# Central characters of the symmetric group: $\sigma$- *vs.* Kerov polynomials

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**Abstract:** Expressions for the central characters of the symmetric group in terms of polynomials in the symmetric power-sums over the contents of the Young diagram that specifies the irreducible representation ("$\sigma$-polynomials") were developed by **Katriel** (1991, 1996).
Expressions in terms of free cumulants that encode the Young diagram ("Kerov polynomials"), were proposed by **Kerov** (2000).
The relation between these procedures is established.

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# Introduction

Both the irreducible representations and the conjugacy-classes of $S_n$ are labelled by partitions of $n$.
The irreducible representations are denoted by $\Gamma = [\lambda_1, \lambda_2, \cdots]$, where $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_i \lambda_i = n$; $\lambda_1, \lambda_2, \cdots$ are non-negative integers. $\Gamma$ is commonly presented as a Young diagram, consisting of left-justified rows of boxes of lengths $\lambda_1, \lambda_2, \cdots$, non-increasing from top to bottom, but other equivalent presentations will be referred to below.
Each conjugacy-class consists of the permutations whose cycle-lengths comprise some partition of $n$.

The irreducible character $\chi^{\Gamma}_{C}$, corresponding to the conjugacy-class $C$ and the irreducible representation $\Gamma$, can be renormalized into the central character

$$\lambda^{\Gamma}_{C} = \chi^{\Gamma}_{C} \frac{|C|}{|\Gamma|} \,,$$

where $|C|$ is the number of group elements in the conjugacy class $C$ and $|\Gamma| = \chi^{\Gamma}_{(1)^n}$ is the dimension of the irreducible representation $\Gamma$. Here, $(1)^n$ stands for the conjugacy-class consisting of the identity. The conjugacy class-sums, $[C] \equiv \sum_{c \in C} c$, span the center of the group-algebra. Acting on the irreducible modules they yield the central characters as eigenvalues.

The single-cycle conjugacy class-sums in $S_n$ generate the center of the group algebra. Therefore, the corresponding central-characters are of special interest.
We will use the shorthand notation $(k)_n$ for the conjugacy class $(k)(1)^{n-k}$ in $S_n$, consisting of a cycle of length $k$ and $n-k$ fixed points (cycles of unit length).
The corresponding conjugacy class-sum will be denoted by $[(k)]_n$.

**Ingram** (1950) cited Frobenius for the expressions

$$\lambda^{\Gamma}_{(2)_n} = \frac{1}{2}M_2 \ ; \ \ \lambda^{\Gamma}_{(3)_n} = \frac{1}{6}M_3 - \frac{n(n-1)}{2} \ ; \ \ \lambda^{\Gamma}_{(4)_n} = \frac{1}{4}M_4 - \frac{2n-3}{2}M_2 \,,$$

and provided a similar expression for $\lambda^{\Gamma}_{(5)_n}$. Here,

$$M_2 = \sum_{j=1}^{k} \Big[(\lambda_j - j)(\lambda_j - j + 1) - j(j-1)\Big] \,,$$

$$M_3 = \sum_{j=1}^{k} \Big[(\lambda_j - j)(\lambda_j - j + 1)(2\lambda_j - 2j + 1) + j(j-1)(2j-1)\Big] \,,$$

$$M_4 = \sum_{j=1}^{k} \Big[(\lambda_j - j)^2(\lambda_j - j + 1)^2 - j^2(j-1)^2\Big] \,.$$

The expressions for $M_i$ ; $i = 2, 3, 4$ do not show enough regularity to suggest a generalization.

The concept of *contents of a Young diagram* was introduced by **Robinson and Thrall** (1953).
Given a Young diagram $\Gamma = [\lambda_1, \lambda_2, \cdots, \lambda_k]$, they considered the set of pairs of integers $(i, j)$ that label the boxes of $\Gamma$, i. e., $\{(i, j) \in \Gamma\}$, where $i$ and $j$ are row and column indices respectively, that satisfy $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$. The contents of the Young diagram form the multiset $\{\{(j - i) \ ; \ (i, j) \in \Gamma\}\}$ (keeping track of repetition of identical members).

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| −1 | 0 | 1 | |
| −2 | −1 | | |
| −3 | −2 | | |
| −4 | | | |

Symmetric power-sums over the contents of a Young diagram

$$\sigma_\ell^\Gamma = \sum_{(i,j)\in\Gamma} (j-i)^\ell \,,$$

were independently introduced by **Jucys** (1974) and by **Suzuki** (1987), who showed that the first and second symmetric power sums can be used to express the central characters for the class of transpositions and for the three-cycles, respectively.

It will be convenient to define $\sigma_0 = n$.

A partition labelling a conjugacy class, stripped of its fixed points, will be referred to as a *reduced partition.*

Two procedures for the evaluation of the central characters, due to **Katriel** (1993,1996) and to **Kerov** (2000), respectively, will now be reviewed. These procedures share the property that they essentially depend on the reduced partition labelling the conjugacy class. The residual dependence on the total degree of the symmetric group considered is simple, in a sense to be explicated below.

**Theorem 1.1.** *Katriel (1991). The central character corresponding to any conjugacy class of the symmetric group $S_n$ can be expressed as a polynomial in the symmetric power-sums $\{\sigma_k^\Gamma\ ;\ k = 1, 2, \cdots, n-1\}$, whose structure depends on the reduced partition labelling the conjugacy class. The coefficient of each term in this polynomial is a polynomial in $n$ that is independent of $\Gamma$.*

On the basis of this Theorem a conjecture was proposed for the construction of single- and multi-cycle central characters **Katriel** (1993, 1996) in terms of the symmetric power-sums over the contents of the Young diagram that specifies the irreducible representation, that will be referred to as the $\sigma$-polynomials.

An essential part of this conjecture was proved by **Poulalhon, Corteel, Goupil and Schaeffer** (2000, 2004).

**Lascoux and Thibon** (2004) obtained expressions for symmetric power-sums over Jucys-Murphy elements in terms of conjugacy class-sums, whose inversion would yield the $\sigma$-polynomials presently discussed. Finally, an alternative derivation, yielding a closed form expression for the central characters in terms of symmetric power sums over the contents, was proposed by **Lassalle** (2008). For a comprehensive exposition we refer to **Ceccherini-Silberstein, Scarabotti and Tolli** (2010).

**Sergei Kerov**, in a talk at Institut Poincaré in Paris (January 2000), presented expressions for central characters of the symmetric group in terms of a family of polynomials in a set of elements called free cumulants. The structure of these polynomials depends on the reduced partitions labelling the conjugacy classes, whose central characters they evaluate, but the dependence on the irreducible representation with respect to which the central character is evaluated enters only via the values that the free cumulants obtain. The free cumulants will be defined below. Here we just mention the rather amazing fact that Kerov's polynomials originate from the asymptotic representation theory of $S_n$ for $n \to \infty$, but turn out to be relevant to finite symmetric groups as well.

Sergei Kerov passed away on July 30, 2000.

It is thanks to Biane that Kerov's work on the central characters found its way into well-presented expositions (2000, 2003). This was followed by considerable research on Kerov's procedure [**Rattan** (2005, 2007), **Biane** (2005), **Féray** (2009), **Petrullo and Senato** (2011), **Dołega and Śniady** (2012)]. A recent masterly exposition was presented by **Cartier** (2013).

**Lassalle** (2008), in his concluding notes, pointed out the desirability of establishing the connection between the expressions for the central characters in terms of the symmetric power sums over the contents, on the one hand, and Kerov's polynomials in terms of the free cumulants, on the other hand. The present paper establishes this connection.

## The single-cycle central characters as $\sigma$-polynomials

We shall denote by $\vdash_{(\ell)}$ a partition whose least part is not smaller than $\ell$. We shall be mainly interested in the case $\ell = 2$.

**Theorem 2.1.** *The central character $\lambda^{\Gamma}_{(k)_n}$ can be expressed as a linear combination of terms specified by the partitions of $k+1$ into parts, none of which is less than 2.*
*The partition*

$$\pi \equiv 2^{n_2}3^{n_3}\cdots(k+1)^{n_{k+1}} \vdash_{(2)} (k+1)\,,$$

*i.e., $2n_2 + 3n_3 + \cdots + (k+1)n_{k+1} = k+1$, yields the term*

$$f_\pi(n)\sigma_1^{n_3}\sigma_2^{n_4}\cdots\sigma_{k-1}^{n_{k+1}}\,,$$

*where $f_\pi(n)$ is a polynomial of degree $n_\pi \leq n_2$ in $n$.*
*$\sigma_i, i = 1, 2, \cdots, k-1$ are the symmetric power sums over the contents of the Young diagram $\Gamma$.*

This Theorem was originally stated as a conjecture, **Katriel** (1993, 1996). It was proved by **Poulalhon, Corteel, Goupil and Schaeffer** (2000, 2004).

The conjecture, as stated in **Katriel** (1996), specifies the degree of the polynomial $f_\pi(n)$ somewhat more precisely, i.e.,

**Conjecture 2.2.**

$$n_\pi = n_2 \,.$$

This refinement is convenient, but not essential for the rest of the argument.

It remains to determine the polynomials $f_\pi(n)$. This is facilitated by the following two Theorems.

**Theorem 2.3.** *The coefficient of the term $\sigma_{k-1}$ in $\lambda^{\Gamma}_{(k)_n}$, that corresponds to the partition of $k+1$ into a single part, is equal to unity.*

**Theorem 2.4.** *If the symmetric power sums $\sigma_i$ are evaluated for a Young diagram with less than $k$ boxes, then $\lambda^{\Gamma}_{(k)_n} = 0$.*

Using these Theorems, more than enough linear equations are generated, allowing the determination of the required polynomials. To clarify the procedure we emphasize that Theorem 2.4 yields a homogeneous system of equations for the desired coefficients.

**Jucys** (1974) and **Suzuki** (1987) obtained

$$\lambda^{\Gamma}_{(2)_n} = \sigma_1 \quad ; \quad \lambda^{\Gamma}_{(3)_n} = \sigma_2 - \frac{n(n-1)}{2}\,.$$

The procedure outlined above yields the following further expressions:

$$\begin{aligned}
\lambda^{\Gamma}_{(4)_n} &= \sigma_3 - (2n-3)\sigma_1 \\
\lambda^{\Gamma}_{(5)_n} &= \sigma_4 - (3n-10)\sigma_2 - 2\sigma_1^2 + \frac{n(n-1)(5n-19)}{6} \\
\lambda^{\Gamma}_{(6)_n} &= \sigma_5 - (4n-25)\sigma_3 - 6\sigma_1\sigma_2 + (6n^2 - 38n + 40)\sigma_1 \\
\lambda^{\Gamma}_{(7)_n} &= \sigma_6 + \left(-5n + \frac{105}{2}\right)\cdot\sigma_4 - 8\cdot\sigma_3\sigma_1 - \frac{9}{2}\cdot\sigma_2^2 \\
&\quad + \left(\frac{21}{2}n^2 - \frac{241}{2}n + 252\right)\cdot\sigma_2 \\
&\quad + (14n - 72)\cdot\sigma_1^2 - \frac{1}{24}n(n-1)(49n^2 - 609n + 1502) \\
&\vdots
\end{aligned}$$

## Kerov's expressions for the characters corresponding to single-cycle conjugacy classes

For the irreducible characters corresponding to the conjugacy class-sum $(k)_n$ of $S_n$ Kerov used the normalization

$$\Sigma_k^\Gamma = \frac{n!}{(n-k)!} \frac{\chi^\Gamma_{(k)_n}}{|\Gamma|} \, .$$

Since $|(k)_n| = \binom{n}{k}(k-1)! = \frac{1}{k}\frac{n!}{(n-k)!}$, we obtain

$$\lambda^\Gamma_{(k)_n} = \frac{1}{k}\Sigma_k^\Gamma \, .$$

By multiplying the length of each row of the Young diagram $\Gamma = [\lambda_1, \lambda_2, \cdots]$ by the positive integer $t$ and repeating it $t$ times we obtain the augmented Young diagram $\Gamma_t = [(t\lambda_1)^t, (t\lambda_2)^t, \cdots]$, representing an irreducible representation of $S_{nt^2}$. Biane (1998) proved that

$$R_{k+1} \equiv \lim_{t\to\infty} \frac{\Sigma_k^{\Gamma_t}}{t^{k+1}}$$

exists, and referred to $R_{k+1}$ (that depends on $\Gamma$) as a free cumulant.

The remarkable property established by Kerov is that for the finite symmetric group $S_n$ the normalized character $\Sigma_k^{\Gamma}$ can be written as a polynomial in the free cumulants $R_2, R_3, \cdots, R_{k+1}$, with constant coefficients, that Kerov conjectured to be positive integers. This property of the coefficients was proved by **Féray** (2009), who proposed their combinatorial interpretation.

The low $k$ Kerov polynomials were given by **Biane** (2003), *i.e.*,

$$\begin{aligned}
\Sigma_1^\Gamma &= R_2 = 2n \\
\Sigma_2^\Gamma &= R_3 \\
\Sigma_3^\Gamma &= R_4 + R_2 \\
\Sigma_4^\Gamma &= R_5 + 5R_3 \\
\Sigma_5^\Gamma &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\
&\vdots
\end{aligned}$$

A general expression was derived by **Goulden and Rattan** (2005, 2007).

The dependence of the free cumulants $R_k$ on the Young diagram $\Gamma$ that specifies the irreducible representation is obtained as follows:
**a.** The Young diagram is specified in terms of a "Russian convention", introduced in Section 4.1, that involves the set of parameters

$$x_1 < y_1 < x_2 < y_2 < \cdots < y_{m-1} < x_m .$$

**b.** The function

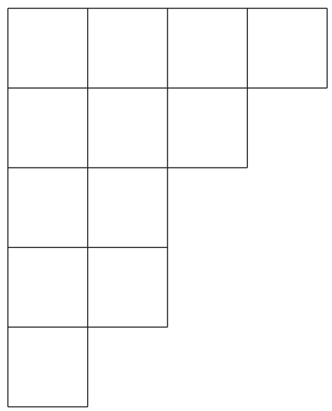
$$H_\omega(z) = \frac{\prod_{i=1}^{m-1}(z - y_i)}{\prod_{i=1}^{m}(z - x_i)}$$

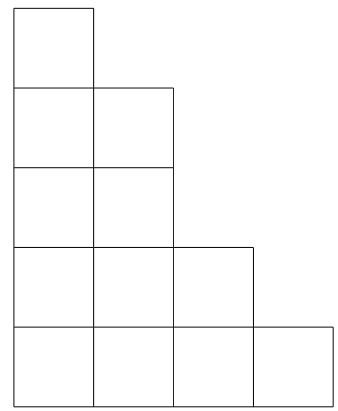
is inverted, yielding the expansion

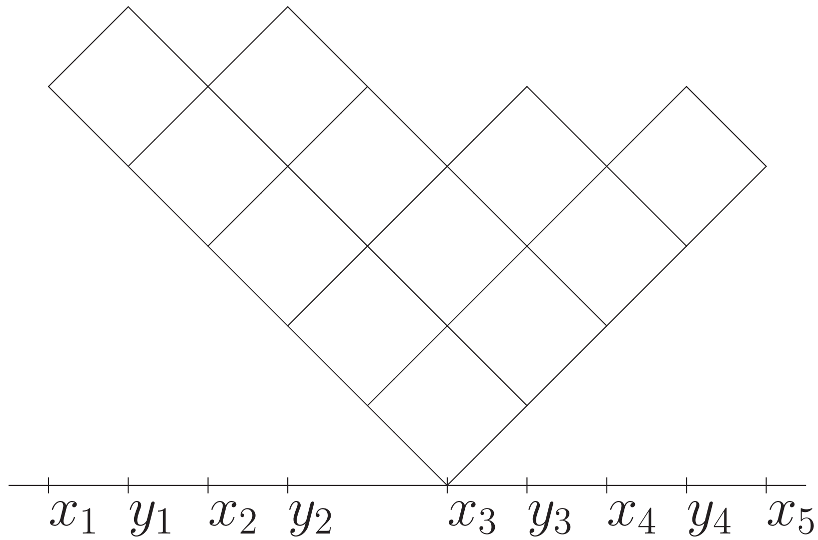
$$H_\omega^{\langle -1 \rangle}(t) = \frac{1}{t} + \sum_{i=1}^{\infty} R_i \cdot t^{i-1}$$

where $R_i$ are the desired free cumulants.

# Relation between the $\sigma$-polynomials and Kerov's polynomials






$x_1$
$y_1$
$x_2$
$y_2$
$x_3$
$y_3$
$x_4$
$y_4$
$x_5$

Kerov's expressions for the central characters are stated in terms of the "Russian" convention, whereas the $\sigma$-polynomials involve the "British" convention. It is therefore necessary to establish the relation between these two conventions. This is done in Section 4.1.

A set of supersymmetric parameters for the rotated Young diagram is defined in Section 4.2 by forming the power sums over the locations of the minima and over the locations of the maxima,

$$X_k = \sum_{i=1}^{m} x_i^k \quad \text{and} \quad Y_k = \sum_{i=1}^{m-1} y_i^k\,,$$

and taking the differences $A_k = X_k - Y_k$.

The symmetric power-sums over the contents are expressed in terms of these supersymmetric parameters of the rotated Young diagram. These relations are inverted in Section 4.3.

In Section 4.4 Kerov's function $H_\omega(z)$ is expanded in terms of a sequence denoted by $G_j, j = 1, 2, \cdots$ and the members of the latter sequence are expressed as polynomials in the supersymmetric parameters of the rotated Young diagram.

In Section 4.5 the inverse of $H_\omega(z)$ is expressed as a series involving the sequence $R_i, i = 1, 2, \cdots$, and the members of this sequence are expressed as polynomials in the $G_j, j = 1, 2, \cdots$.

In Section 4.6 Kerov's sequence $R_i, i = 1, 2, \cdots$ is expressed in terms of the supersymmetric parameters of the rotated Young diagram.

Finally, in Section 4.7 Kerov's expressions for the central characters are expressed in terms of the supersymmetric parameters of the rotated Young diagram, allowing a detailed comparison with the $\sigma$-polynomials.

## 4.1

## The rotated ("Russian") and the "British" Young diagram

The locations of the minima and of the intertwining maxima,

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{m-1} < y_{m-1} < x_m \,,$$

satisfy

$$\sum_{j=1}^{m-1} y_j = \sum_{j=1}^{m} x_j \,.$$

Here, we specify the Young diagram $\Gamma$ in terms of its distinct row lengths $\lambda_{(i)}$ and their multiplicities $\mu_i$, i.e., $\Gamma = [\lambda_{(1)}^{\mu_1}, \lambda_{(2)}^{\mu_2}, \cdots]$, where $\lambda_{(1)} > \lambda_{(2)} > \cdots$ and $\sum_i \lambda_{(i)}\mu_i = n$. Specifying $\Gamma$ columnwise (or specifying the conjugate Young diagram) we similarly define $[{\lambda'_{(1)}}^{\mu'_1}, {\lambda'_{(2)}}^{\mu'_2}, \cdots]$.

Since the number of distinct rows (and of distinct columns) is the number of maxima in the "Russian" notation, i.e., $k = m - 1$, it is easy to see that

$$\begin{aligned} x_\ell &= \lambda_{(m+1-\ell)} - \lambda^{'}_{(\ell)} \; ; \; \ell = 1, 2, \cdots, m \,, \\ & \text{and} \\ y_\ell &= \lambda_{(m-\ell)} - \lambda^{'}_{(\ell)} \; ; \; \ell = 1, 2, \cdots, m - 1 \,. \end{aligned} \tag{1}$$

The $2k$ parameters $\{\lambda_{(\ell)}, \lambda^{'}_{(\ell)} \; ; \; \ell = 1, 2, \cdots, k\}$ determine the $2k+1 = 2m-1$ parameters $\{x_\ell \; ; \; \ell = 1, 2, \cdots, m\} \cup \{y_\ell \; ; \; \ell = 1, 2, \cdots, m-1\}$ (recall that $\sum_{\ell=1}^{m} x_\ell = \sum_{\ell=1}^{m-1} y_\ell$).

The relations 1 can be inverted into

$$\lambda'_{(\ell)} = \sum_{i=1}^{\ell-1} y_i - \sum_{i=1}^{\ell} x_i$$

$$\lambda_{(m-\ell)} = \sum_{i=1}^{\ell} (y_i - x_i) = \lambda'_{(\ell)} + y_\ell$$

where $\ell = 1, 2, \cdots, m-1$.

**4.2**

## Symmetric power sums over the "contents" in terms of the symmetric "Russian" parameters

Let

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{m-1} < y_{m-1} < x_m$$

specify a Young diagram, $\Gamma$.

The symmetric power sums over the minima and over the maxima are

$$X_k = \sum_{i=1}^{m} x_i^k \,, \qquad Y_k = \sum_{i=1}^{m-1} y_i^k \,.$$

The differences $A_k = X_k - Y_k$ will be referred to as the supersymmetric power sums. It was noted above that $A_1 = 0$.

First, we consider a Young diagram consisting of a single row of length x. This diagram is specified by the “Russian” parameters $x_1 = -1$, $y_1 = x - 1$ and $x_2 = x$. For this Young diagram we obtain

$$A_k = (-1)^k + x^k - (x-1)^k = -\sum_{j=1}^{k-1} \binom{k}{j} \cdot x^j \cdot (-1)^{k-j} .$$

This set of linear relations between $x, x^2, \cdots, x^{k-1}$ and $A_2, A_3, \cdots, A_k$ can be inverted to obtain

**Lemma 4.1.**

$$x^k = \frac{1}{k+1} \sum_{j=0}^{k-1} (-1)^j \cdot B_j \cdot \binom{k+1}{j} \cdot A_{k+1-j} .$$

The symmetric power sums over the contents that correspond to the single-row Young diagram considered above are

$$\sigma_k = \sum_{i=0}^{x-1} i^k = \frac{1}{k+1} \cdot \sum_{j=1}^{k+1} \binom{k+1}{j} \cdot B_{k+1-j} \cdot x^j \,,$$

where $B_k$ are the Bernoulli numbers.
Using Lemma 4.1 we obtain

**Lemma 4.2.** *For a single-row Young diagram of length $x$*

$$\sigma_k = -\frac{1}{(k+1)\cdot(k+2)} \cdot \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} B_{2n} \cdot (2n-1) \cdot \binom{k+2}{2n} \cdot A_{k+2-2n} \,.$$

Finally,

**Theorem 4.3.** *The expression for $\sigma_k$ presented in Lemma 4.2 holds for arbitrary Young diagrams.*

Theorem 4.3 means that the supersymmetric power sums $\{A_2, A_3, \cdots\}$ determine the symmetric power sums over the contents, $\{\sigma_1, \sigma_2, \cdots\}$. The latter determine the multiset of contents, hence the Young diagram.

Using the theorem we obtain

$$
\begin{aligned}
\sigma_0 &= n = \frac{1}{2} \cdot A_2 \\
\sigma_1 &= \frac{1}{6} \cdot A_3 \\
\sigma_2 &= \frac{1}{12} \cdot A_4 - \frac{1}{12} \cdot A_2 \\
\sigma_3 &= \frac{1}{20} \cdot A_5 - \frac{1}{12} \cdot A_3 \\
\sigma_4 &= \frac{1}{30} \cdot A_6 - \frac{1}{12} \cdot A_4 + \frac{1}{20} \cdot A_2 \\
\sigma_5 &= \frac{1}{42} \cdot A_7 - \frac{1}{12} \cdot A_5 + \frac{1}{12} \cdot A_3 \\
\sigma_6 &= \frac{1}{56} \cdot A_8 - \frac{1}{12} \cdot A_6 + \frac{1}{8} \cdot A_4 - \frac{5}{84} \cdot A_2 \\
\sigma_7 &= \frac{1}{72} \cdot A_9 - \frac{1}{12} \cdot A_7 + \frac{7}{40} \cdot A_5 - \frac{5}{36} \cdot A_3 \\
\sigma_8 &= \frac{1}{90} \cdot A_{10} - \frac{1}{12} \cdot A_8 + \frac{7}{30} \cdot A_6 - \frac{5}{18} \cdot A_4 + \frac{7}{60} \cdot A_2
\end{aligned}
$$

**4.3**

## The supersymmetric parameters of the rotated Young diagram in terms of the symmetric power sums over the contents

The relations in Theorem 4.3 can be inverted to obtain

**Lemma 4.4.**

$$A_k = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} 2 \cdot \binom{k}{2i} \cdot \sigma_{k-2i} \,.$$

The inverted relations are illustrated by

$$\begin{aligned}
A_2 &= 2 \cdot \sigma_0 = 2n \\
A_3 &= 6 \cdot \sigma_1 \\
A_4 &= 12 \cdot \sigma_2 + 2 \cdot \sigma_0 \\
A_5 &= 20 \cdot \sigma_3 + 10 \cdot \sigma_1 \\
A_6 &= 30 \cdot \sigma_4 + 30 \cdot \sigma_2 + 2 \cdot \sigma_0 \\
A_7 &= 42 \cdot \sigma_5 + 70 \cdot \sigma_3 + 14 \cdot \sigma_1
\end{aligned}$$

**4.4**

## Expansion of $H_\omega(z)$

Let

$$H_\omega(z) \equiv \frac{\prod_{i=1}^{m-1}(z-y_i)}{\prod_{i=1}^{m}(z-x_i)}\,. \tag{2}$$

Here, to adhere with accepted notation, $\omega$ denotes the Young diagram specified by the sets of extrema $\{x_1, x_2, \cdots, x_m\}$ and $\{y_1, y_2, \cdots, y_{m-1}\}$, that is denoted by $\Gamma$ in the rest of the paper.

Writing (2) in the form

$$\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \cdot G_{j+1} \cdot \prod_{i=1}^{m}(z-x_i) = \prod_{i=1}^{m-1}(z-y_i)\,,$$

and equating coefficients of equal powers of $z$ up to the $(m-1)$'s power, we obtain

$$\sum_{j=0}^{\ell}(-1)^j \cdot X^{(\ell-j)} \cdot G_{j+1} = Y^{(\ell)} \quad ; \quad \ell = 0, 1, \cdots, m-1. \tag{3}$$

For $\ell = 0$ (3) yields $G_1 = 1$, and for $\ell = 1$ it yields $G_2 = X_1 - Y_1 = 0$. Hence, $H_\omega(z)$ is of the form

$$H_\omega(z) = \frac{1}{z} + \sum_{j=3}^{\infty} G_j \frac{1}{z^j}\,.$$

Proceeding, we obtain

$$\begin{aligned}
G_3 &= \frac{1}{2} \cdot A_2 = n \\
G_4 &= \frac{1}{3} \cdot A_3 \\
G_5 &= \frac{1}{4} \cdot A_4 + \frac{1}{8} \cdot A_2^2 \\
G_6 &= \frac{1}{5} \cdot A_5 + \frac{1}{6} \cdot A_2 \cdot A_3 \\
G_7 &= \frac{1}{6} \cdot A_6 + \frac{1}{18} \cdot A_3^2 + \frac{1}{8} \cdot A_2 \cdot A_4 + \frac{1}{48} A_2^3 \\
&\vdots
\end{aligned}$$

The coefficients $\{G_i\}$ evaluated above depend on the min-max coordinates only via their supersymmetric sums, $\{A_j\ ;\ j = 2, 3, \cdots\}$.
It is obvious that the numerator of $H_\omega(z)$ is a symmetric polynomial in $y_1, y_2, \cdots, y_{m-1}$, and the denominator is a symmetric polynomial in $x_1, x_2, \cdots, x_m$. It follows that the coefficients $G_i$ depend on two such sets of symmetric power sums.
Allowing the parameters $\{x_i; \quad i = 1, 2, \cdots, m\}$ and $\{y_i; \quad i = 1, 2, \cdots, m-1\}$ to be continuous we note that taking the limit $y_k \to x_k$ for a particular $k$ we obtain an expression for an $H_\omega(z)$ corresponding to a Young diagram with $m-1$ minima (and $m-2$ maxima), which depends on the corresponding symmetric power sums of the remaining min-max coordinates. Consistency requires that the symmetric power sums appear only in the combinations $X_\ell - Y_\ell\ ;\ \ell = 2, 3, \cdots$.

**Theorem 4.5.**

$$G_k = \sum_{Q \vdash_2 (k-1)} \frac{A_Q}{|Q|},$$

*where* $Q$ *is a partition of* $k-1$ *into parts none of which is less then 2,* $Q = (2)^{q_2}(3)^{q_3}\cdots$ *such that* $\sum_i i \cdot q_i = k-1$,
$A_Q \equiv A_2^{q_2} \cdot A_3^{q_3} \cdots$ *and* $|Q| \equiv 2^{q_2} \cdot q_2! \cdot 3^{q_3} \cdot q_3! \cdots$.

4.5

## Inversion of $H_\omega(z)$

It is easy to see that the inverse of $H_\omega(z)$ is of the form

$$H_\omega^{\langle -1\rangle}(t) = \frac{1}{t} + \sum_{i=1}^{\infty} R_i \cdot t^{i-1}\,. \tag{4}$$

From (2) it follows that

$$H_\omega^{\langle -1\rangle}\left(H_\omega(z)\right) = z\,. \tag{5}$$

Substituting $t = H_\omega(z)$ in (4), multiplying by $H_\omega(z)$ and using (5) it follows that

$$zH_\omega(z) = 1 + \sum_{i=1}^{\infty} R_i \Big(H_\omega(z)\Big)^i\,.$$

Establishing the recurrence relation

$$R_m = G_{m+1} - \sum_{\ell=0}^{m-3} \sum_{Q \vdash_3 (m-\ell)} \frac{(\ell+k)!}{\ell!} \cdot R_{\ell+k} \cdot \frac{G_Q}{[Q]}, \tag{6}$$

where $k = \ell_3 + \ell_4 + \cdots$ and $[Q] = \ell_3! \cdot \ell_4! \cdots$, we obtain

$$\begin{aligned}
R_1 &= 0 \\
R_2 &= G_3 = n \\
R_3 &= G_4 \\
R_4 &= G_5 - 4 \cdot \frac{1}{2!} \cdot G_3^2 \\
R_5 &= G_6 - 5 \cdot G_3 \cdot G_4 \\
R_6 &= G_7 - 6 \cdot \left( G_3 \cdot G_5 + \frac{1}{2!} \cdot G_4^2 \right) + 7 \cdot G_3^3 \\
R_7 &= G_8 - 7 \cdot (G_3 \cdot G_6 + G_4 \cdot G_5) + 28 \cdot G_3^2 \cdot G_4
\end{aligned}$$

The expressions obtained above suggest

**Conjecture 4.6.**

$$R_k = -\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \cdot \frac{(k+i-2)!}{(k-1)!} \cdot S_i(k+i)\,,$$

*where*

$$S_p(K) = \sum_{Q \vdash_3^p K} \frac{G_Q}{[Q]} \;\;;\;\; K \geq 3p\,,$$

$Q$ is a partition of $K$ into $p$ parts each of which is not less than 3, i.e., $Q = (3)^{q_3}(4)^{q_4}\cdots$ where $q_3 + q_4 + \cdots = p$ and $3 \cdot q_3 + 4 \cdot q_4 + \cdots = K$. Finally, $G_Q = G_3^{q_3} \cdot G_4^{q_4} \cdots$ and $[Q] = q_3! \cdot q_4! \cdots$.

Conjecture 4.6 is not used below because a direct expression for $R_k$ in terms of the supersymmetric parameters $A_i$ is established in the following section.

**4.6**

## Kerov's sequence $R_i, i = 1, 2, \cdots$ in terms of the supersymmetric parameters of the rotated Young diagram

Since the coefficients $G_i$ are determined by the supersymmetric parameters $A_i$, the same holds for the coefficients $R_i$. Thus, using (6) and Theorem 4.5, we obtain

$$
\begin{aligned}
R_1 &= 0 \\
R_2 &= \frac{1}{2} \cdot A_2 = n \\
R_3 &= \frac{1}{3} \cdot A_3 \\
R_4 &= \frac{1}{4} \cdot A_4 - \frac{3}{8} \cdot A_2^2 \\
R_5 &= \frac{1}{5} \cdot A_5 - \frac{2}{3} \cdot A_2 \cdot A_3 \\
R_6 &= \frac{1}{6} \cdot A_6 - \frac{5}{8} \cdot A_2 \cdot A_4 - \frac{5}{18} \cdot A_3^2 + \frac{25}{48} \cdot A_2^3 \\
&\vdots
\end{aligned}
$$

The expressions obtained above suggest the general form

**Proposition 4.7.** *The relationship between the $R_i$ and $A_i$ is*

$$R_k = \sum_{Q \vdash_2 k} (-1)^{p(Q)-1} \frac{(k-1)^{p(Q)-1}}{|Q|} \cdot A_Q \,. \tag{7}$$

*Here, $Q$, $|Q|$ and $A_Q$ are defined as in Theorem 4.5. $p(Q) = \sum_i q_i$ is the number of parts in $Q$.*

*Proof.* Use Lagrange inversion. $\square$

**4.7**

## The central characters

Using Proposition 4.7, Kerov's expressions for the central characters yield

$$
\begin{aligned}
\lambda_{[(2)]_n} &= \frac{1}{2}\cdot\Sigma_2 = \frac{1}{2}\cdot R_3 = \frac{1}{6}\cdot A_3\\
\lambda_{[(3)]_n} &= \frac{1}{3}\cdot\Sigma_3 = \frac{1}{3}\cdot(R_4+R_2) = \frac{1}{12}\cdot A_4 - \frac{1}{8}\cdot A_2^2 + \frac{1}{6}\cdot A_2\\
\lambda_{[(4)]_n} &= \frac{1}{4}\cdot\Sigma_4 = \frac{1}{4}\cdot R_5 + \frac{5}{4}\cdot R_3 = \frac{1}{20}\cdot A_5 - \frac{1}{6}\cdot A_3\cdot A_2 + \frac{5}{12}\cdot A_3\\
\lambda_{[(5)]_n} &= \frac{1}{5}\cdot\Sigma_5 = \frac{1}{5}\cdot R_6 + 3\cdot R_4 + \frac{8}{5}\cdot R_2 + R_2^2\\
&= \frac{1}{30}\cdot A_6 - \frac{1}{8}\cdot A_4\cdot A_2 - \frac{1}{18}\cdot A_3^2 + \frac{5}{48}\cdot A_2^3 + \frac{3}{4}\cdot A_4 - \frac{7}{8}\cdot A_2^2 + \frac{4}{5}\cdot A_2\\
&\ \vdots
\end{aligned}
$$

The corresponding expressions in **Katriel** (1996) are

$$\begin{aligned}
\lambda_{[(2)]_n} &= \sigma_1 = \frac{1}{6} \cdot A_3 \\
\lambda_{[(3)]_n} &= \sigma_2 - \frac{1}{2} \cdot n \cdot (n-1) = \frac{1}{12} \cdot A_4 - \frac{1}{8} \cdot A_2^2 + \frac{1}{6} \cdot A_2 \\
\lambda_{[(4)]_n} &= \sigma_3 - (2n-3) \cdot \sigma_1 = \frac{1}{20} \cdot A_5 - \frac{1}{6} \cdot A_3 \cdot A_2 + \frac{5}{12} \cdot A_3 \\
&\vdots
\end{aligned}$$

To obtain the expressions in terms of $\{A_i \ ; \ i = 2, 3, \cdots\}$ we used Theorem 4.3.

**Conclusion:** The connection between the expressions for the central characters of the one-cycle conjugacy classes in the symmetric group in terms of $\sigma$-polynomials, and the expressions in terms of Kerov's polynomials, has been established.

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