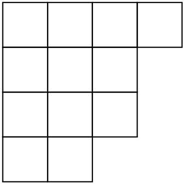
A q-Nekrasov–Okounkov formula for type $\widetilde{C}$

Séminaire Lotharigien de Combinatoire n°87
Saint-Paul-en-Jarez
04–06 April, 2022

David Wahiche
Université Lyon 1 – Institut Camille Jordan
05/04/2022

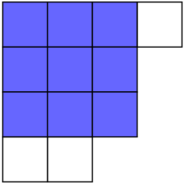
Ferrers diagram and hooks of partitions



$(4, 3, 3, 2) \in \mathcal{P}$

$$|\lambda| = 4 + 3 + 3 + 2 = 12$$

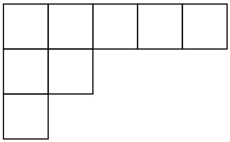
Ferrers diagram and hooks of partitions



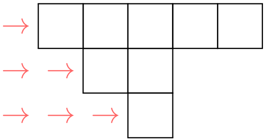
$(4, 3, 3, 2) \in \mathcal{P}$

Durfee square

Ferrers diagram and hooks of partitions

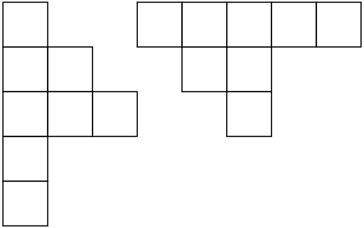


$(5, 2, 1) \in \mathcal{D}$

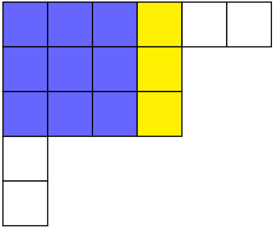


$(5, 2, 1) \in \mathcal{D}$

Ferrers diagram and hooks of partitions

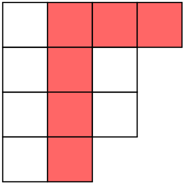


Twice $(5, 2, 1) \in \mathcal{D}$



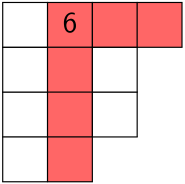
$(6, 4, 4, 1, 1) \in \mathcal{DD}$

Ferrers diagram and hooks of partitions



$(4, 3, 3, 2) \in \mathcal{P}$

Ferrers diagram and hooks of partitions



$(4, 3, 3, 2) \in \mathcal{P}$

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1		

$(4, 3, 3, 2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) :=$ {hook-lengths}

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1	$\mathcal{H}_3$	

$(4,3,3,2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- for $t \in \mathbb{N}^*$, $\mathcal{H}_{\mathbf{t}}(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$

Ferrers diagram and hooks of partitions

8	6	4	1
6	4	2	
5	3	1	
3	1	$\mathcal{H}_3$	
1			

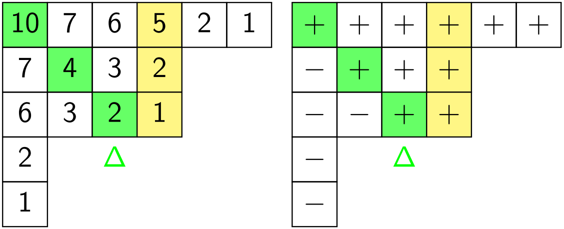
$(4,3,3,2,1) \in \mathcal{P}$

10	7	6	5	2	1
7	4	3	2		
6	3	2	1		
2		$\mathcal{H}_3$			
1					

$(6,4,4,1,1) \in \mathcal{DD}$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- for $t \in \mathbb{N}^*$, $\mathcal{H}_{\mathbf{t}}(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$

Ferrers diagram and hooks of partitions



(a) $(6,4,4,1,1) \in \mathcal{DD}$ (b) ε_s

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- $\varepsilon_s = \begin{cases} -1 \text{ if } s \text{ is a box strictly below } \Delta \\ 1 \text{ otherwise} \end{cases}$

A q-Nekrasov–Okounkov formula

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z^2}{h^2}\right) = (T;T)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

A q-Nekrasov–Okounkov formula

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z^2}{h^2}\right) = (T;T)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

Note $z = t \in \mathbb{N}^* \Rightarrow \lambda$ are t-cores.

A q-Nekrasov–Okounkov formula

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z^2}{h^2}\right) = (T;T)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

Note $z = t \in \mathbb{N}^* \Rightarrow \lambda$ are t-cores.

Dehaye–Han (2011), Rains–Warnaar (2018), Carlsson–Rodriguez-Villegas (2018)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1-uq^h)(1-u^{-1}q^h)}{(1-q^h)^2} = \prod_{k,r \geq 1} \frac{(1-uq^r T^k)^r (1-u^{-1}q^r T^k)^r}{(1-q^{r-1}T^k)^r (1-q^{r+1}T^k)^r}$$

A q-Nekrasov–Okounkov formula

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z^2}{h^2}\right) = (T;T)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$
Note $z = t \in \mathbb{N}^* \Rightarrow \lambda$ are t-cores.
Dehaye–Han (2011), Rains–Warnaar (2018),
Carlsson–Rodriguez-Villegas (2018)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1-uq^h)(1-u^{-1}q^h)}{(1-q^h)^2}$$

$$= \prod_{k,r \geq 1} \frac{(1-uq^rT^k)^r(1-u^{-1}q^rT^k)^r}{(1-q^{r-1}T^k)^r(1-q^{r+1}T^k)^r}$$

Special case $u = q^z$ and $q \to 1$

Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$.

1. **Partitions**:

$$\lambda \in \mathcal{P} \longleftrightarrow (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$.

1. **Partitions**:

$$\lambda \in \mathcal{P} \longleftrightarrow (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

2. **Double distinct partitions**:
   (a) for t odd:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t-1)/2}$$

   (b) for t even:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}, \kappa) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t/2-1)} \times \mathcal{SC}$$

Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$.

1. **Partitions**:

$$\lambda \in \mathcal{P} \longleftrightarrow (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

2. **Double distinct partitions**:
   (a) for t odd:

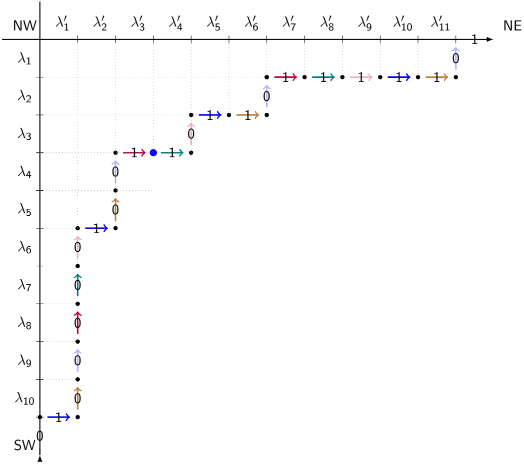
$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t-1)/2}$$

   (b) for t even:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}, \kappa) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t/2-1)} \times \mathcal{SC}$$

Tools: $\lambda \longleftrightarrow s(\lambda)$ bi-infinite word of 0's and 1's

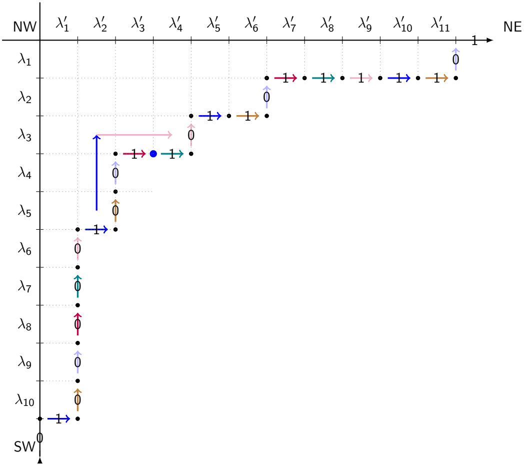
An example for $\omega \in \mathcal{DD}_{(6)}$



$\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1, 1)$ and its binary correspondence

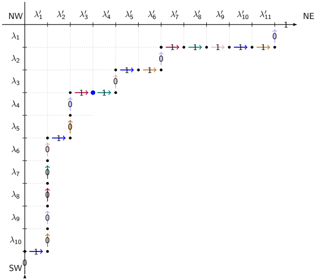
An example for $\omega \in \mathcal{DD}_{(6)}$

$$\mathcal{DD}_{(t)} := \{\omega \in \mathcal{DD} \mid \mathcal{H}_t(\omega) = \emptyset\}.$$



$\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1, 1)$ and its binary correspondence

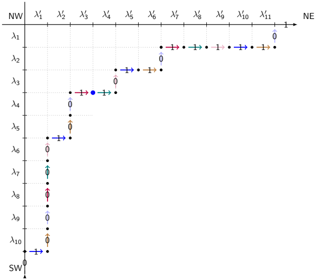
An example for $\omega \in \mathcal{DD}_{(6)}$



$$s\left(w_0\right) = \cdots 000|111 \cdots$$
$$s\left(w_1\right) = \cdots 000|011 \cdots$$
$$s\left(w_2\right) = \cdots 011|111 \cdots$$
$$s\left(w_3\right) = \cdots 000|111 \cdots$$
$$s\left(w_4\right) = \cdots 000|001 \cdots$$
$$s\left(w_5\right) = \cdots 001|111 \cdots$$

$$s(\omega) = \cdots 0000001000001001|10110111110111111111 \cdots$$

An example for $\omega \in \mathcal{DD}_{(6)}$

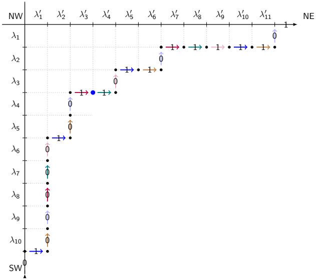


$$s(w_0) = \cdots 000|111 \cdots$$
$$s(w_1) = \cdots 000|011 \cdots$$
$$s(w_2) = \cdots 011|111 \cdots$$
$$s(w_3) = \cdots 000|111 \cdots$$
$$s(w_4) = \cdots 000|001 \cdots$$
$$s(w_5) = \cdots 001|111 \cdots$$

Garvan–Kim–Stanton (1990): $\omega \in \mathcal{P}_{(t)} \leftrightarrow (n_0, \ldots, n_{t-1}) \in \mathbb{Z}^t$ such that $\sum_{i=0}^{t-1} n_i = 0$. Here

$$(n_0, n_1, n_2, n_3, n_4, n_5) = (0, 1, -2, 0, 2, -1) \in \mathbb{Z}^6$$

An example for $\omega \in \mathcal{DD}_{(6)}$



$$s(w_0) = \cdots 000|111 \cdots$$
$$s(w_1) = \cdots 000|011 \cdots$$
$$s(w_2) = \cdots 011|111 \cdots$$
$$s(w_3) = \cdots 000|111 \cdots$$
$$s(w_4) = \cdots 000|001 \cdots$$
$$s(w_5) = \cdots 001|111 \cdots$$

Garvan–Kim–Stanton (1990): $\omega \in \mathcal{P}_{(t)} \leftrightarrow (n_0, \ldots, n_{t-1}) \in \mathbb{Z}^t$ such that $\sum_{i=0}^{t-1} n_i = 0$. Here

$$(n_0, n_1, n_2, n_3, n_4, n_5) = (0, 1, -2, 0, 2, -1) \in \mathbb{Z}^6$$

$$\omega \in \mathcal{DD}_{(6)} \leftrightarrow (1, -2) \in \mathbb{Z}^2$$

t-cores and vectors of integers

Garvan–Kim–Stanton (1990):
$\omega \in \mathcal{DD}_{(2t+2)} \leftrightarrow (n_0, \ldots, n_{t-1}) \in \mathbb{Z}^t$.

t-cores and vectors of integers

Garvan–Kim–Stanton (1990):
$\omega \in \mathcal{DD}_{(2t+2)} \leftrightarrow (n_0, \ldots, n_{t-1}) \in \mathbb{Z}^t$.

A vector of integers $(v_1, \ldots, v_t) \in \mathbb{Z}^t$ is called a V_t - coding if:

1. $\#\{v_i - i \pmod{2t+2}, i = 0, \cdots, t-1\} = t$
2. $v_i \not\equiv 0 \pmod{2t+2}$, $v_i \not\equiv t+1 \pmod{2t+2}$
3. $v_1 > v_1 > \cdots > v_t > 0$

Hook-length formulas for $\mathcal{DD}_{(t)}$

Theorem [W.,2022]

Set t a strictly positive integer and let τ be a function defined over $\mathbb{N}$.
Then there is a bijection $\phi_t : \omega \mapsto (v_0, \cdots, v_{t-1})$ from $\mathcal{DD}_{(2t+2)}$ to V_t- codings such that:

$$\frac{|\omega|}{2} = f(\underline{v})$$

Hook-length formulas for $\mathcal{DD}_{(t)}$

Theorem [W.,2022]

Set t a strictly positive integer and let τ be a function defined over $\mathbb{N}$.
Then there is a bijection $\phi_t : \omega \mapsto (v_0, \cdots, v_{t-1})$ from $\mathcal{DD}_{(2t+2)}$ to V_t- codings such that:

$$\frac{|\omega|}{2} = f(\underline{v})$$

And for $\beta_i(\omega) := \#\{h \in \mathcal{H}(\omega) \mid h = 2t+2-i\}$:

$$\prod_{\substack{s\in\omega \\ h_s\in\mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s(2t+2))}{\tau(h_s)} = \prod_{i=1}^{2t+1} \left(\frac{\tau(-i)}{\tau(i)}\right)^{\beta_i(\omega)}$$

$$\times \prod_{i=1}^{t} \frac{\tau(v_i)}{\tau(i)} \prod_{1\leq i<j\leq t} \frac{\tau(v_i - v_j)}{\tau(i-j)} \frac{\tau(v_i + v_j)}{\tau(i+j)}$$

A q-Nekrasov–Okounkov analogue for type $\widetilde{C}$

$\delta_\lambda := (-1)^d$

Theorem [W., 2022]

$$\sum_{\lambda \in \mathcal{DD}} \delta_\lambda T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - q^{h_s - 2\varepsilon_s} u^{-2\varepsilon_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + u q^{h_s/2+1}}{1 + u^{-1} q^{h_s/2-1}}$$

$$= \prod_{m \geq 1} (1 - T^m) \left[\prod_{r \geq 1} \left[\frac{(1 - u q^r T^m)^r (1 - u^{-1} q^r T^m)^r}{(1 - q^{r+1} T^m)^r (1 - q^{r-1} T^m)^r} \right.\right.$$

$$\times \frac{(1 - u^2 q^r T^m)^{r - \lfloor r/2 \rfloor - 1} (1 - u^{-2} q^{-r} T^m)^{r - \lfloor r/2 \rfloor - 1}}{(1 - u q^{r+1} T^m)^r (1 - u^{-1} q^{-r-1} T^m)^r}$$

$$\left.\left. \times (1 - q^r T^m)^{\lfloor r/2 \rfloor} (1 - q^{-r} T^m)^{\lfloor r/2 \rfloor} \right]\right]$$

A q-Nekrasov–Okounkov analogue for type $\widetilde{C}$

$\delta_\lambda := (-1)^d$

Theorem [W., 2022]

$$\sum_{\lambda \in \mathcal{DD}} \delta_\lambda T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - q^{h_s - 2\varepsilon_s} u^{-2\varepsilon_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + u q^{h_s/2+1}}{1 + u^{-1} q^{h_s/2-1}}$$
$$= \prod_{m \geq 1} (1 - T^m) \left[\prod_{r \geq 1} \left[\frac{(1 - u q^r T^m)^r (1 - u^{-1} q^r T^m)^r}{(1 - q^{r+1} T^m)^r (1 - q^{r-1} T^m)^r} \right.\right.$$
$$\times \frac{(1 - u^2 q^r T^m)^{r - \lfloor r/2 \rfloor - 1} (1 - u^{-2} q^{-r} T^m)^{r - \lfloor r/2 \rfloor - 1}}{(1 - u q^{r+1} T^m)^r (1 - u^{-1} q^{-r-1} T^m)^r}$$
$$\left.\left. \times (1 - q^r T^m)^{\lfloor r/2 \rfloor} (1 - q^{-r} T^m)^{\lfloor r/2 \rfloor} \right]\right]$$

$u = q^z$ and $q \to 1$ yields Pétréolle's Nekrasov–Okounkov type formula (2016).

Sketch of proof

1. Macdonald (1972), Stanton (1989), Rosengren–Schlosser (2006): specialization of the analogues of Weyl denominator formula for affine root systems

$$\sum_{w \in W} \det(w) e^{w(\rho)-\rho} = \prod_{\substack{a>0 \\ a \in R}} \left(1 - e^{-a}\right)$$

specialization in type $\widetilde{C}$ $x_i = e^{-\epsilon_i}, T = e^{-1}$

Sketch of proof

1. Macdonald (1972), Stanton (1989), Rosengren–Schlosser (2006): specialization of the analogues of Weyl denominator formula for affine root systems

$$\sum_{w \in W} \det(w) e^{w(\rho)-\rho} = \prod_{\substack{a>0 \\ a \in R}} (1 - e^{-a})$$

specialization in type $\widetilde{C}$ $x_i = e^{-\epsilon_i}, T = e^{-1}$

2. specialization with $x_i = q^i$ and $\tau(h) = 1 - q^h$ in the last Theorem

Special role of Δ

Sketch of proof

1. Macdonald (1972), Stanton (1989), Rosengren–Schlosser (2006): specialization of the analogues of Weyl denominator formula for affine root systems

$$\sum_{w \in W} \det(w) e^{w(\rho)-\rho} = \prod_{\substack{a>0 \\ a \in R}} \left(1 - e^{-a}\right)$$

   specialization in type $\widetilde{C}$ $x_i = e^{-\epsilon_i}, T = e^{-1}$

2. specialization with $x_i = q^i$ and $\tau(h) = 1 - q^h$ in the last Theorem

   Special role of Δ

3. proof using polynomiality seen as Laurent polynomials