(1-q^5)(1-q^7)} = \frac{q^2(1-q^3)(1-q^7) + q^5 - q^{11}}{(1-q^3)(1-q^5)(1-q^7)} \\
&= \frac{q^2}{1-q^5} + \frac{q^5(1+q^3)}{(1-q^5)(1-q^7)} = \frac{q^2}{1-q^5} + \frac{q^5(1-q^6)}{(1-q^3)(1-q^5)(1-q^7)} \succeq 0,
\end{aligned}
$$

as desired. ■

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