

Bijective proof of Cauchy Identity for q -Whittaker polynomials

SLC — April 6th 2022

Matteo Mucciconi — based on a joint work with Takashi Imamura and Tomohiro Sasamoto

Cauchy Identities

$$\sum_{\lambda} b_{\lambda}(q, t) P_{\lambda}(x; q, t) P_{\lambda}(y; q, t) = \prod_{k \geq 0} \prod_{i,j} \frac{1 - tx_i y_j q^k}{1 - x_i y_j q^k}$$

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- The Cauchy Identity can be used to define $P_{\lambda}(x; q, t)$ [Macdonald's book]
- There are combinatorial definitions of Macdonald polynomials: tableaux expansion [Haglund-Haiman-Loher], Vertex models [Cantini-De Gier-Wheeler, Borodin-Wheeler, Garbali-Wheeler,...], alcove walks [Ram-Yip],...

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 - $t = 0$ YES : [Imamura-M-Sasamoto'21]
 - $q, t \neq 0$???

Cauchy Identity for Schur polynomials

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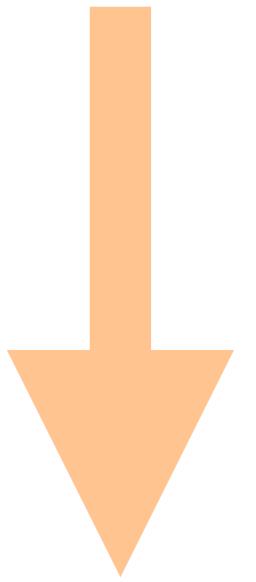
$$\frac{1}{1 - x_i y_j} = \sum_{M_{i,j}=0,1,2,\dots} (x_i y_j)^{M_{i,j}}$$

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$$\frac{1}{1 - x_i y_j} = \sum_{M_{i,j}=0,1,2,\dots} (x_i y_j)^{M_{i,j}}$$

$$\sum_{\lambda} \sum_{P,Q \in \text{SSYT}(\lambda)} x^P y^Q = \sum_{M \in \mathbb{M}_{n \times n}} \prod_{i,j=1}^n (x_i y_j)^{M_{i,j}}$$

Bijective proof:

$$(P, Q) \xleftrightarrow{\text{RSK}} M$$

RSK correspondence

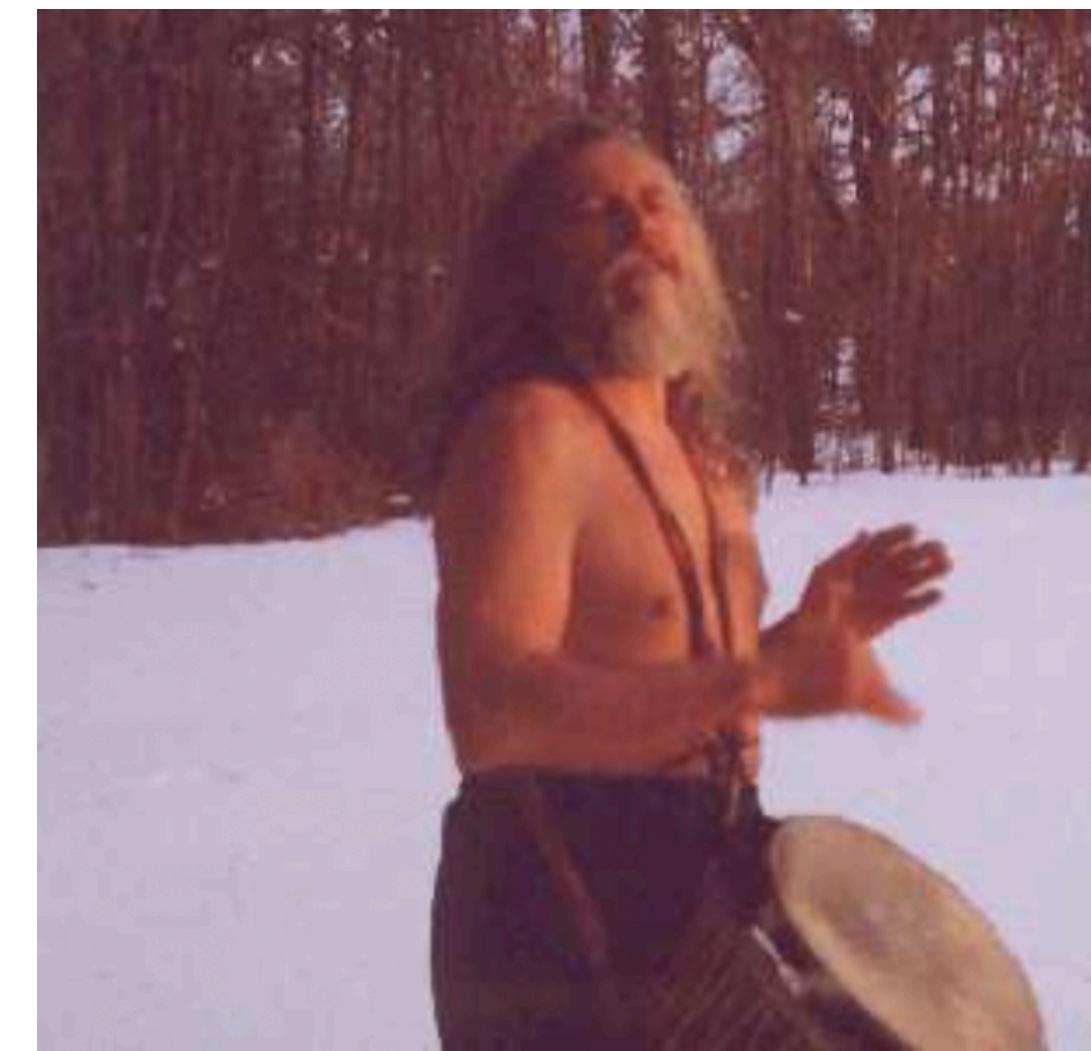
$$\left(M_{i,j} \right)_{i,j=1}^n \longleftrightarrow (P, Q)$$

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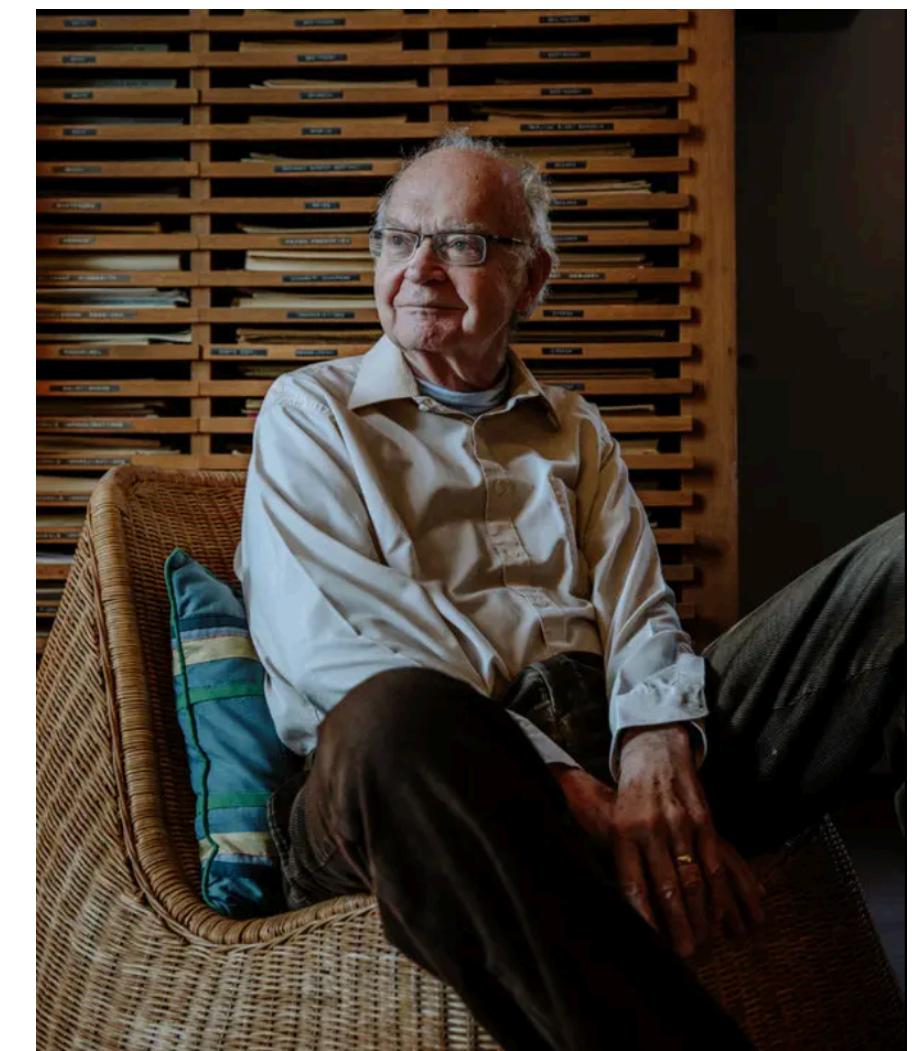
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Robinson



Schensted



Knuth

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Example

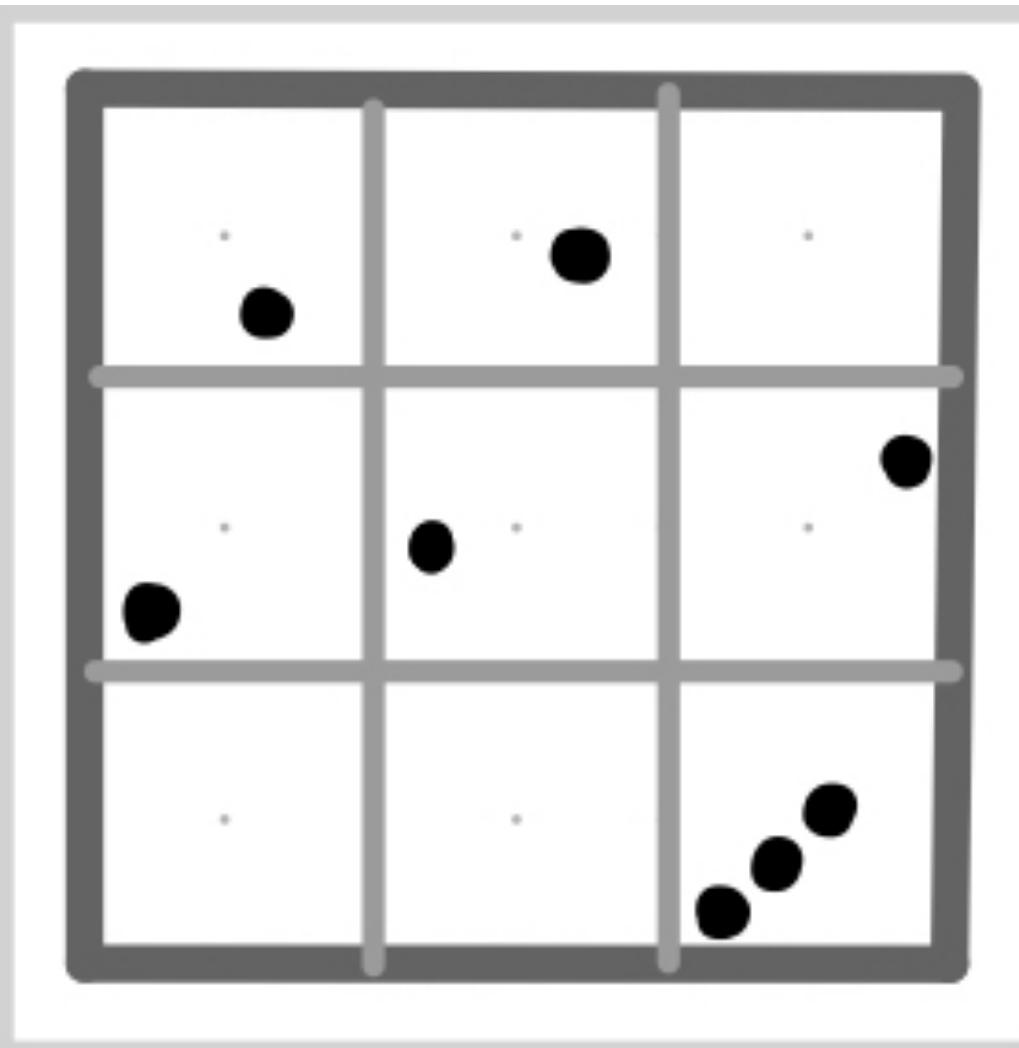
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow ? , ?$$

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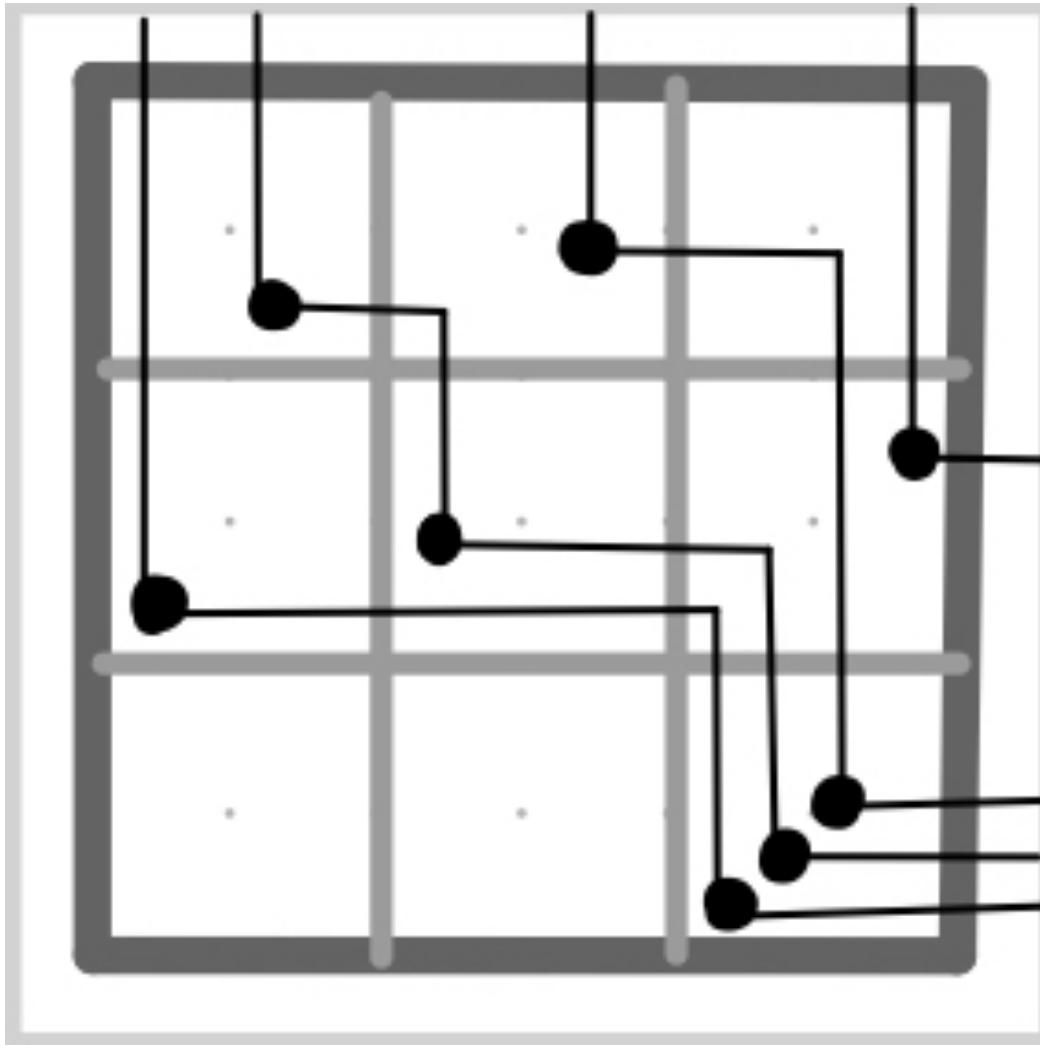


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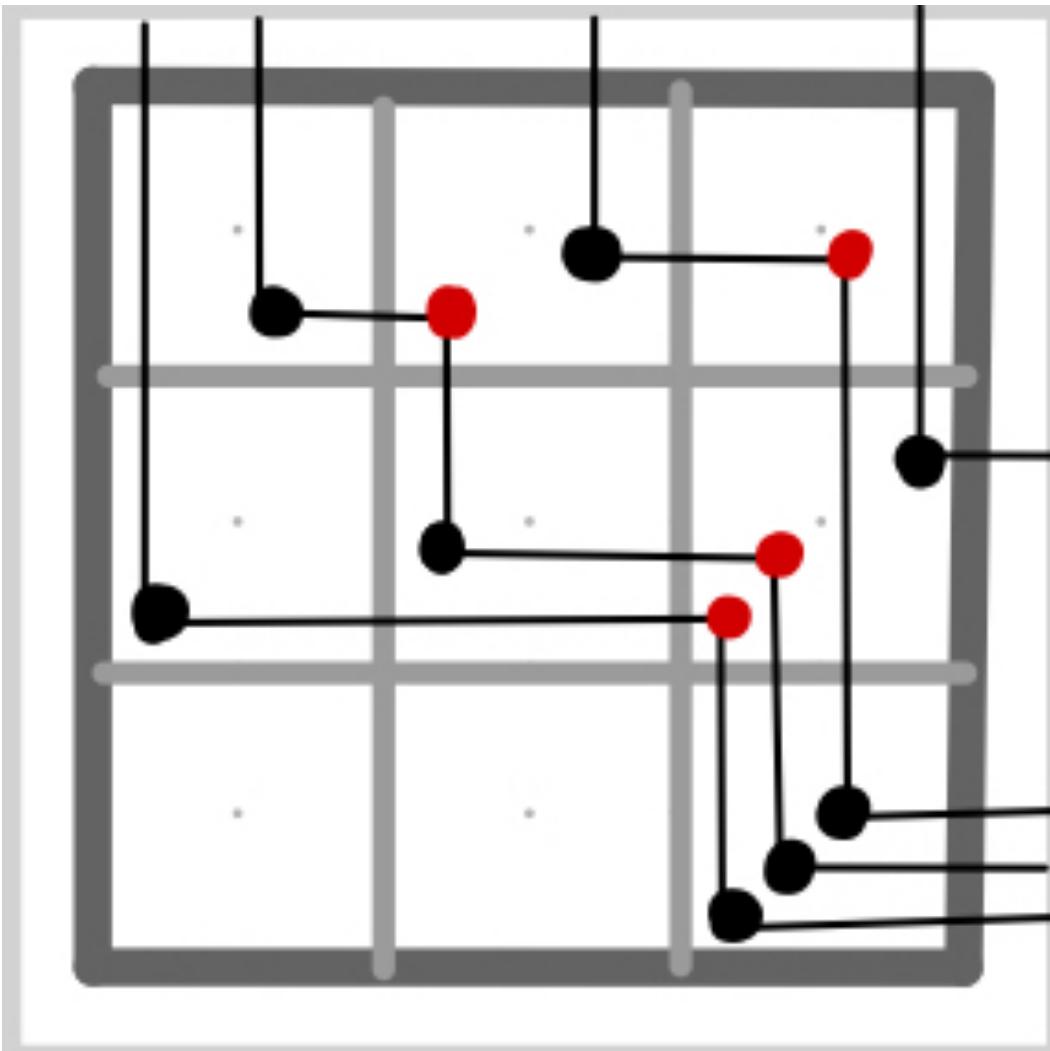
X.G. Viennot

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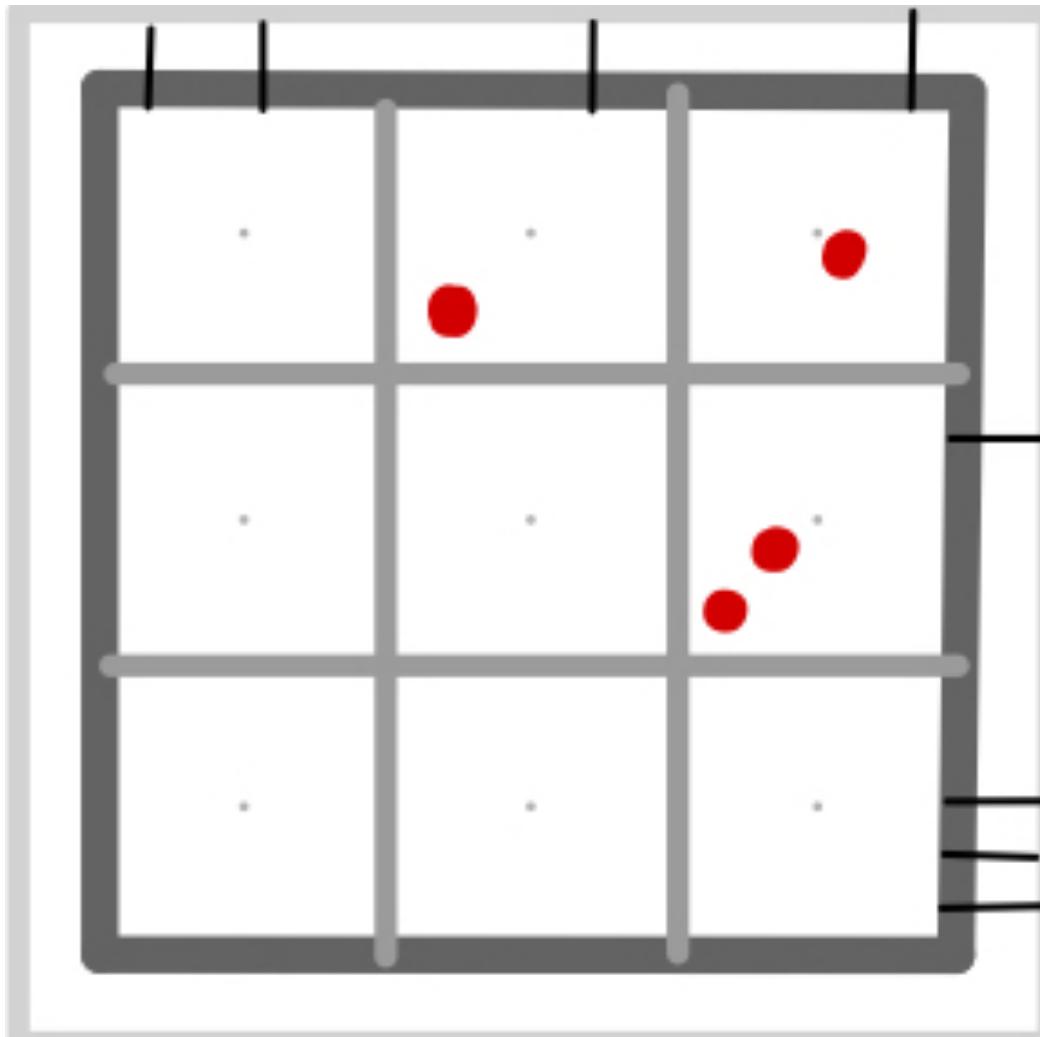


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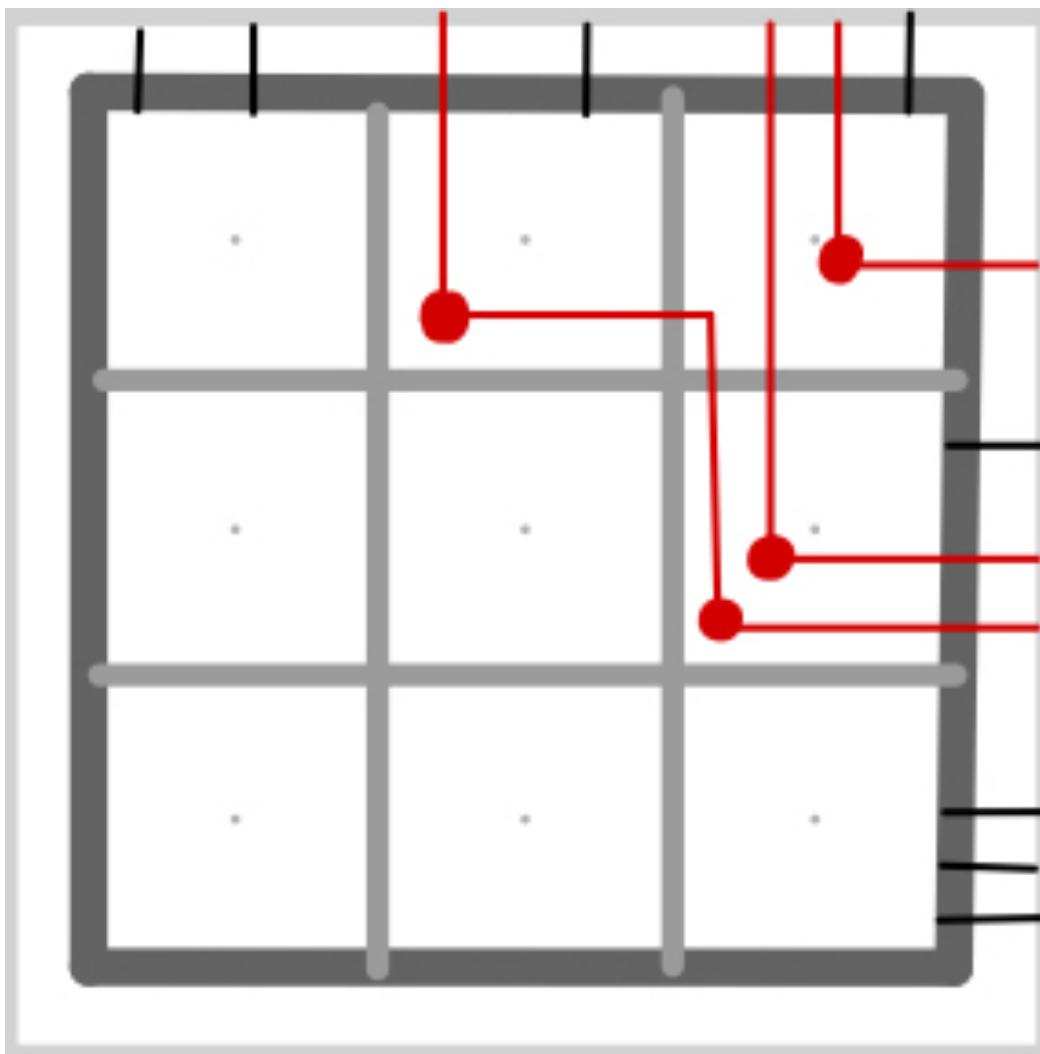


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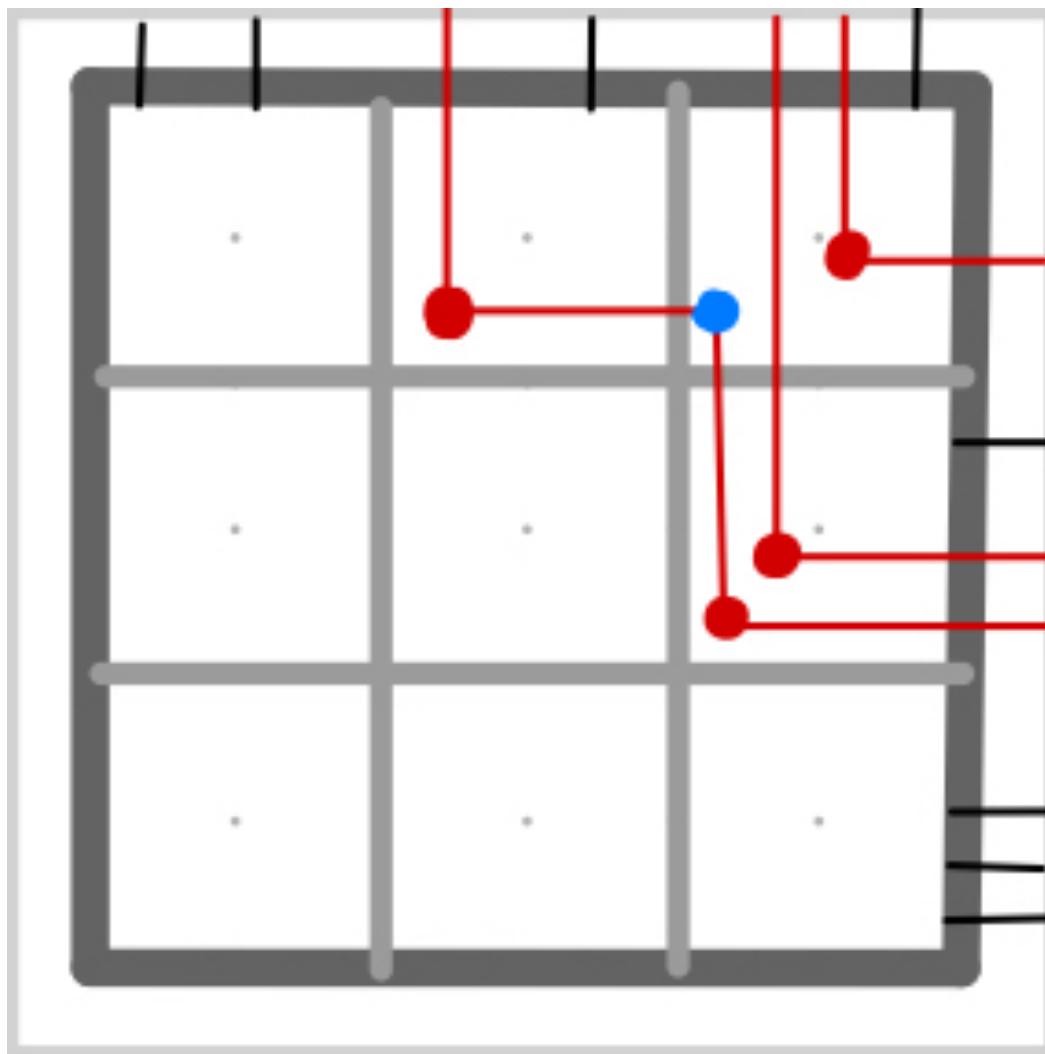


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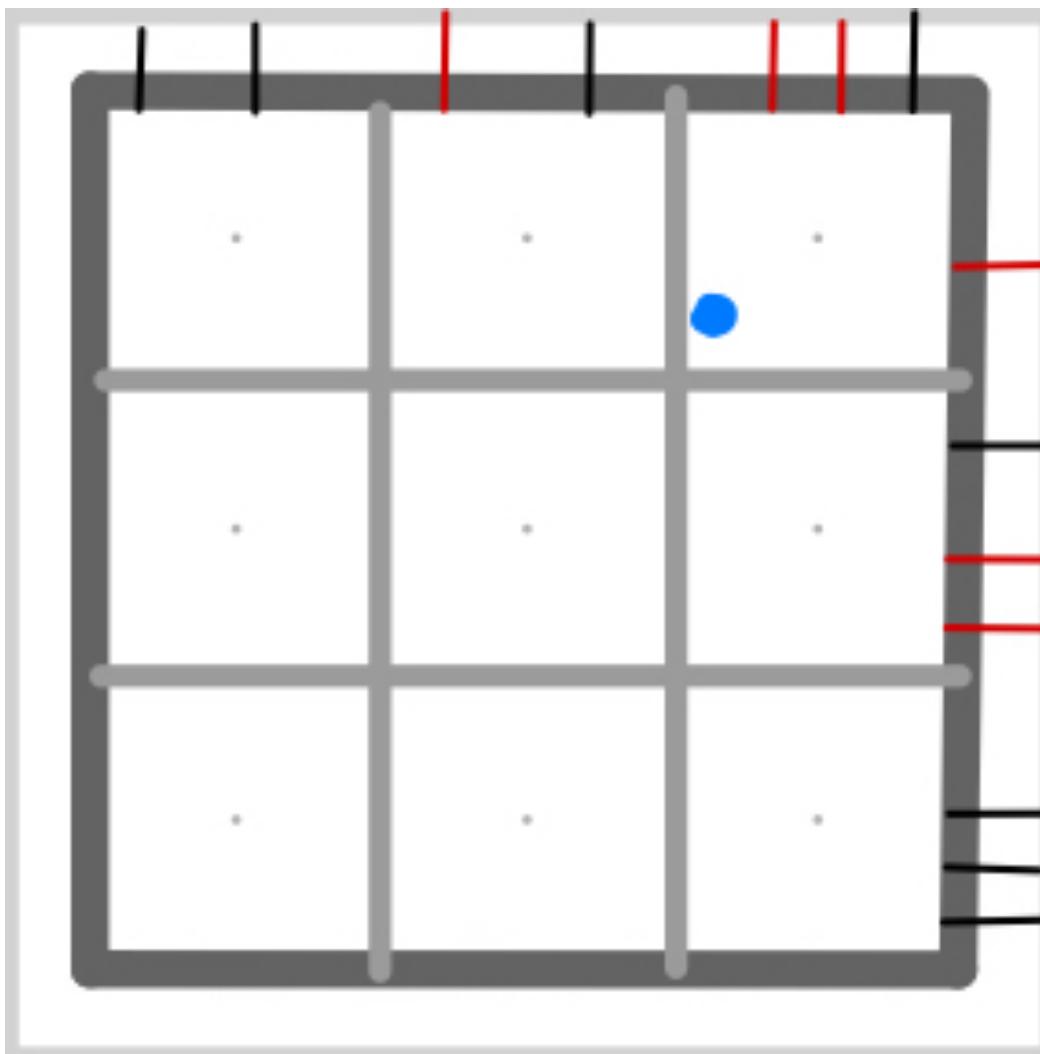


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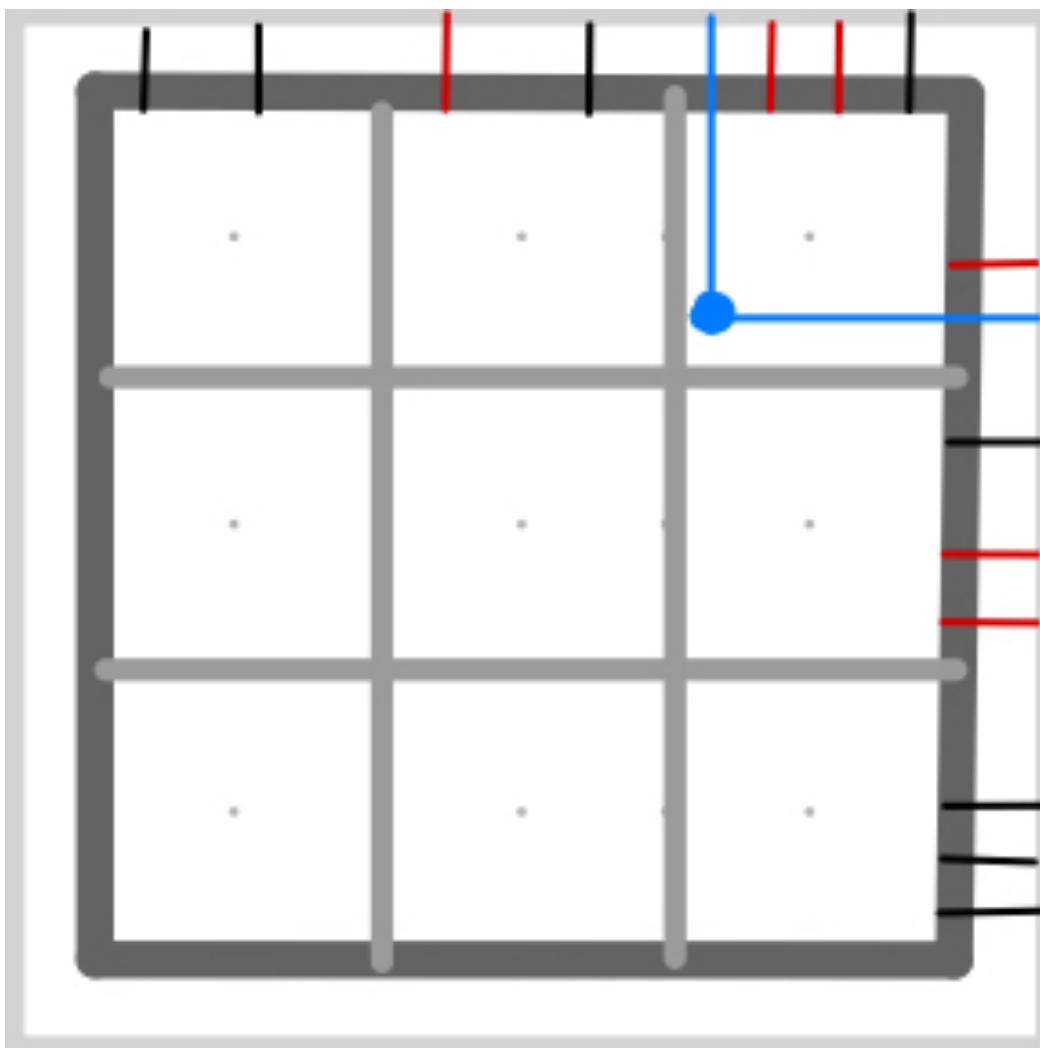


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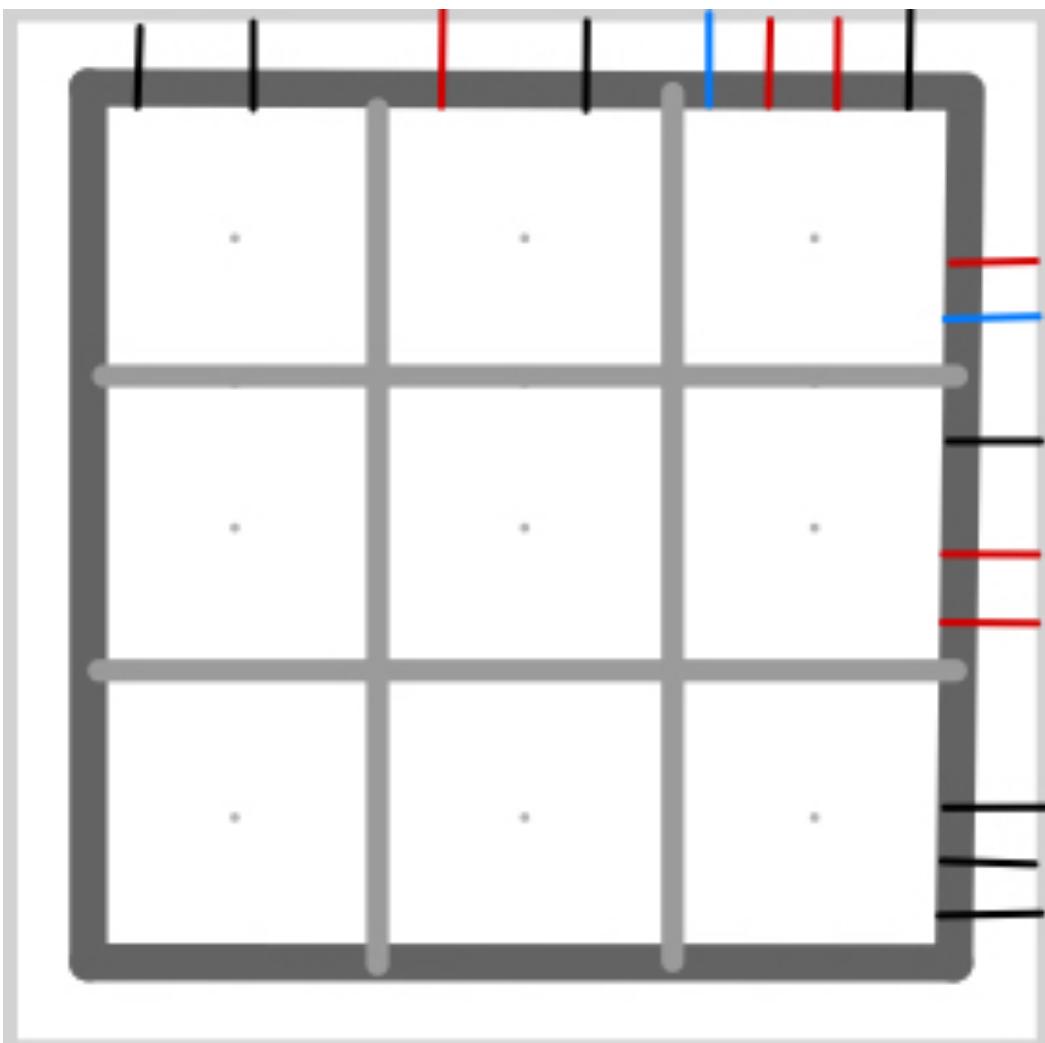


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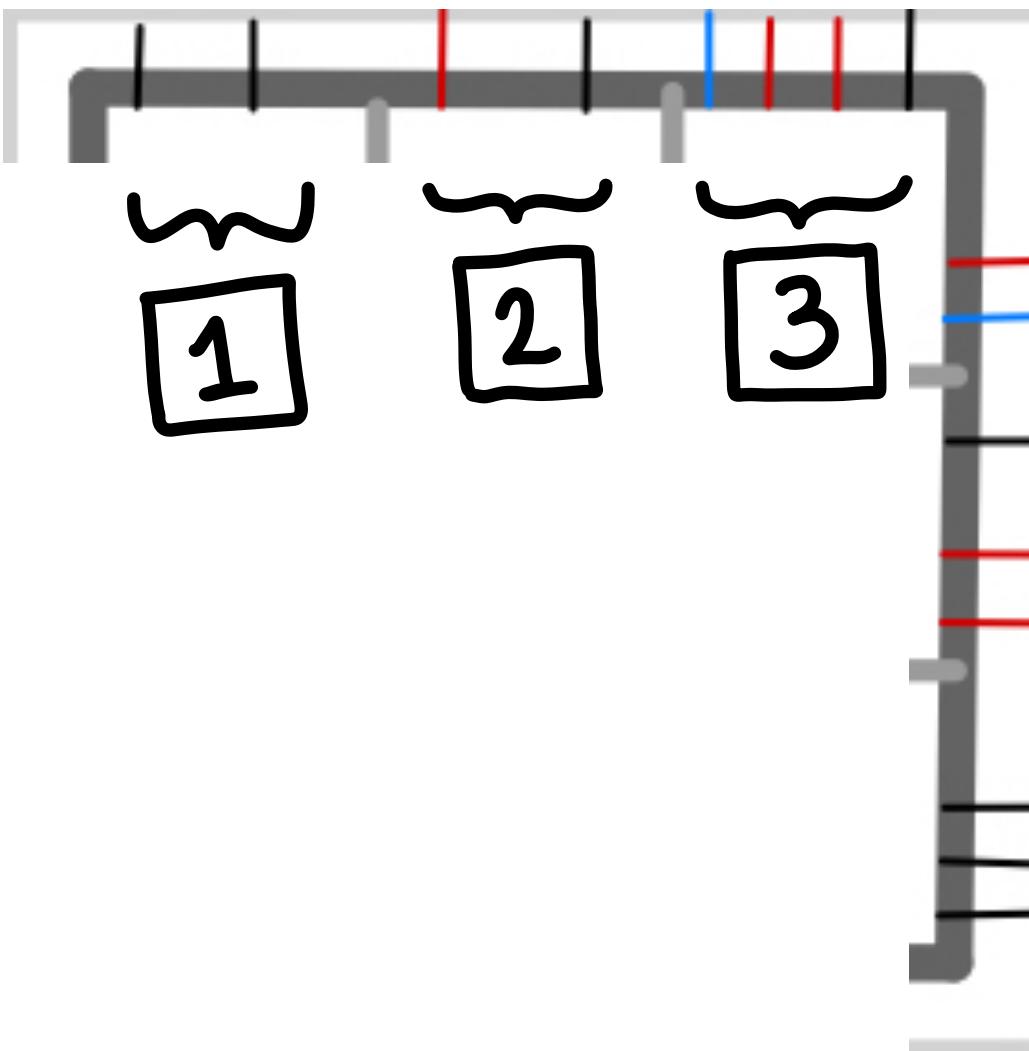


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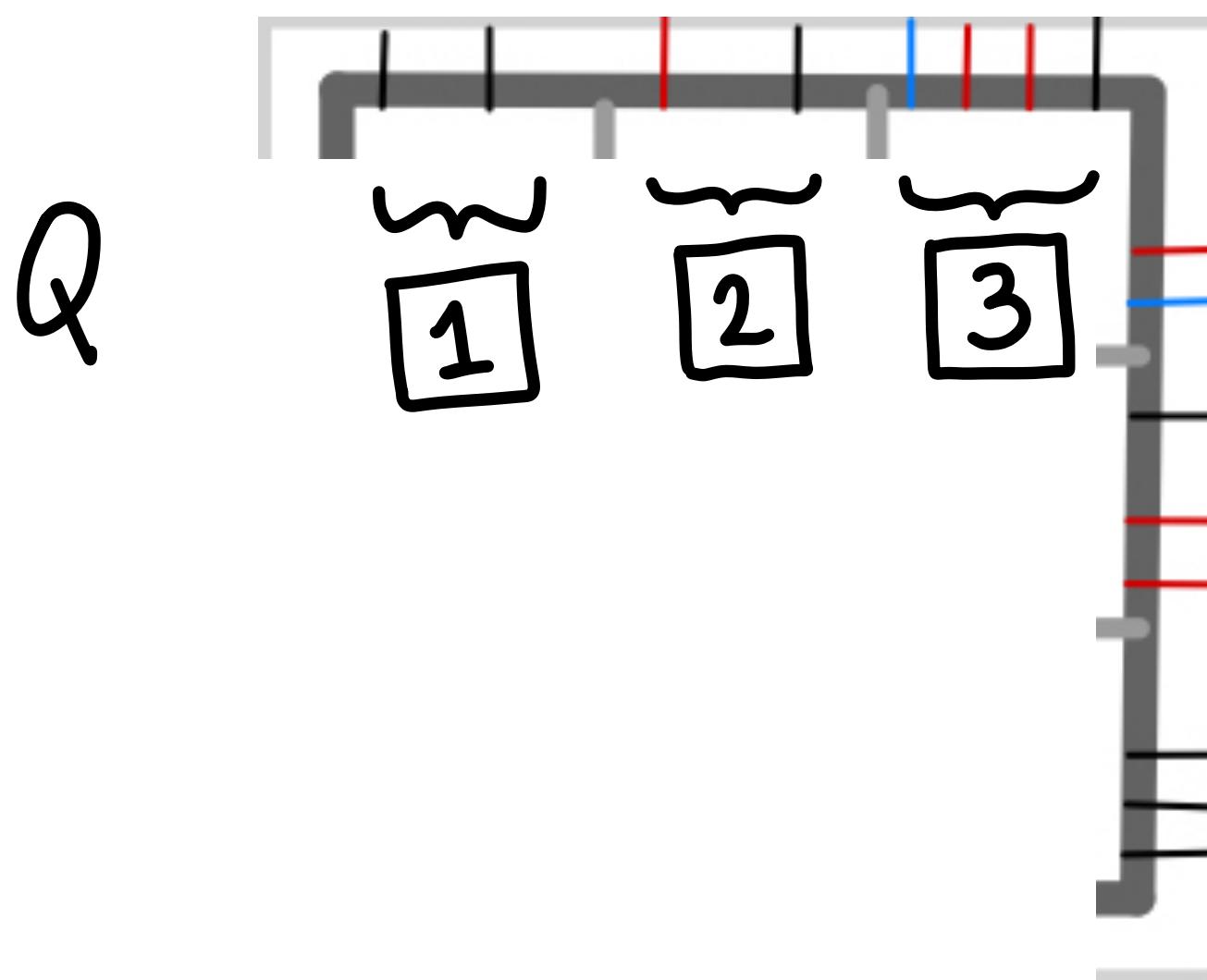
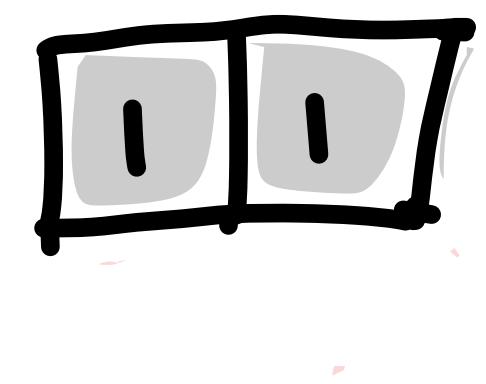
— 1st row
— 2nd row
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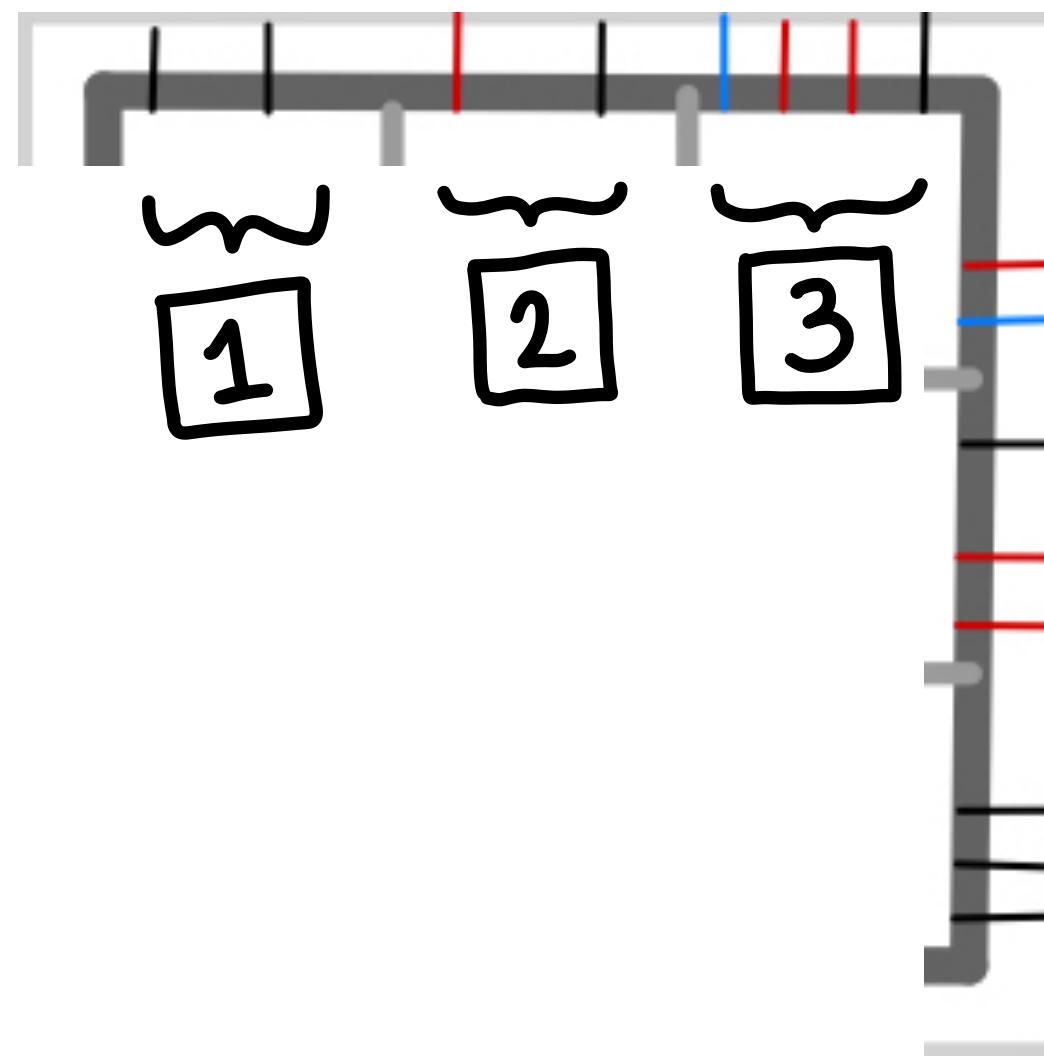
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0	0	2
2		



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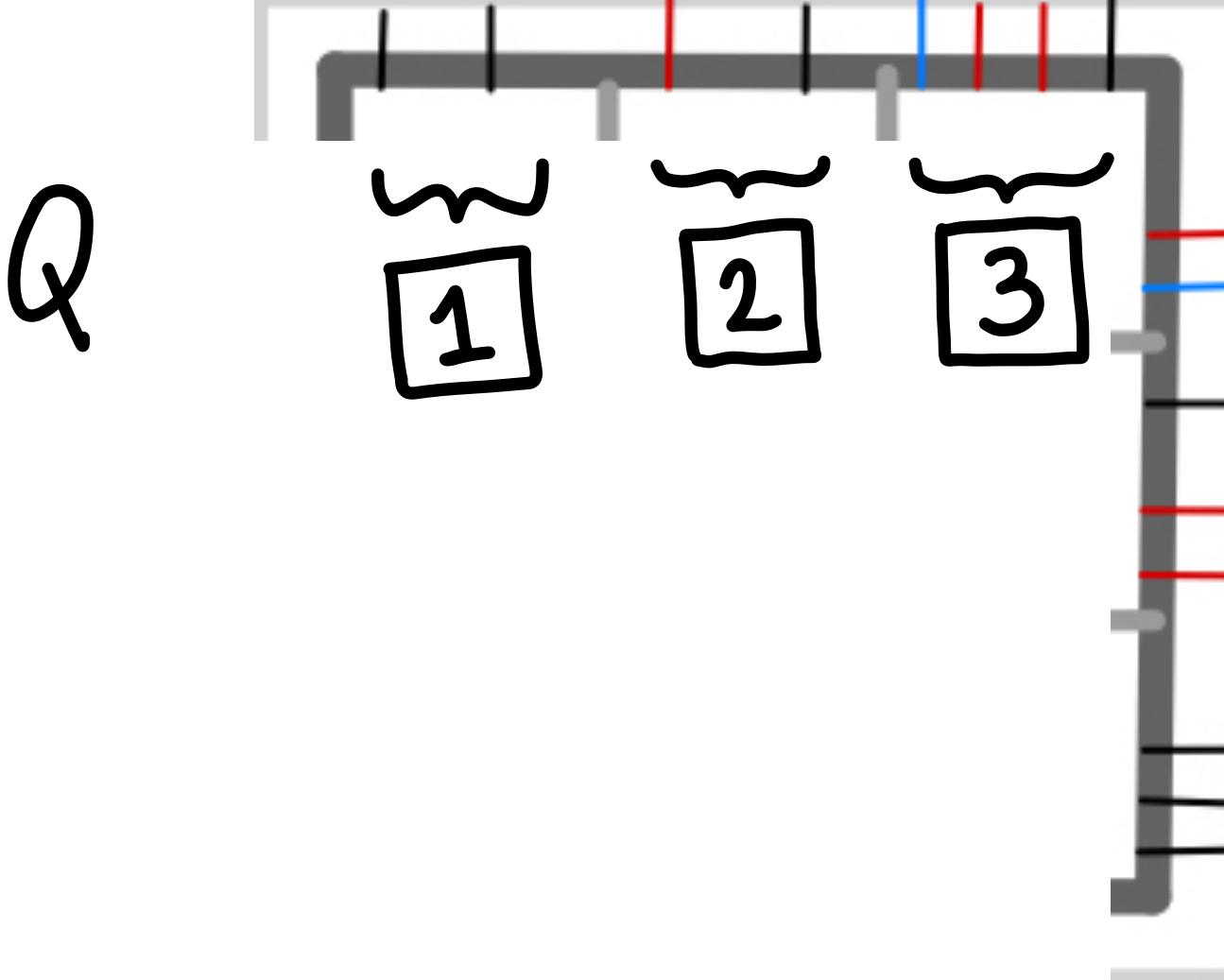
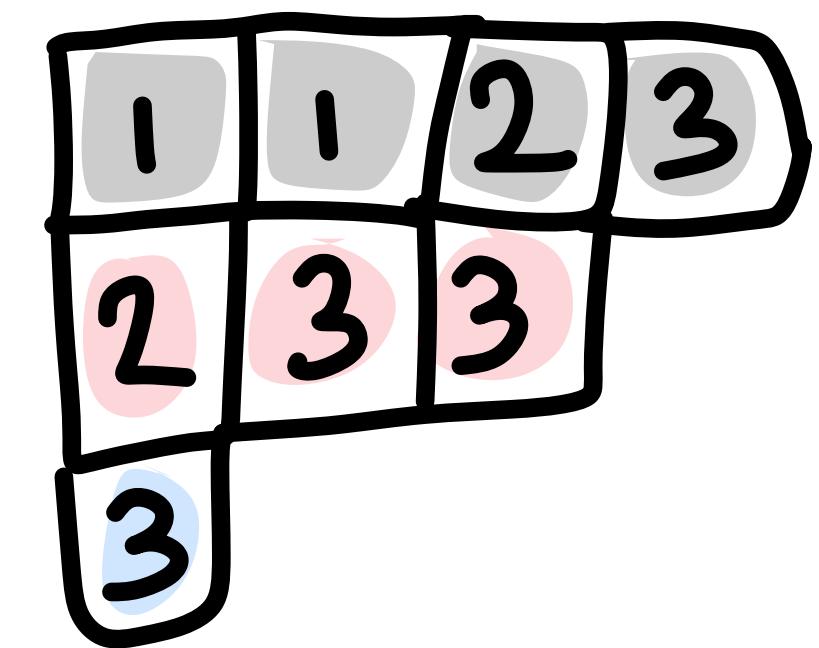
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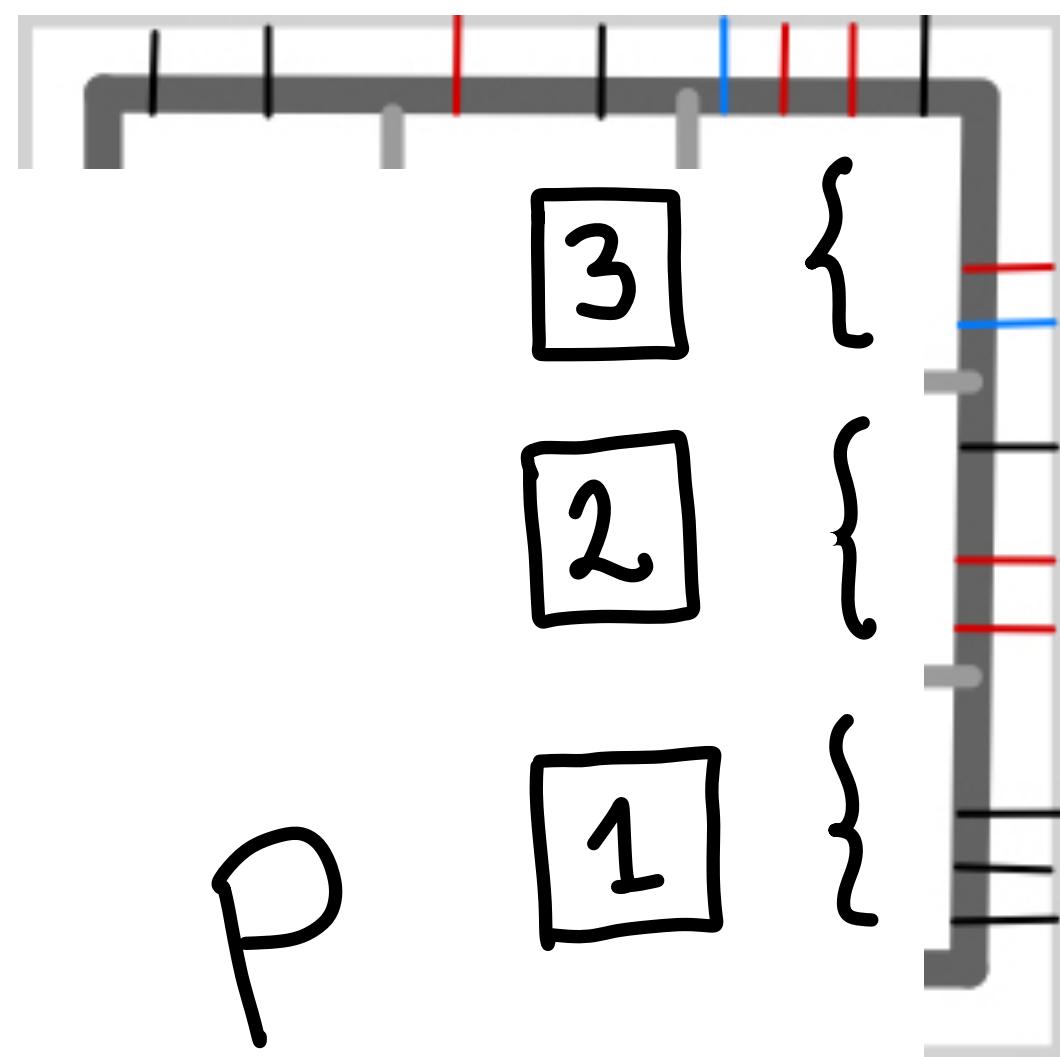
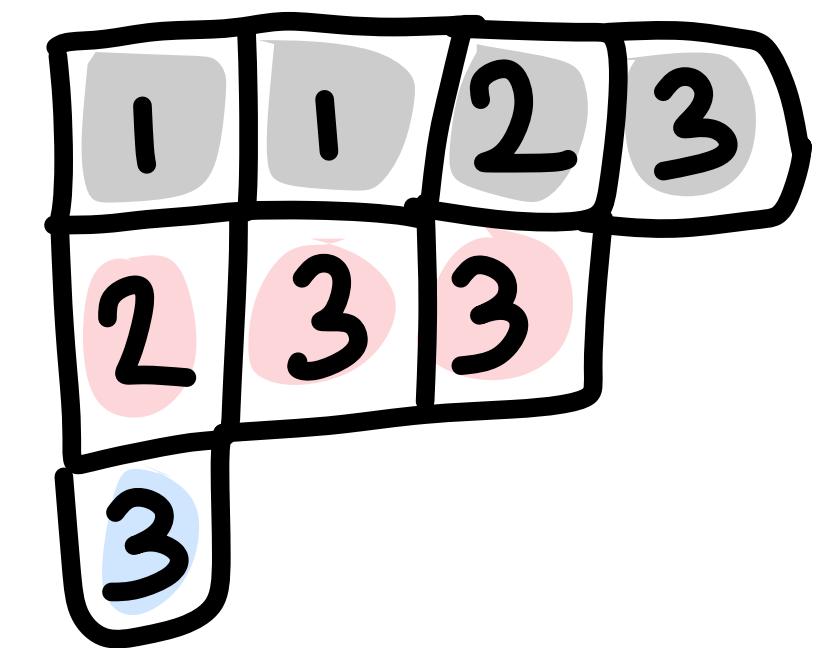
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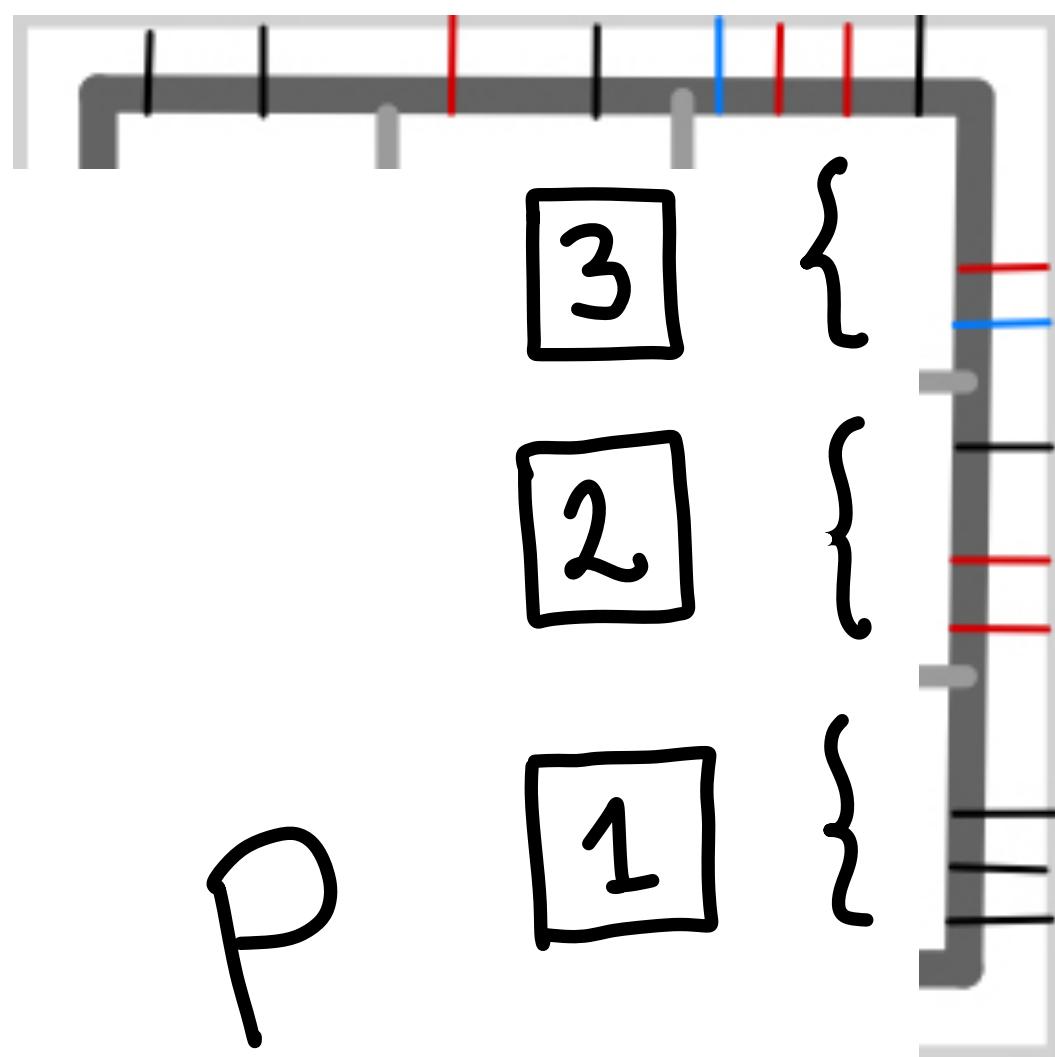
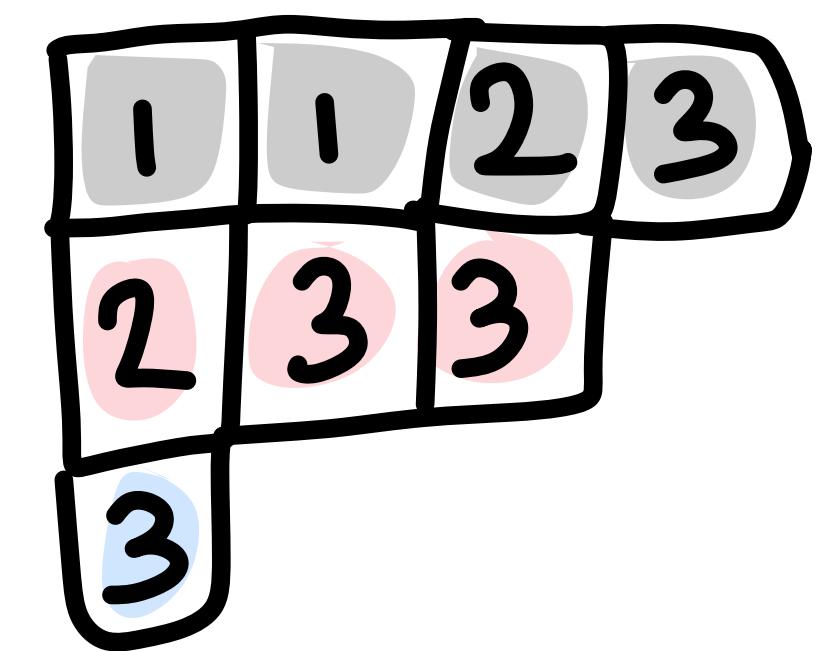
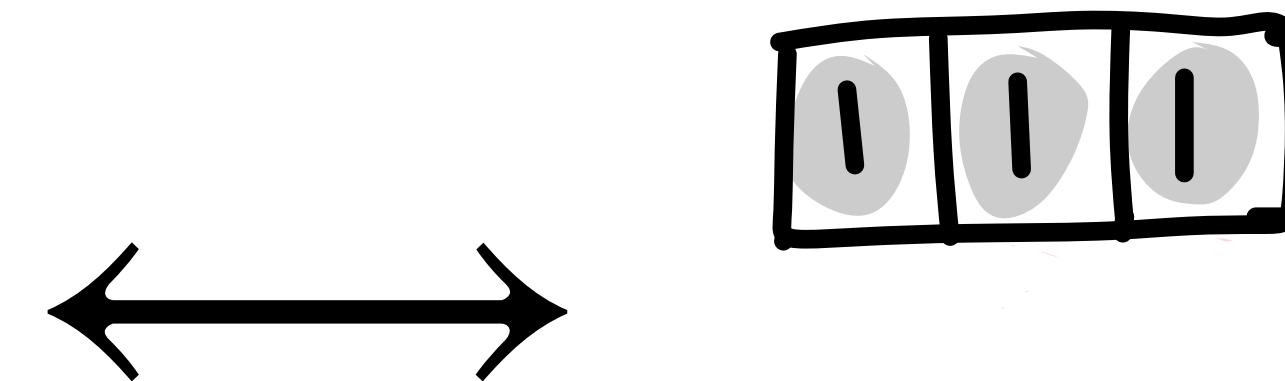
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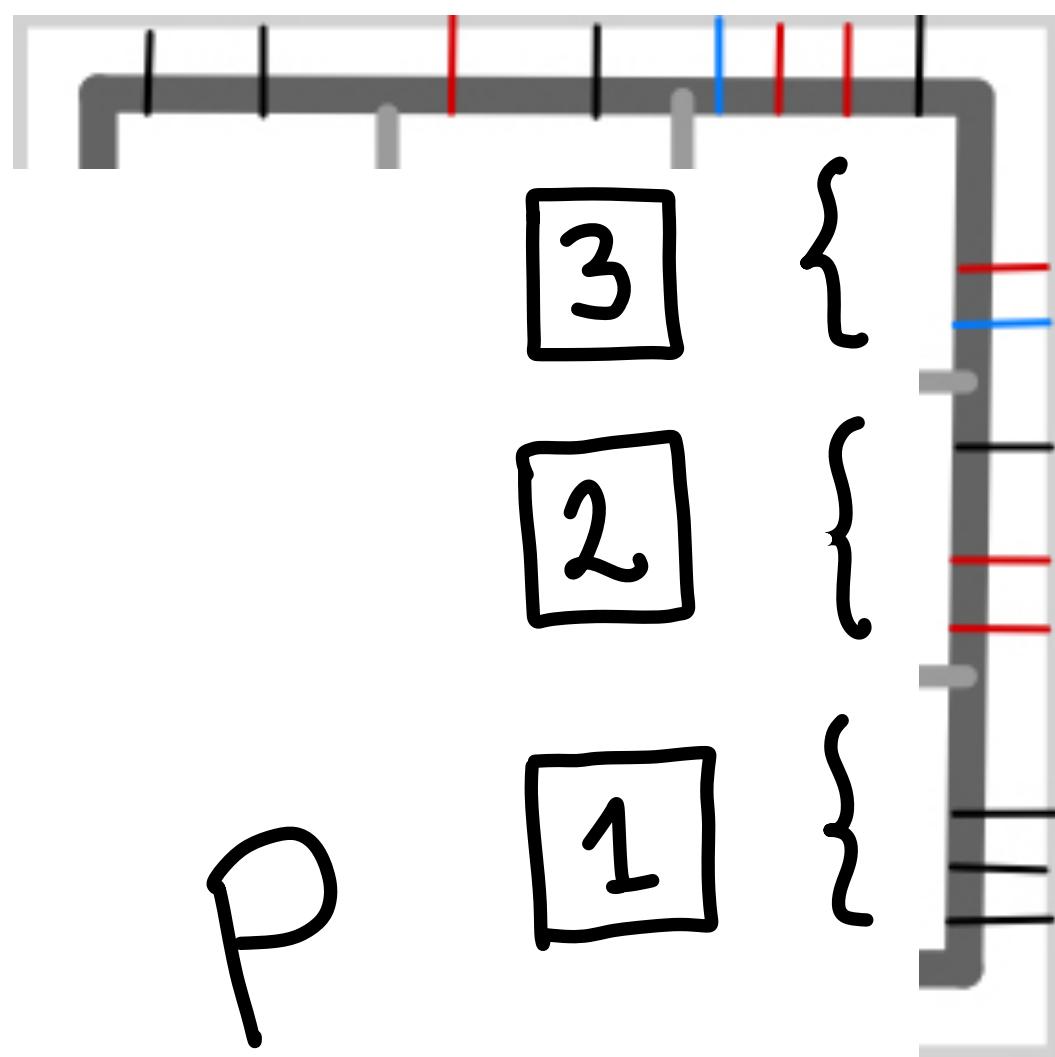
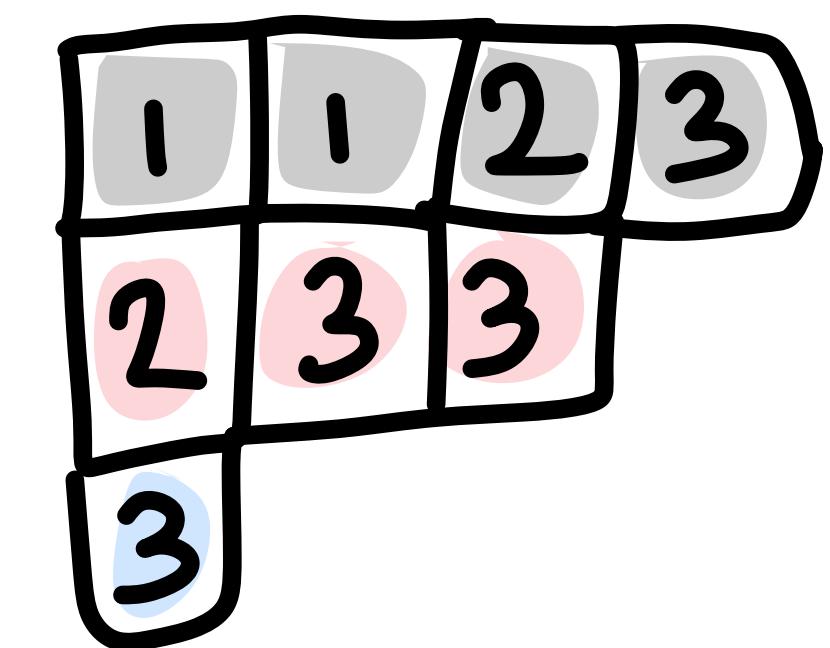
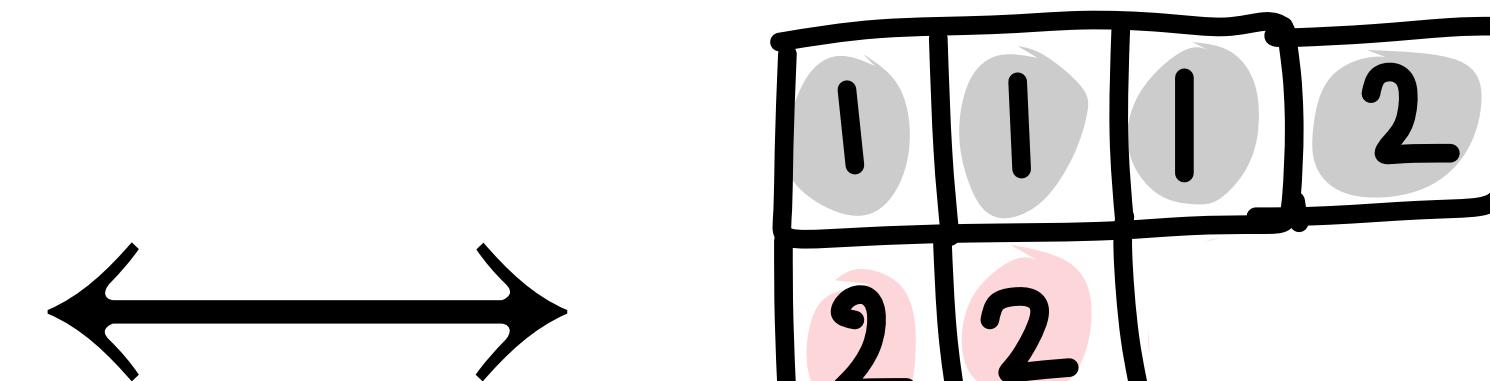
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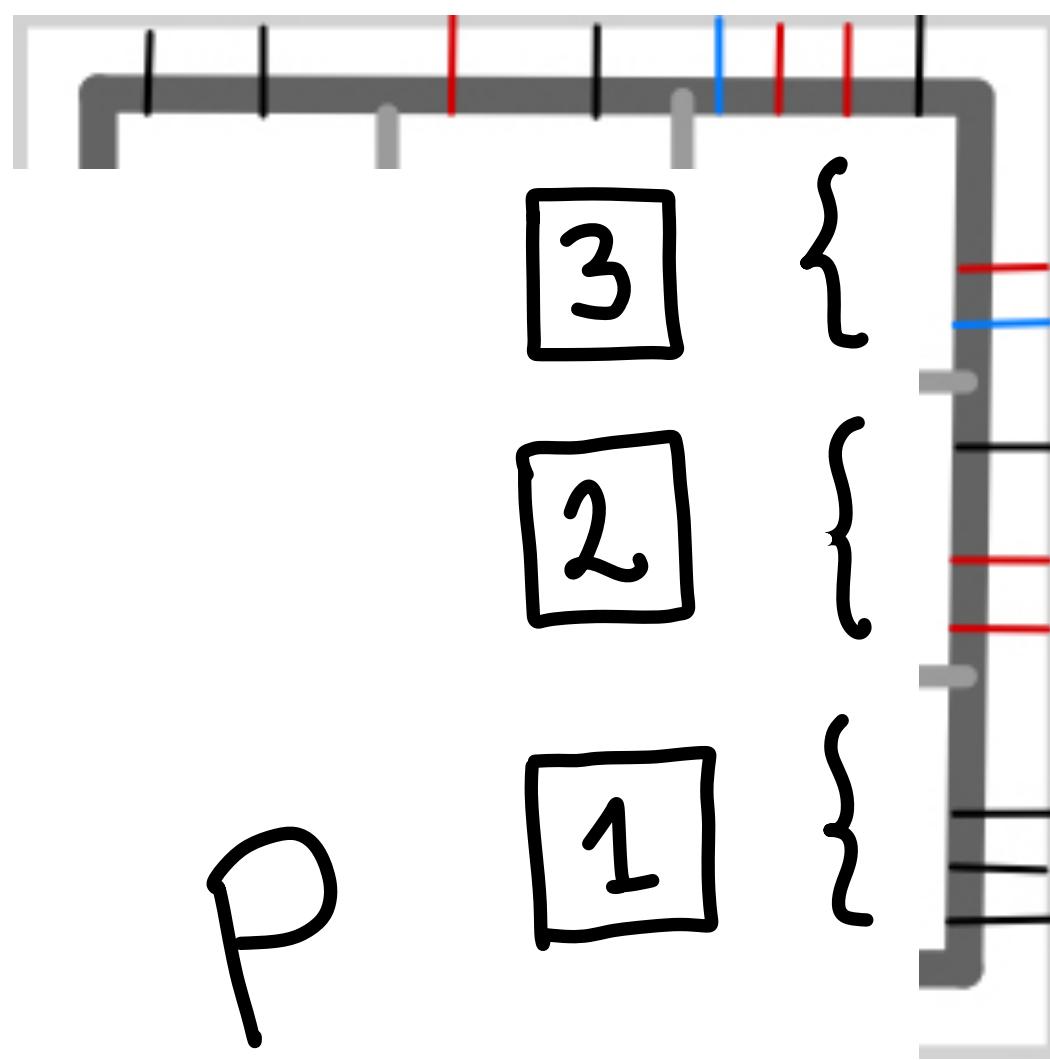
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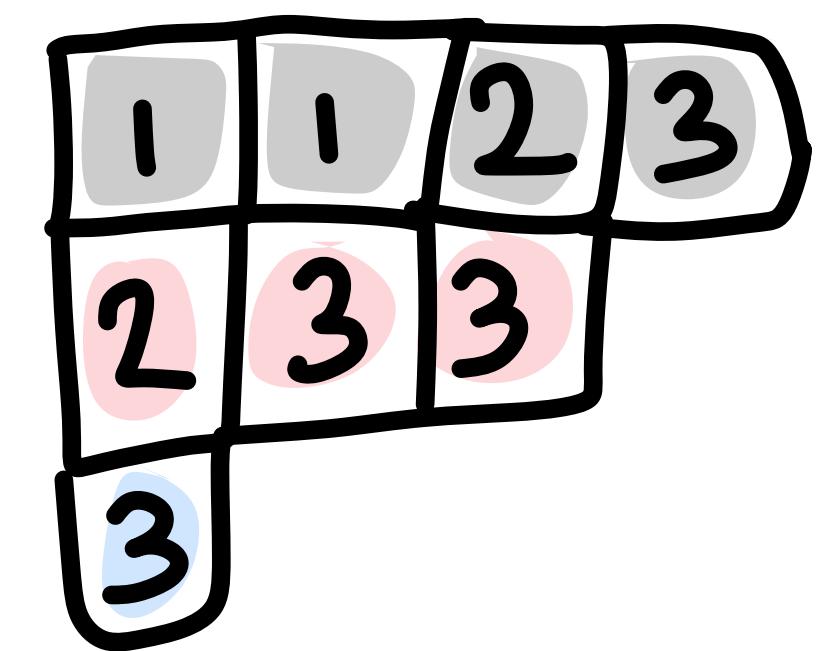
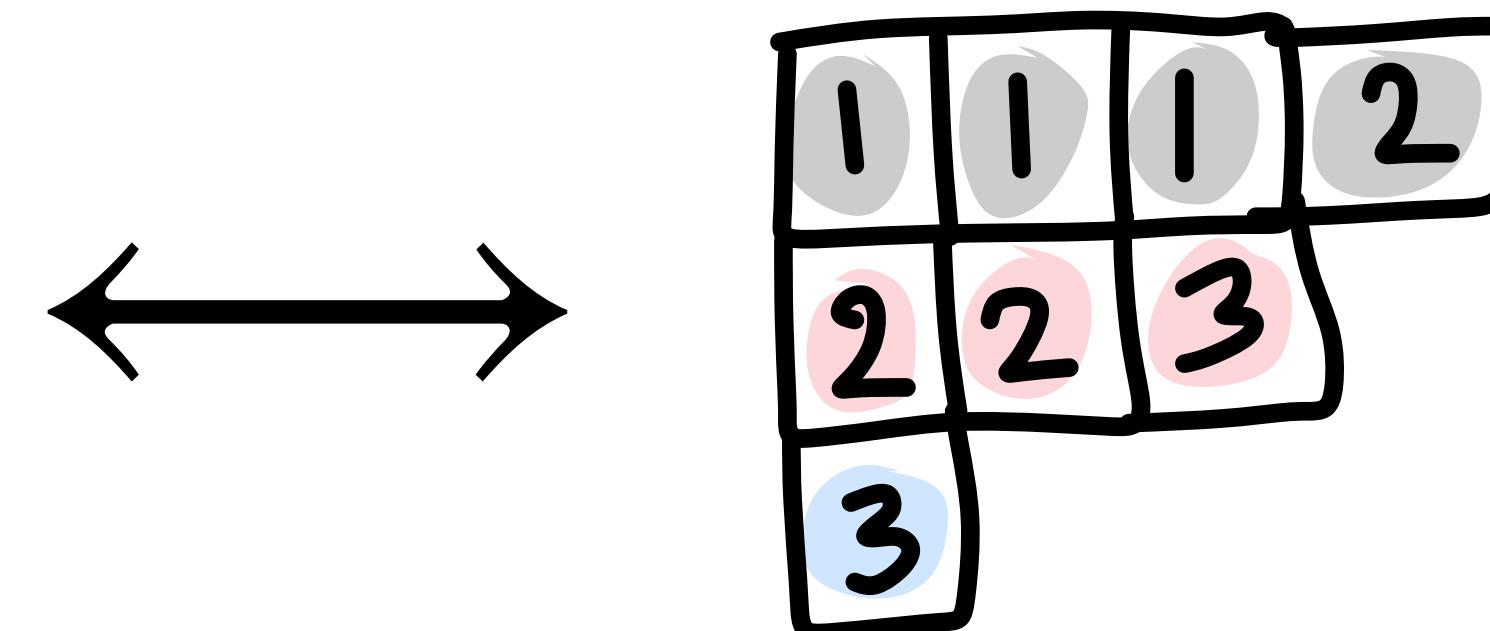
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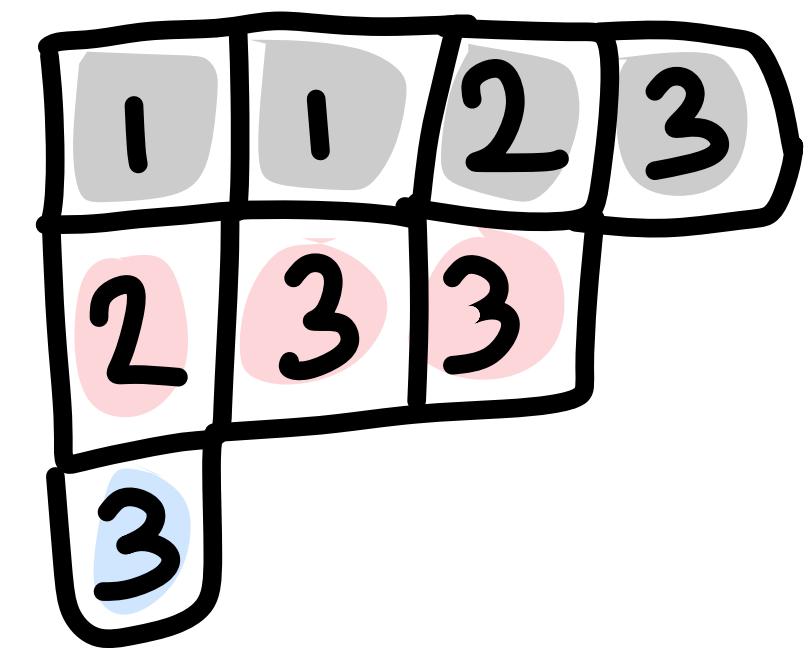
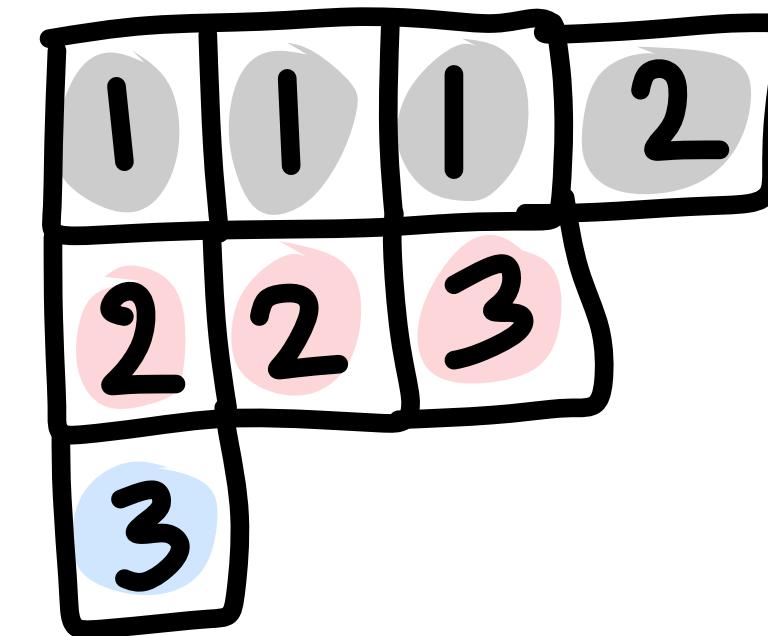
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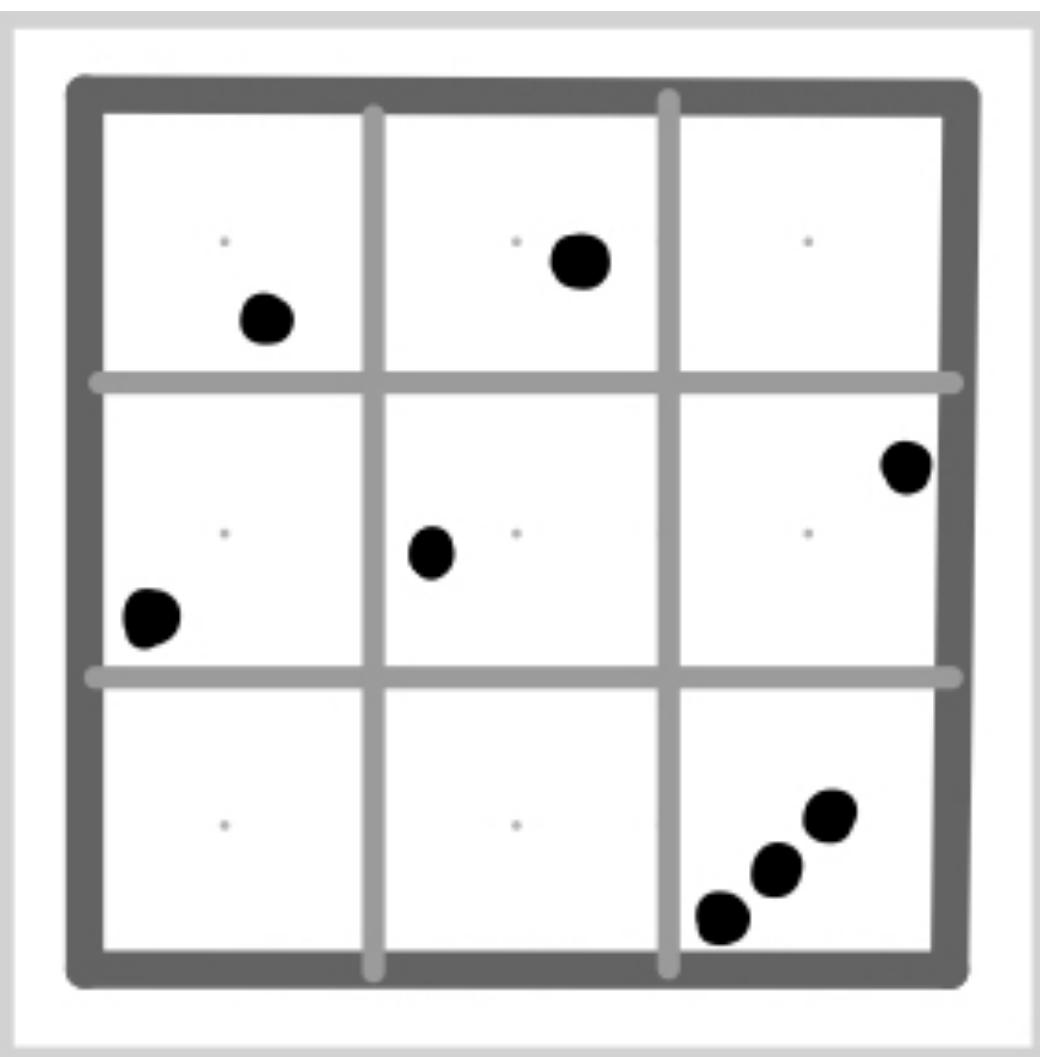
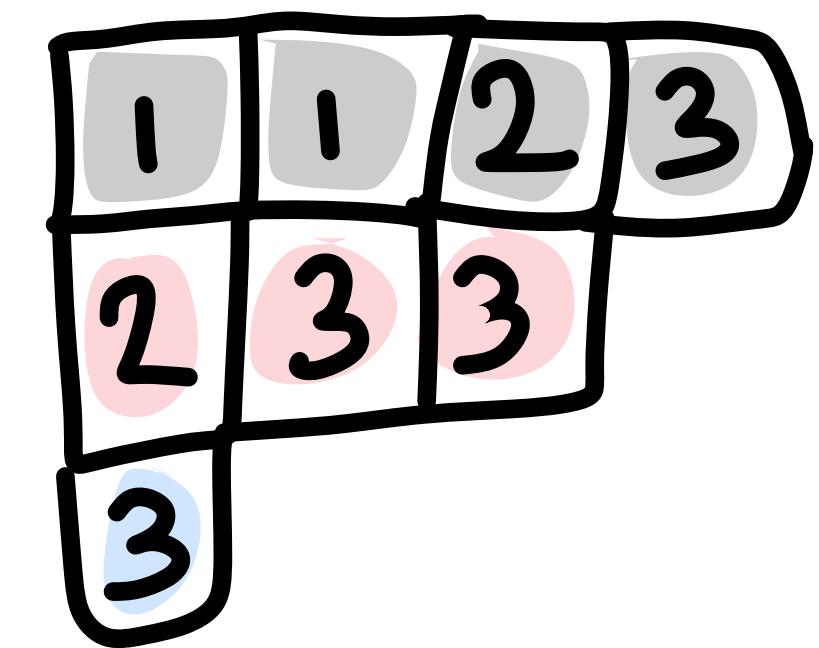
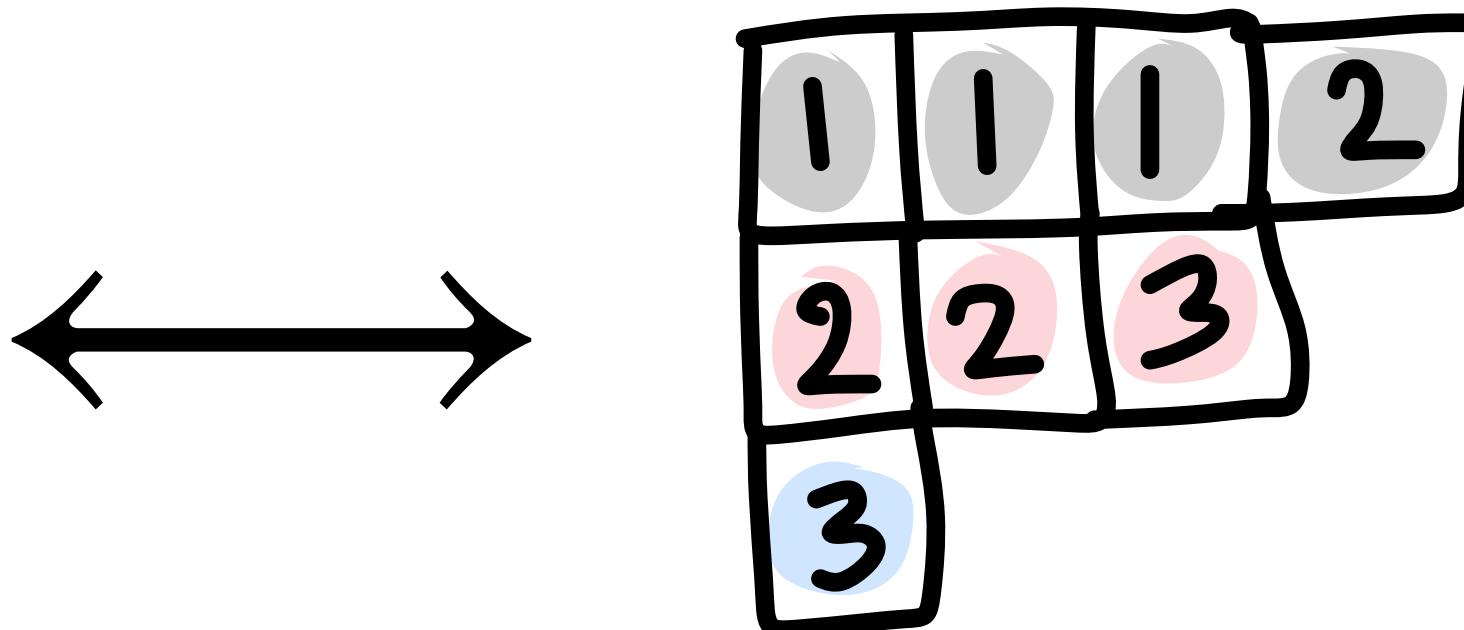
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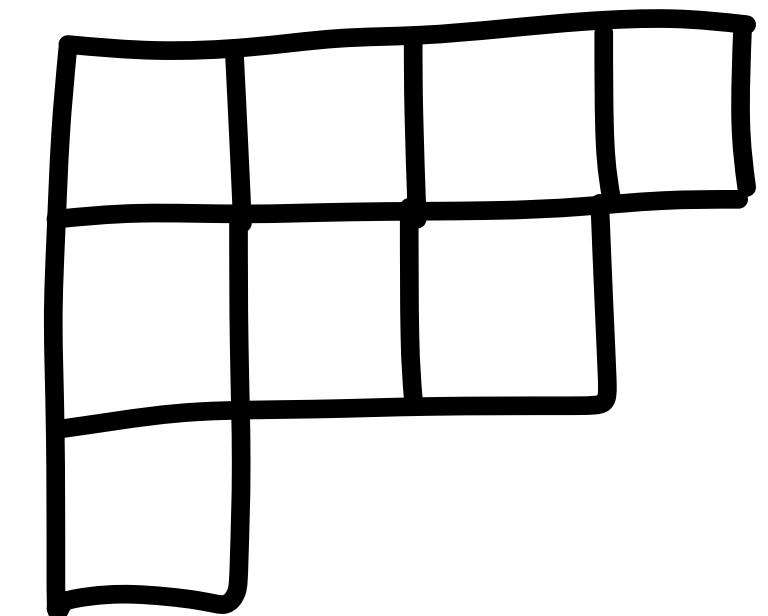
$$\left(M_{i,j} \right)_{i,j=1}^n \longleftrightarrow (P, Q)$$

Example

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$$\lambda =$$

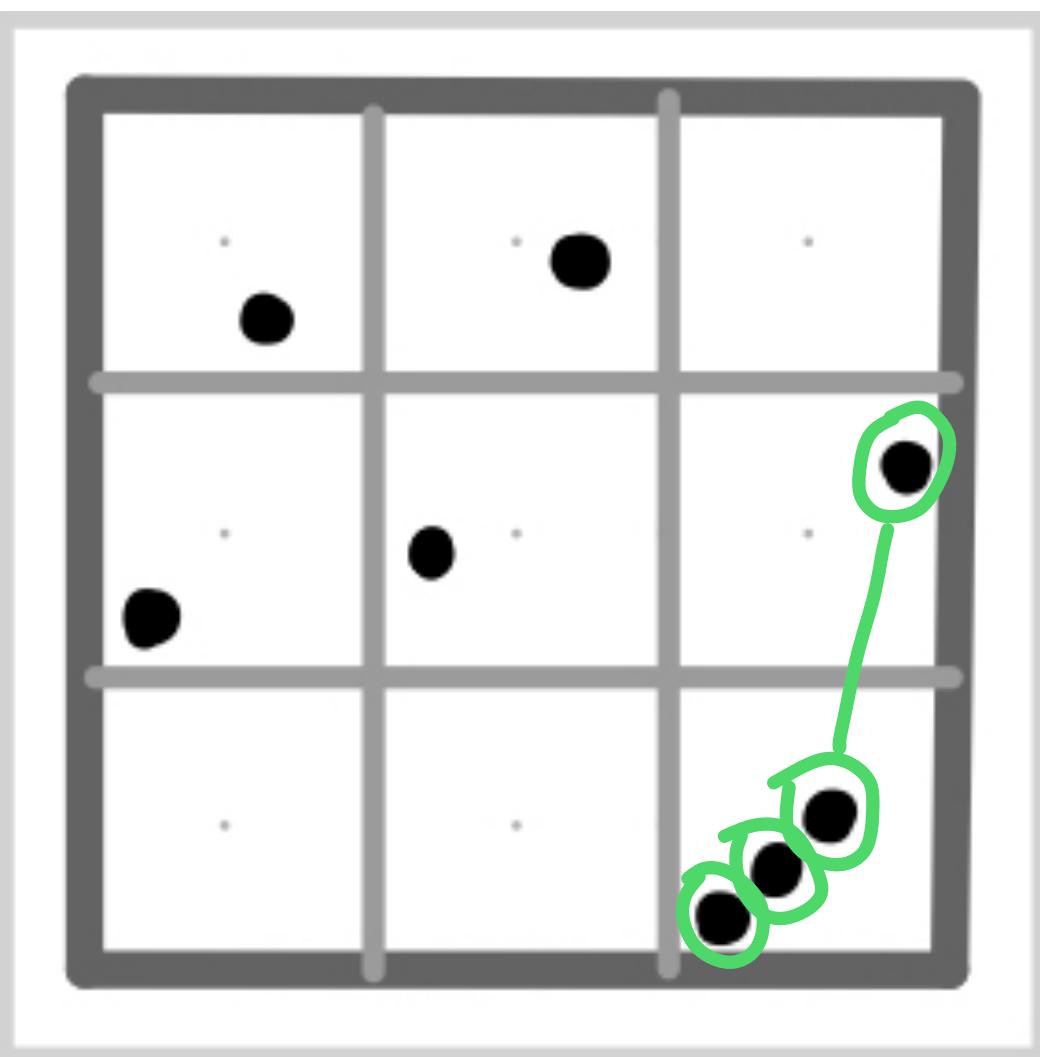
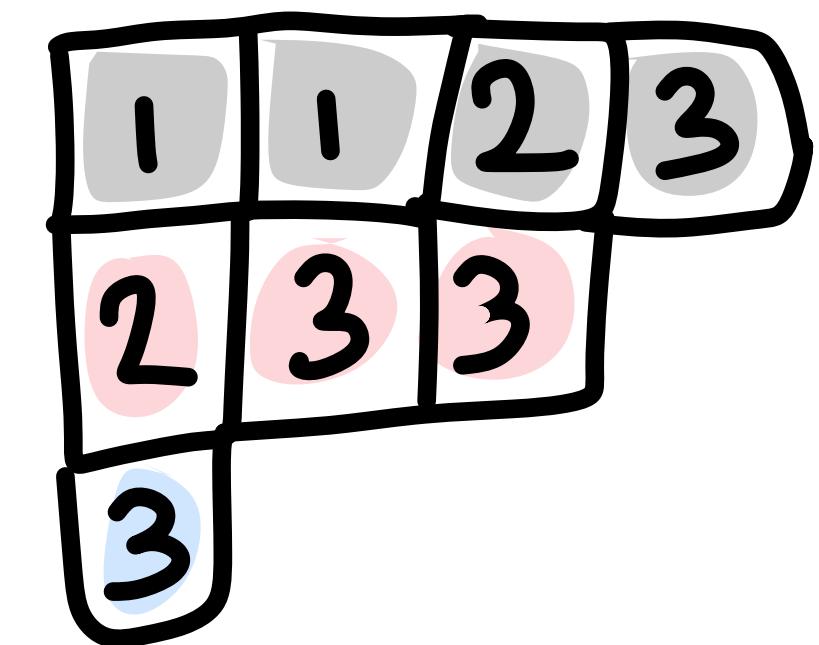
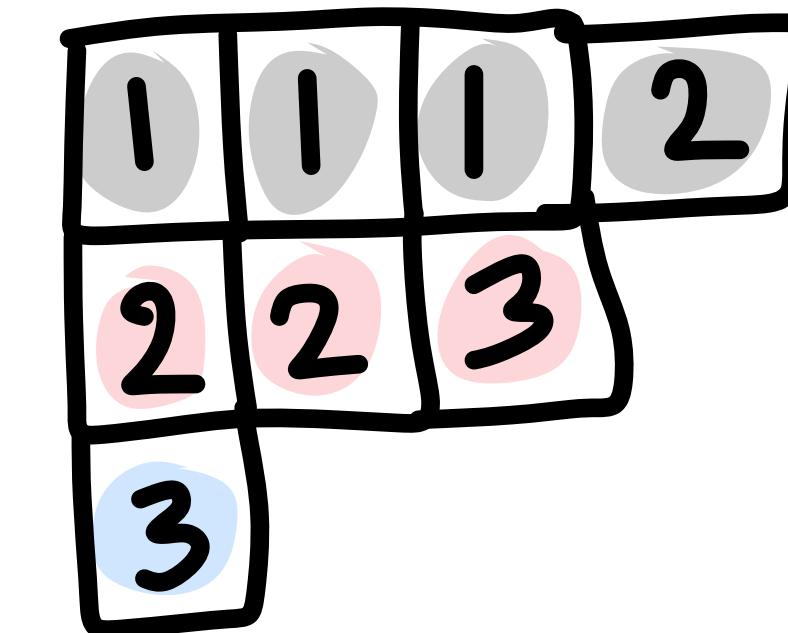


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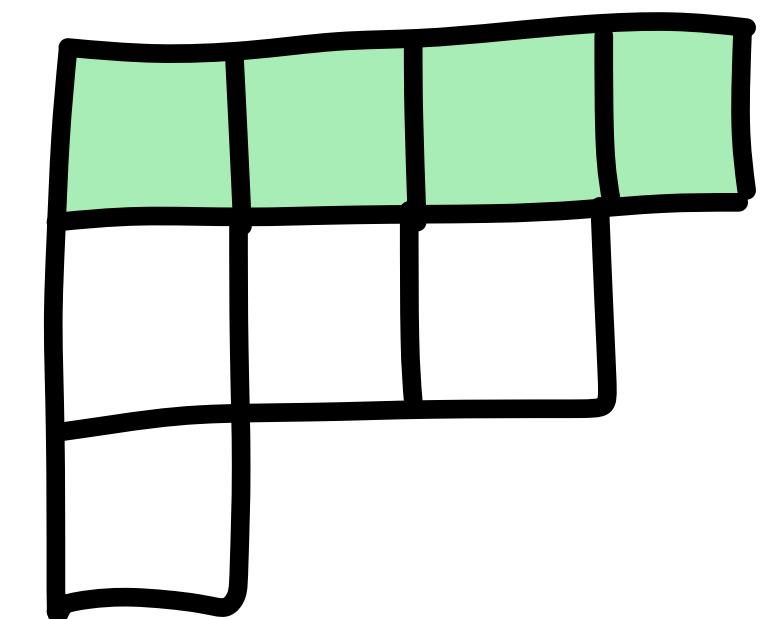
Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow$$



$$\lambda_1 = \text{LIS}_1$$

$$\lambda =$$



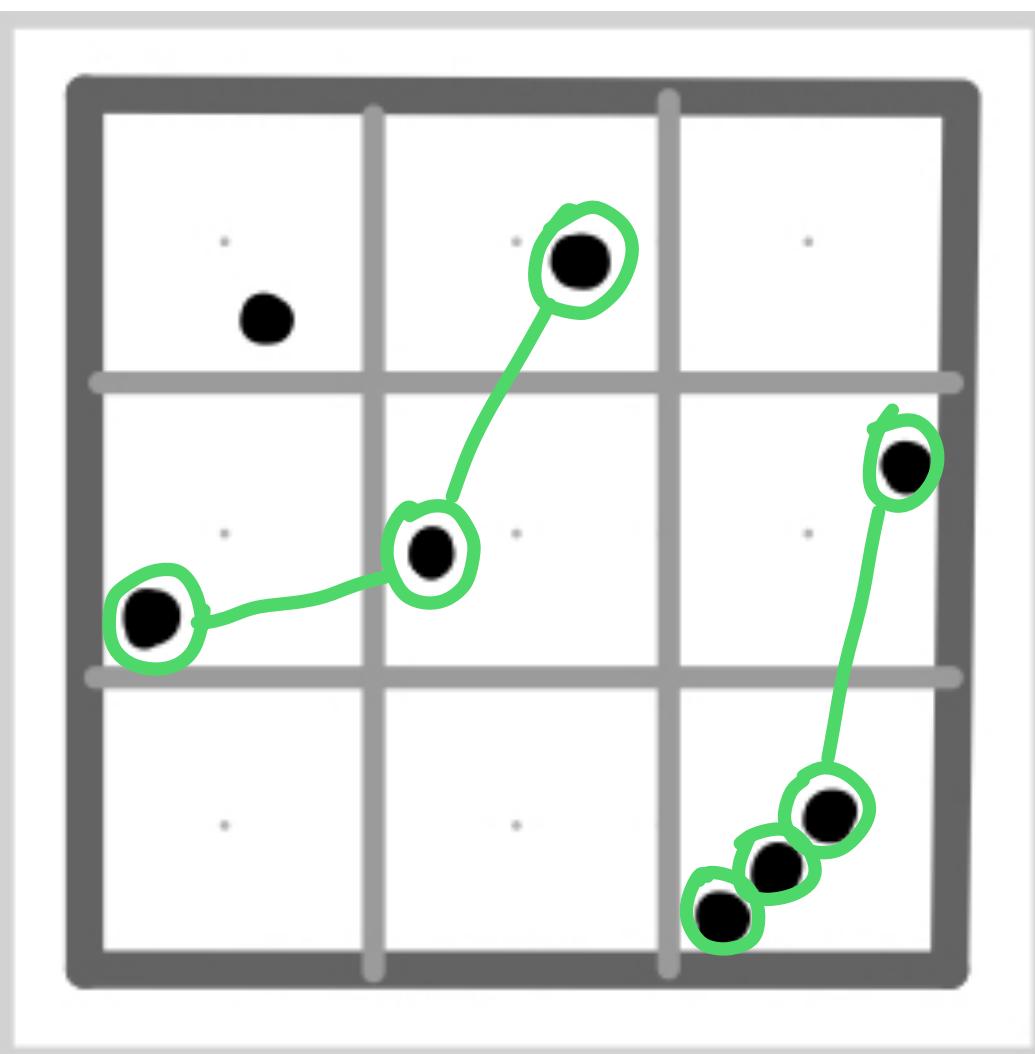
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Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow \begin{array}{c} \begin{matrix} 0 & 0 & 0 & 2 \\ 2 & 2 & 3 \\ 3 \end{matrix} \\ \begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 \end{matrix} \end{array}$$

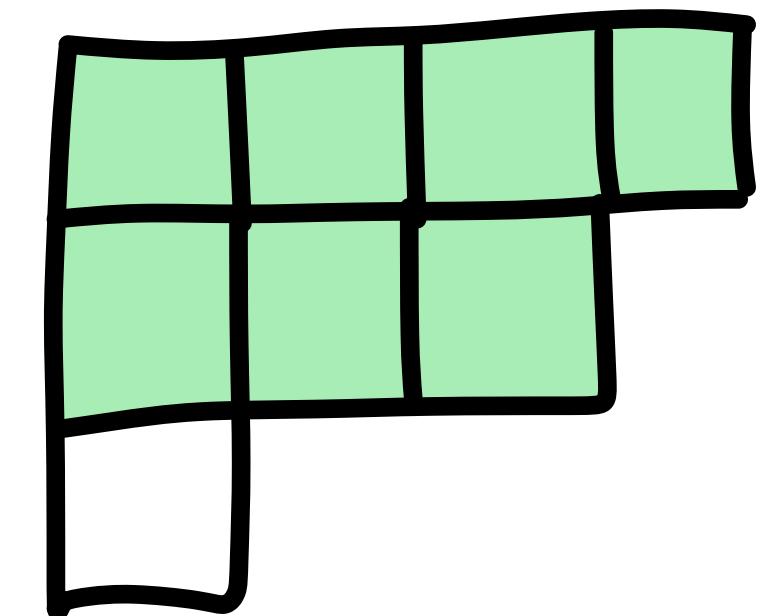
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$$\lambda_1 = \text{LIS}_1$$

$$\lambda_1 + \lambda_2 = \text{LIS}_2$$

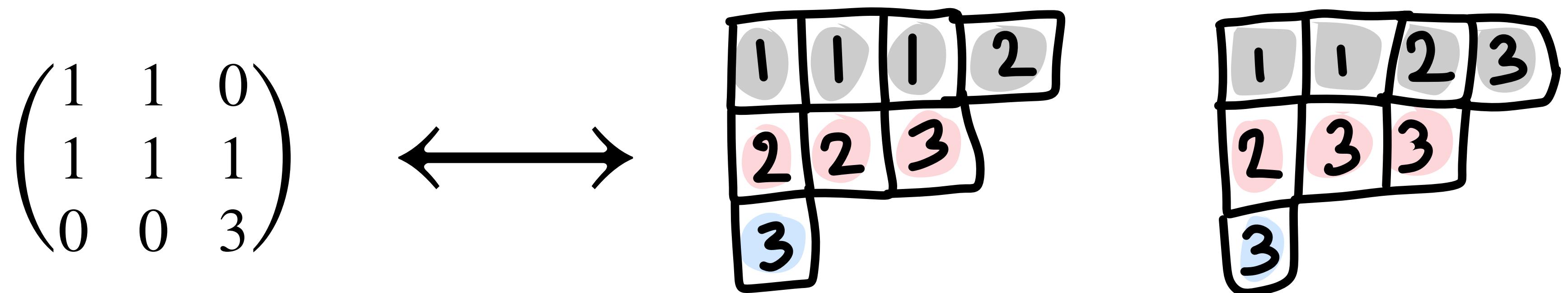
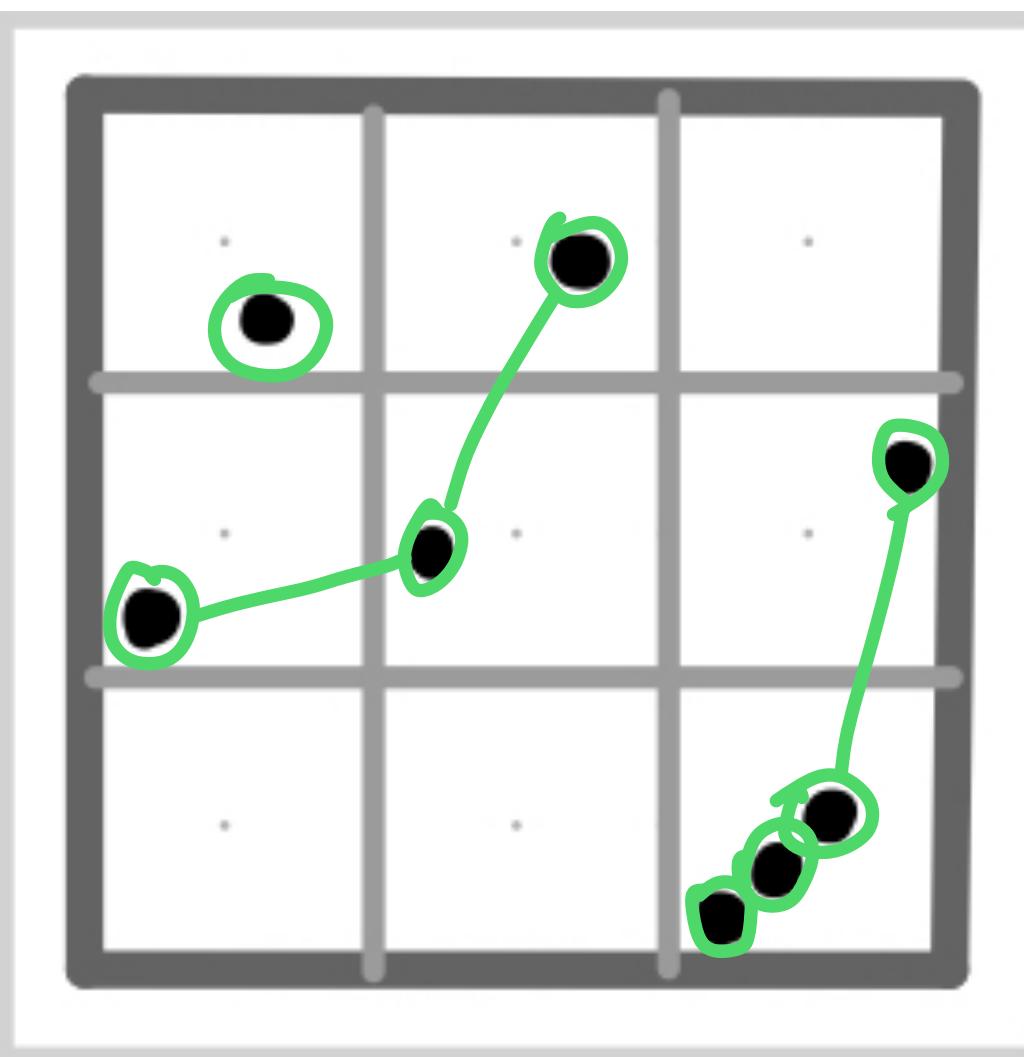
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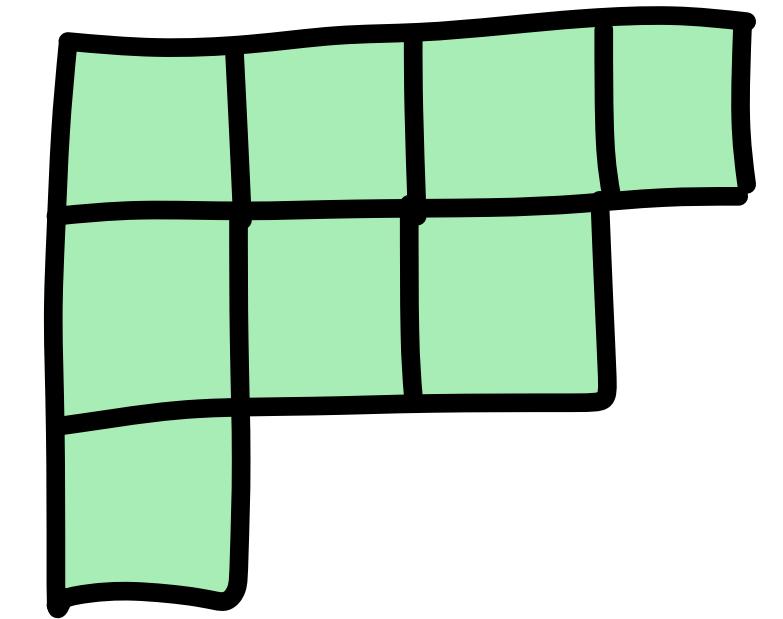
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Example



$$\begin{aligned} \lambda_1 &= \text{LIS}_1 \\ \lambda_1 + \lambda_2 &= \text{LIS}_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &= \text{LIS}_3 \end{aligned} \quad \lambda =$$

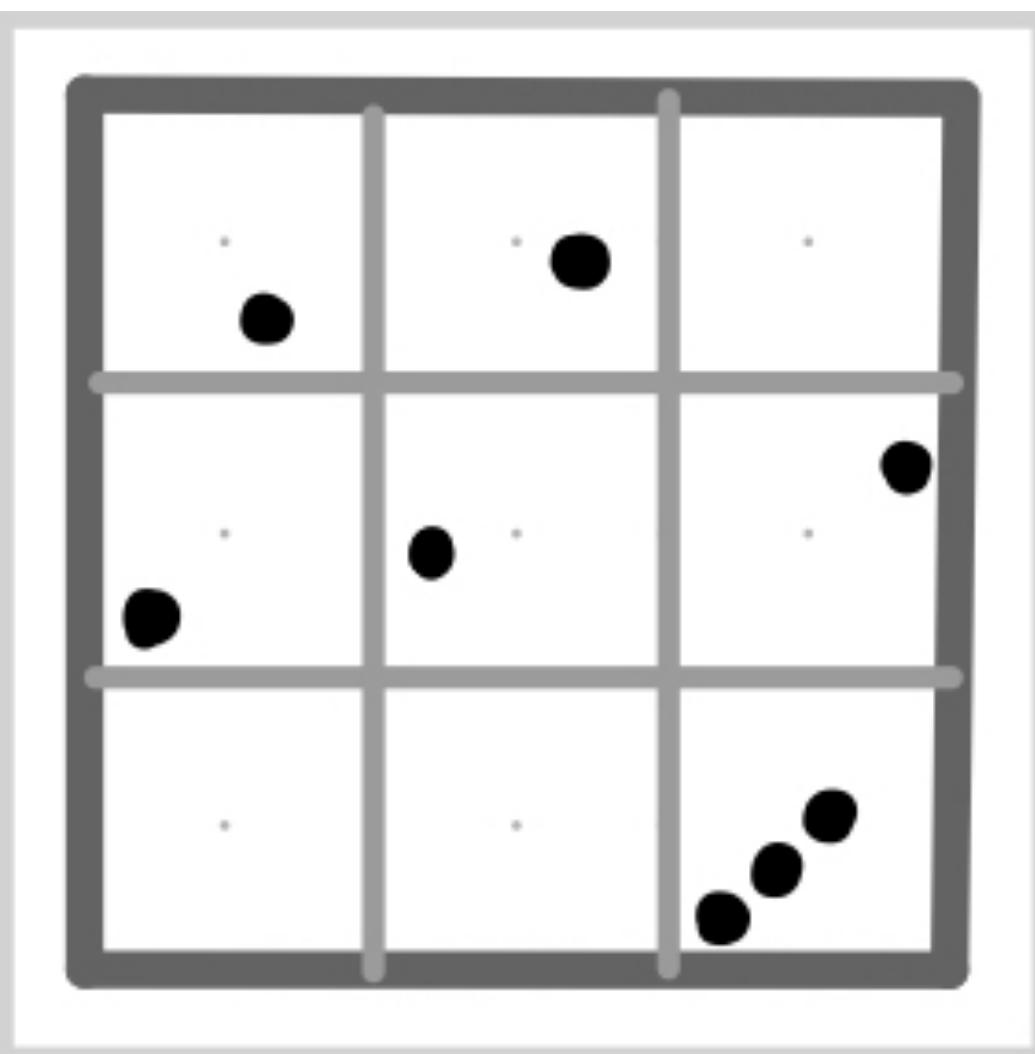
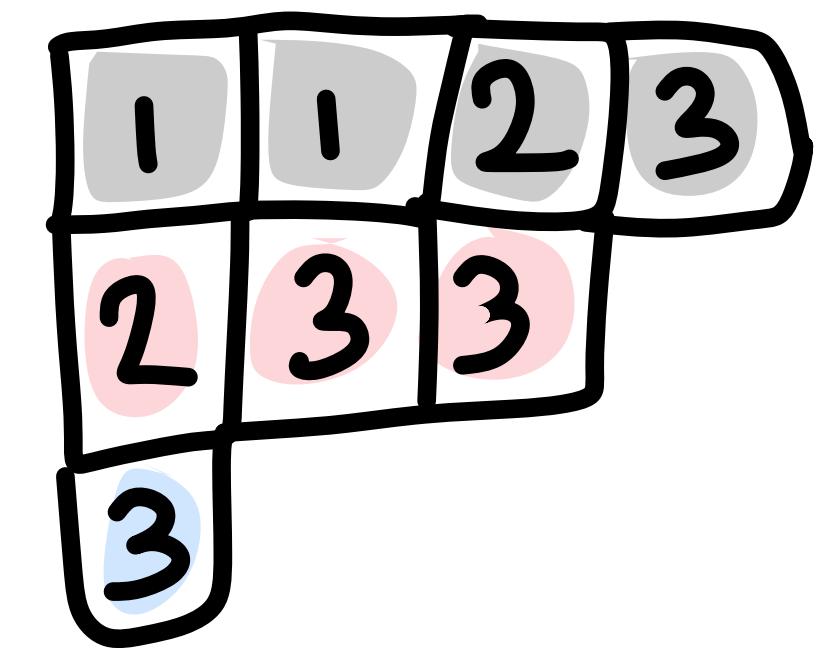
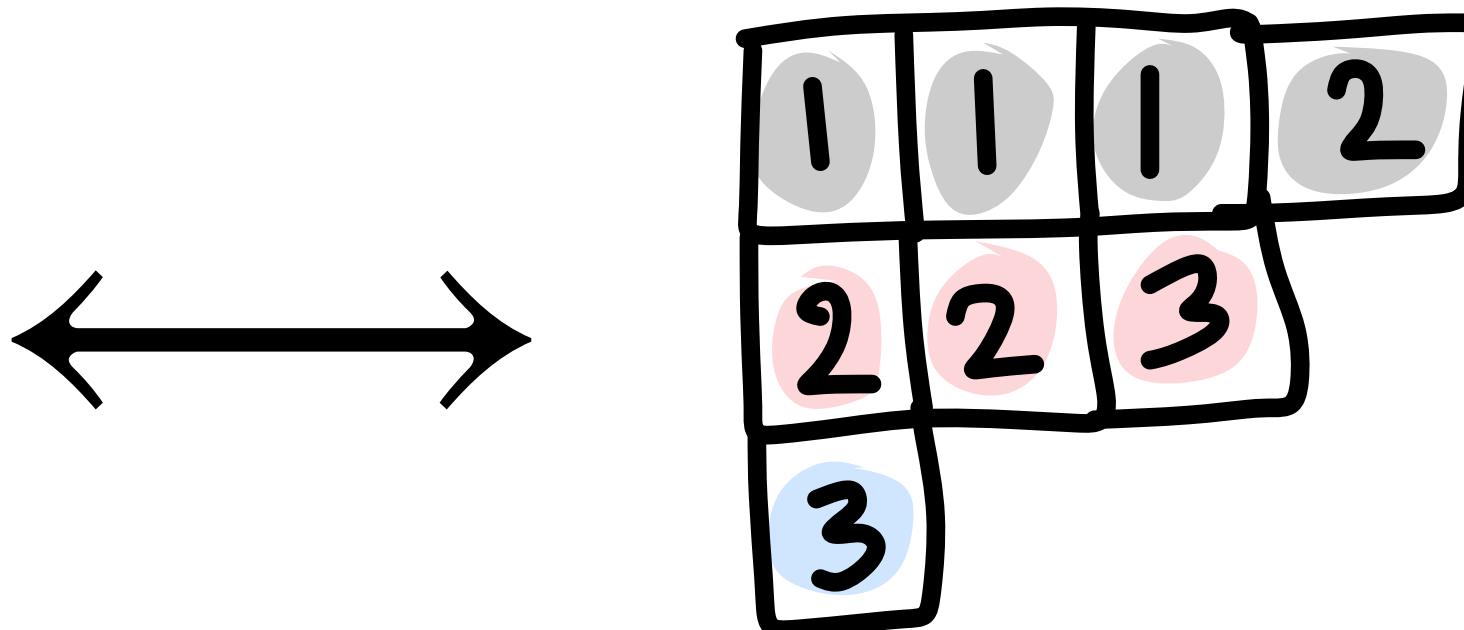


RSK correspondence

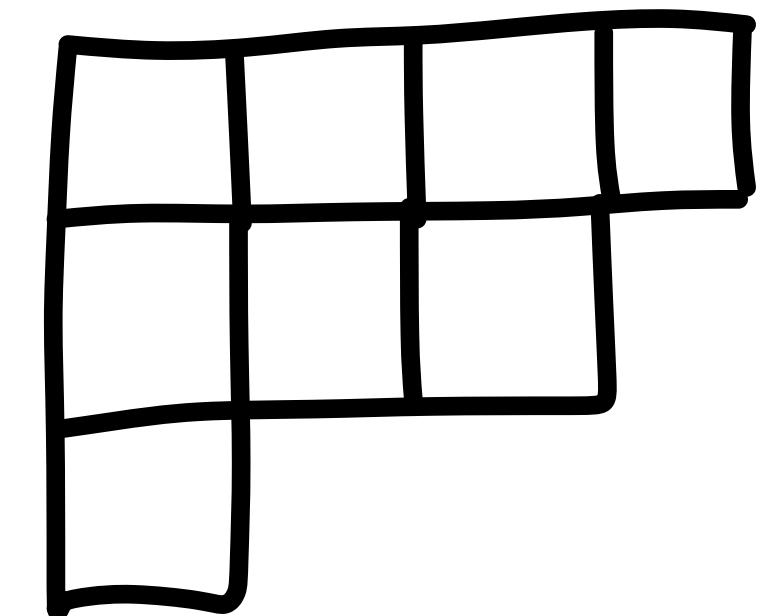
$$\left(M_{i,j} \right)_{i,j=1}^n \longleftrightarrow (P, Q)$$

Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$



$$\lambda =$$

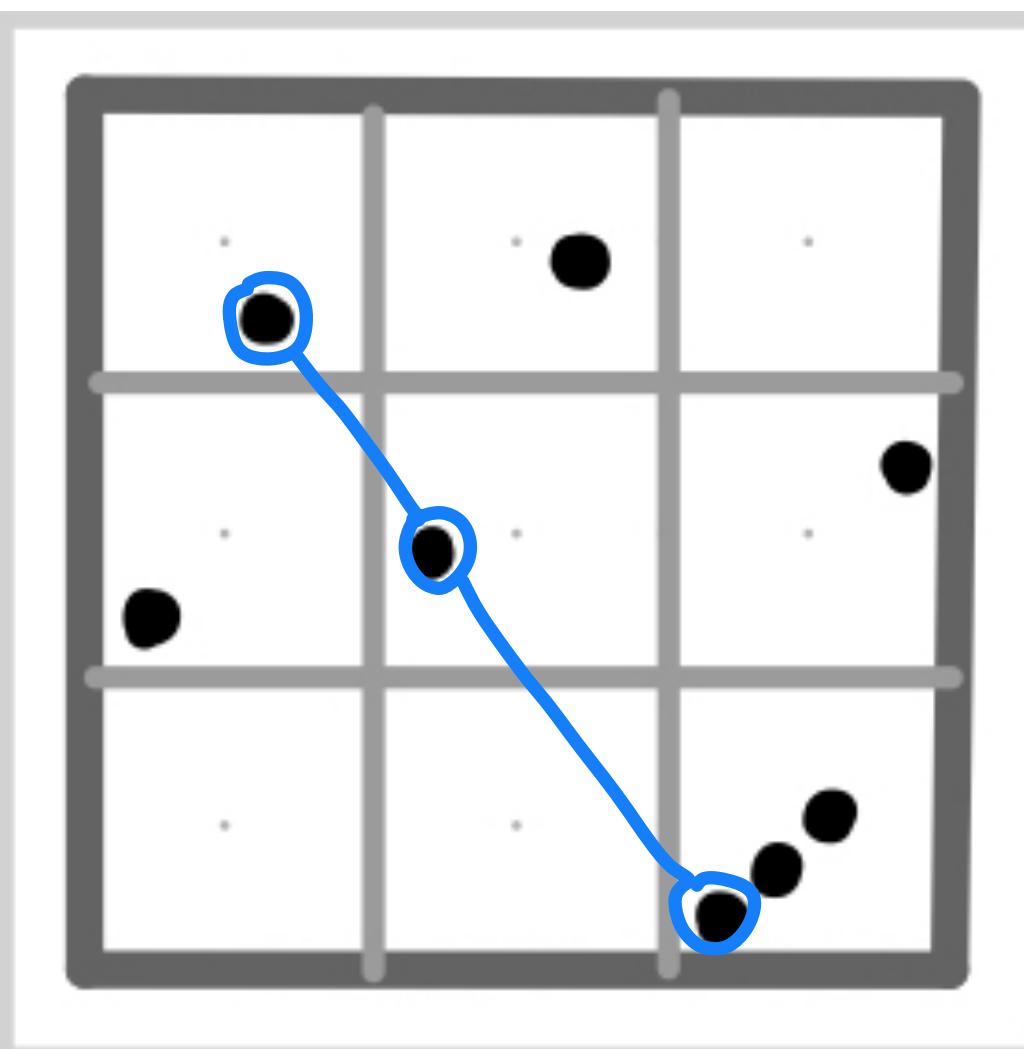


RSK correspondence

$$\left(M_{i,j} \right)_{i,j=1}^n \longleftrightarrow (P, Q)$$

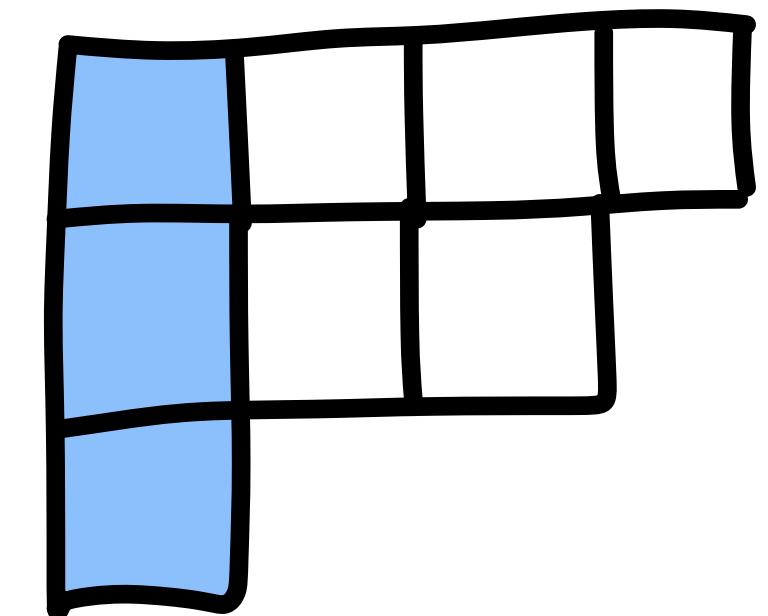
Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow \begin{array}{c} \begin{array}{cccc} 0 & 0 & 0 & 2 \\ 2 & 2 & 3 \\ 3 \end{array} \\ \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 \end{array} \end{array}$$



$$\lambda'_1 = \text{LDS}_1$$

$$\lambda =$$



RSK correspondence

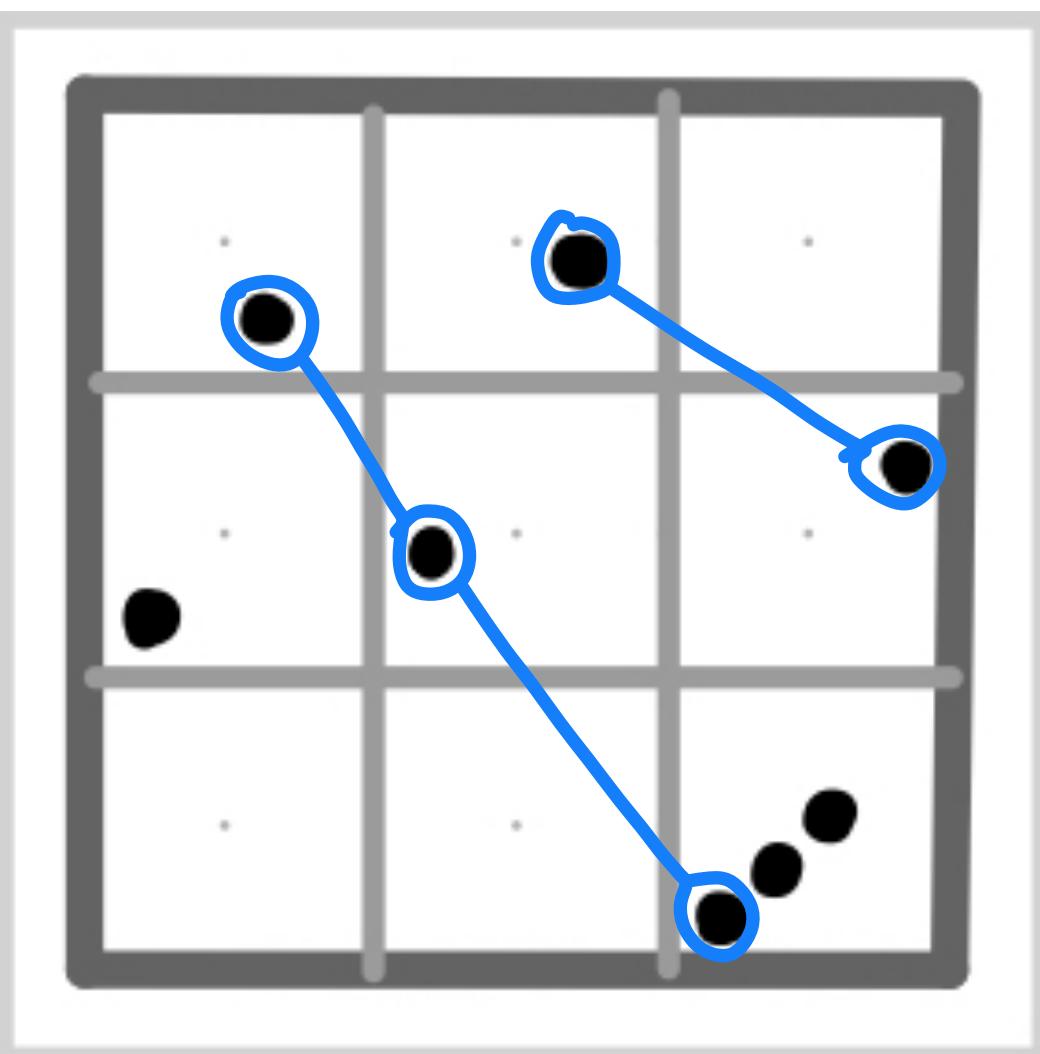
$$\left(M_{i,j} \right)_{i,j=1}^n \longleftrightarrow (P, Q)$$

Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow$$

0	0	0	2
2	2	3	
	3		

1	1	2	3
2	3	3	
	3		



$$\lambda'_1 = \text{LDS}_1$$

$$\lambda'_1 + \lambda'_2 = \text{LDS}_2$$

$$\lambda =$$

blue	white
blue	white
blue	white

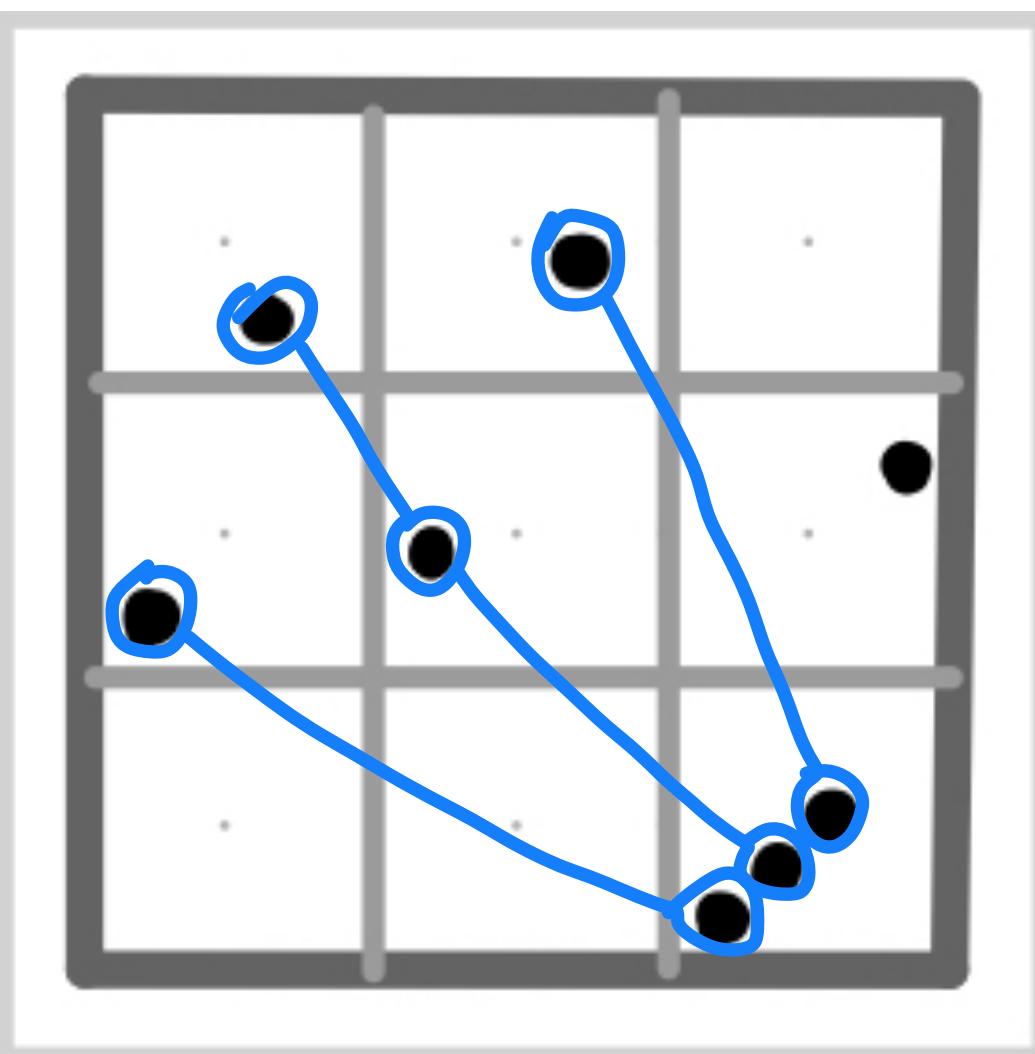
RSK correspondence

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Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow \begin{array}{c} \begin{array}{cccc} 0 & 0 & 0 & 2 \\ 2 & 2 & 3 & \\ \hline 3 & & & \end{array} \\ \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ \hline 3 & & \end{array} \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & \\ \hline 3 & & & \end{array} \\ \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ \hline 3 & & \end{array} \end{array}$$

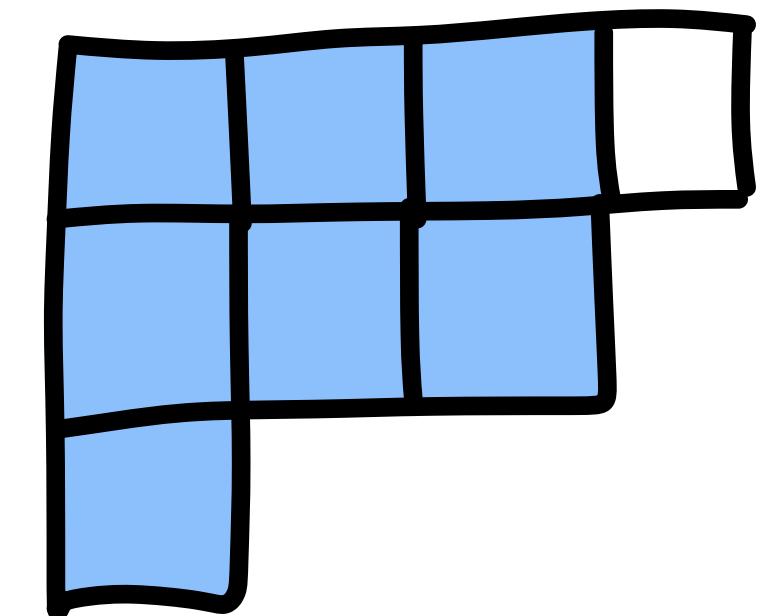


$$\lambda'_1 = \text{LDS}_1$$

$$\lambda'_1 + \lambda'_2 = \text{LDS}_2$$

$$\lambda'_1 + \lambda'_2 + \lambda'_3 = \text{LDS}_3$$

$$\lambda =$$



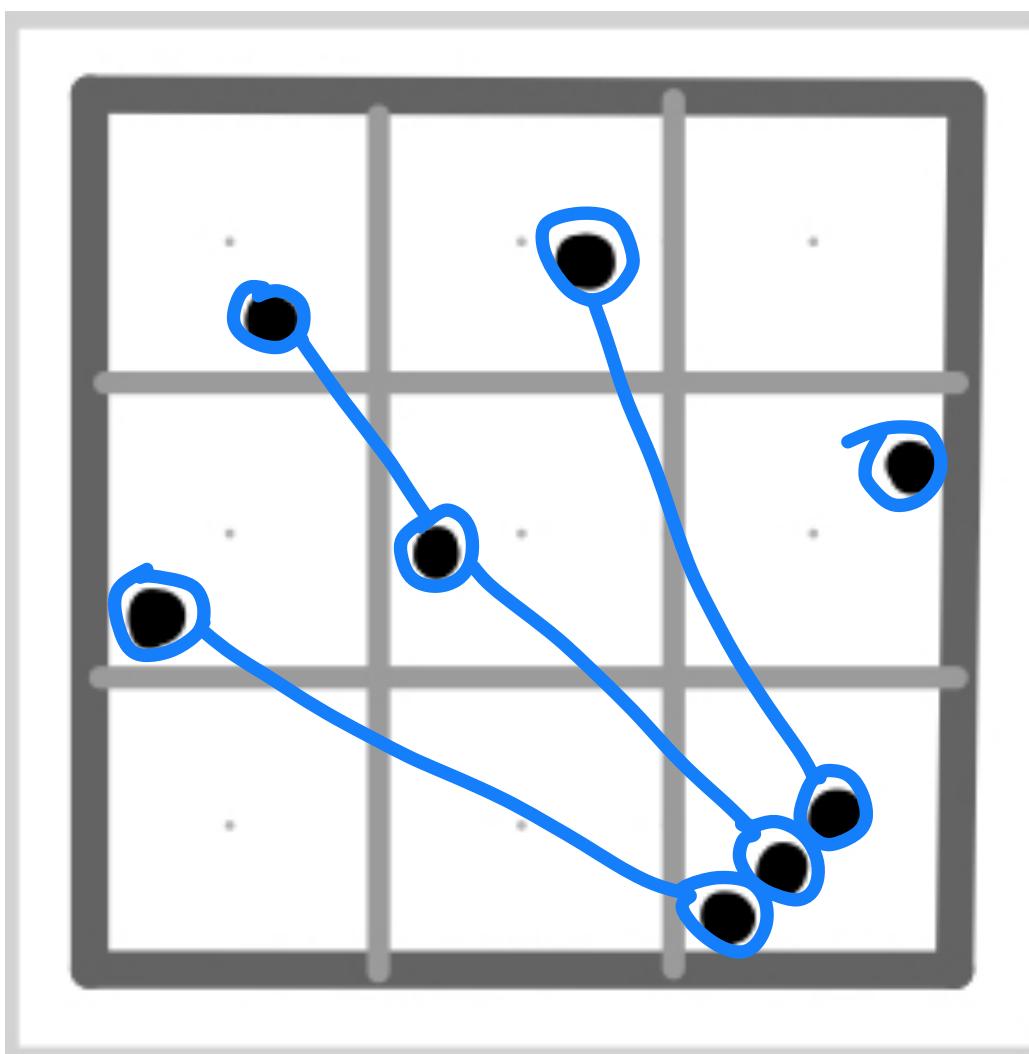
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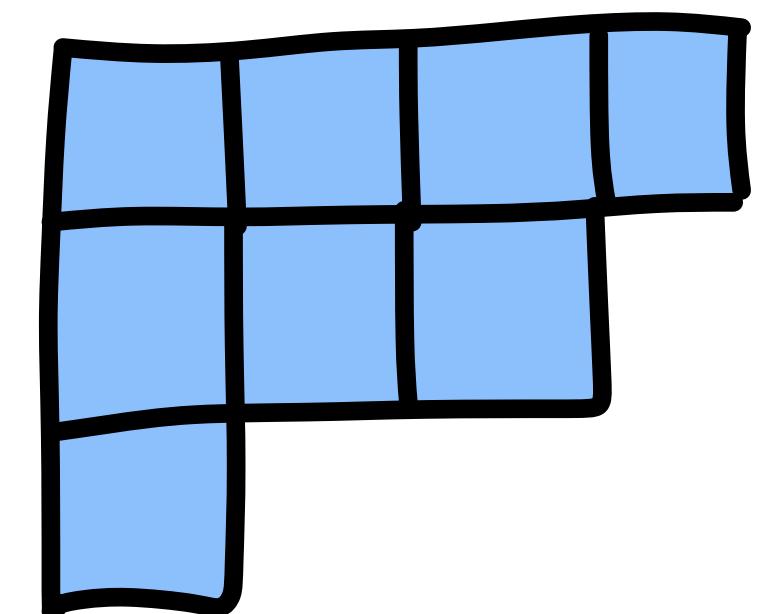
$$\lambda'_1 = \text{LDS}_1$$

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$$\lambda'_1 + \lambda'_2 + \lambda'_3 = \text{LDS}_3$$

$$\lambda'_1 + \lambda'_2 + \lambda'_3 + \lambda'_4 = \text{LDS}_4$$

$$\lambda =$$



Cauchy Identity for q -Whittaker polynomials

Cauchy Identity for q -Whittaker polynomials

$$\sum_{\mu} b_{\mu}(q) P_{\mu}(x; q) P_{\mu}(y; q) = \prod_{k \geq 0} \prod_{i,j=1}^n \frac{1}{1 - x_i y_j q^k}$$

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q -Whittaker polynomials

$$P_{\mu}(x; q) = \sum_{V \in VST(\mu)} q^{\mathcal{H}(V)} x^V$$

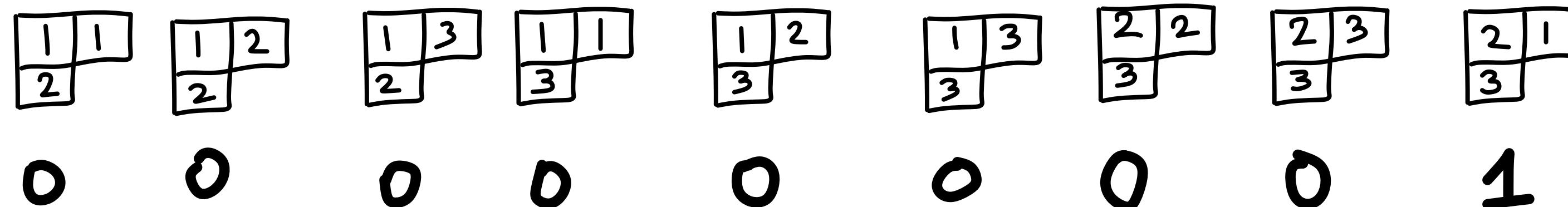
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$$P_{\mu}(x; q) = \sum_{V \in VST(\mu)} q^{\mathcal{H}(V)} x^V$$

\mathcal{H} = intrinsic energy (almost defined in Lecouvey's lecture)



$$\mathcal{P}_{\mu}(x; q) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + q x_1 x_2 x_3$$

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$$\mathcal{K}(\mu) = \{\kappa = (\kappa_1, \dots, \kappa_{\mu_1}) : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}$$

$$\mu = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \quad \kappa_4 \geq \kappa_5$$

$$\kappa_1 \geq \kappa_2 \geq \kappa_3$$

Cauchy Identity for q -Whittaker polynomials

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$$\frac{1}{1 - x_i y_j q^k} = \sum_{M_{i,j}^k = 0, 1, 2, \dots} (x_i y_j q^k)^{M_{i,j}^k}$$

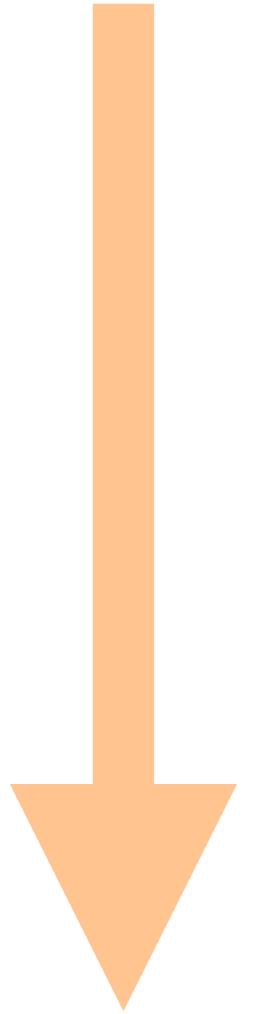
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$$\sum_{\mu} \sum_{\kappa \in \mathcal{K}(\mu)} \sum_{V, W \in VST(\mu)} q^{|\kappa| + \mathcal{H}(V) + \mathcal{H}(W)} x^V y^W = \sum_{M \in \overline{\mathbb{M}}_{n \times n}} \prod_{i,j=1}^n (x_i y_j q^k)^{M_{i,j}^k}$$

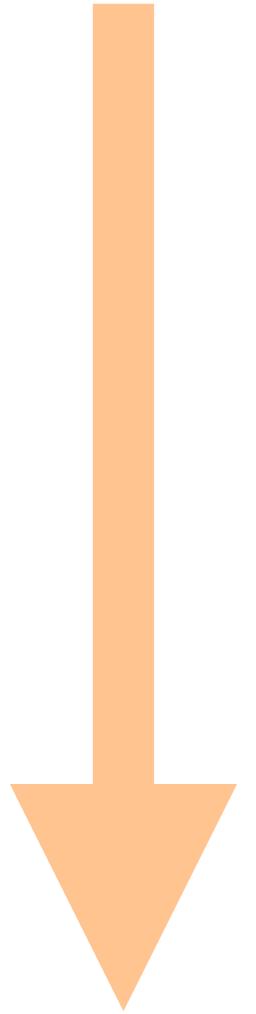
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Bijective proof: $M \xleftrightarrow{\Upsilon} (V, W; \kappa) : \sum_{i,j=1}^n \sum_{k>0} k M_{i,j}^k = |\kappa| + \mathcal{H}(V) + \mathcal{H}(W)$

Construction of Υ

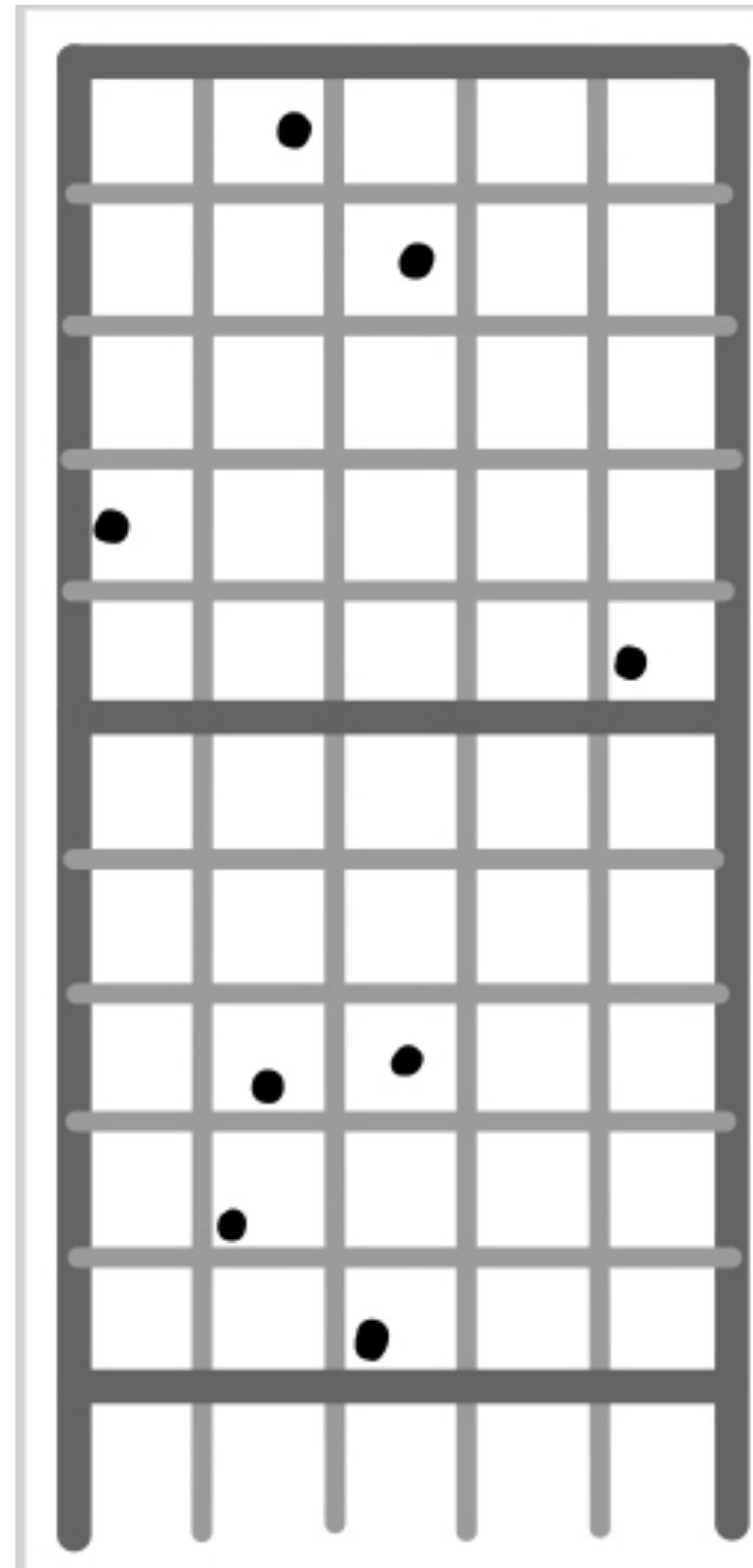
$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

Construction of Υ

$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$

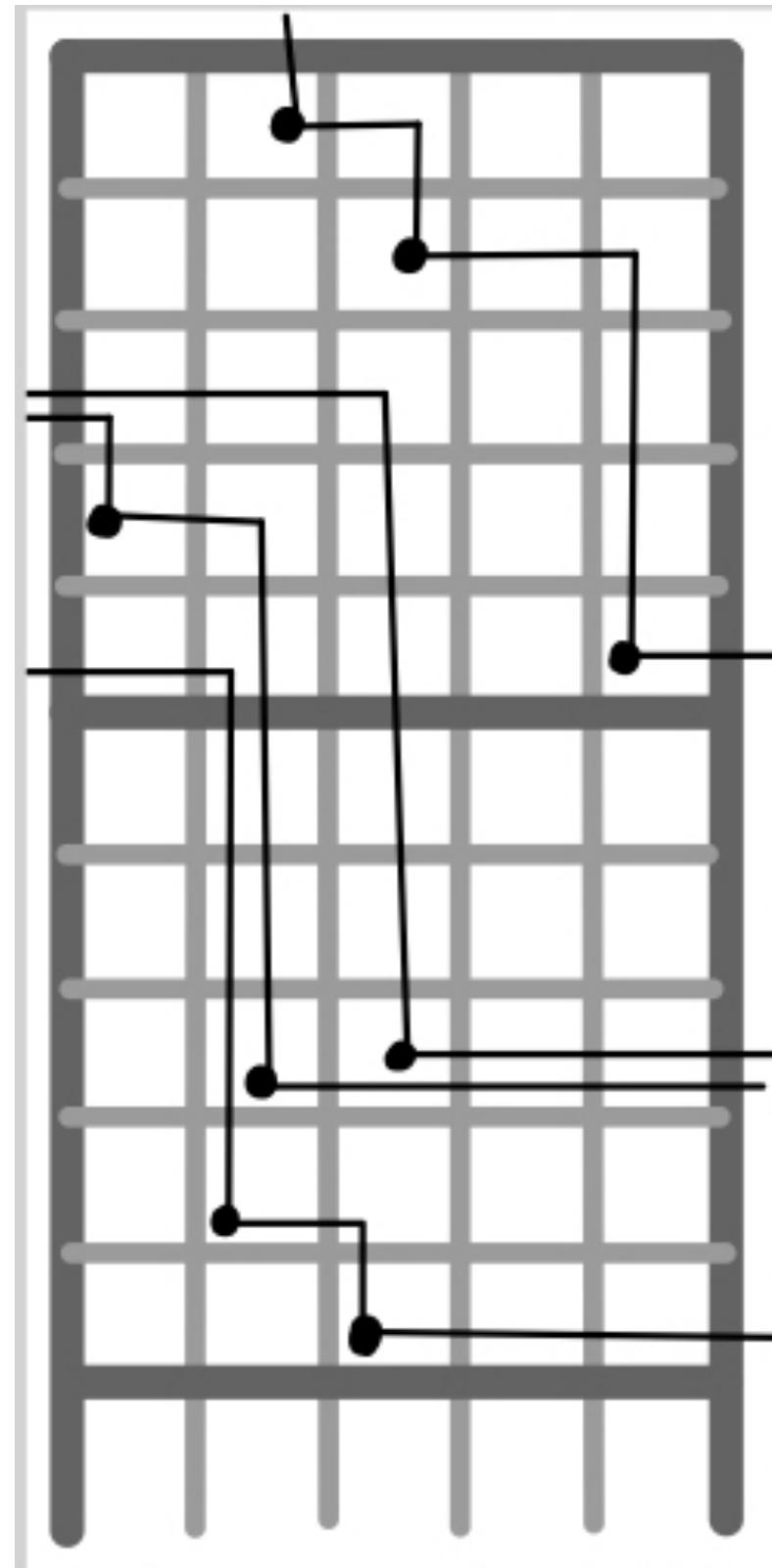
$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$\longleftrightarrow ?$



Construction of Υ

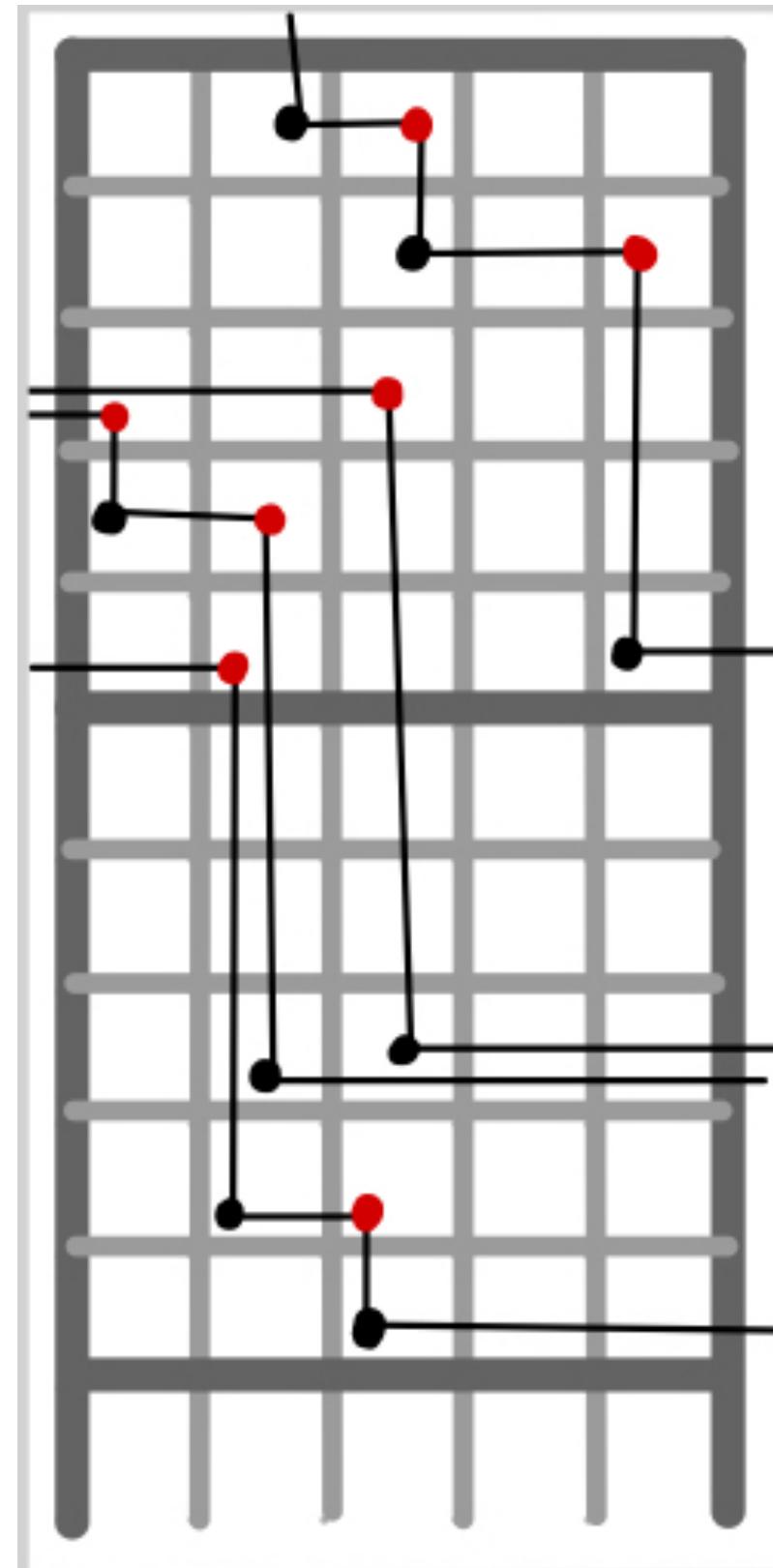
$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{matrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{matrix} \right) \quad \longleftrightarrow \quad ?$$



Periodic boundary conditions

Construction of Υ

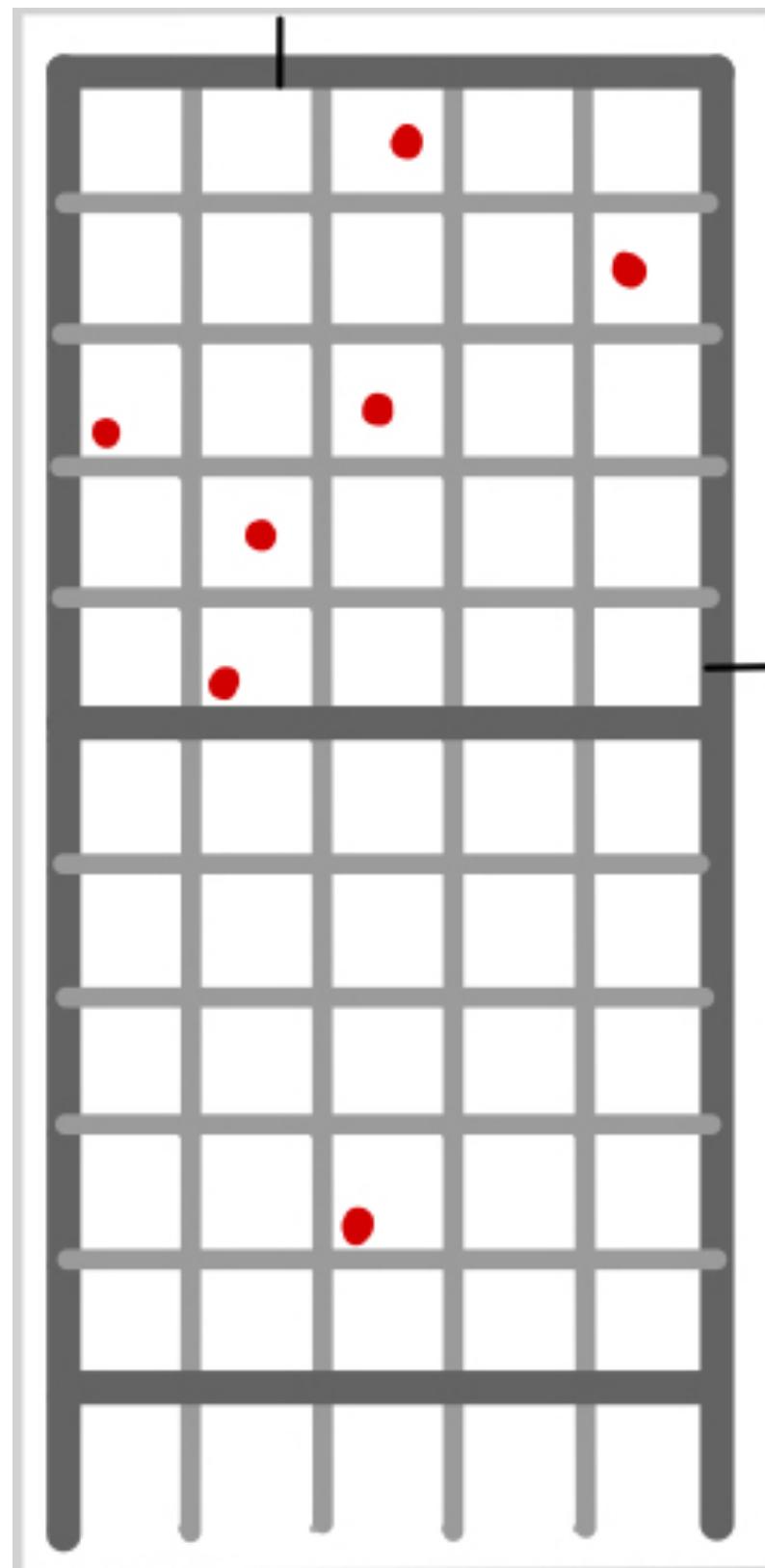
$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \quad \longleftrightarrow \quad ?$$



Periodic boundary conditions

Construction of Υ

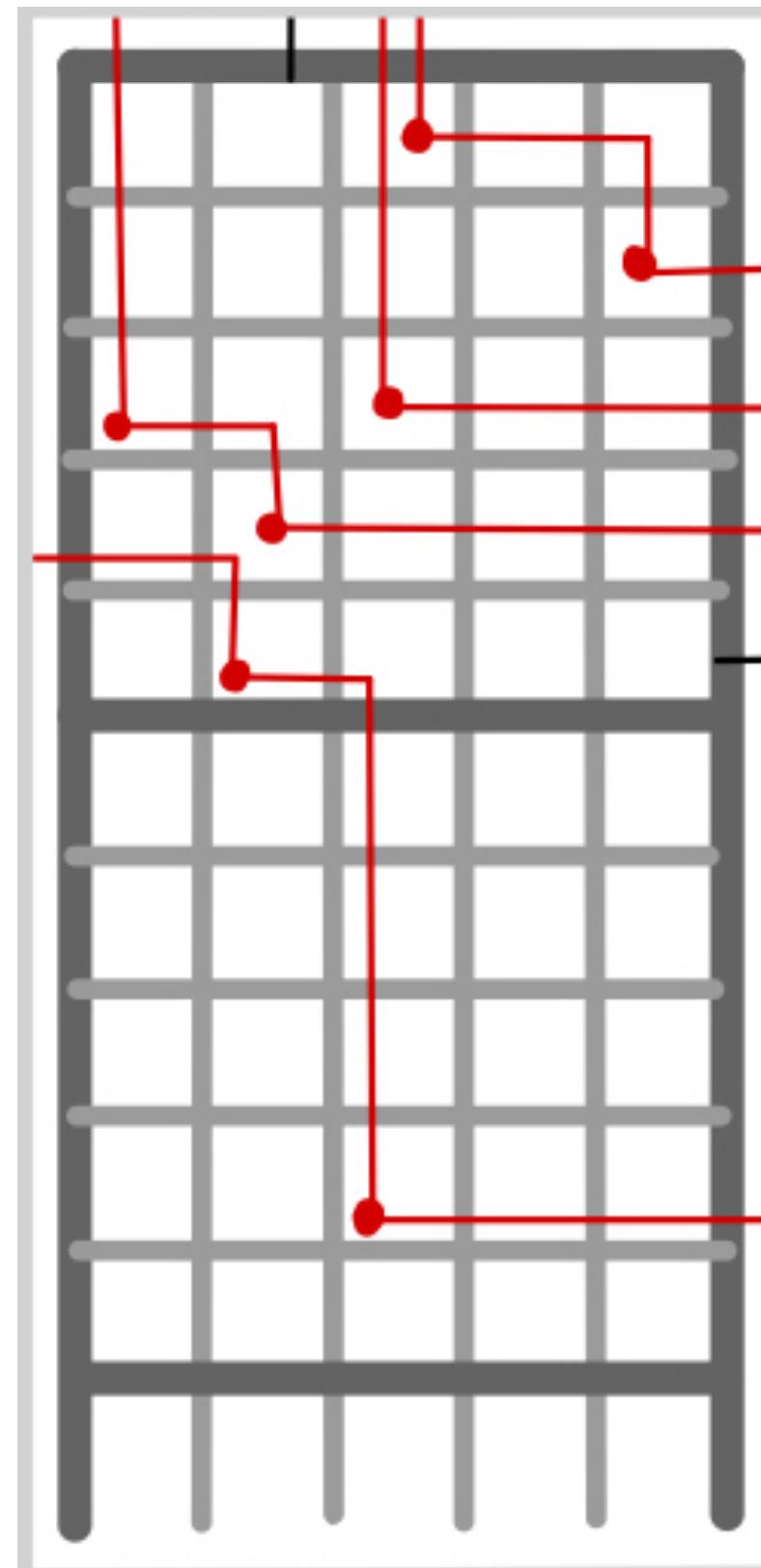
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Periodic boundary conditions

Construction of Υ

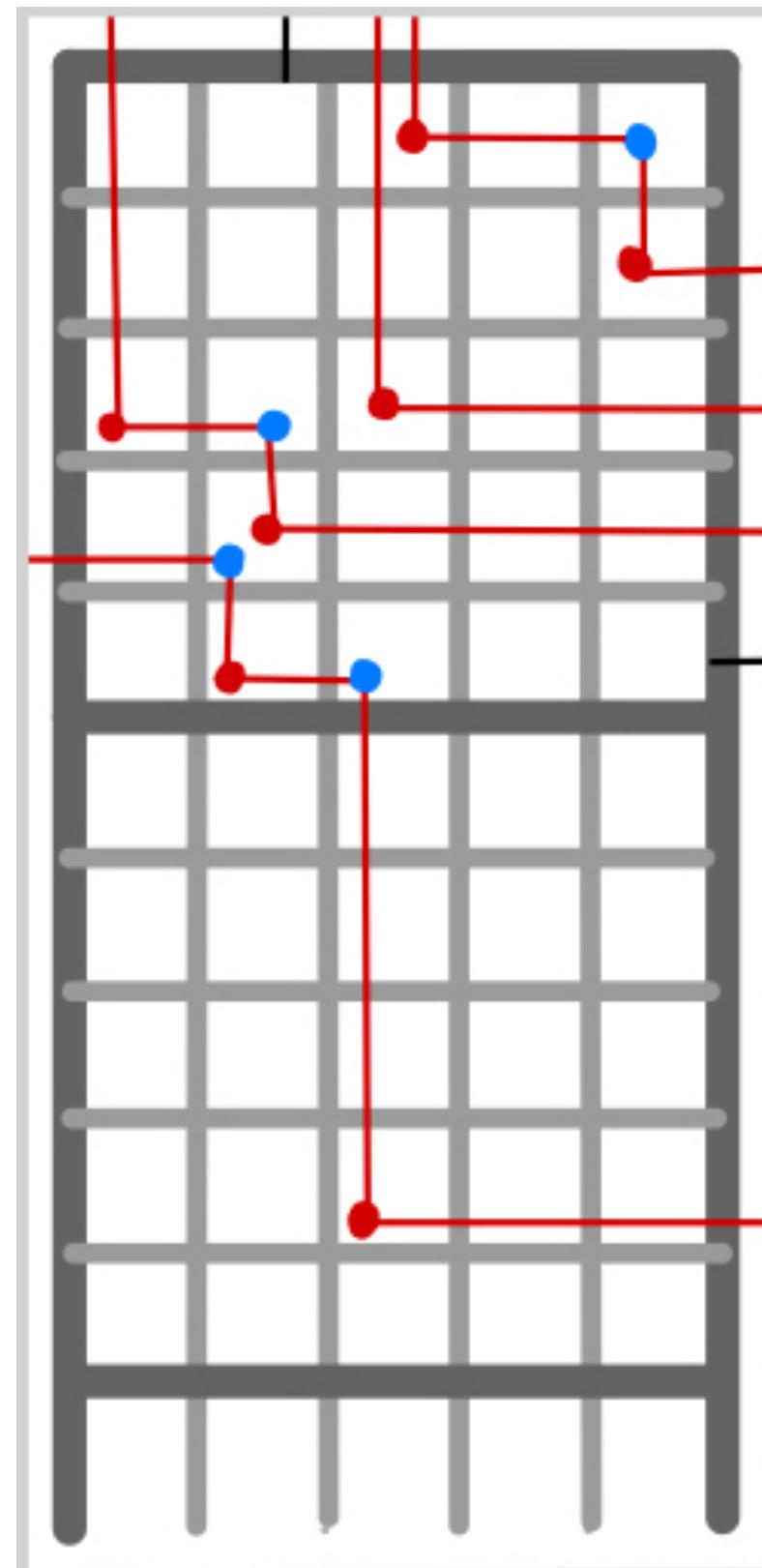
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Periodic boundary conditions

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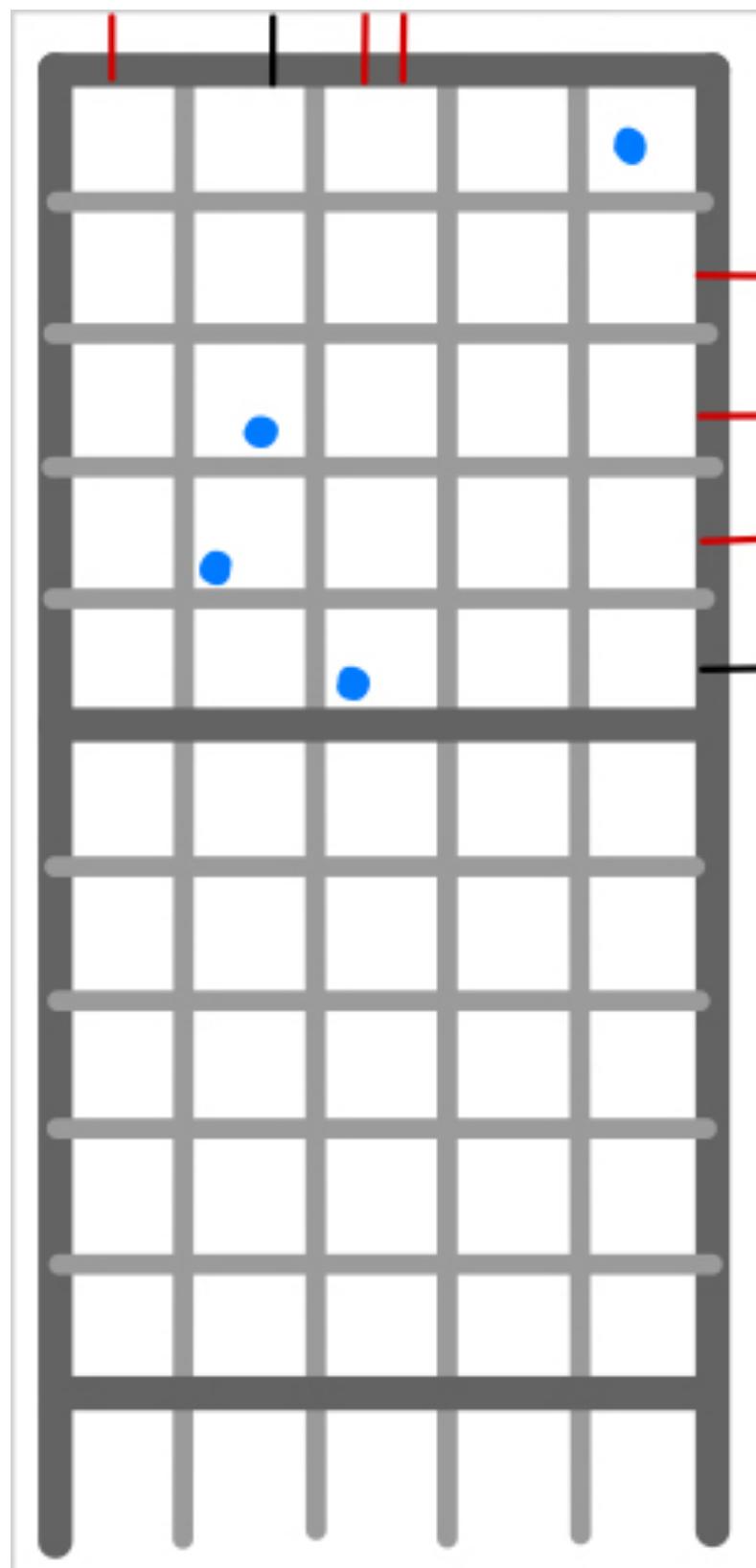
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Periodic boundary conditions

Construction of Υ

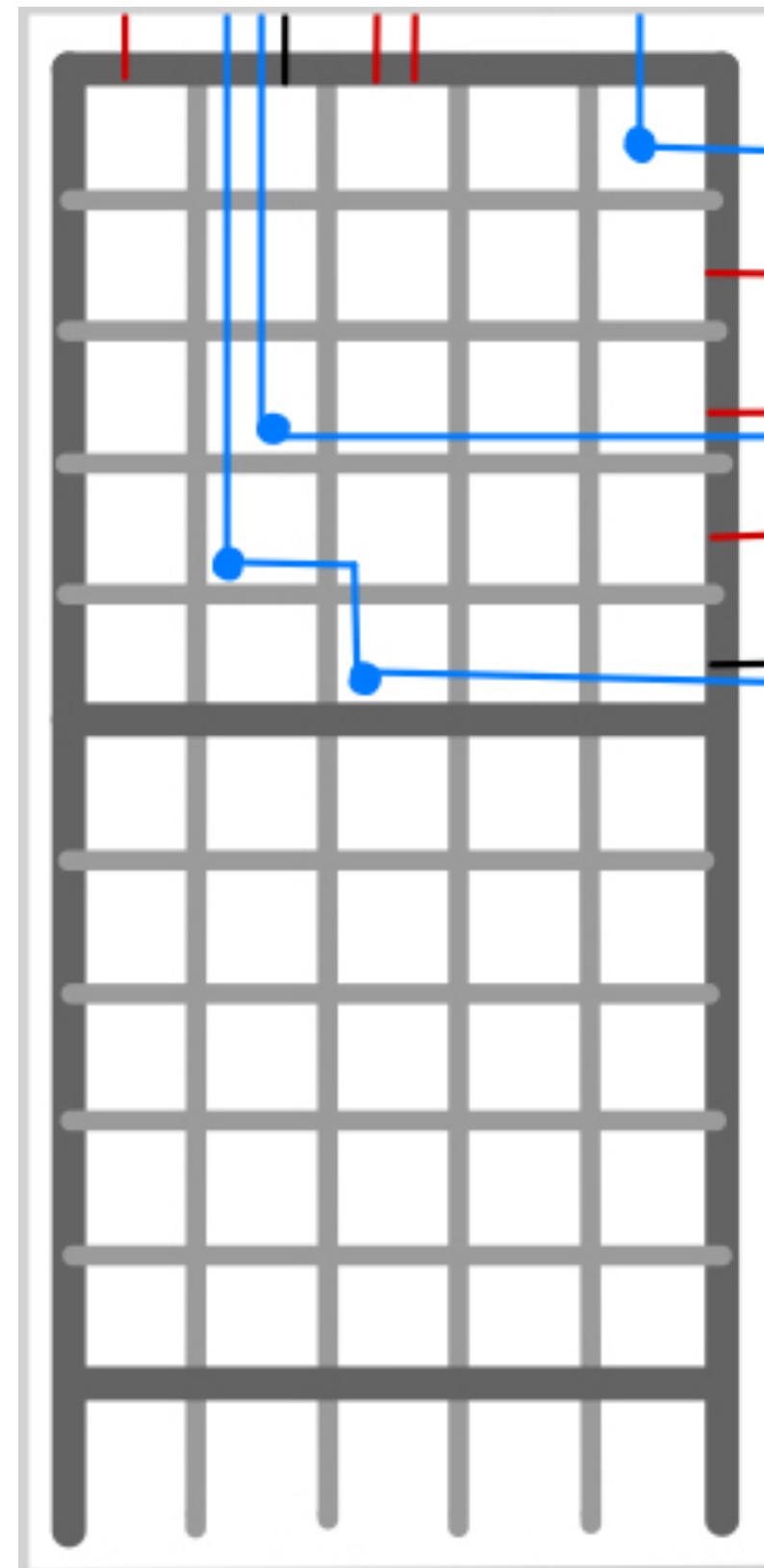
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Periodic boundary conditions

Construction of Υ

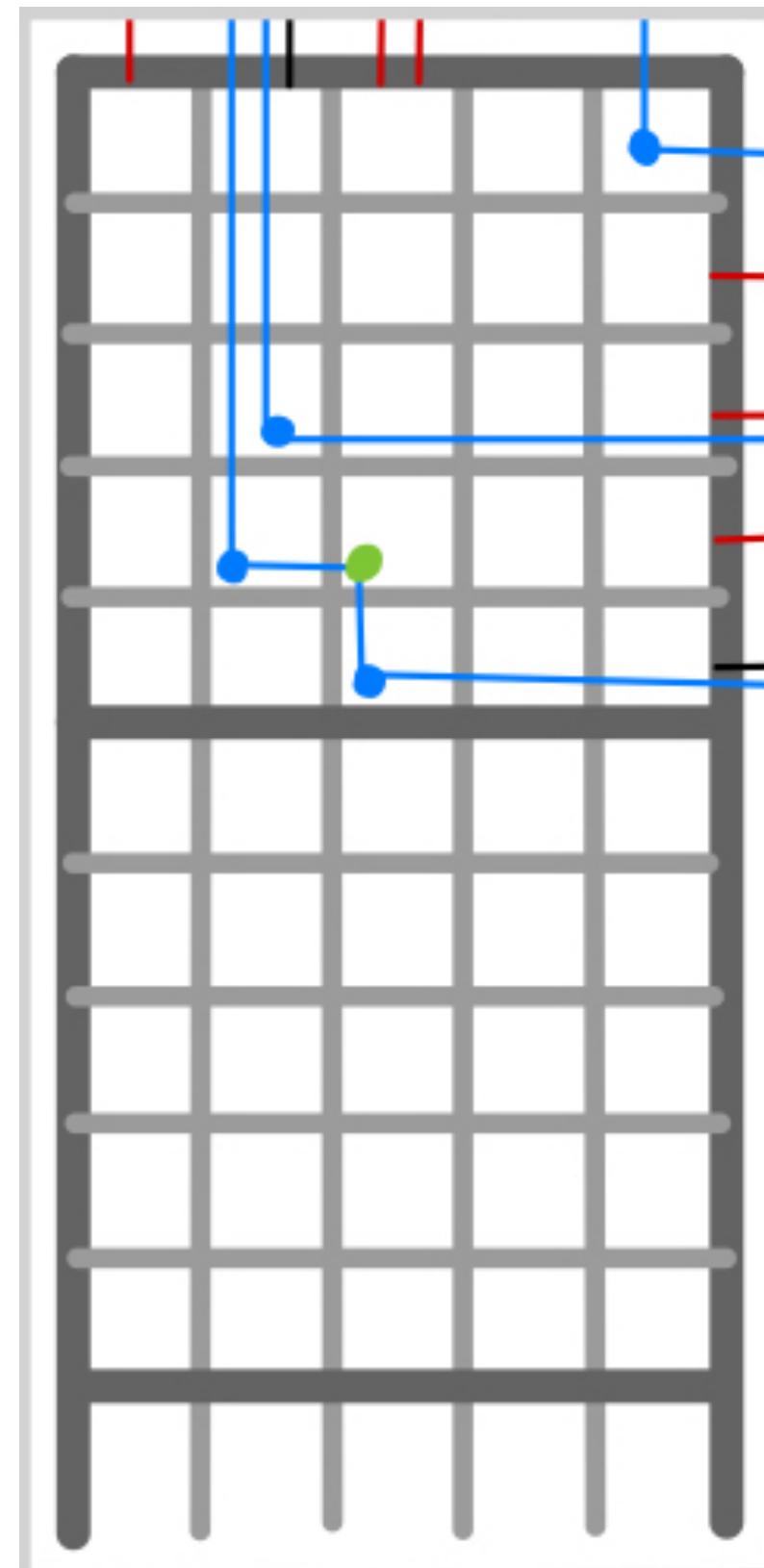
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Periodic boundary conditions

Construction of Υ

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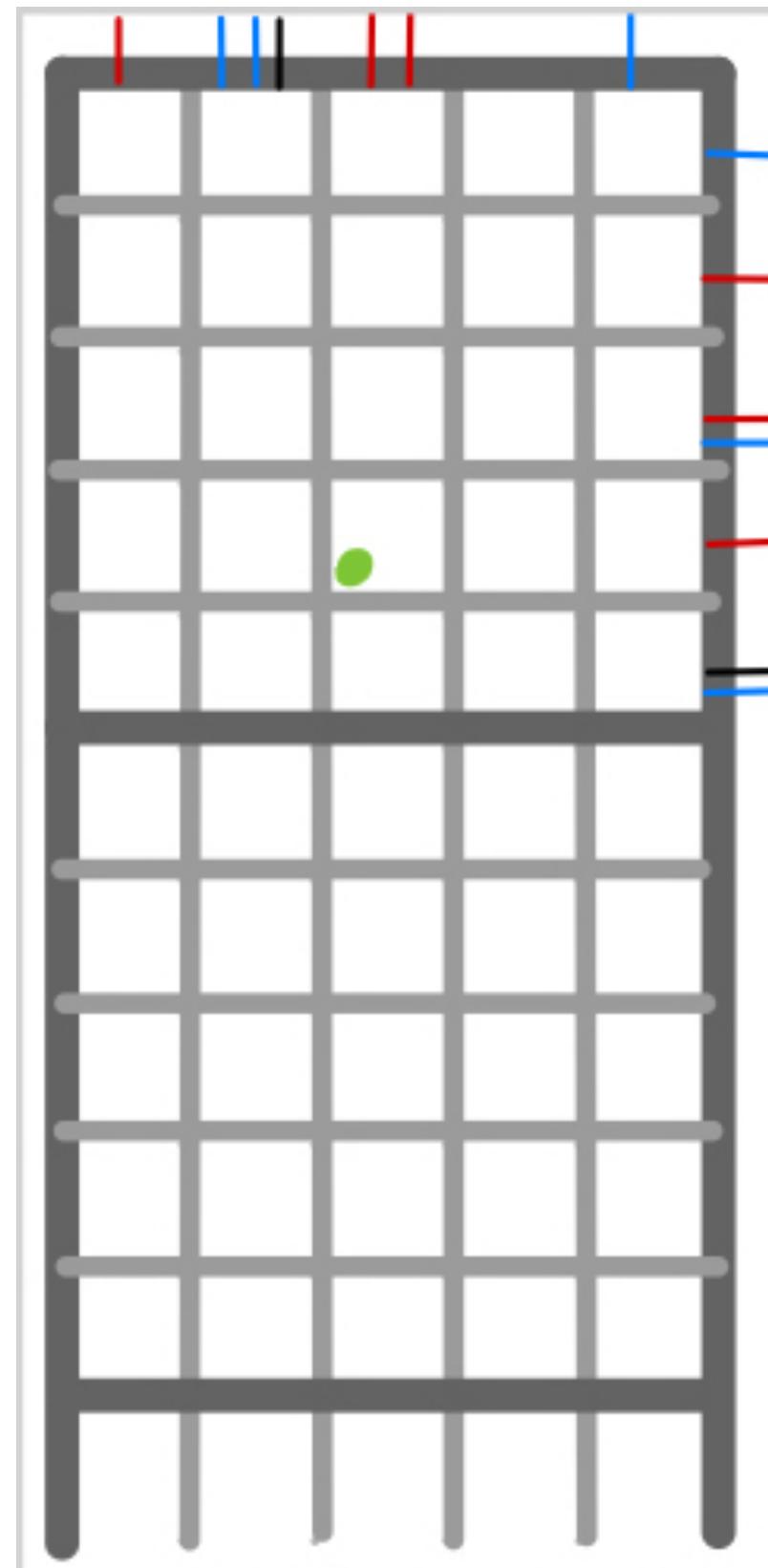
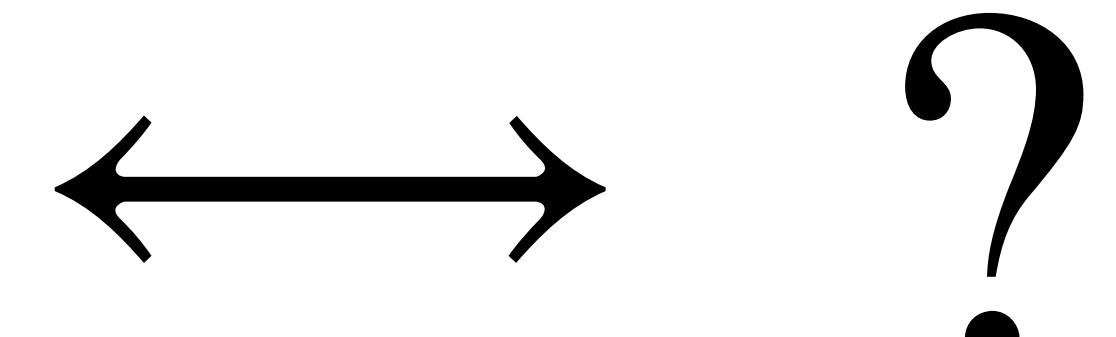


Periodic boundary conditions

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

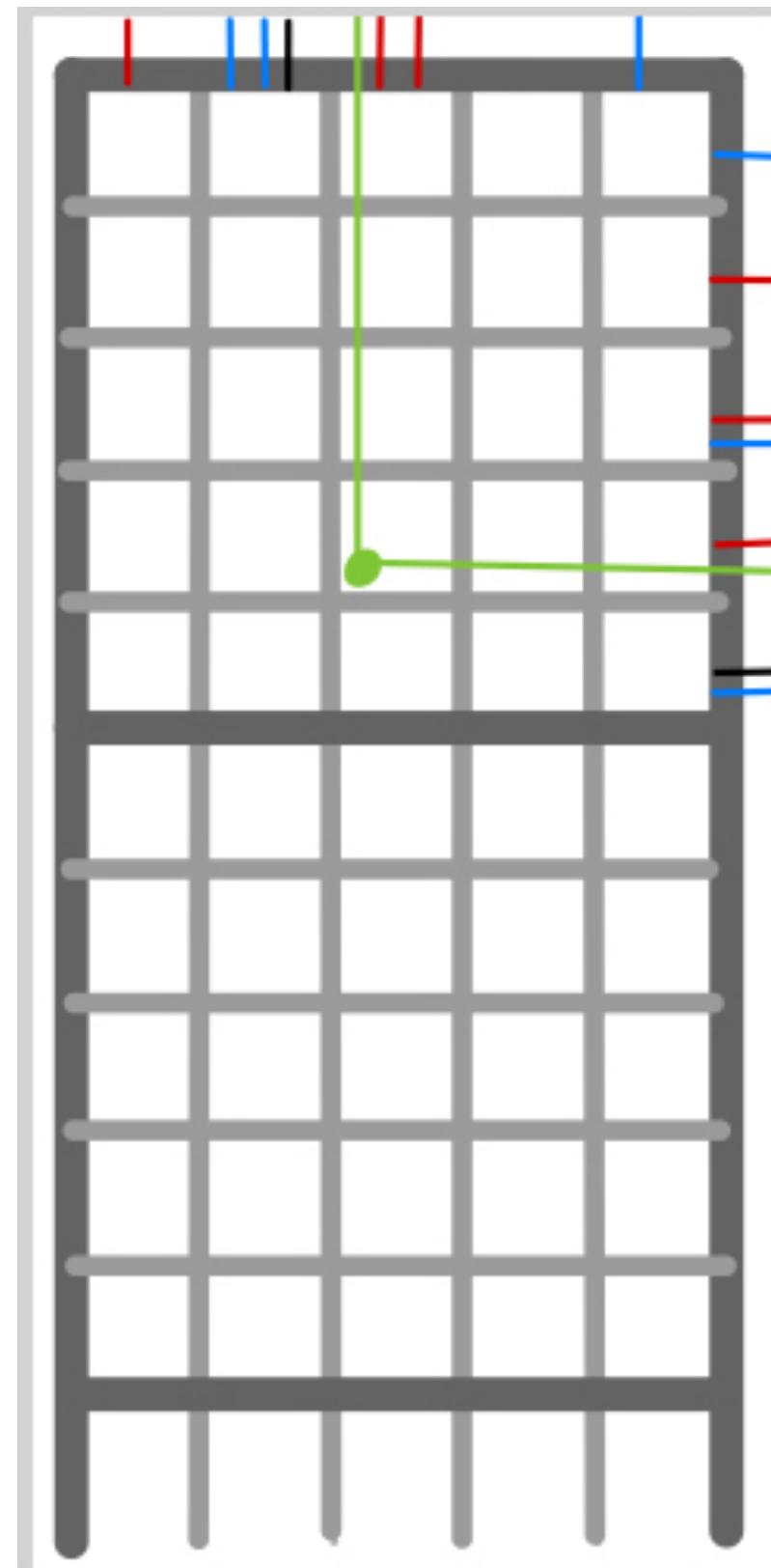
$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



Periodic boundary conditions

Construction of Υ

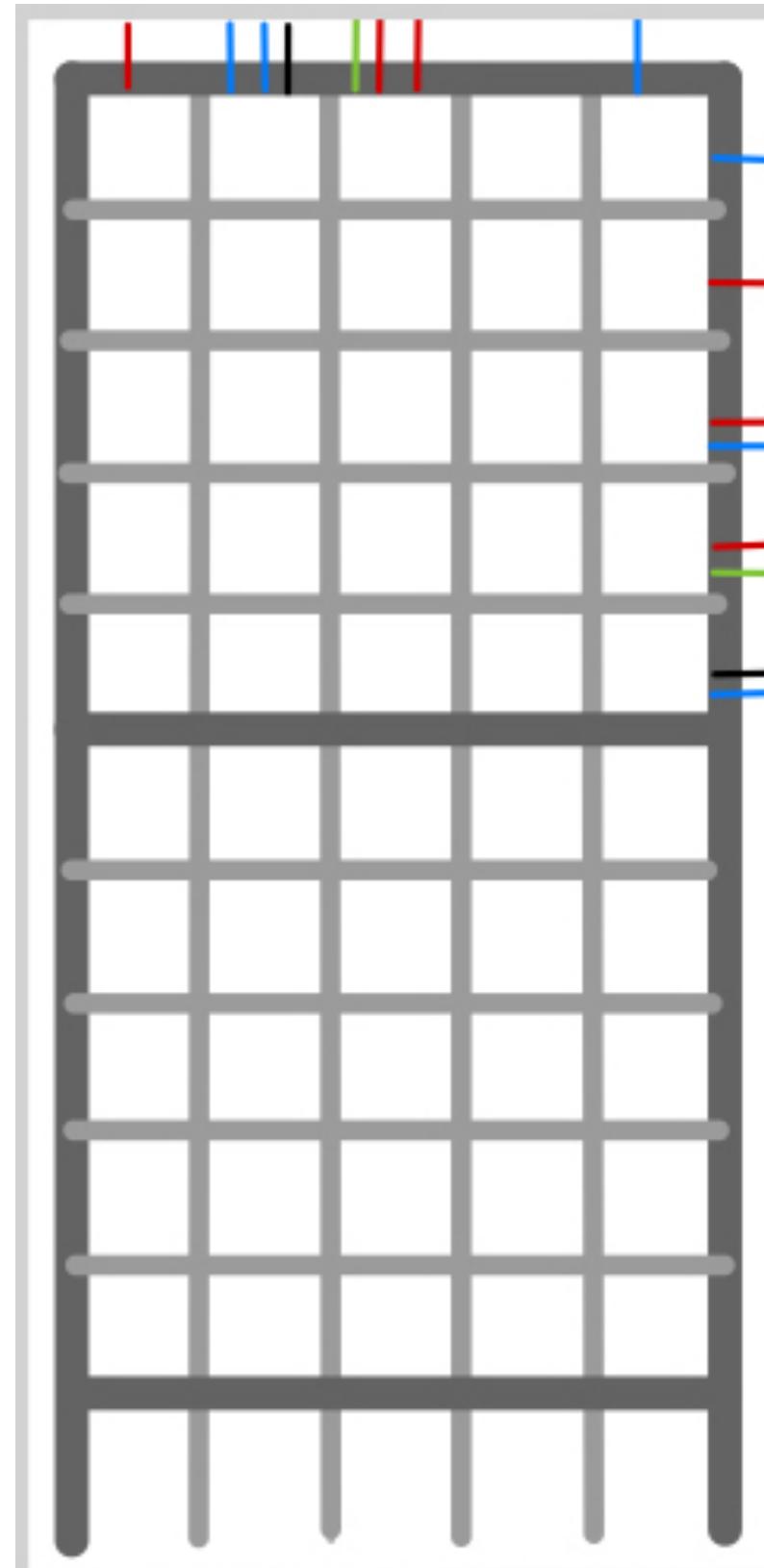
$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \quad \longleftrightarrow \quad ?$$



Periodic boundary conditions

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \quad \longleftrightarrow \quad ?$$

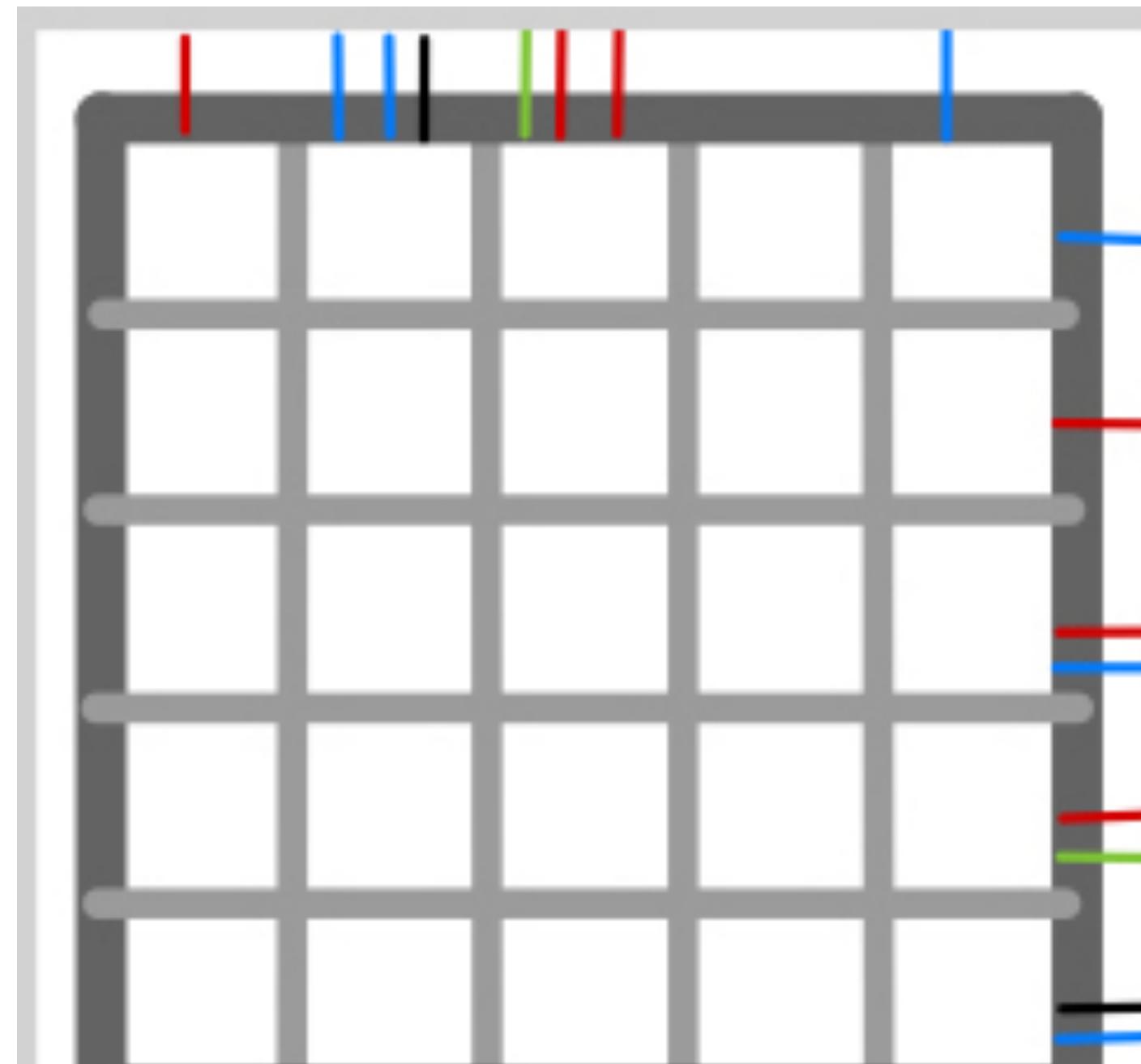
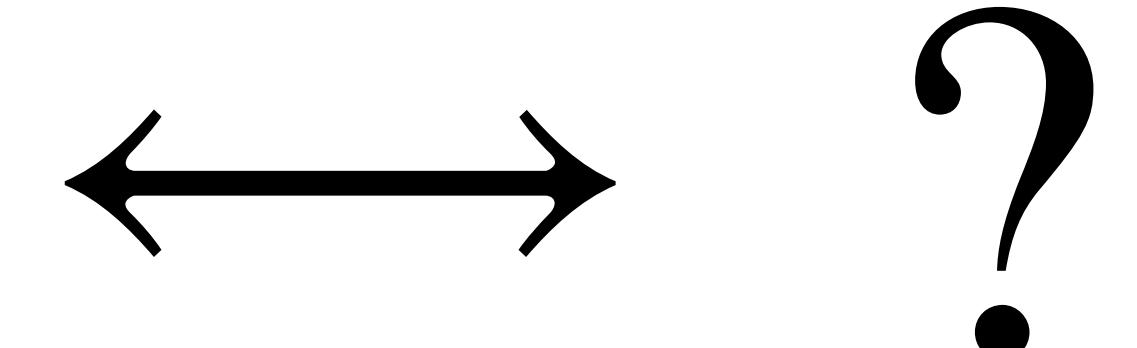


Periodic boundary conditions

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



— 1st row

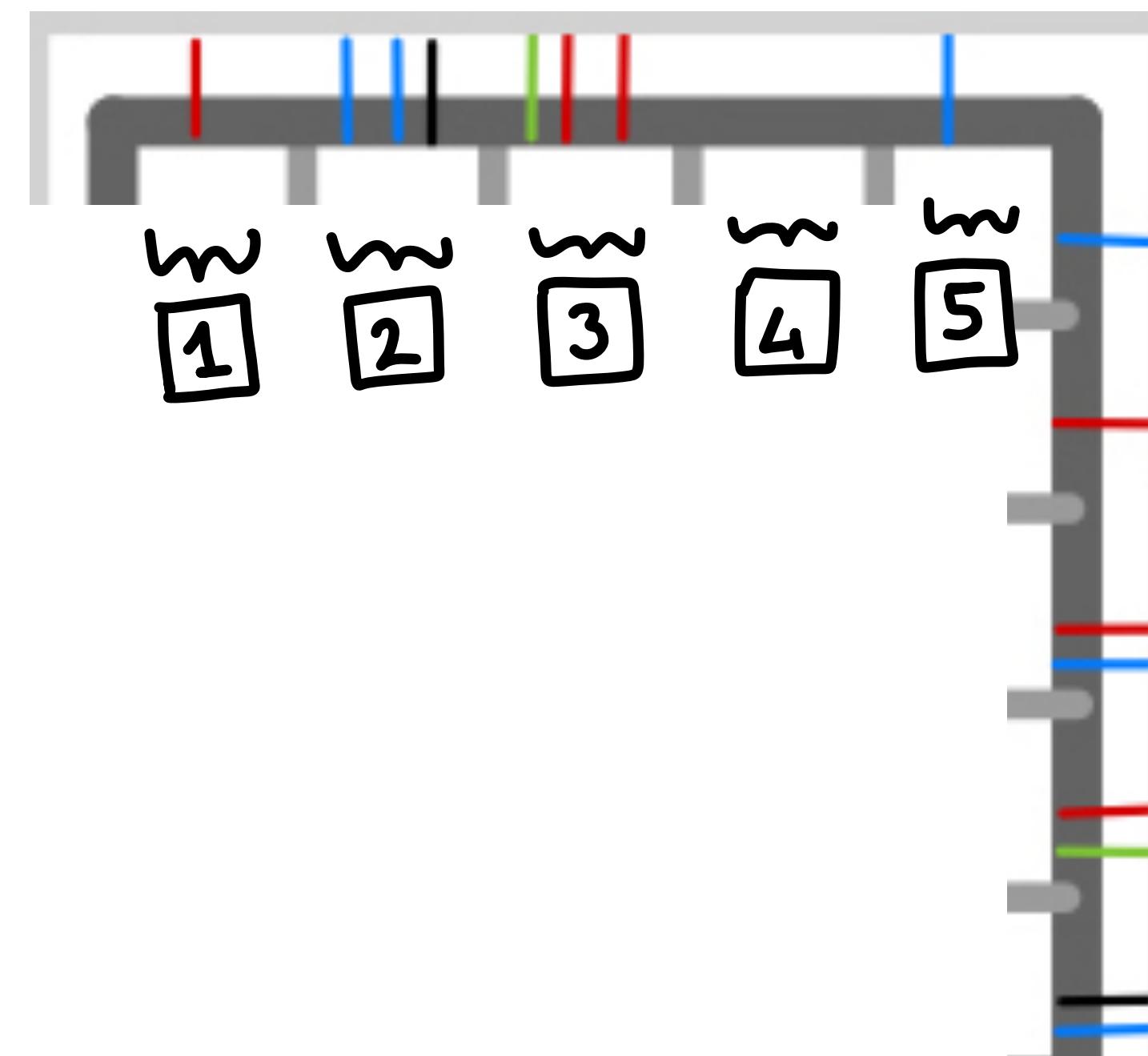
— 2nd row

— 3rd row

— 4th row

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

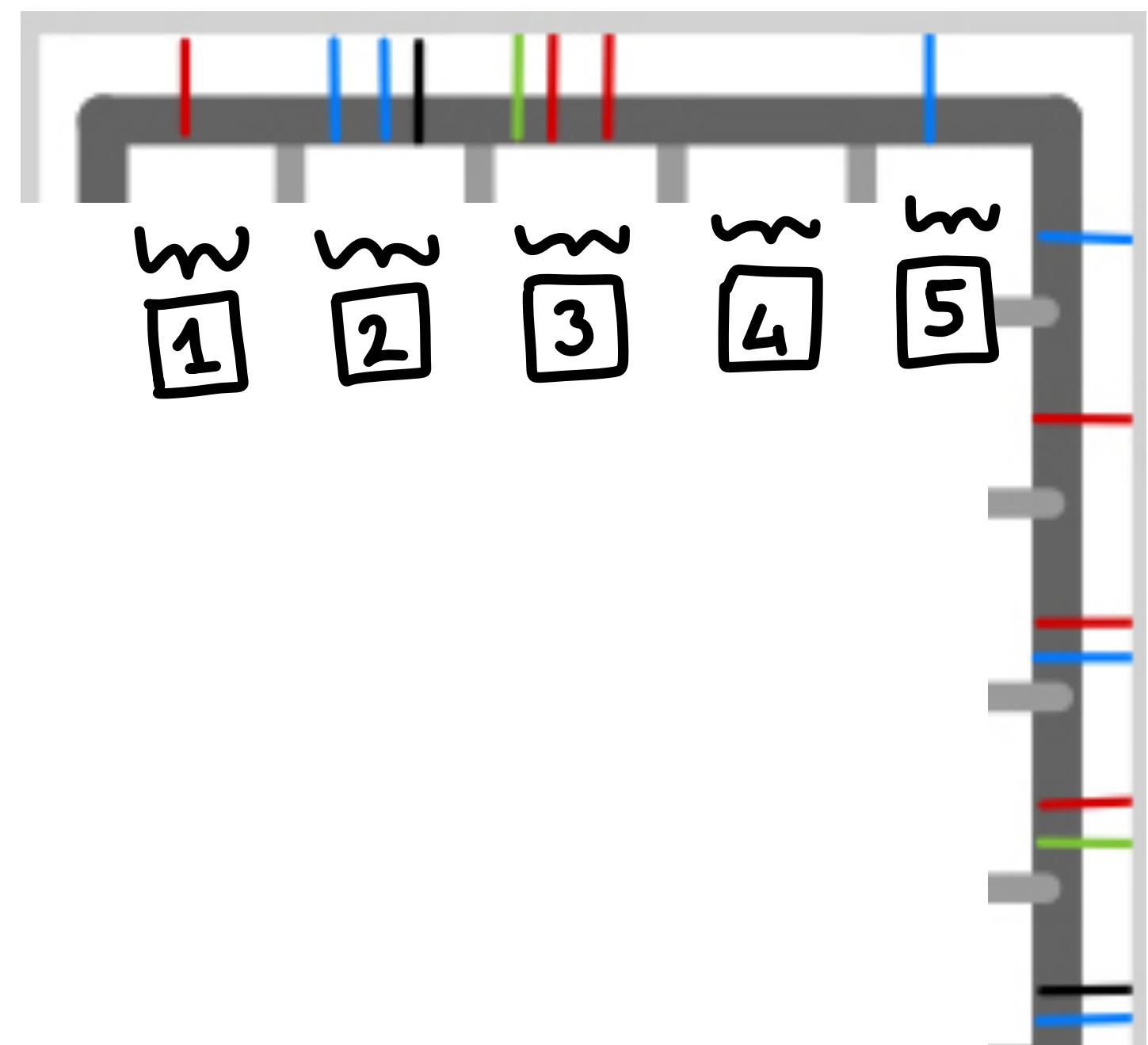
— 4th row

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$

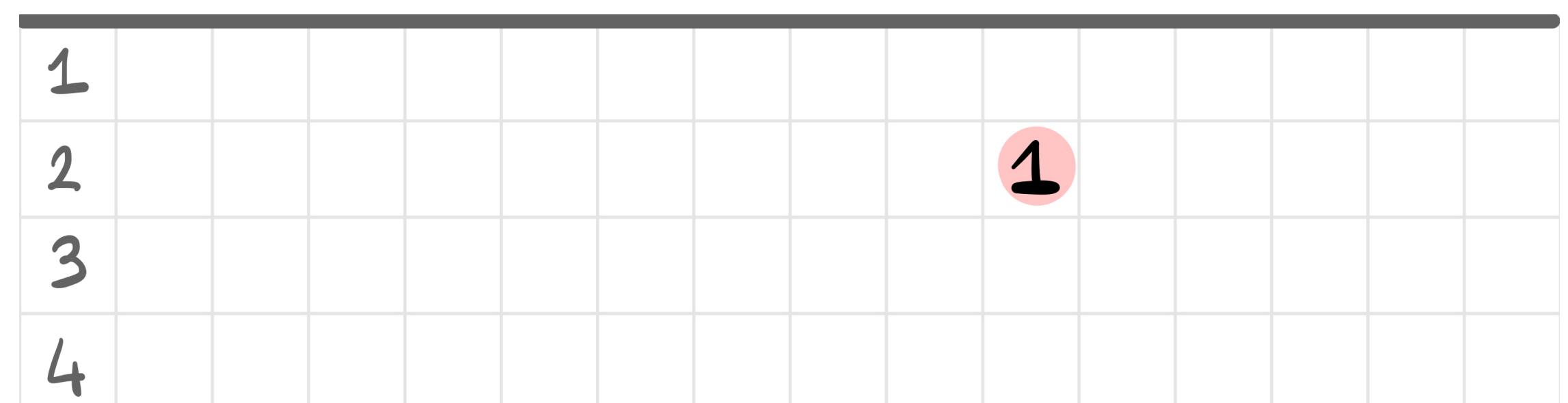


— 1st row

— 2nd row

— 3rd row

— 4th row

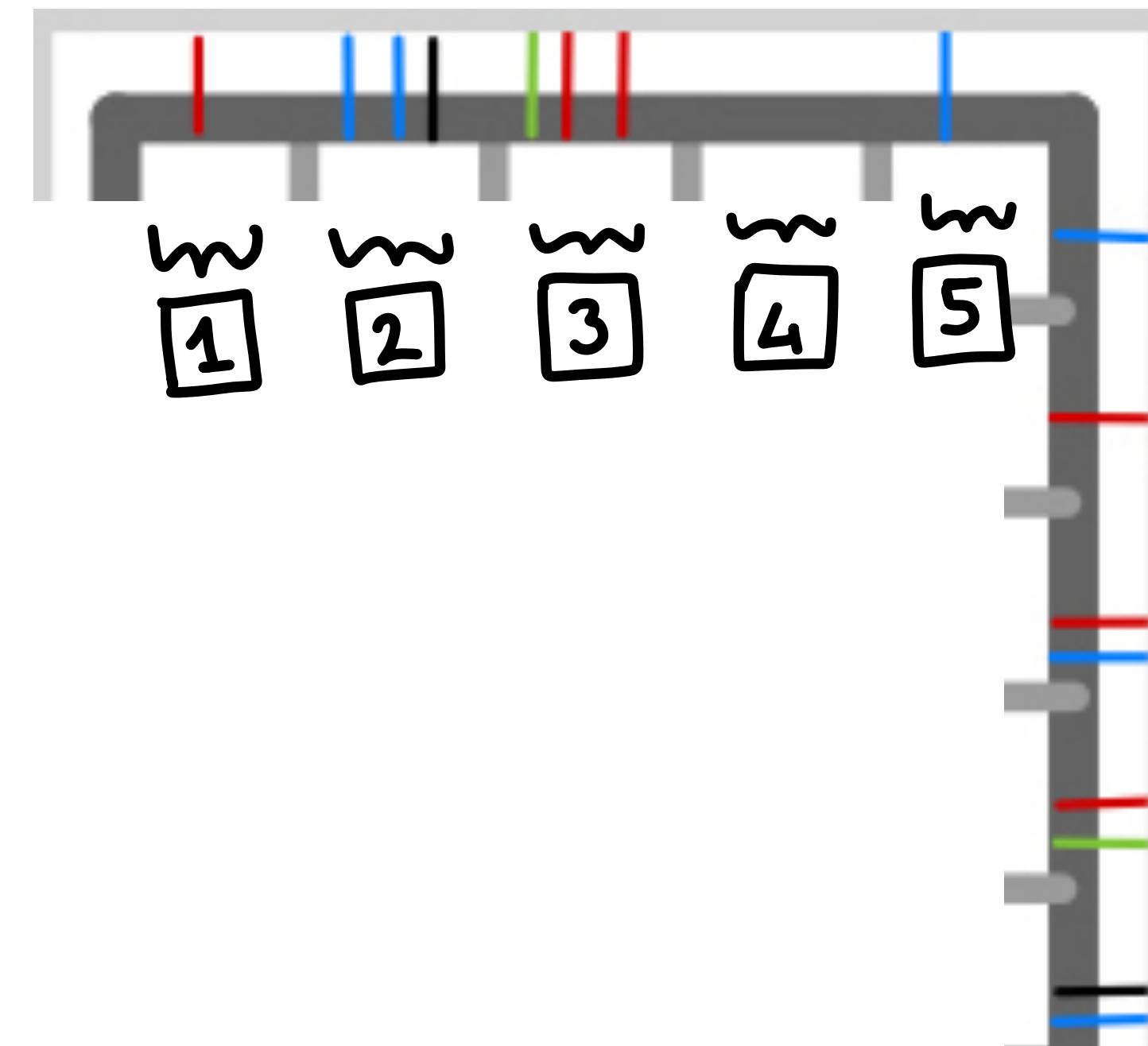


Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

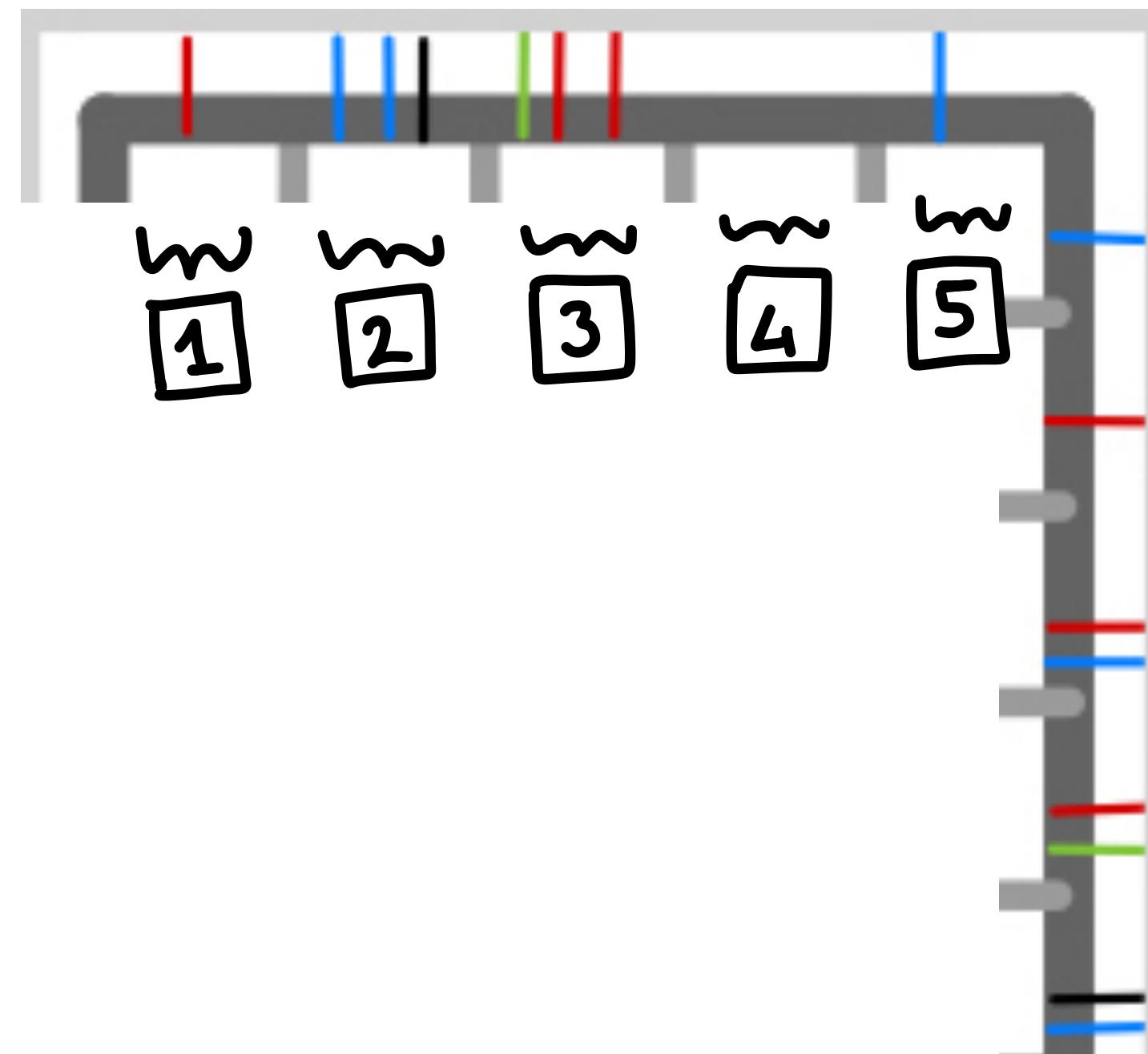
1		2	
2		1	
3			2
4			2

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

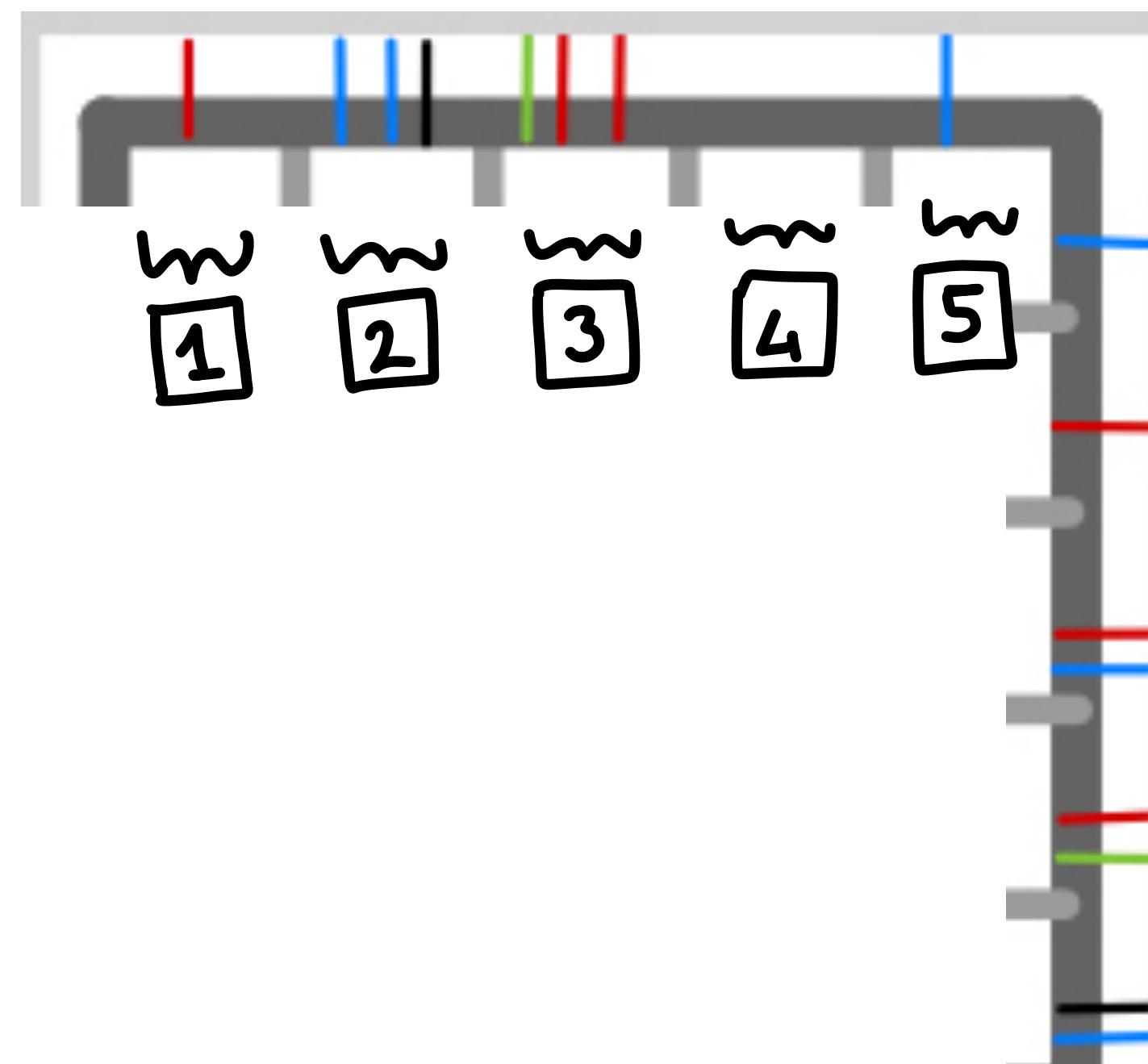
1			2	
2			1	3
3			2	2
4			3	

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

1					
2					
3					
4					

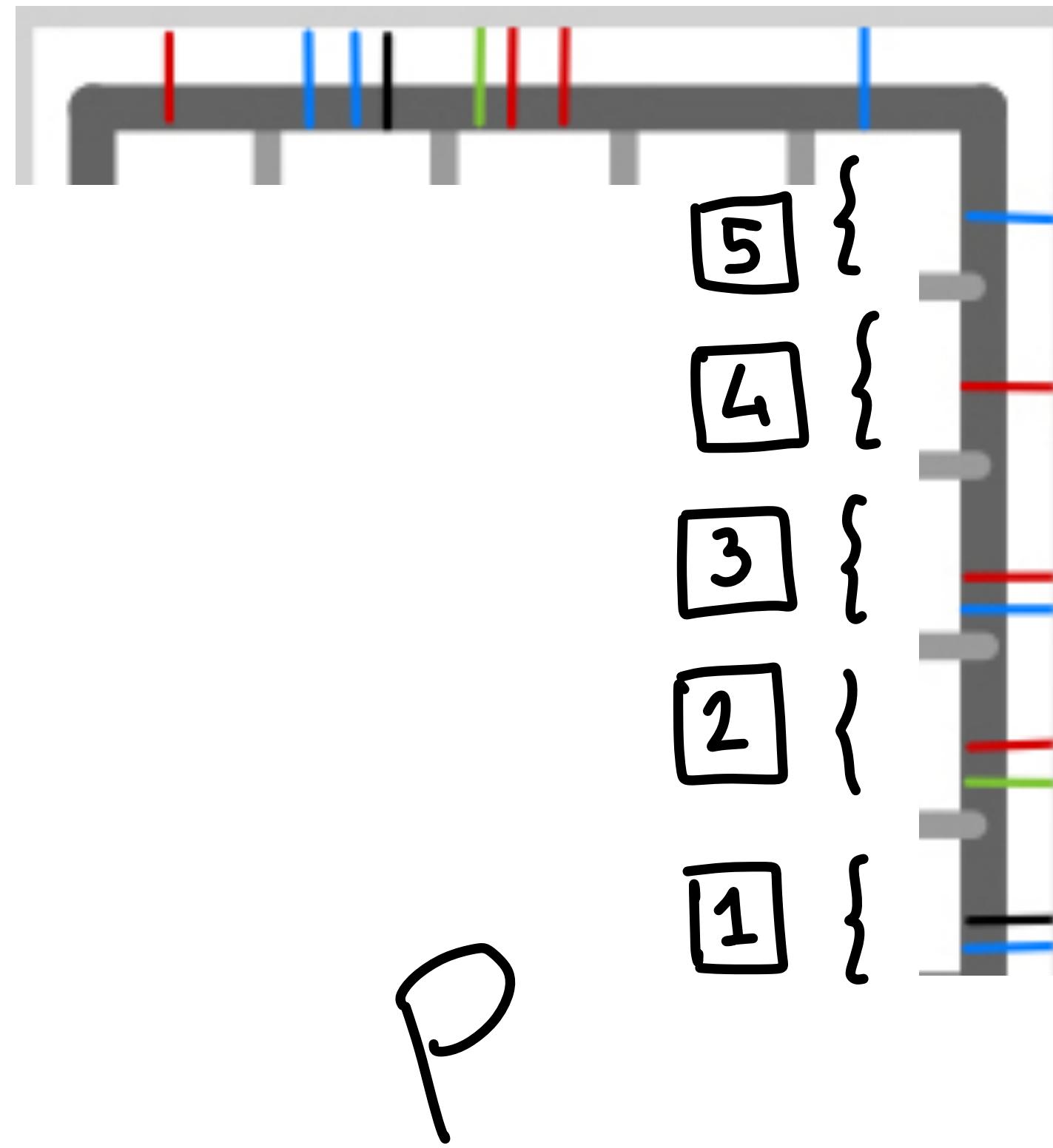
Diagram showing a 4x6 grid with numbers 1 through 4 in the first four columns and 2 through 5 in the last two columns. The numbers are colored: 2 (grey), 1 (pink), 3 (pink), 2 (blue), 2 (blue), 5 (blue), and 3 (green). The last two columns are empty.

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

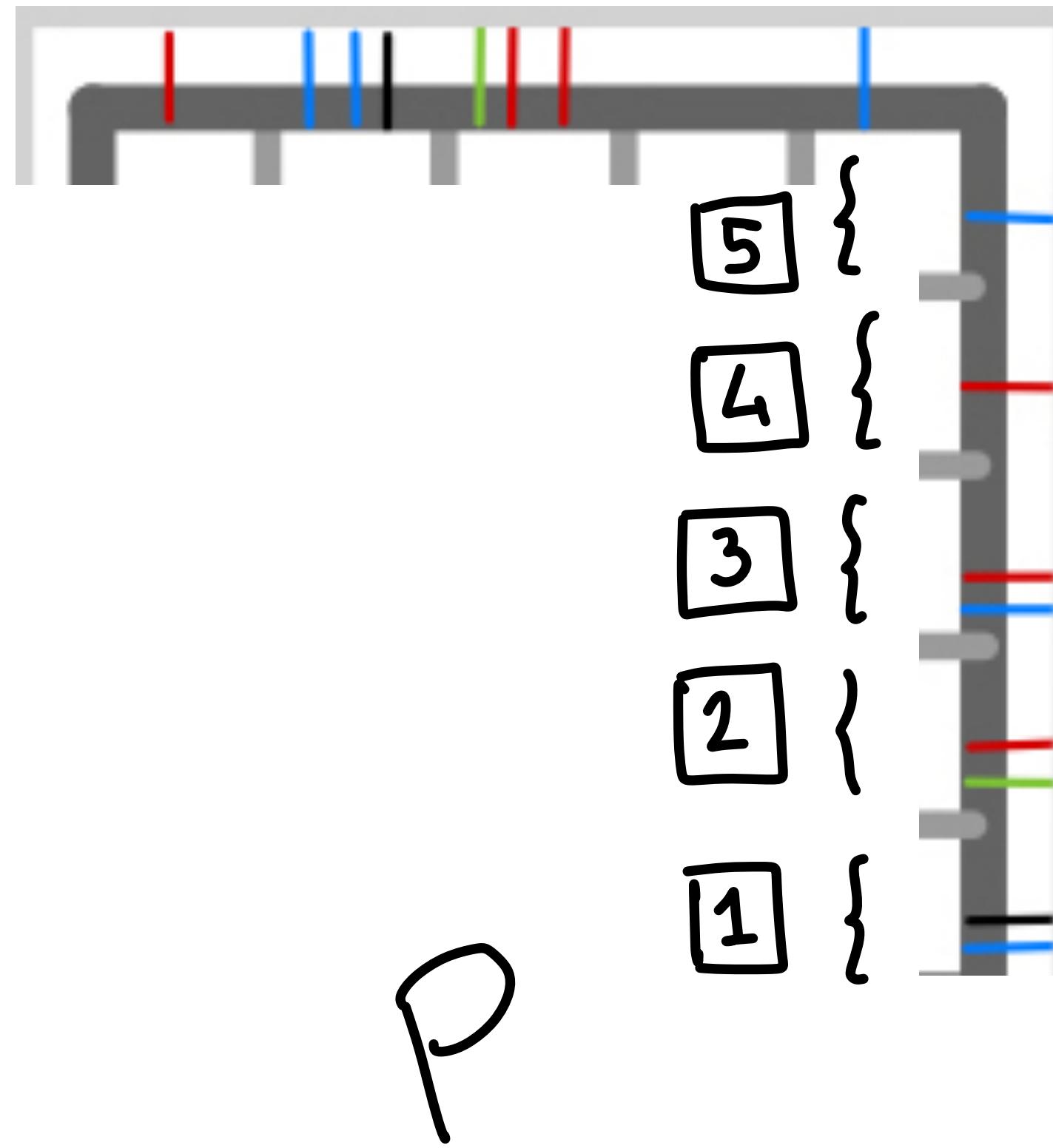
1	1	2
2		1 3 3
3	1	2 2 5
4		3

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

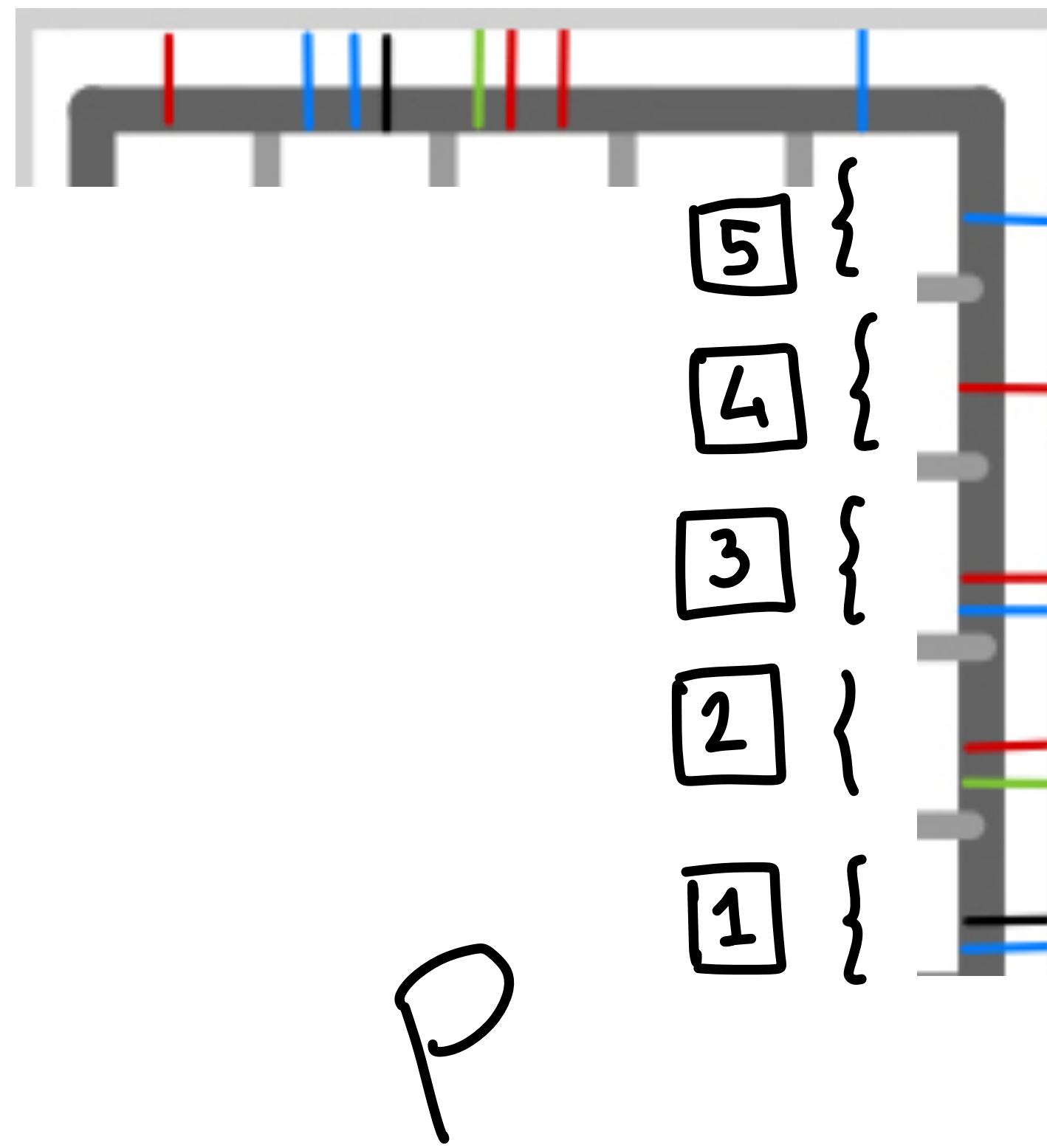
1	1	2
2	2	1 3 3
3	1	2 2 5
4	2	3

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

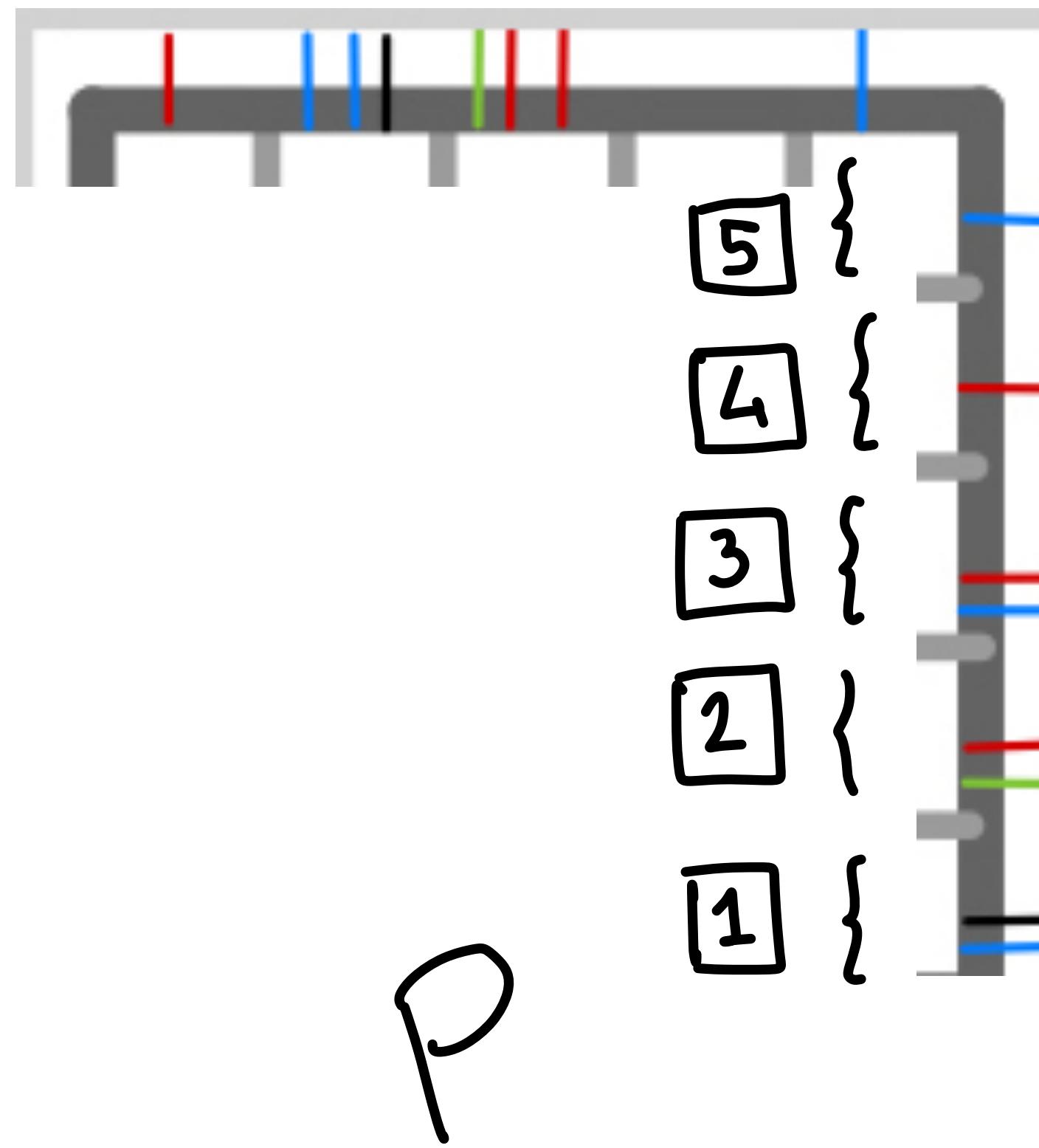
1	1	2
2	2 3	1 3 3
3	1 3	2 2 5
4	2	3

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

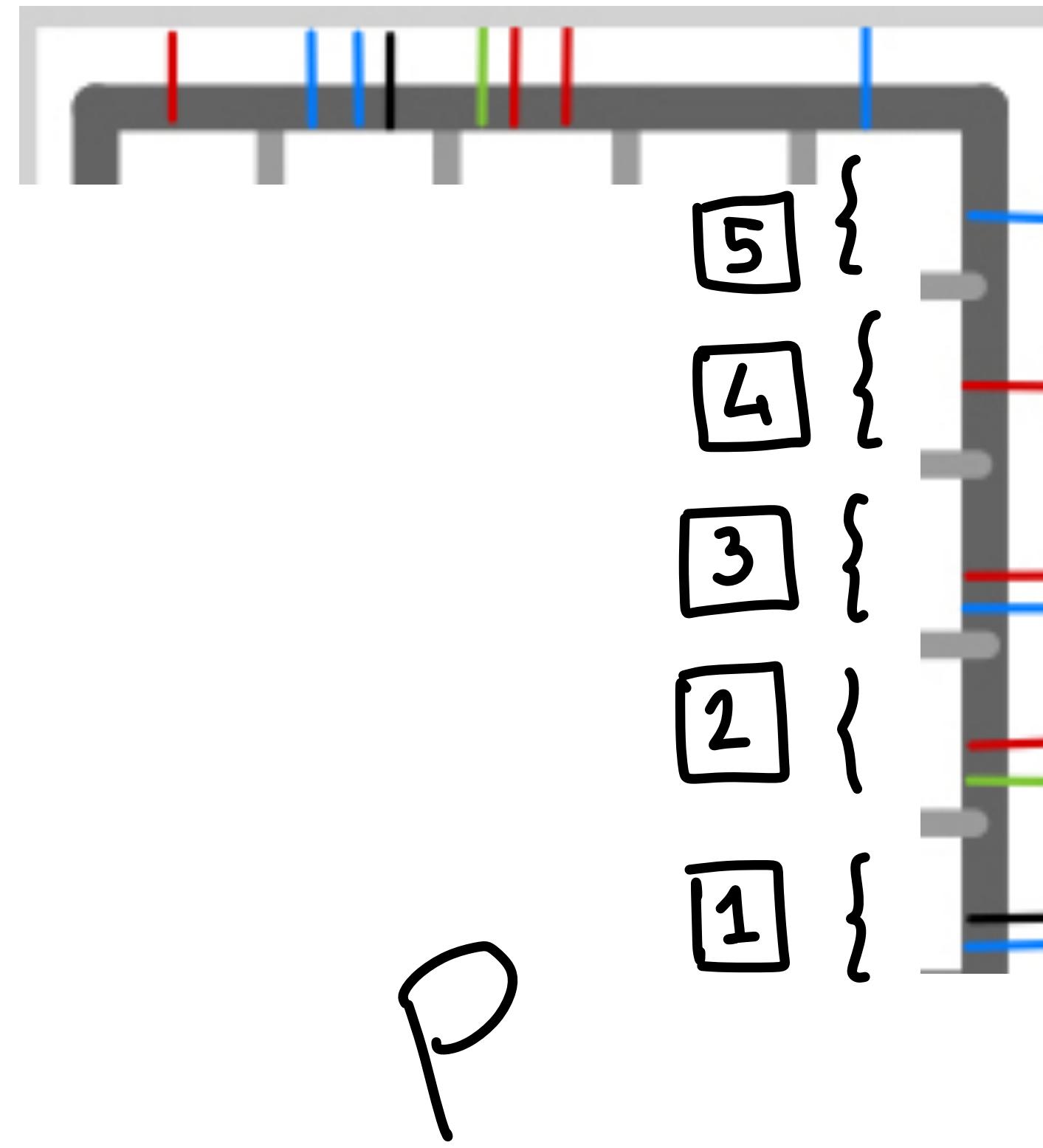
1	1	2
2	2 3 4	1 3 3
3	1 3	2 2 5
4	2	3

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

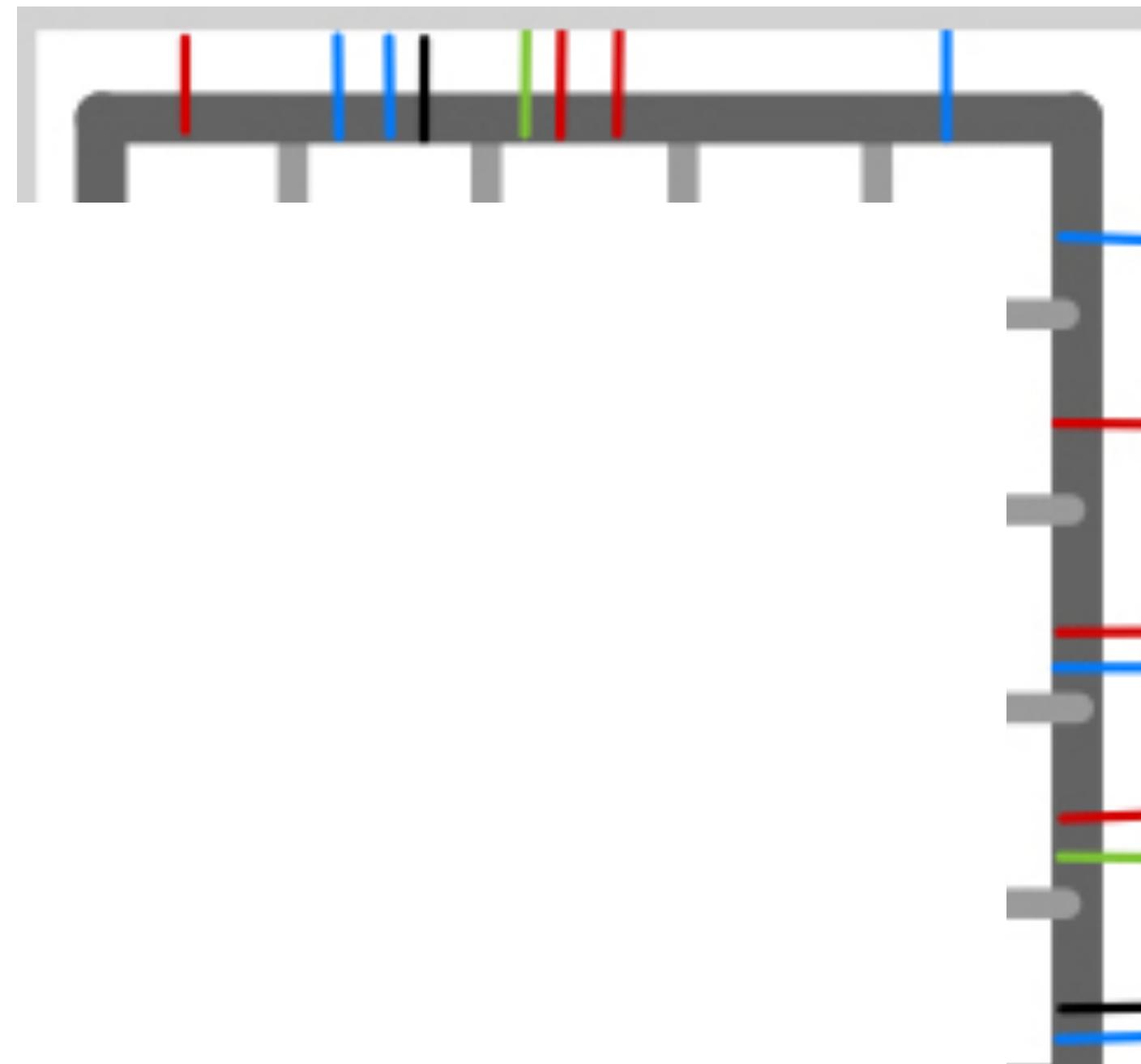
1	1	2
2	2 3 4	1 3 3
3	1 3 5	2 2 5
4	2	3

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

1	1	2
2	2 3 4	1 3 3
3	1 3 5	2 2 5
4	2	3

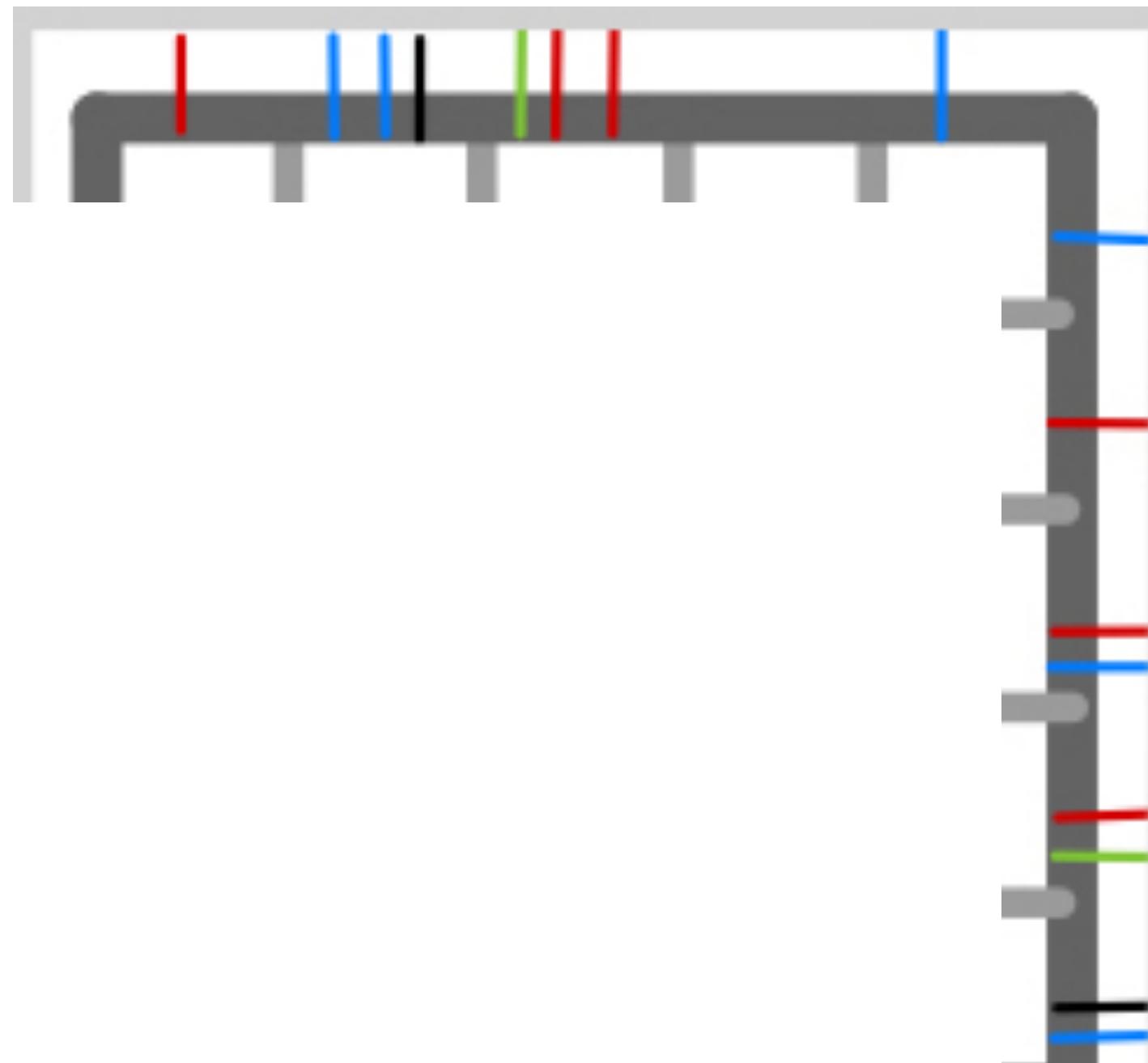
Produce a pair of semi-standard skew tableaux

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

1				1		2
2		2	3	4		1 3 3
3	1	3	5		2 2 5	3
4	2					

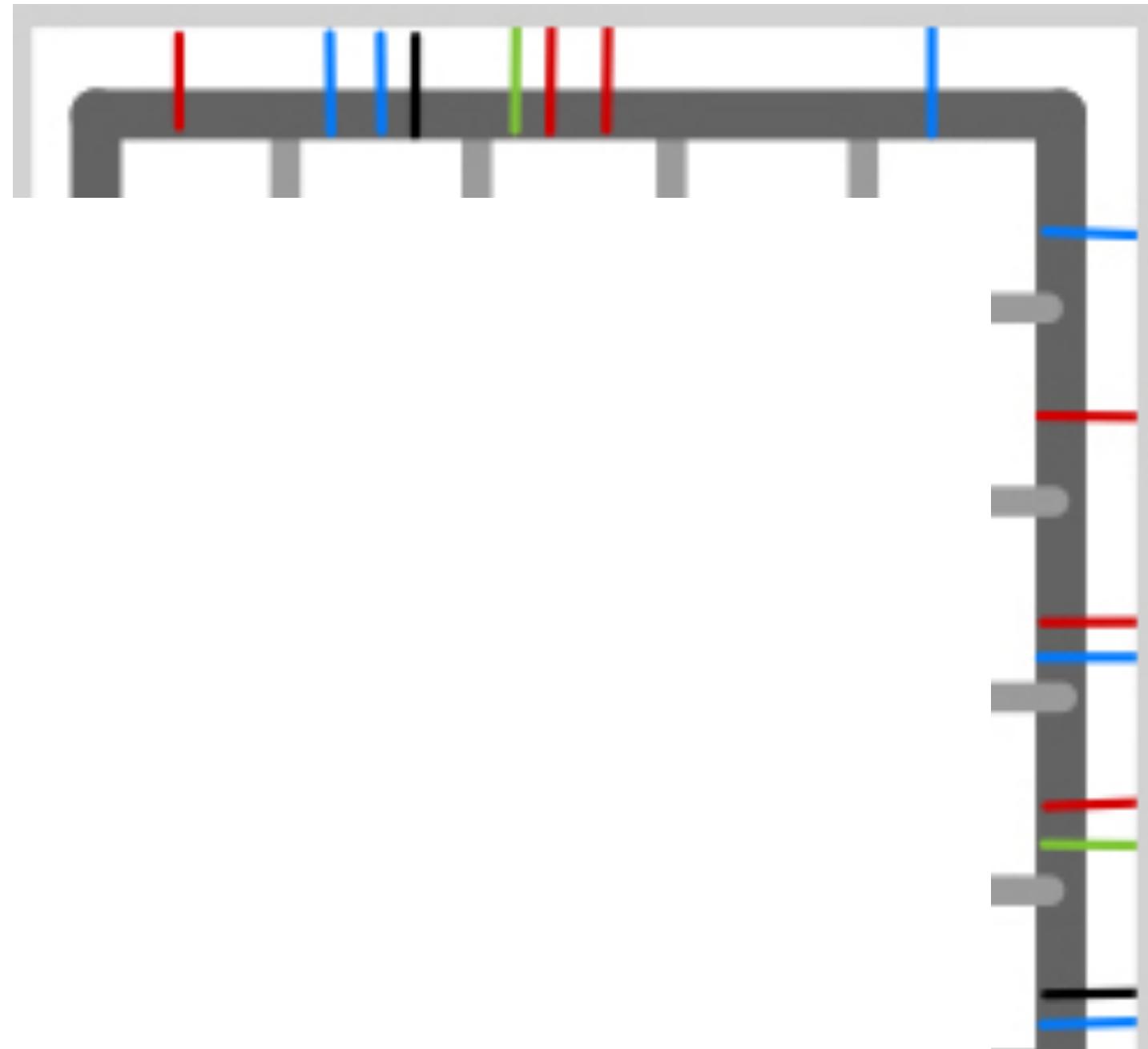
Produce a pair of semi-standard
skew tableaux

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



— 1st row

— 2nd row

— 3rd row

— 4th row

1			1		2
2		2	3	4	1 3 3
3		1	3	5	2 2 5
4		2			3

$$P, Q$$

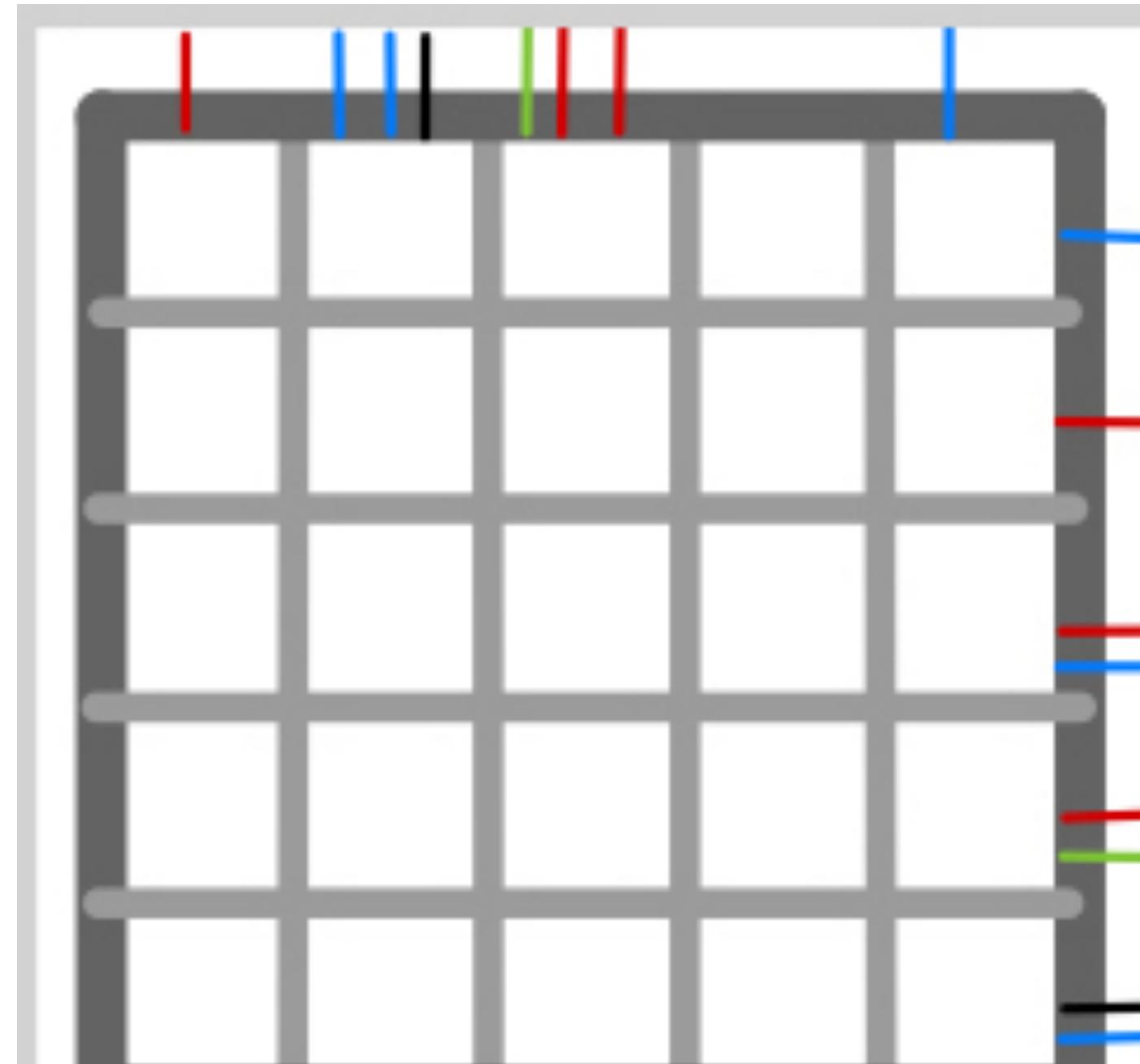
Produce a pair of semi-standard
skew tableaux

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



1			1		2
2		2	3	4	1 3 3
3		1	3	5	2 2 5
4		2			3

$$P, Q$$

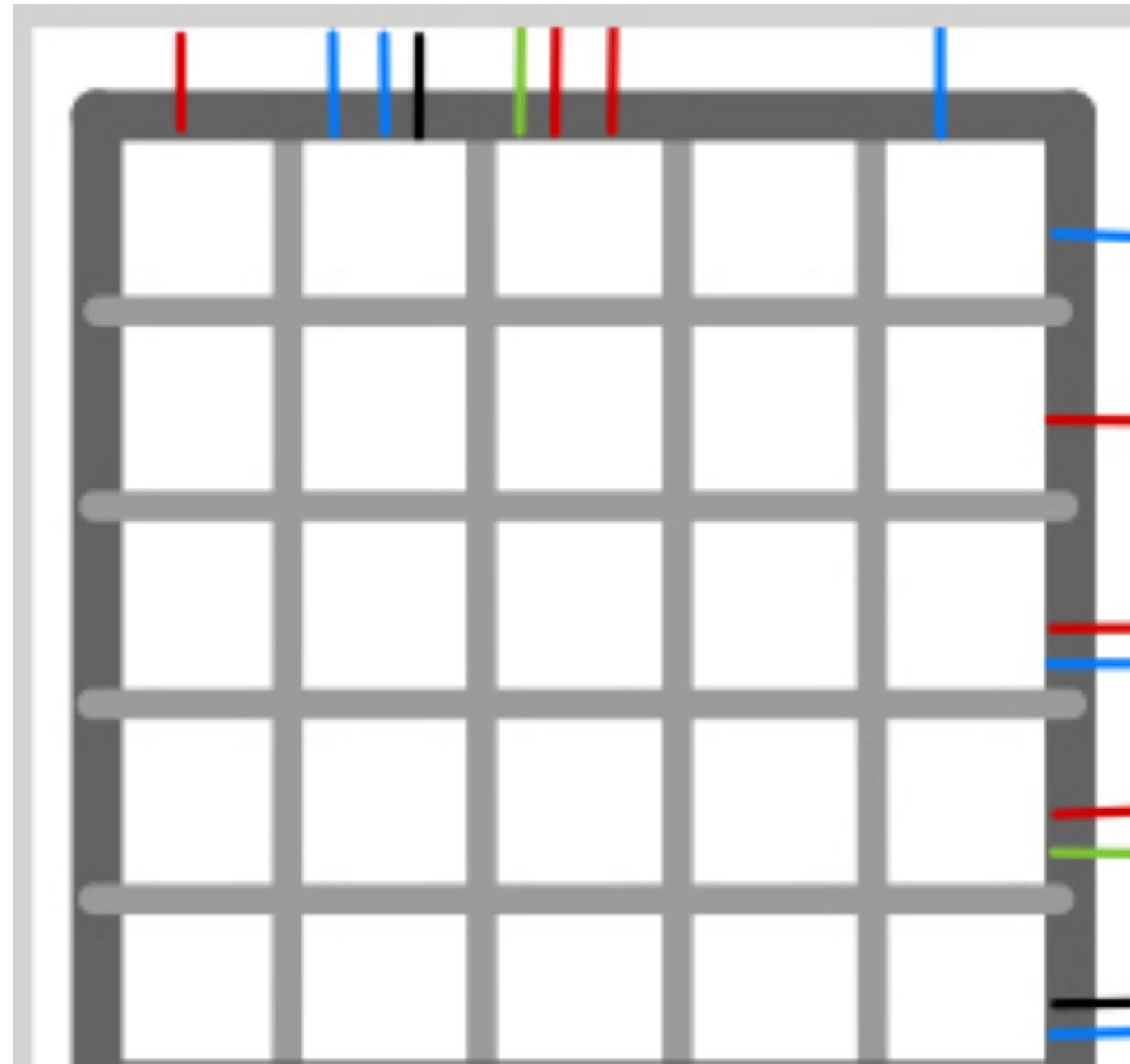
Fact: if ρ = empty shape of P, Q , then $|\rho| = \sum_{k>0} \sum_{i,j=1}^n k M_{i,j}^k$ [Sagan-Stanley'89]

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$



1				1		2
2			2	3	4	1 3 3
3			1	3	5	2 2 5
4			2			3

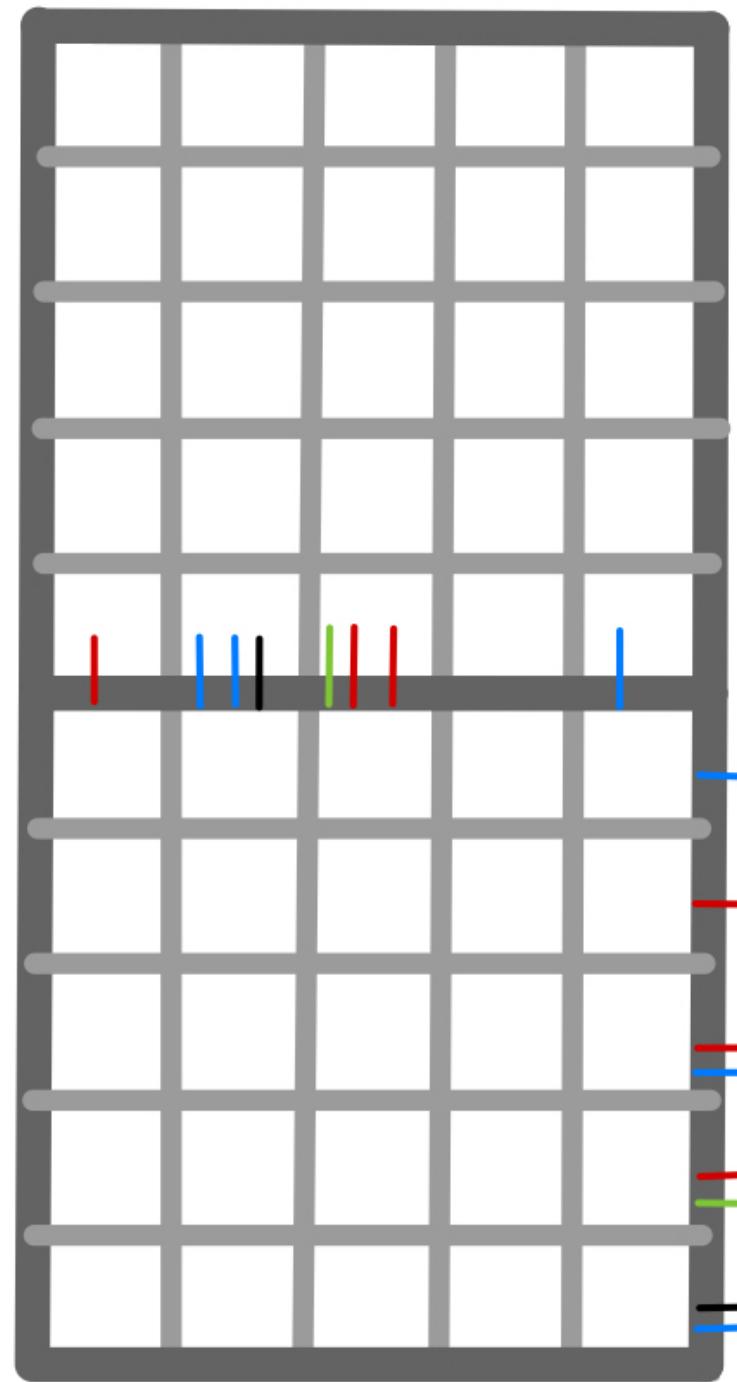
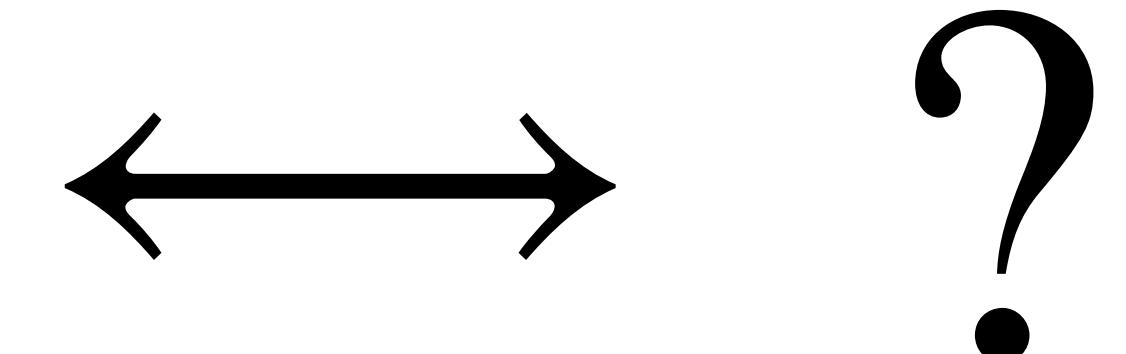
$$P, Q$$

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



1			1		2
2		2	3	4	1 3 3
3		1	3	5	2 2 5
4		2			3

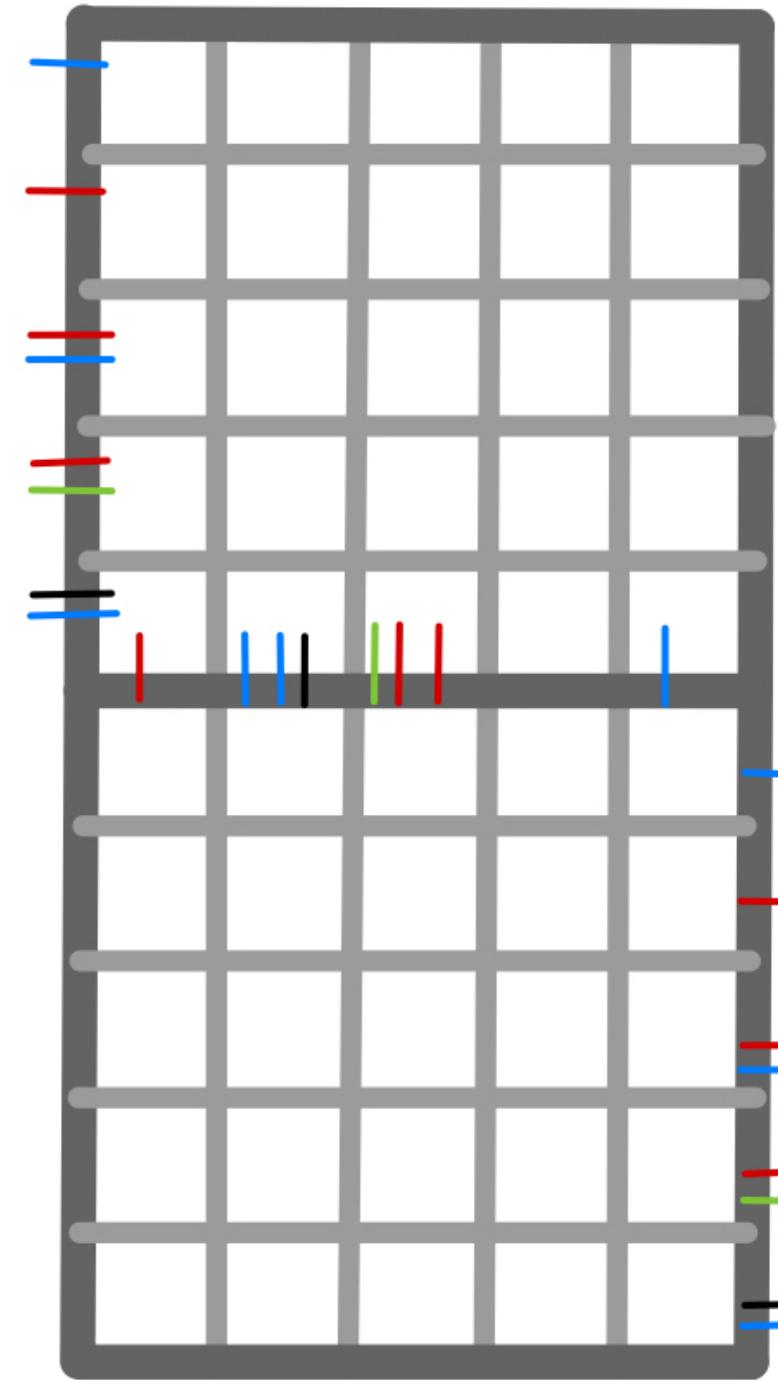
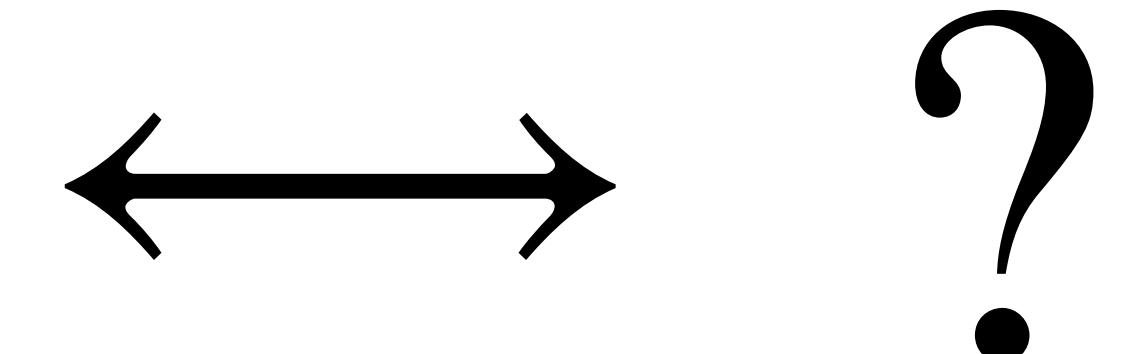
$$P, Q$$

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



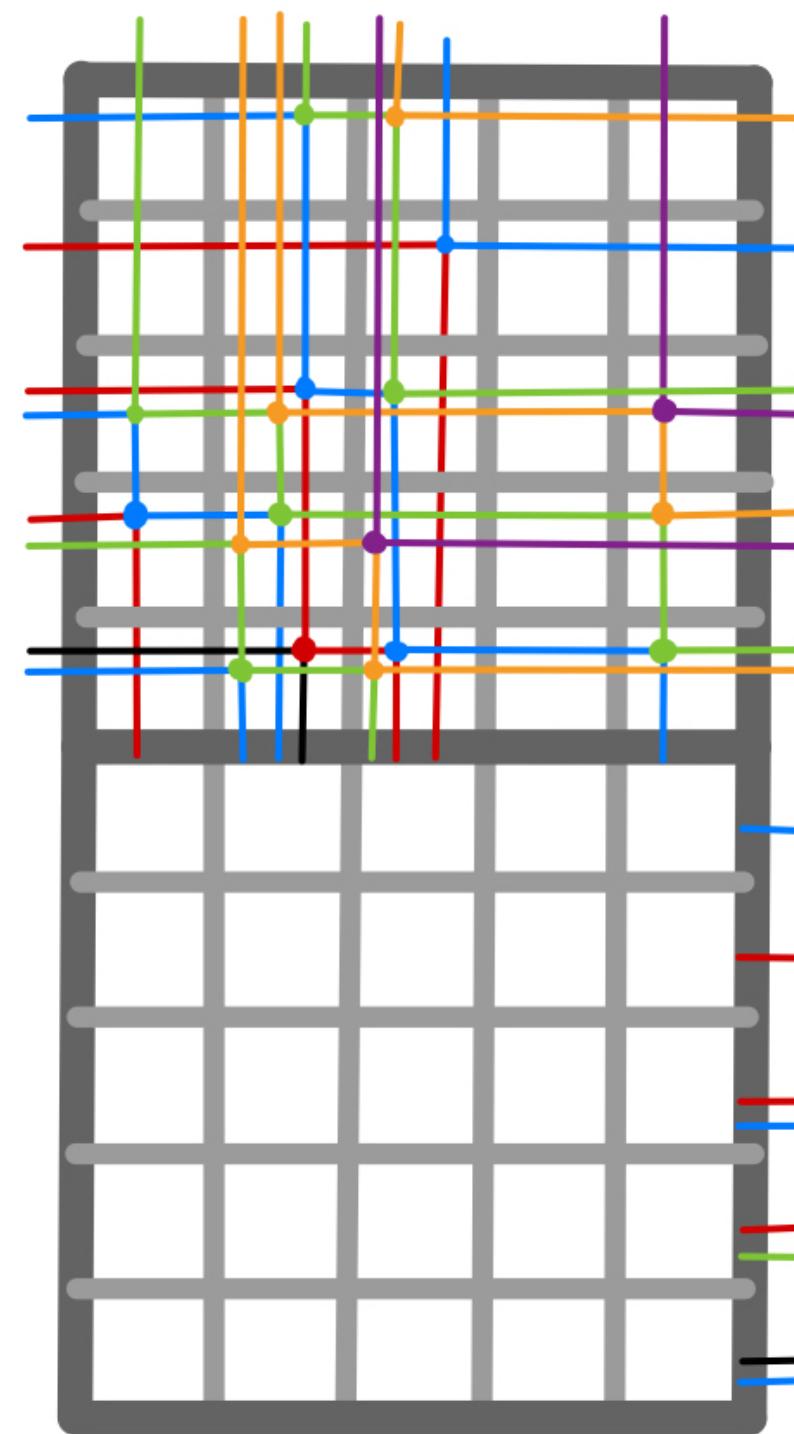
1			1		2
2		2	3	4	1 3 3
3		1	3	5	2 2 5
4		2			3

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$$\longleftrightarrow ?$$

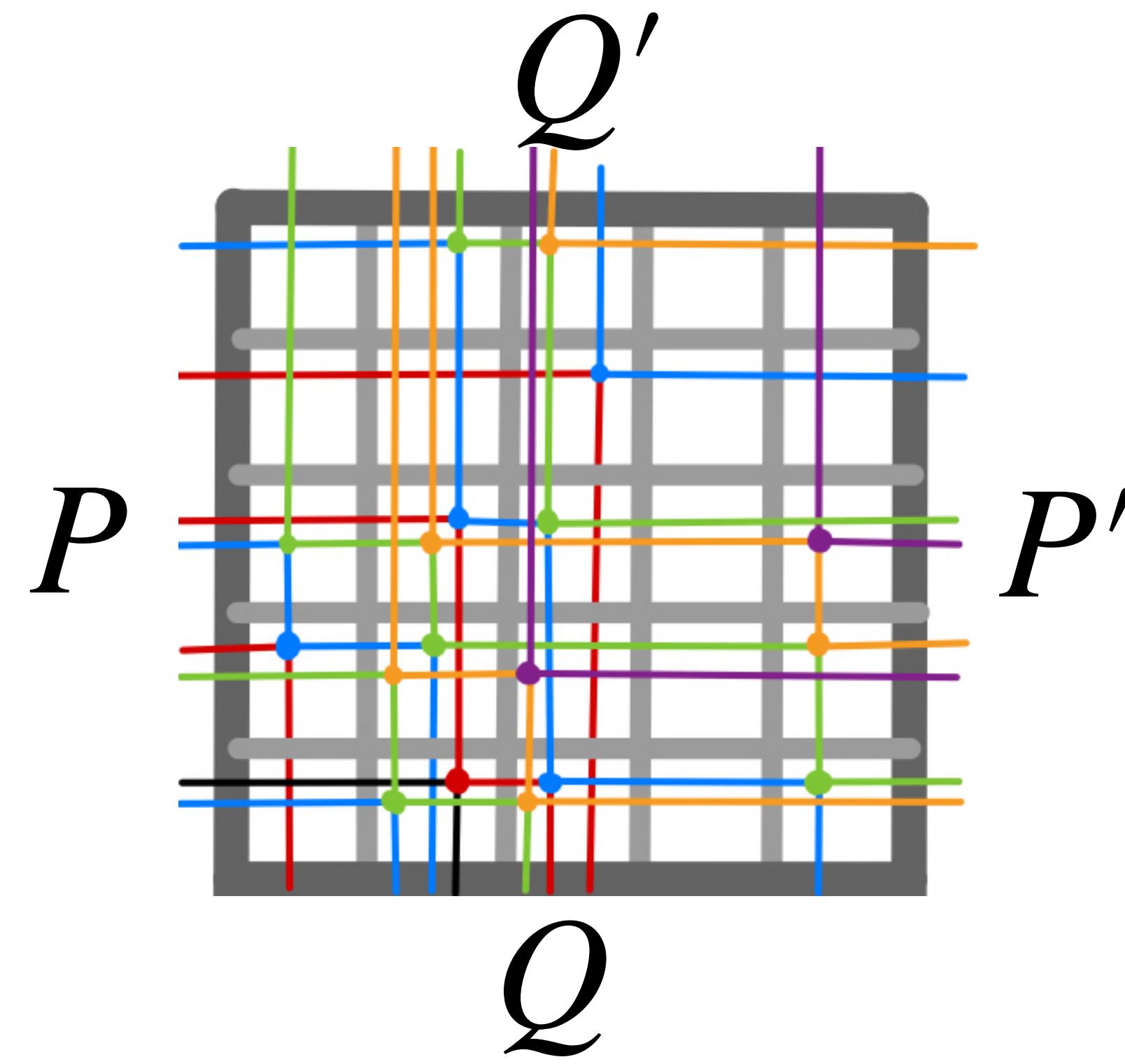
1			1		2
2		2	3	4	1 3 3
3		1	3	5	2 2 5
4		2			3

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$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

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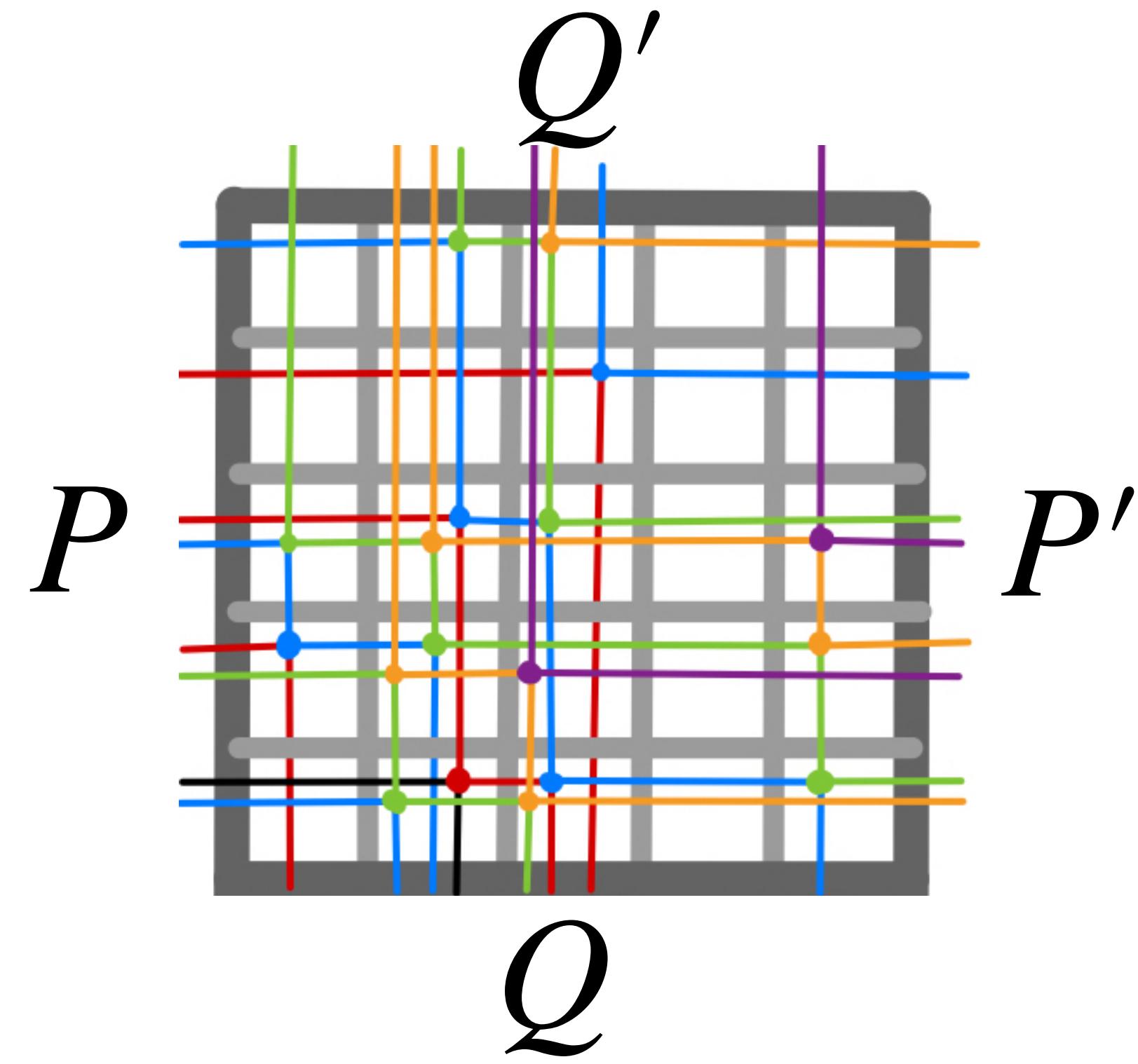
1				1	2
2		2	3	4	1 3 3
3		1	3	5	2 2 5
4		2			3

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$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$

$$\longleftrightarrow ?$$

1		1		2
2		2 3 4		
3	1 3 5		1 3 3	
4	2		2 2 5	
			3	

1		1		3
2			4	
3				1 2
4	1 3			2 2 3
5	1 2 5			3 5
6	2 3			

$$P \quad , \quad Q$$

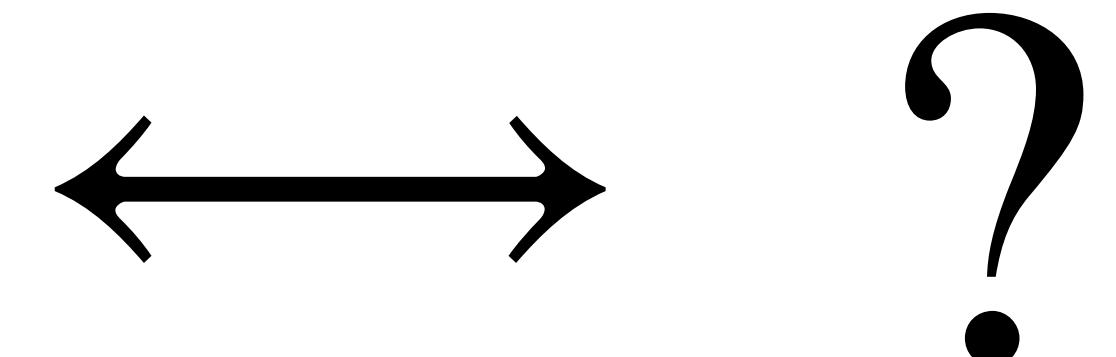
$$P' \quad , \quad Q'$$

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



1		1		
2		2	3	4
3	1	3	5	
4	2		2	5

	1	3	3
	2	2	5
	3		

$$P \quad , \quad Q$$

RSK
→

1								
2								
3								
4								
5								
6								

$$P' \quad , \quad Q'$$

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \longleftrightarrow ?$$

$$(P, Q) \xrightarrow{\text{RSK}} (P', Q')$$

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \longleftrightarrow ?$$

$$(P, Q) \xrightarrow{\text{RSK}} (P', Q') \rightarrow \dots \rightarrow (P^{(n)}, Q^{(n)})$$

Skew RSK dynamics

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Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \quad \left(\begin{array}{ccccc} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{array} \right) \longleftrightarrow ?$$

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Skew RSK dynamics

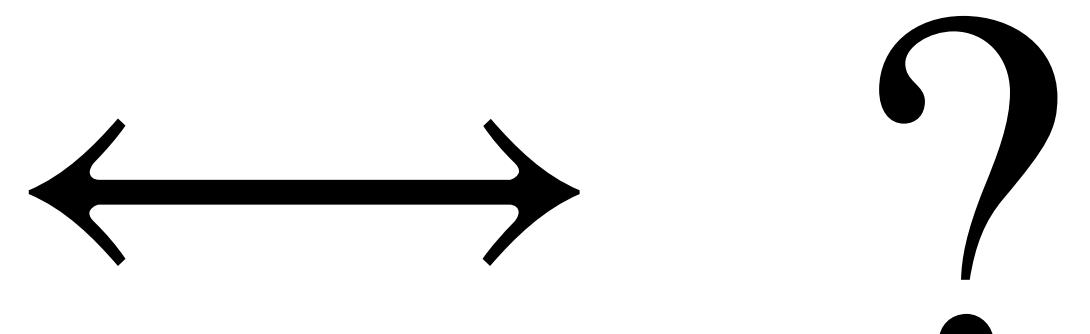
Fact: Asymptotically the tableaux $P^{(n)}, Q^{(n)}$ become “stable”

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Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



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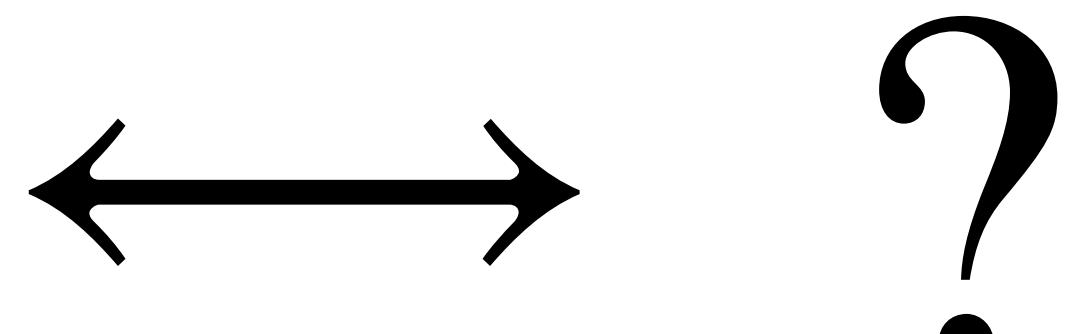
12	4	3
12	4	3
22	3	2
23	1 5	2 3
24	1	5
31	1	1
32	2	2
33	3	3

$$P^{(10)}, Q^{(10)}$$

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



$$(P, Q) \xrightarrow{\text{RSK}} (P', Q') \rightarrow \dots \rightarrow (P^{(n)}, Q^{(n)})$$

Skew RSK dynamics

Fact: Asymptotically the tableaux $P^{(n)}, Q^{(n)}$ become “stable”

From stable configurations we determine
vertically strict tableaux V, W

12	4	3
12	4	3
22	3	2
23	1 5	2 3
24	1	5
1		
31	1	1
32	2	2
33	3	3

$$P^{(10)}, Q^{(10)}$$

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



1	1	3	4
2	1	5	
3			

1	2	2	3
2	5	3	
3			

; κ

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$P^{(10)}, Q^{(10)}$

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$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



1	1	3	4
2	1	5	
3			

1	2	2	3
2	5	3	
3			

; κ

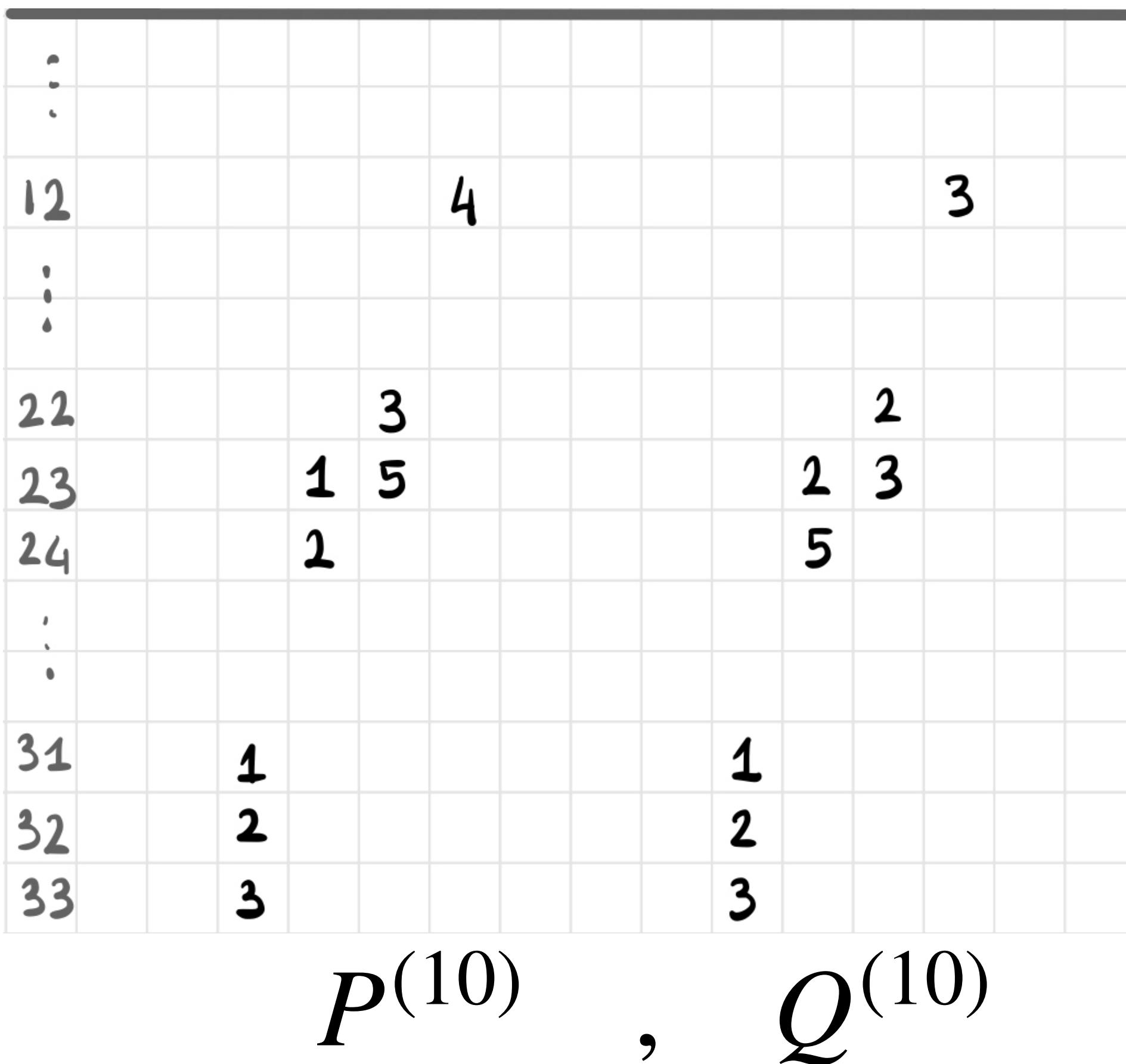
$$(P, Q) \xrightarrow{\text{RSK}} (P', Q') \rightarrow \dots \rightarrow (P^{(n)}, Q^{(n)})$$

Skew RSK dynamics

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To determine κ we need to study the scattering of
the skew RSK dynamics (no time today)



Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix}$$



1	1	3	4
2	1	5	
3			

1	2	2	3
2	5	3	
3			

; κ

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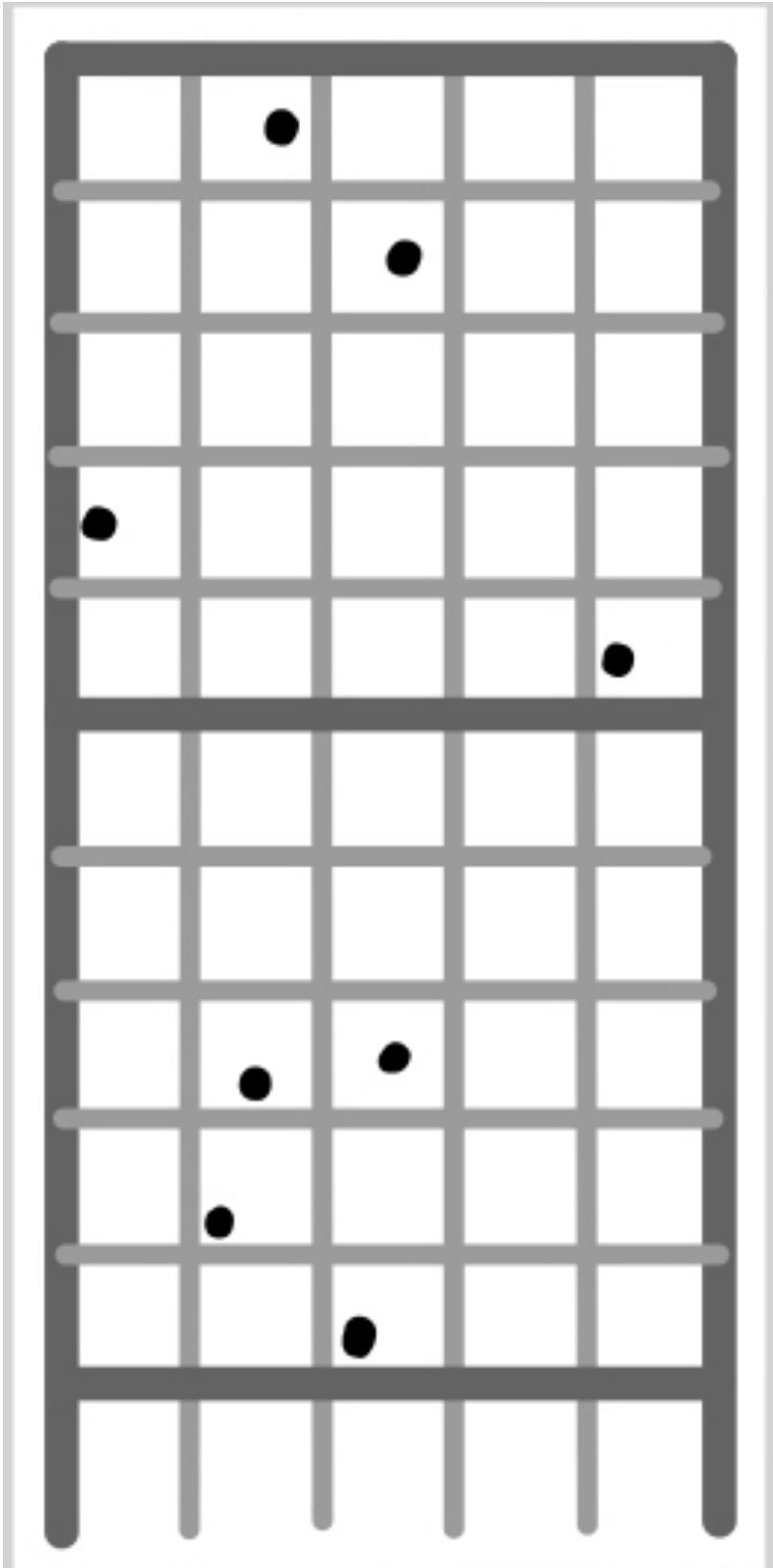
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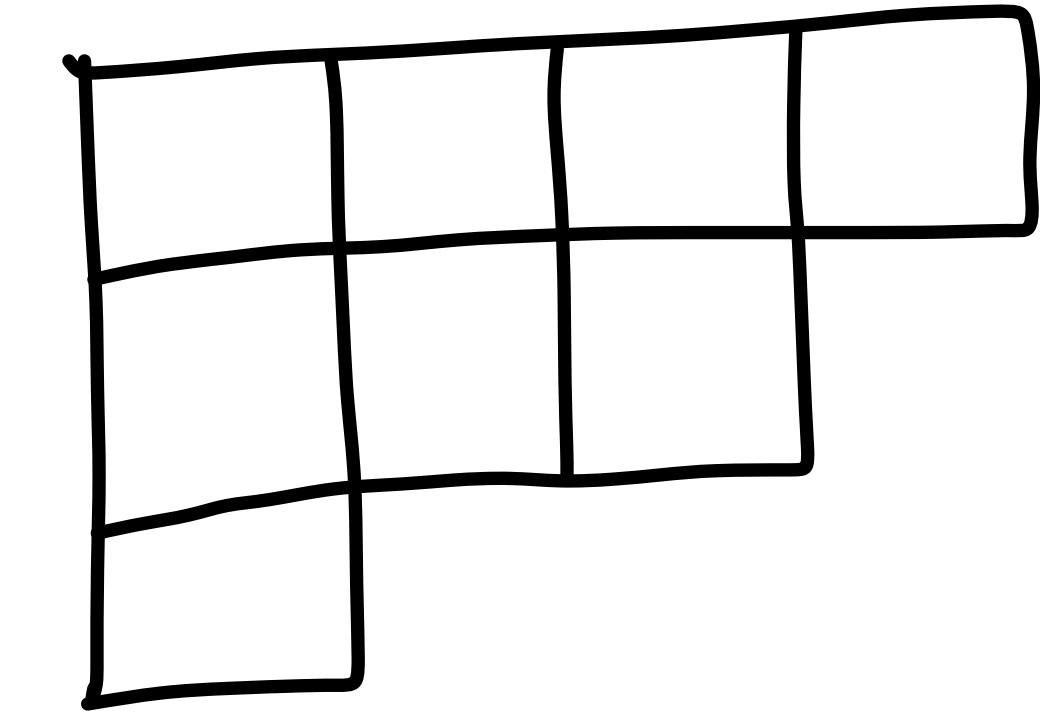
Question: Can we characterize the shape μ of V, W in terms of the matrix $M_{i,j}^k$?

Greene invariants

- **Definition:**

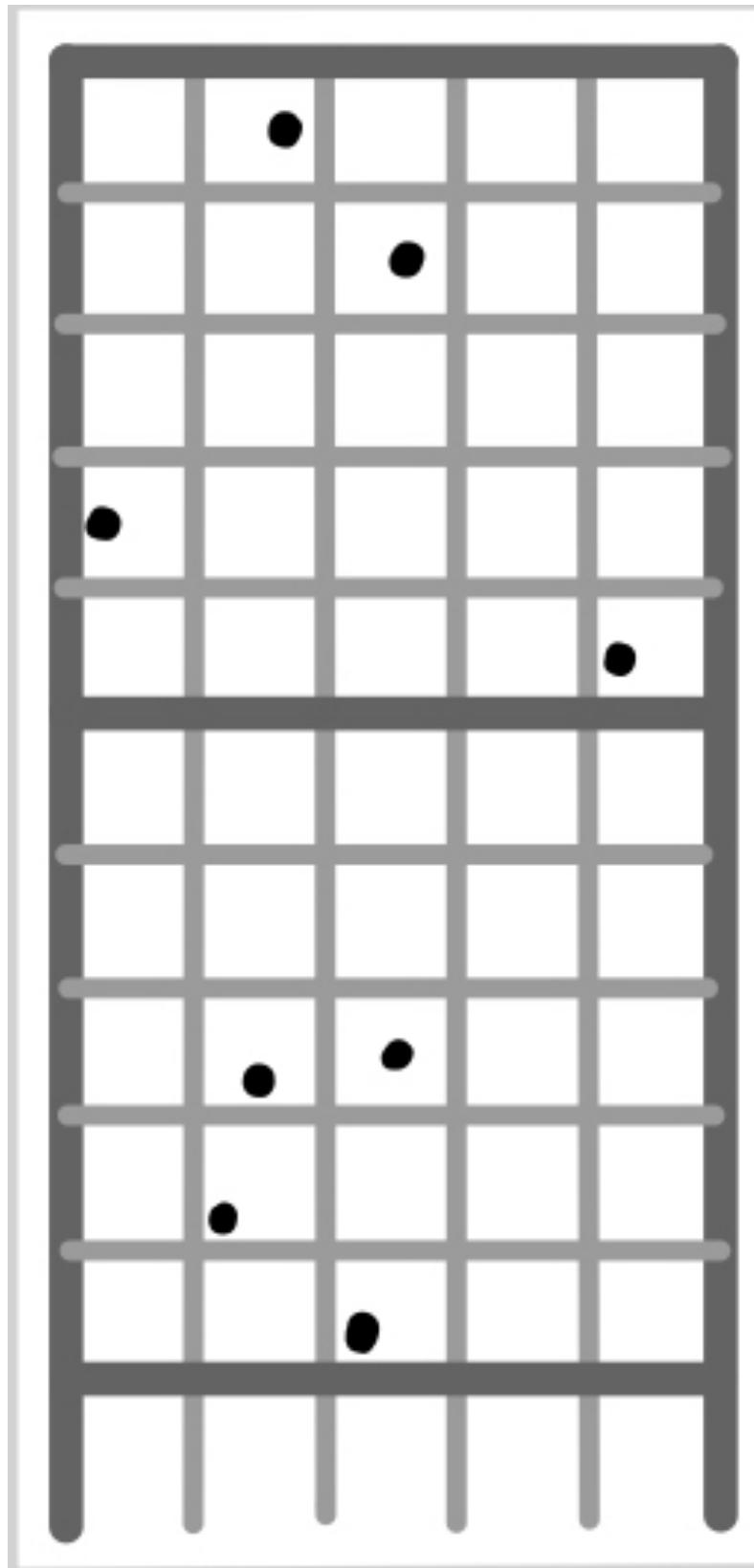


- LDS_k = maximal number of bullets in k nonintersecting loops
- LIS_k = maximal number of bullets in k nonintersecting up-right paths

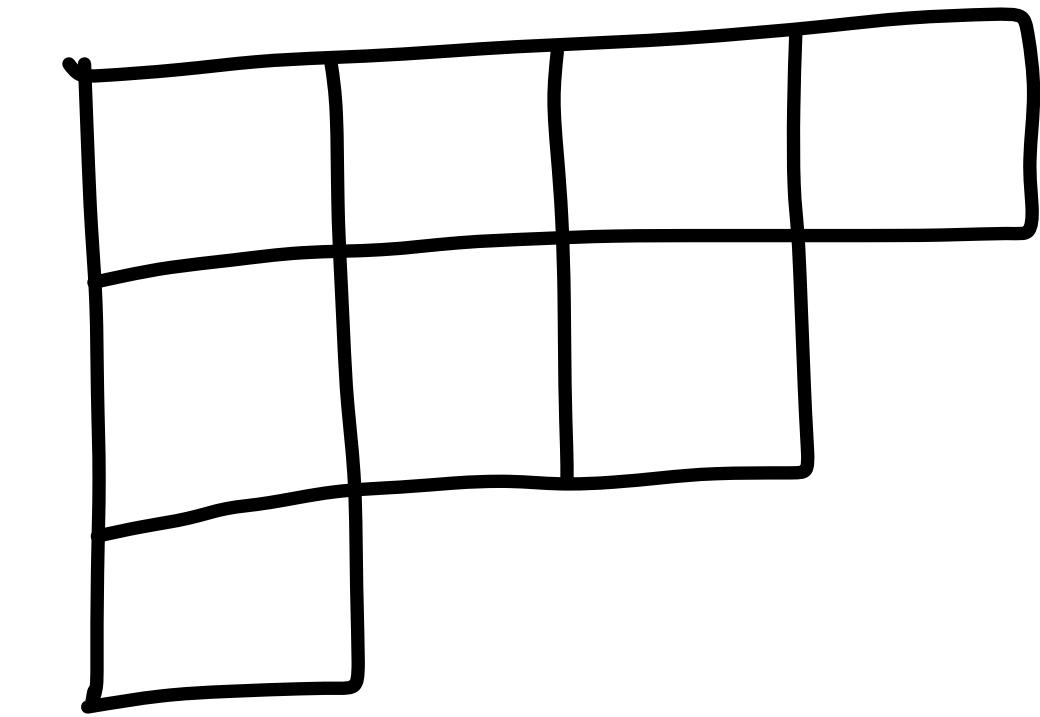


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Theorem [IMS'21]

LDS_k and LIS_k determine the shape μ of (V, W)

$$\mu_1 = \text{LIS}_1$$

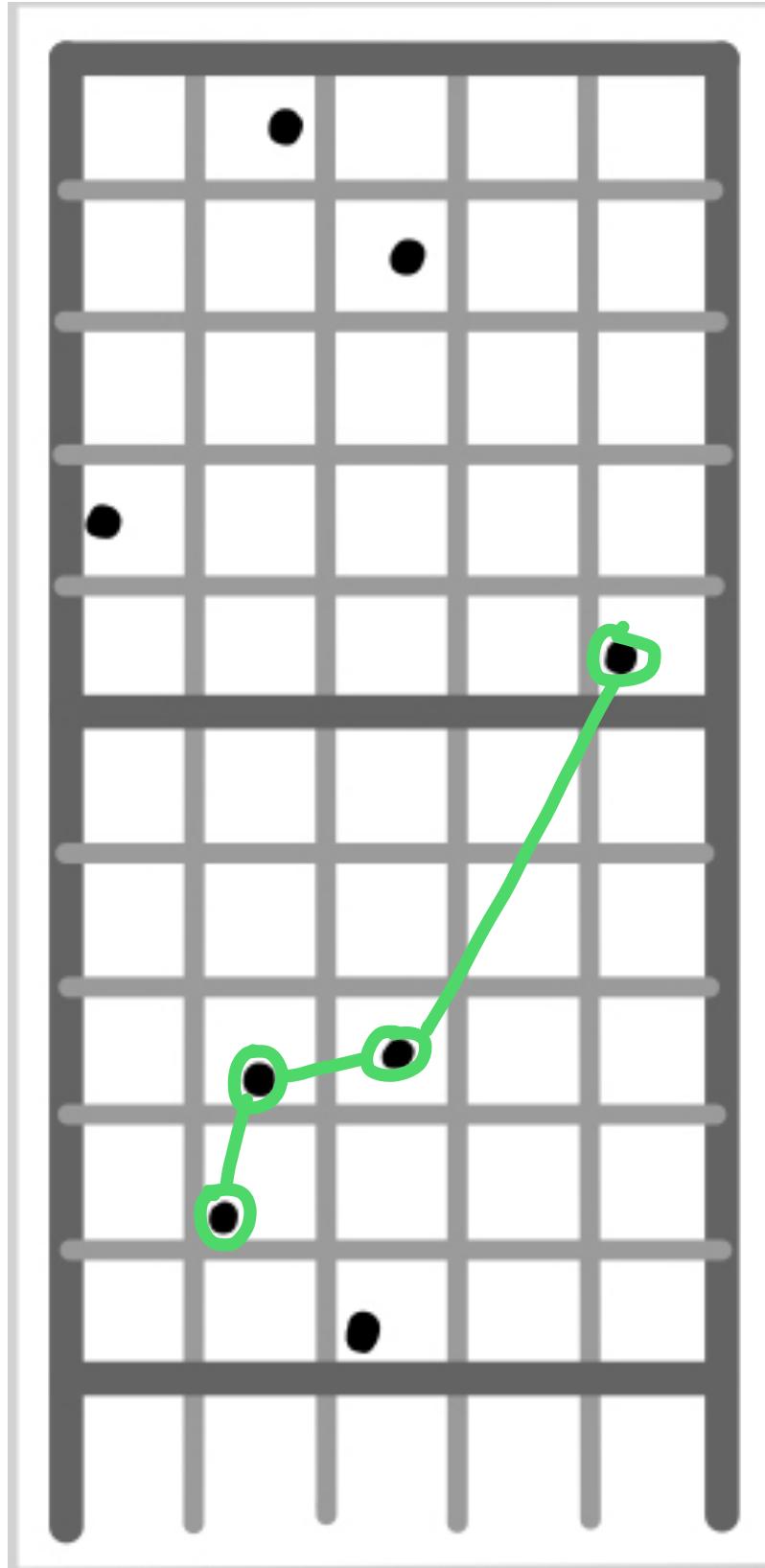
$$\mu'_1 = \text{LDS}_1$$

$$\mu_1 + \cdots + \mu_\ell = \text{LIS}_\ell$$

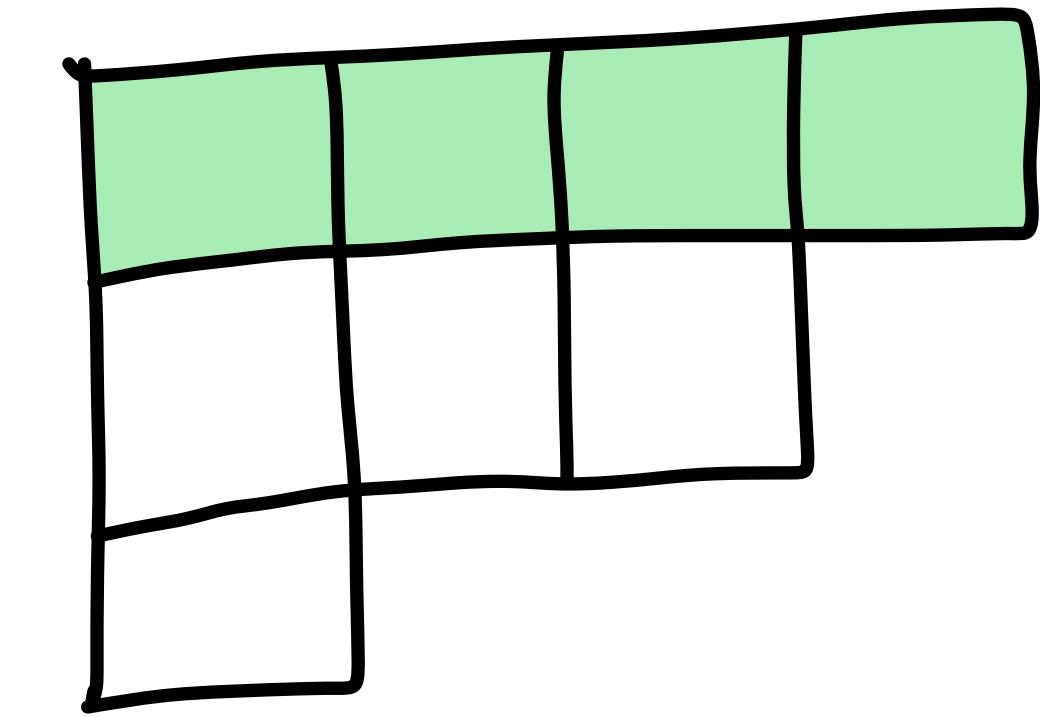
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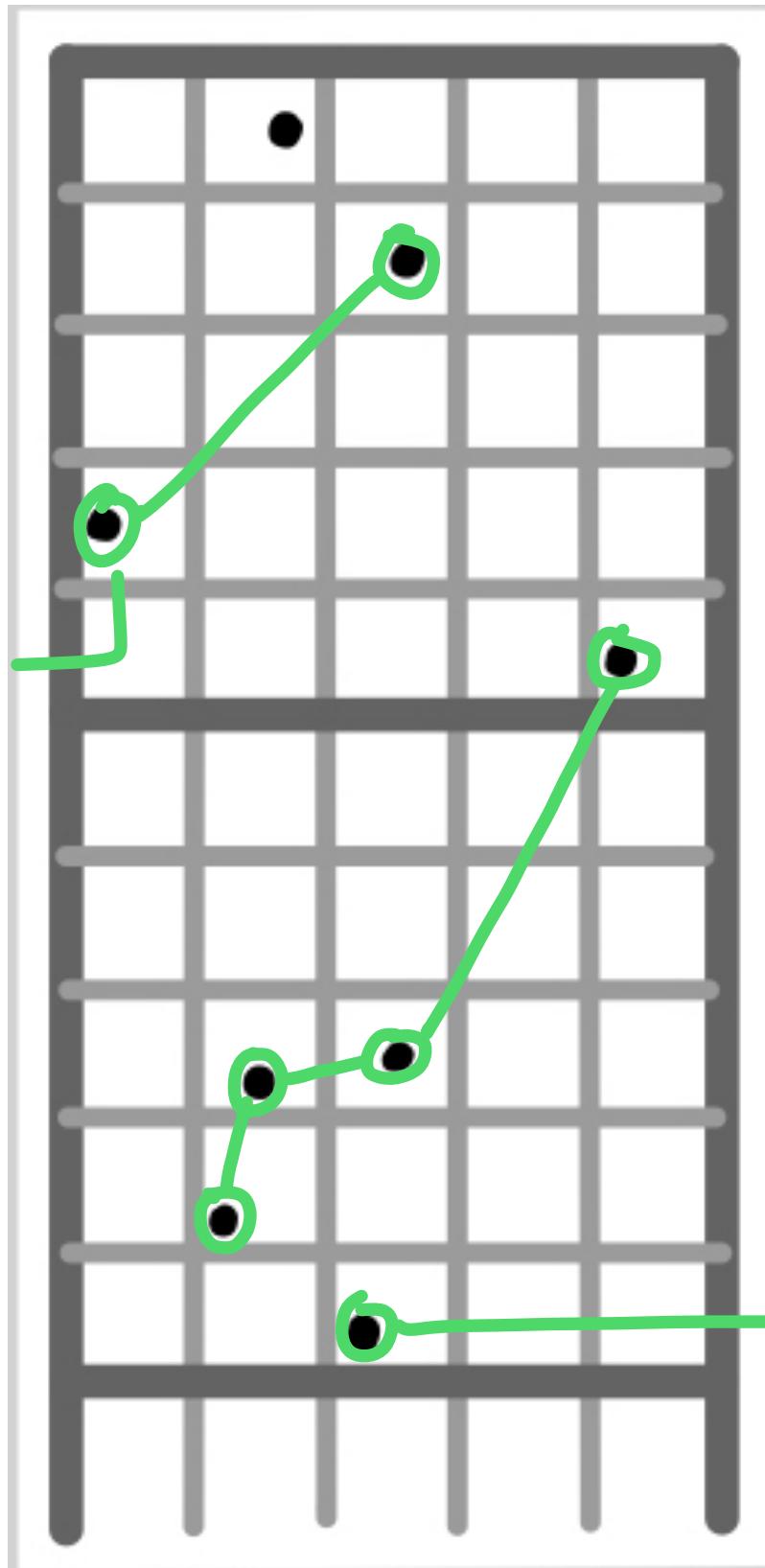
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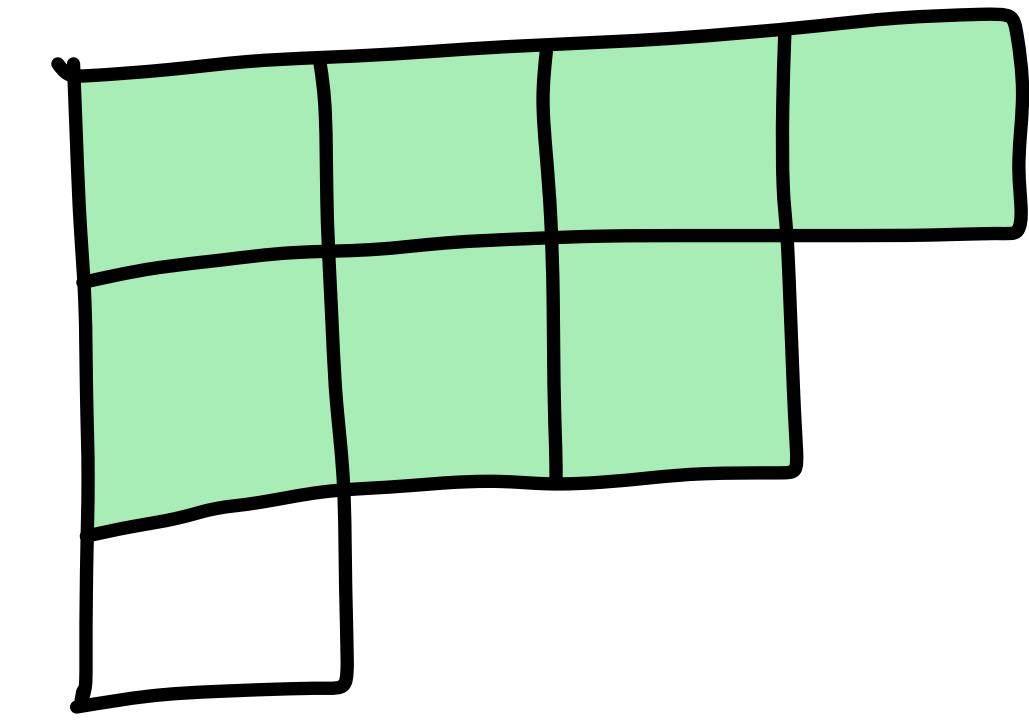
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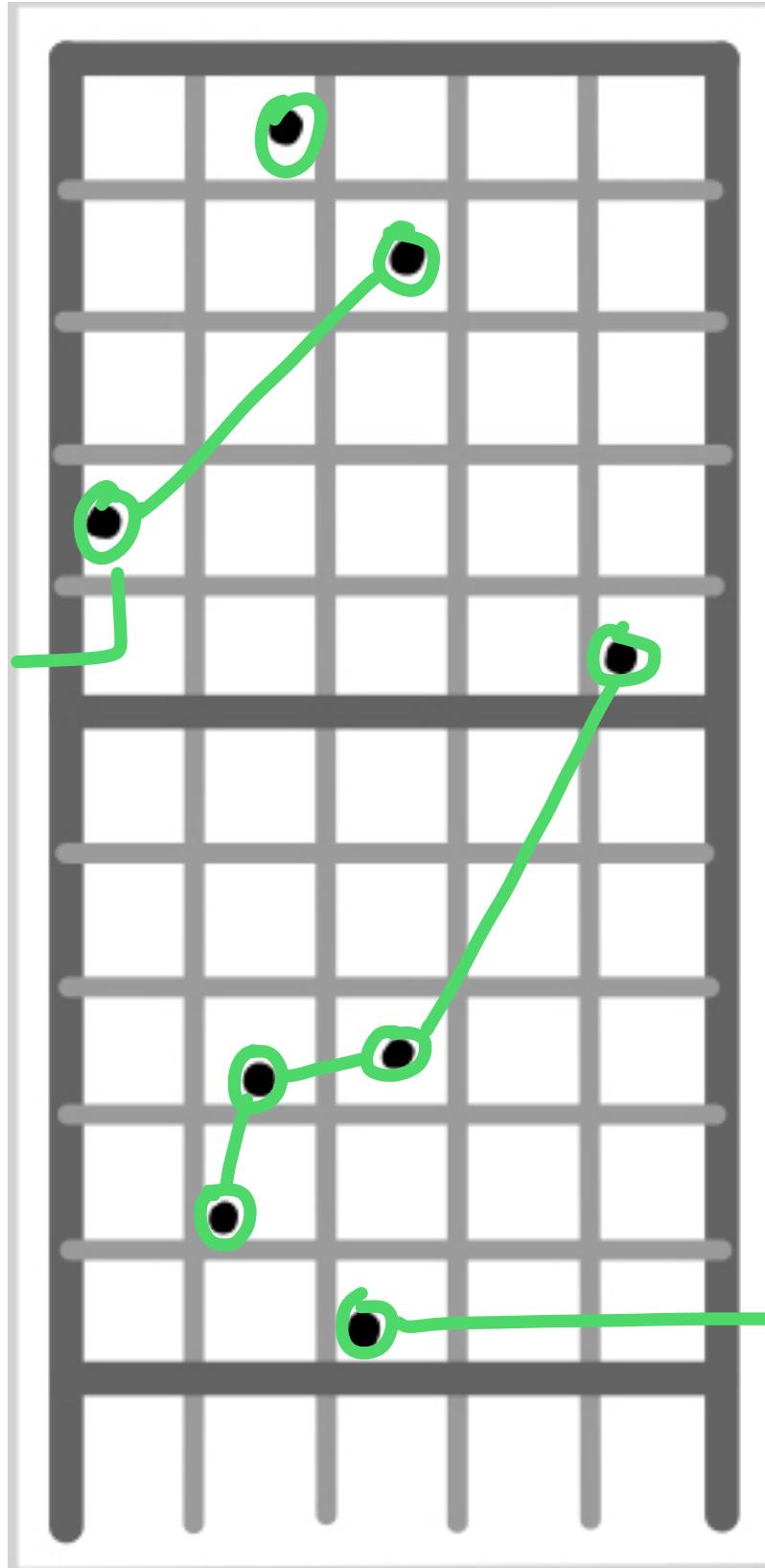
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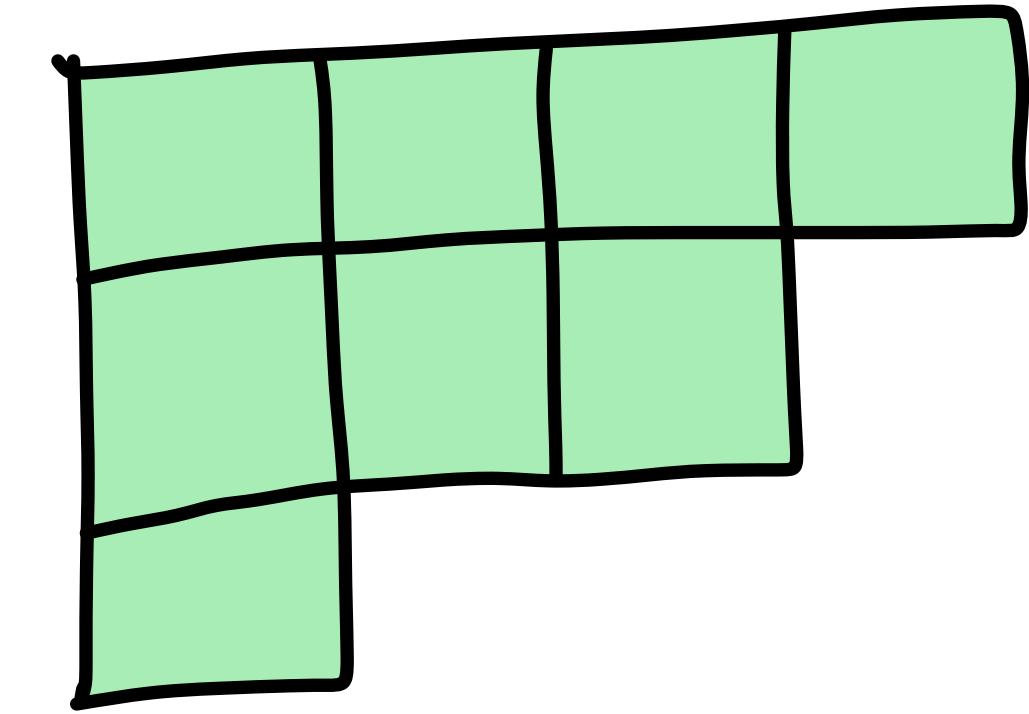
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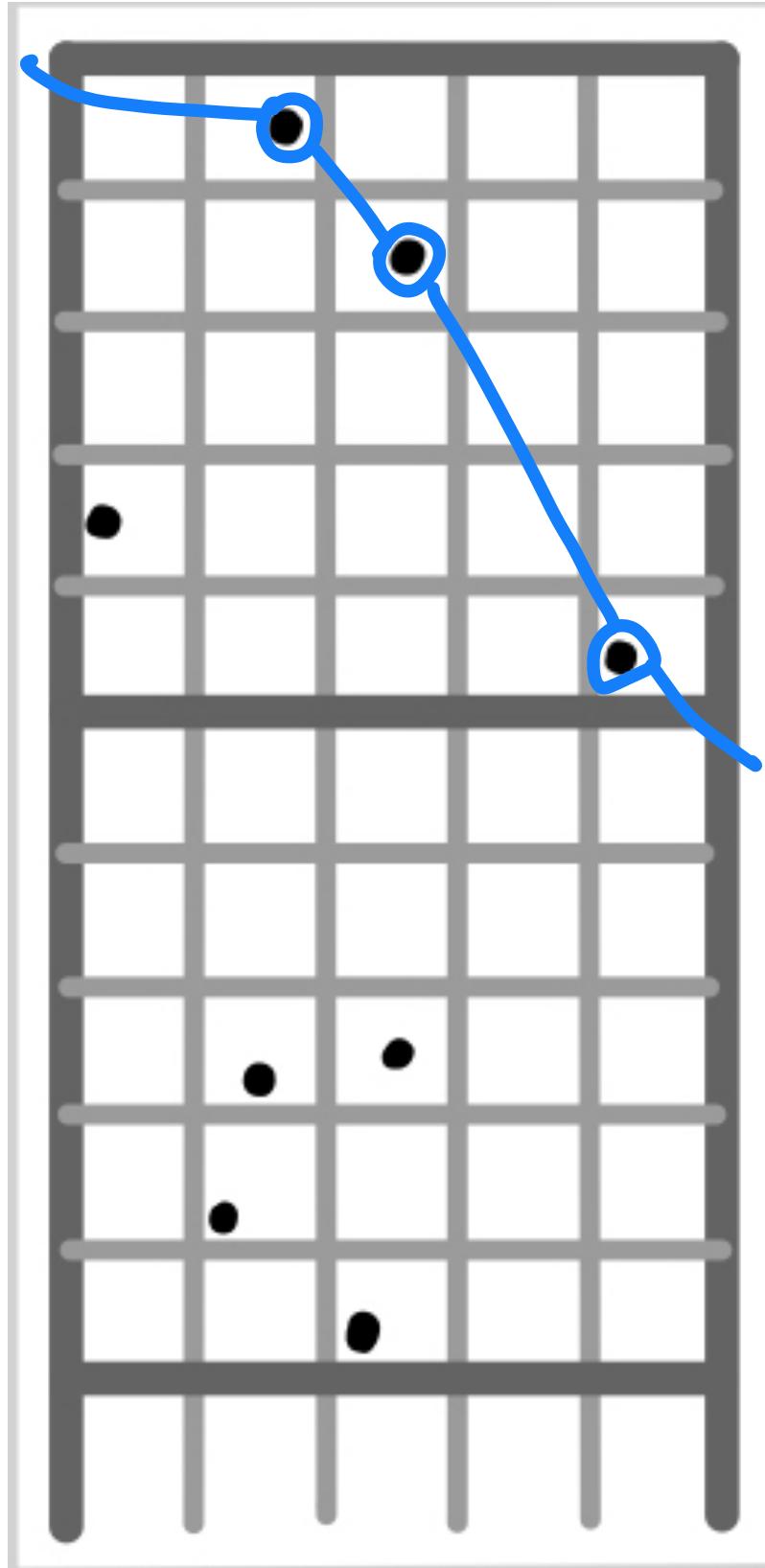
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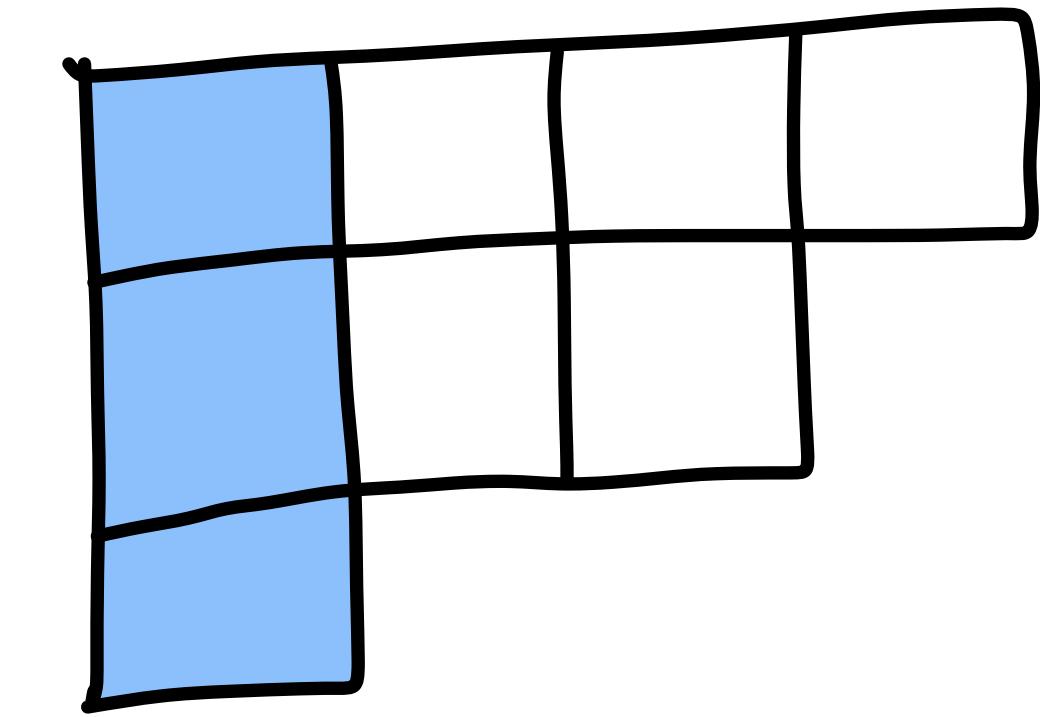
$$\mu_1 + \dots + \mu_\ell = \text{LIS}_\ell \quad \mu'_1 + \dots + \mu'_\ell = \text{LDS}_\ell$$

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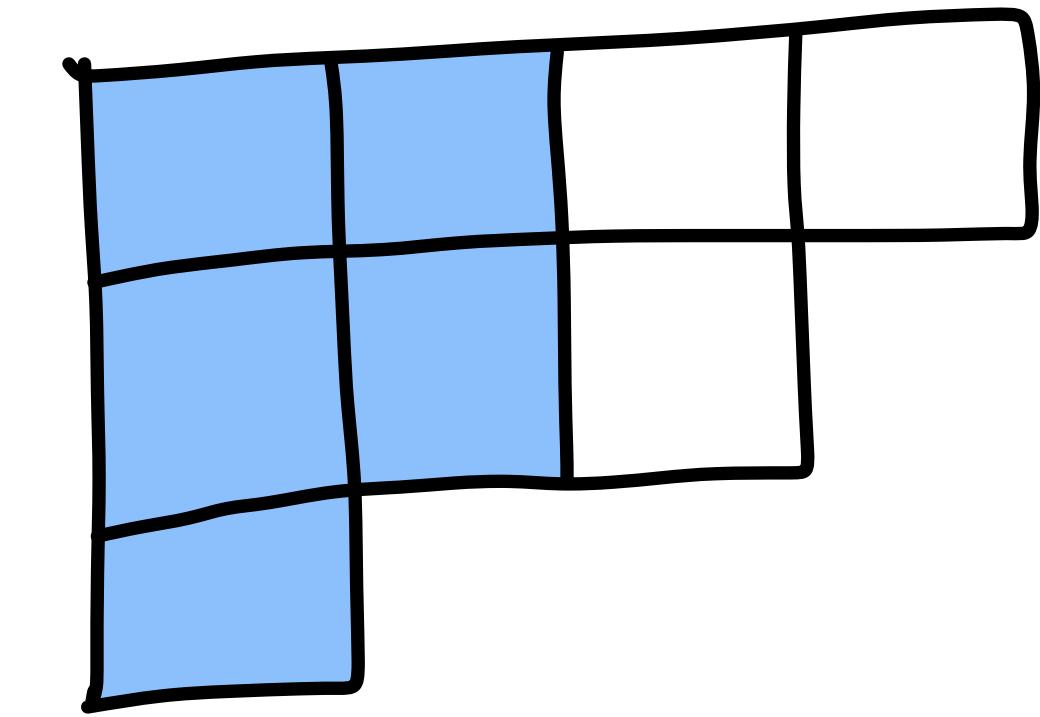
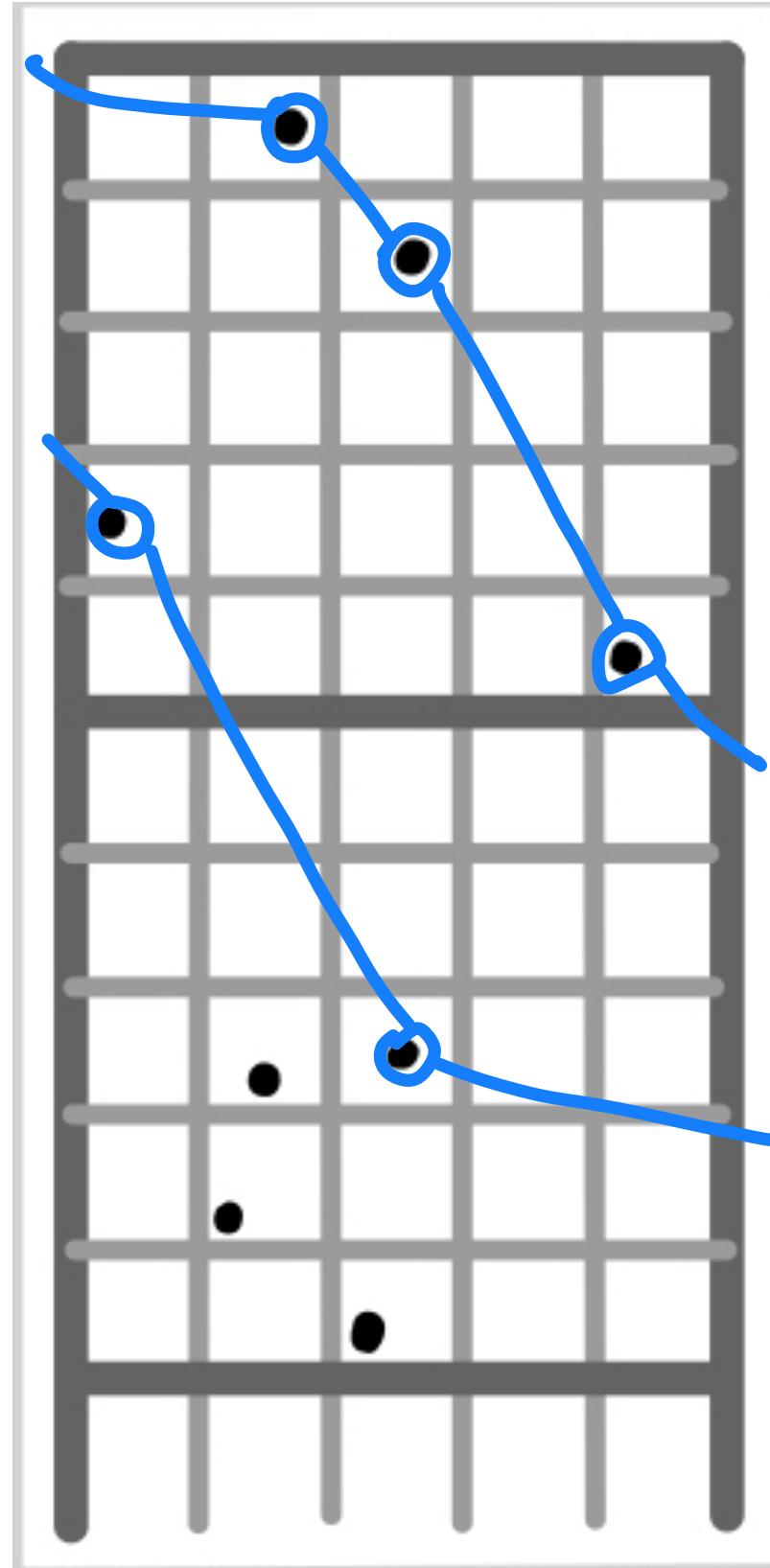
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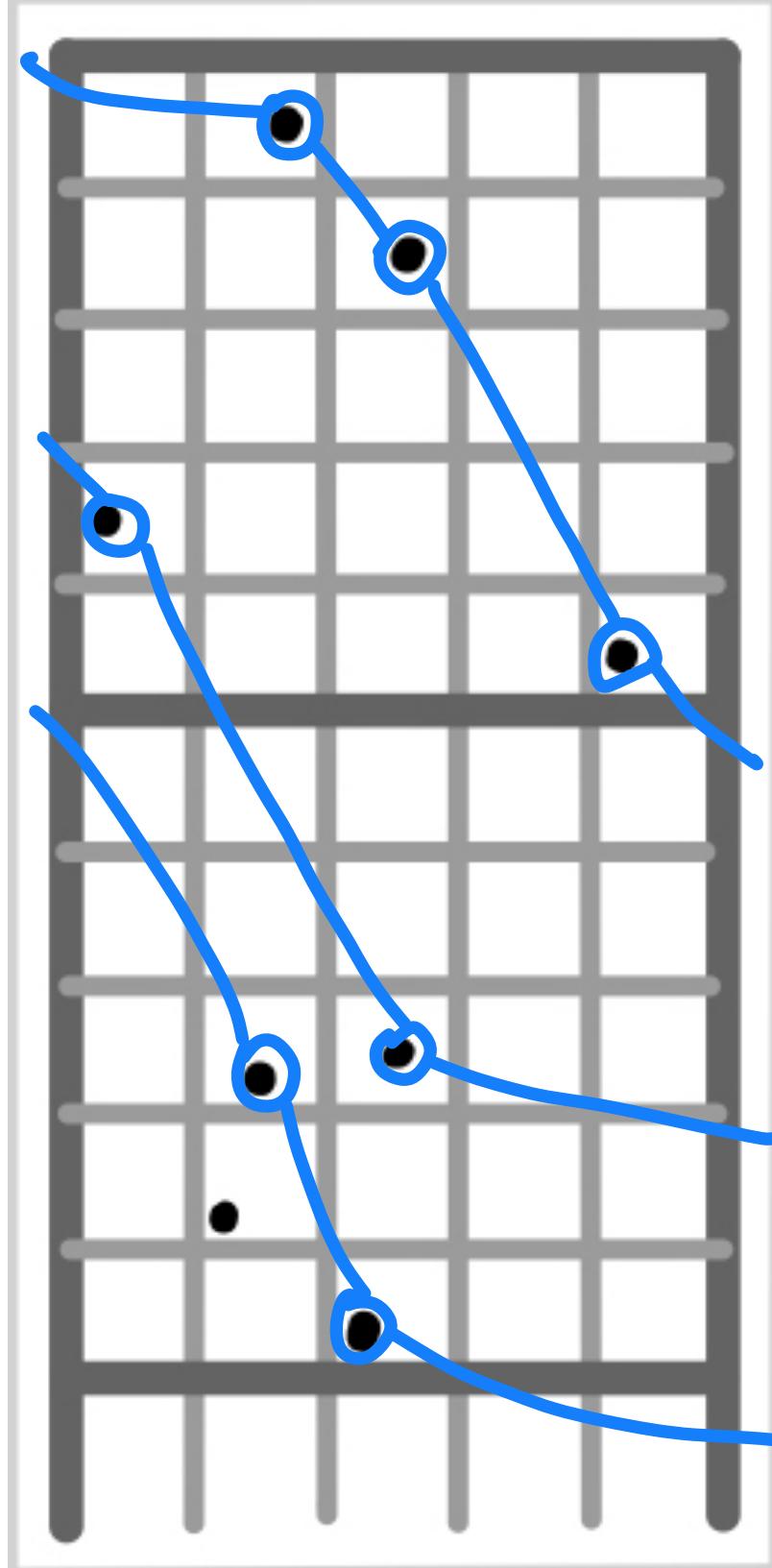
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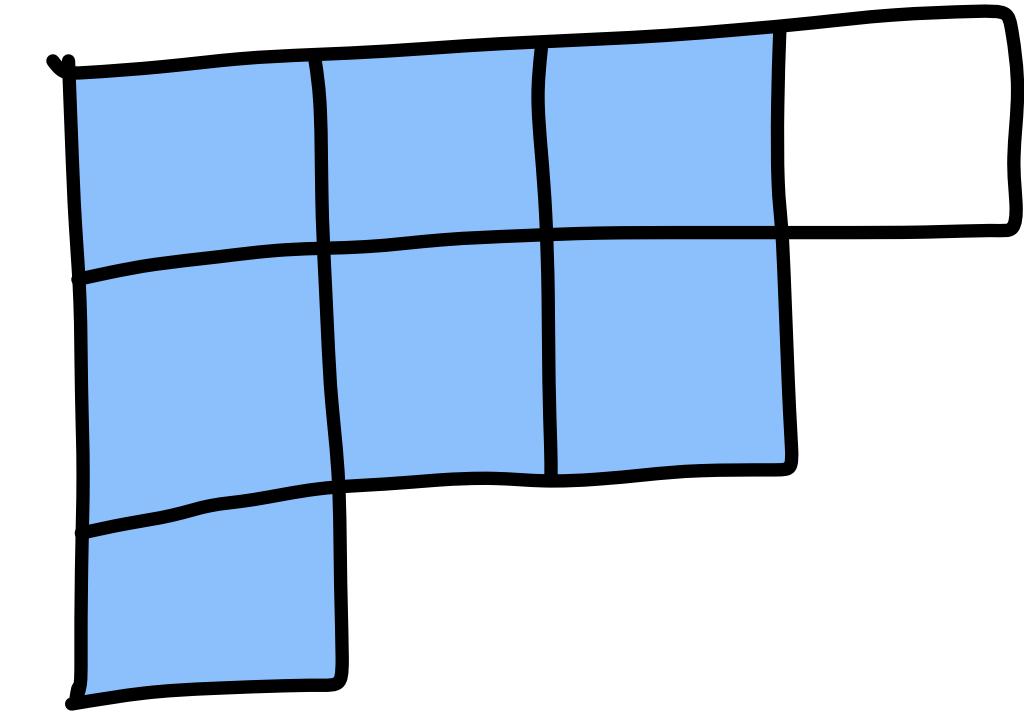
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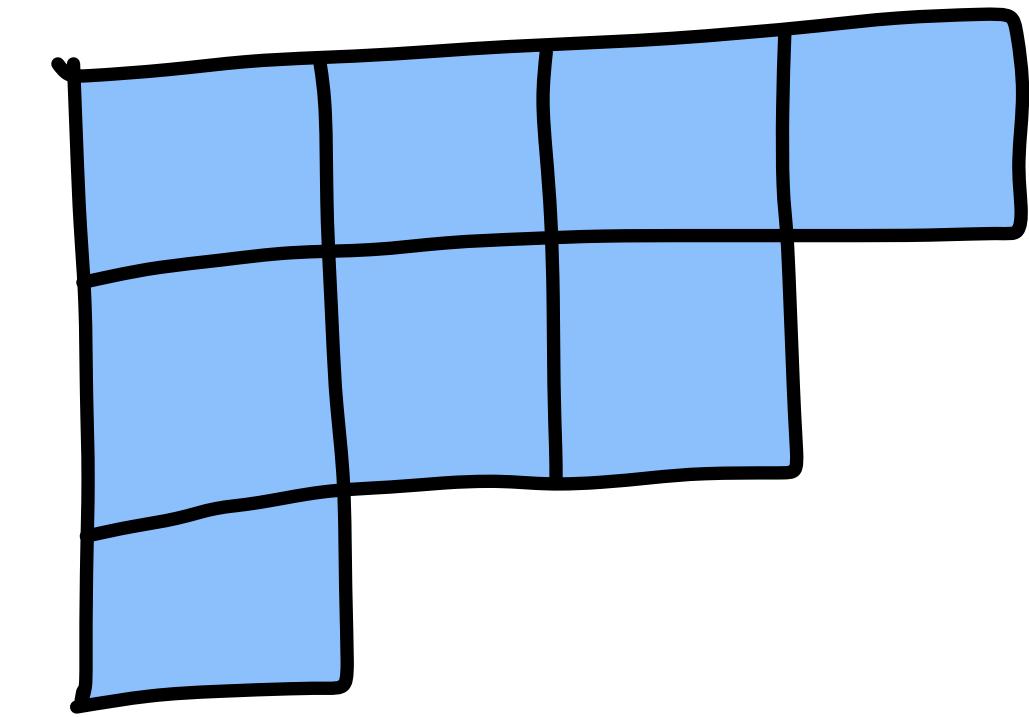
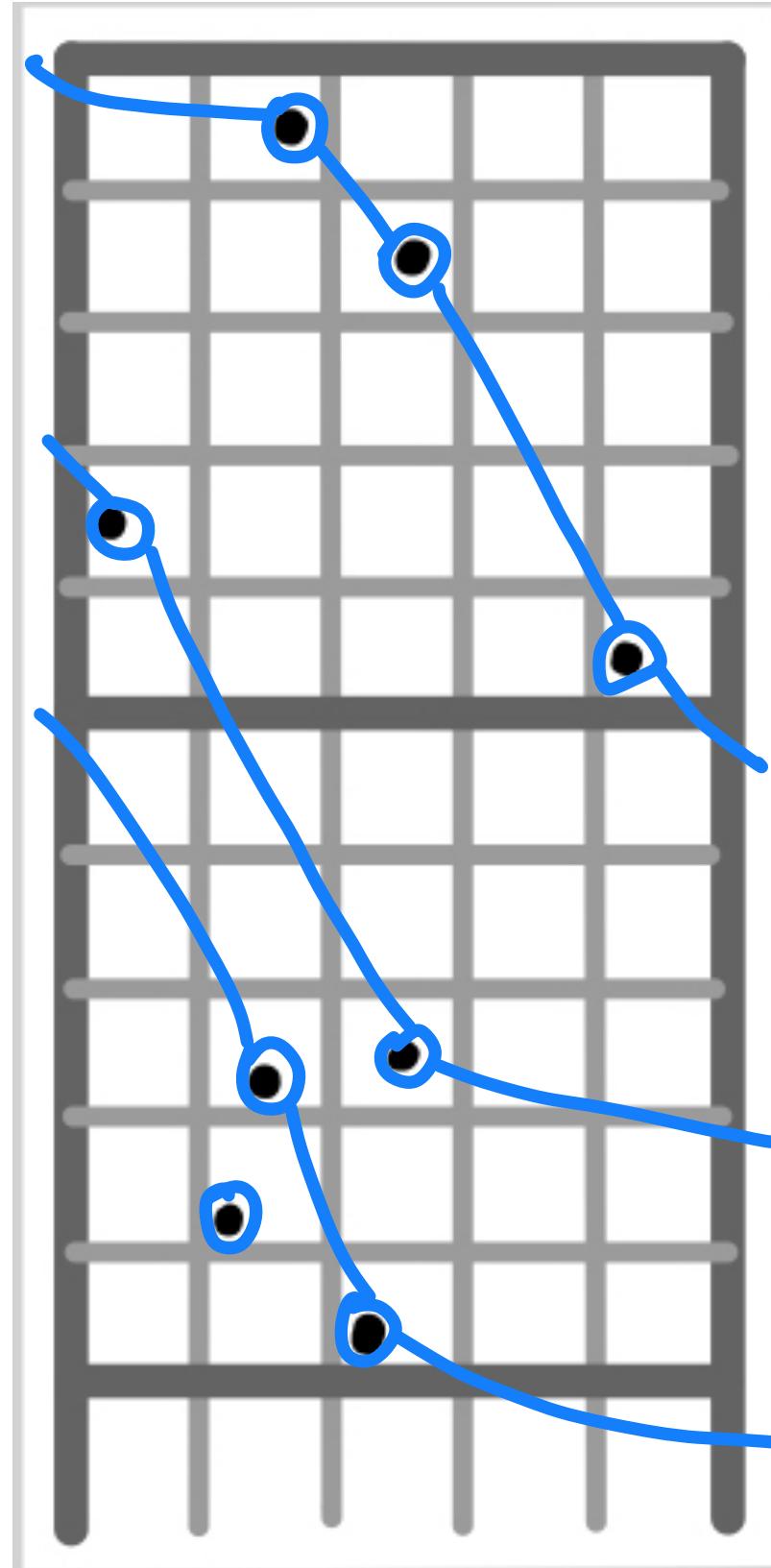
$$\mu_1 + \dots + \mu_\ell = \text{LIS}_\ell$$

$$\mu'_1 + \dots + \mu'_\ell = \text{LDS}_\ell$$

Greene invariants

- **Definition:**

- LDS_k = maximal number of bullets in k nonintersecting loops
- LIS_k = maximal number of bullets in k nonintersecting up-right paths



Theorem [IMS'21]

LDS_k and LIS_k determine the shape μ of (V, W)

$$\mu_1 = \text{LIS}_1$$

$$\mu_1 + \dots + \mu_\ell = \text{LIS}_\ell$$

$$\mu'_1 = \text{LDS}_1$$

$$\mu'_1 + \dots + \mu'_\ell = \text{LDS}_\ell$$

Conclusion

- We construct a bijective q -extension of the RSK correspondence Υ
- With Υ we can prove bijectively the Cauchy identities (CI) for q -Whittaker polynomials
- It is the first time a bijective proof is given for CI of Macdonald polynomials outside of the Schur case
- Symmetries of the bijection Υ allow to prove Littlewood identities, Kawanaka identities and skew Gessel identities
- We extend in periodic setting the notion of Greene invariants