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NORMAL ANTI-INVARIANT SUBMANIFOLDS OF PARAQUATERNIONIC KÄHLER MANIFOLDS

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Abstract. We introduce normal anti-invariant submanifolds of paraquaternionic Kähler manifolds and study the geometric structures induced on them. We obtain necessary and sufficient conditions for the integrability of the distributions defined on a normal anti-invariant submanifold. Also, we present characterizations of local (global) anti-invariant products.

1 Introduction

The paraquaternionic Kähler manifolds have been introduced and studied by Garcia-Rio, Matsushita and Vazquez-Lorenzo [4]. We think of a paraquaternionic Kähler manifold as a semi-Riemannian manifold endowed with two local almost product structures and a local almost complex structure satisfying some compatibility conditions. Several classes of submanifolds of a Kähler manifolds have been investigated according to the behavior of the geometric structures of the ambient manifold on a submanifold (see Bejancu [1]). The same idea we follow for the case when the ambient manifold is a paraquaternionic Kähler manifold.

In the present paper we define the normal anti-invariant submanifolds of a paraquaternionic Kähler manifold and obtain some basic results on their differential geometry. First we show that the tangent bundle of a normal antiinvariant submanifold N of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$ admits the decomposition (8) where$\mathcal{D}$ and $\mathcal{D}^{\perp}$ are complementary orthogonal distributions on N. Then we obtain necessary and sufficient conditions for the integrability of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ (see Theorems 4 and 7). We also prove that the foliations determined by $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are totally geodesic (see Theorem 8). Finally, we study the existence of local (global) normal anti-invariant products (Corollaries 9 and 12, Theorem 11). As examples, we show that totally geodesic normal anti-invariant submanifolds are local normal anti-invariant products (Corollary 10).

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2 Preliminaries

Throughout the paper all manifolds are smooth and paracompact. If M is a smooth manifold then we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $F(M)$-module of smooth sections of the tangent bundleTM of M. Similar notations will be used for any other manifold or vector bundle. If not stated otherwise, we use indices: $a, b, c, ... \in \{1,2,3\}$ and $i, j, k, ... \in \{1, ..., n\}$.

Let M be a manifold endowed with a *paraquaternionic structure* $\mathbf{V}$, that is, $\mathbf{V}$ is a rank-3 subbundle of $End(TM)$ which has a local basis $\{J_1, J_2, J_3\}$ on a coordinate neighborhood $\mathcal{U} \subset M$ satisfying (see Garcia-Rio-Matsushita-Vazquez-Lorenzo [4])

$$\begin{array}{ll} (a) & J_a^2 = \lambda_a I,\ a \in \{1,\ 2,\ 3\}, \\ (b) & J_1 J_2 = -J_2 J_1 = J_3, \\ (c) & \lambda_1 = \lambda_2 = -\lambda_3 = 1. \end{array} \tag{1}$$

A semi-Riemannian metric g on M is said to be *adapted* to the paraquaternionic structure $\mathbf{V}$ if it satisfies

$$g(X,Y) + \lambda_a g(J_a X, J_a Y) = 0, \forall a \in \{1,2,3\}, \tag{2}$$

for any $X, Y \in \Gamma(TM)$, and any local basis $J_1,\ J_2,\ J_3$ of $\mathbf{V}$. From relation1 and relation2 it follows that

$$g(J_a X, Y) + g(X, J_a Y) = 0, \forall X, Y \in \Gamma(TM), a \in \{1,2,3\}. \tag{3}$$

Now, suppose $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ is a local basis of $\mathbf{V}$ on $\tilde{\mathcal{U}} \subset M$ and $\mathcal{U} \cap \widetilde{U} \neq \emptyset$. Then we have

$$\tilde{J}_a = \sum_{b=1}^{3} A_{ab} J_b, \tag{4}$$

where the 3×3 matrix $[A_{ab}]$ is an element of the pseudo-orthogonal group $SO(2,1)$. From 1 and 2 it follows that M is of dimension $4m$ and g is of neutral signature $(2m, 2m)$.

Next, we denote by $\tilde{\nabla}$ the Levi-Civita connection on (M, g). Then the triple $(M, \mathbf{V}, g)$is called a *paraquaternionic Kähler manifold* if $\mathbf{V}$ is a parallel bundle with respect to $\tilde{\nabla}$. This means that for any local basis $\{J_1, J_2, J_3\}$ of $\mathbf{V}$ on $\mathcal{U} \subset M$ there exist the 1-forms $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ on $\mathcal{U}$ such that (cf. Garcia-Rio-Matsushita-Vazquez-Lorenzo [4])

$$\begin{array}{llcl} \text{(a)} & (\tilde{\nabla}_X J_1)Y & = & \mathbf{q}(X) J_2 Y - \mathbf{r}(X) J_3 Y, \\ \text{(b)} & (\tilde{\nabla}_X J_2)Y & = & -\mathbf{q}(X) J_1 Y - \mathbf{p}(X) J_3 Y, \\ \text{(c)} & (\tilde{\nabla}_X J_3)Y & = & -\mathbf{r}(X) J_1 Y - \mathbf{p}(X) J_2 Y\ ,\quad \forall X, Y \in \Gamma(T\mathcal{U}). \end{array} \tag{5}$$

Now, we consider a non-degenerate submanifold N of $(M, \mathbf{V}, g)$ of codimension n. Then we say that N is a *normal anti-invariant submanifold* of $(M, \mathbf{V}, g)$ if the normal bundle $TN^{\perp}$ of N is anti-invariant with respect to any local basis $\{J_1, J_2, J_3\}$ of $\mathbf{V}$ on $\mathcal{U}$, that is, we have

$$J_a(T_xN^{\perp}) \subset T_xN, \quad \forall a \in \{1,2,3\}, x \in \mathcal{U}^* = \mathcal{U} \cap N. \tag{6}$$

A large class of normal anti-invariant submanifolds is given in the next proposition.

Proposition 1. *Any non-degenerate real hypersurface N of $(M,\ g)$ is a normal anti-invariant submanifold of $(M,\ \mathbf{V},\ g)$.*

Proof. From 3 we deduce that $g(J_aU, U) = 0,$ for any $U \in \Gamma(TN^{\perp})$ and $a \in \{1, 2, 3\}$. Hence $J_aU \in \Gamma(TN)$, which proves 6. □

Next, we examine the structures that are induced on the tangent bundle of a normal anti-invariant submanifold N of $(M,\ \mathbf{V},\ g)$. First, we put $\mathcal{D}_{ax} = J_a\left(T_xN^{\perp}\right)$ and note that $\mathcal{D}_{1x}$,$\mathcal{D}_{2x}$ and $\mathcal{D}_{3x}$ are mutually orthogonal nondegenerate n-dimensional vector subspaces of T_xN, for any $x \in N$. Indeed, by using 3, (1b) and 6 we obtain

$$g\left(J_1X, J_2Y\right) = -g\left(X, J_1J_2Y\right) = -g\left(X, J_3Y\right) = 0, \forall X, Y \in \Gamma(TN^{\perp}),$$

which shows that $\mathcal{D}_{1x}$ and $\mathcal{D}_{2x}$ are orthogonal. By a similar reason we conclude that $\mathcal{D}_{ax}$ and $\mathcal{D}_{bx}$ are orthogonal for any $a \neq b$. Then we can state the following.

Proposition 2. *Let N be a normal anti-invariant submanifold of $(M,\ \mathbf{V},\ g)$ of codimension n. Then we have the assertions:*

(i) The subspaces $\mathcal{D}_{ax}$ of T_xN satisfy the following

$$J_a(\mathcal{D}_{ax}) = T_xN^{\perp} \text{ and } J_a(\mathcal{D}_{bx}) = \mathcal{D}_{cx},$$

for any $x \in \mathcal{U}^*$, $a \in \{1, 2, 3\}$, and any permutation $(a, b,\ c)$ of $(1,\ 2,\ 3)$.

(ii) The mapping

$$\mathcal{D}^{\perp} : x \in N \to \mathcal{D}^{\perp}_x = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x},$$

defines a non-degenerate distribution of rank $3n$ on N.

(iii) The complementary orthogonal distribution $\mathcal{D}$ to $\mathcal{D}^{\perp}$ in TN is invariant with respect to the paraquaternionic structure$\mathbf{V}$, that is, we have

$$J_a(\mathcal{D}_x) = \mathcal{D}_x, \forall x \in \mathcal{U}^*, a \in \{1, 2, 3\}.$$

Proof. First, by using 1 we obtain the assertion (i). Next, by 4 and taking into account that J_a, $a \in \{1,2,3\}$, are automorphisms of $\Gamma(TM)$ and $\mathcal{D}_{ax}$, $a \in \{1,2,3\}$ are mutually orthogonal subspaces we get the assertion (ii). Now, we note that the tangent bundle of M along N has the following orthogonal decompositions:

$$TM = TN \oplus TN^{\perp} = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus TN^{\perp}. \tag{7}$$

Then we take $Y \in \Gamma\left(\mathcal{D}^{\perp}\right)$ and by the assertion (i) we deduce that

$$J_a Y \in \Gamma\left(\mathcal{D}^{\perp} \oplus TN^{\perp}\right), \forall a \in \{1,2,3\}.$$

On the other hand, if $Y \in \Gamma\left(TN^{\perp}\right)$, by 6 and the assertion (ii) we infer that

$$J_a Y \in \Gamma\left(\mathcal{D}^{\perp}\right), \forall a \in \{1,2,3\}$$

Thus by using 3 and the second equality in 7 we obtain

$$g\left(J_a X, Y\right) = -g\left(X, J_a Y\right) = 0, \forall a \in \{1,2,3\},$$

for any $X \in \Gamma(\mathcal{D})$ and $Y \in \Gamma\left(\mathcal{D}^{\perp} \oplus TN^{\perp}\right)$. Hence $J_a X \in \Gamma(\mathcal{D})$ for any $a \in \{1,2,3\}$ and $X \in \Gamma(\mathcal{D})$, that is, $\mathcal{D}$ is invariant with respect to the paraquaternionic structure $\mathbf{V}$. This completes the proof of the proposition. □

By assertion (iii) of the above proposition we are entitled to call $\mathcal{D}$ the *paraquaternionic distribution* on N. Also, we note that the paraquaternionic distribution in non-trivial, that is $\mathcal{D} \neq \{0\}$, if and only if $dimN > 3n$.

3 Integrability of the Distributions on a Normal Anti-Invariant Submanifold

Let N be a normal anti-invariant submanifold of codimension n of a $4m$-dimensional paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then according to the definitions of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ we have the orthogonal decomposition

$$TN = \mathcal{D} \oplus \mathcal{D}^{\perp} \tag{8}$$

Then we consider a local field of orthonormal frames $\{U_1, ..., U_n\}$ of the normal bundle $TN^{\perp}$, and define

$$E_{ai} = J_a U_i, \ a \in \{1,2,3\}, i \in \{1,...,n\}. \tag{9}$$

Taking into account 6 and the assertion (ii) of Proposition 2 we deduce that $\{E_{ai}\}, a \in \{1,2,3\}, i \in \{1,...,n\}$, is a local field of orthonormal frames of $\mathcal{D}^{\perp}$. Thus we can put

$$X = PX + \sum_{b=1}^{3}\sum_{i=1}^{n} \omega_{bi}(X) E_{bi}, \quad \forall X \in \Gamma\left(TN^{\perp}\right), \tag{10}$$

where P is the projection morphism of TN on $\mathcal{D}$ with respect to the decomposition 8, and ω_{bi} are 1-forms given by

$$\omega_{bi}(X) = \varepsilon_{bi} g\left(X, E_{bi}\right), \quad \varepsilon_{bi} = g\left(E_{bi}, E_{bi}\right). \tag{11}$$

Now, we apply J_a, $a \in \{1,2,3\}$ to 10 and by using 9 and 1 we obtain

$$\begin{aligned}
&\text{(a)} \quad J_1 X = J_1 PX + \sum_{i=1}^{n} \left\{\omega_{2i}(X)E_{3i} + \omega_{3i}(X)E_{2i} + \omega_{1i}(X)U_i\right\}, \\
&\text{(b)} \quad J_1 X = J_1 PX - \sum_{i=1}^{n} \left\{\omega_{1i}(X)E_{3i} + \omega_{3i}(X)E_{1i} - \omega_{2i}(X)U_i\right\}, \\
&\text{(c)} \quad J_1 X = J_1 PX - \sum_{i=1}^{n} \left\{\omega_{1i}(X)E_{2i} - \omega_{2i}(X)E_{1i} + \omega_{3i}(X)U_i\right\}.
\end{aligned} \tag{12}$$

Next, we consider the Gauss equation (cf. Chen [3])

$$\tilde{\nabla}_X Y = \nabla_x Y + h\left(X, Y\right), \quad \forall X, Y \in \Gamma\left(TN\right), \tag{13}$$

where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections on (M, g) and (N, g) respectively, and h is the second fundamental form of N. Also, we have the Weingarten equation

$$\tilde{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \quad \forall X \in \Gamma\left(TN\right), U \in \Gamma\left(TN^{\perp}\right), \tag{14}$$

where A_U is the shape operator of N with respect to the normal section U, and $\nabla^{\perp}$ is the normal connection on $TN^{\perp}$. Moreover, h and A_U are related by

$$g(h(X,Y),U) = g(A_U X, Y), \quad \forall X, Y \in \Gamma\left(TN\right), U \in \Gamma\left(TN^{\perp}\right). \tag{15}$$

Proposition 3. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold $(M,\ \mathbf{V},\ g)$. Then we have*

$$h\left(X, J_a Y\right) = \lambda_a \sum_{i=1}^{n} \left\{\omega_{ai}(\nabla_X Y)U_i\right\}, \tag{16}$$

for any $X, Y \in \Gamma\left(\mathcal{D}\right)$ and $a \in \{1,2,3\}$.

Proof. By direct calculations using (13) and (12a) in (5a) we deduce that

$$\nabla_x J_1 Y + h\left(X, J_1 Y\right) = J_1 P\left(\nabla_x Y\right)$$

$$+ \sum_{i=1}^{n} \left\{\omega_{2i}(\nabla_X Y)E_{3i} + \omega_{3i}(\nabla_X Y)E_{2i} + \omega_{1i}(\nabla_X Y)U_i\right\}$$

$$+ J_1 h\left(X, Y\right) + \mathbf{q}(X) J_2 Y - \mathbf{r}(X) J_3 Y.$$

Then taking the normal parts in the above equality we obtain (16) for $a = 1$. In a similar way follows (16) for $a = 2$ and $a = 3$. □

Now, we say that N is $\mathcal{D}$*-geodesic* if its second fundamental form h satisfies (see Bejancu [1])

$$h(X,Y)=0, \quad \forall X,Y\in\Gamma(\mathcal{D}), \tag{17}$$

Then by using (13) and (17) we deduce that N is $\mathcal{D}$-geodesic if and only if any geodesic of (N,g) passing through each $x\in N$ and tangent to $\mathcal{D}_x$ is a geodesic of (M,g).

Theorem 4. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold$(M,\ \mathbf{V},\ g)$. Then the following assertions are equivalent:*

(i) The second fundamental form h of N satisfies

$$h(X,J_aY)=h(Y,J_aX), \quad \forall X,Y\in\Gamma(\mathcal{D}),\ a\in\{1,2,3\} \tag{18}$$

(ii) N is $\mathcal{D}$-geodesic.

(iii) The paraquaternionic distribution $\mathcal{D}$ is integrable.

Proof. (i) =) (ii). By using (18) and (1b) we deduce that

$$h(J_3X,Y)=h(X,J_3Y)=h(X,J_1(J_2Y))=h(J_1X,J_2Y)$$
$$=h(J_2(J_1X),Y)=-h(J_3X,Y), \quad \forall X,Y\in\Gamma(\mathcal{D})$$

which implies $h(J_3X,Y)=0$. Taking into account that J_3 is an automorphism of $\Gamma(\mathcal{D})$ we obtain (17). Hence N is $\mathcal{D}$-geodesic.

(ii) =) (iii). By using (17) and (11) in (16) we infer that

$$g(\nabla_xY,E_{ai})=0, \quad \forall X,Y\in\Gamma(\mathcal{D}),\ a\in\{1,2,3\},i\in\{1,...,n\}. \tag{19}$$

Hence $\nabla_xY\in\Gamma(\mathcal{D})$, which implies

$$[X,Y]=\nabla_xY-\nabla_YX\in\Gamma(\mathcal{D})$$

Thus $\mathcal{D}$ is integrable.

(iii) =) (i). By using (16) and (11), and taking into account that ∇ is a torsion-free connection, we obtain

$$h(X,J_aY)-h(Y,J_aX)=\sum_{i=1}^{n}\{g([X,Y],E_{ai})U_i\}=0,$$

for any $X,Y\in\Gamma(\mathcal{D})$ and $a\in\{1,2,3\}$. This completes the proof of the theorem. □

Proposition 5. *The shape operators A_i with respect to the normal sections U_i, $i\in\{1,...,n\}$, satisfy the identities:*

$$A_iE_{aj}=A_jE_{ai}, \quad \forall a\in\{1,2,3\},\ i,j\in\{1,...,n\}. \tag{20}$$

Proof. We take $X \in \Gamma(TN)$ and $Y = E_{1i}$ in (5a) and by using (13), (14), (9) and (1) we obtain

$$-A_iX + \nabla_X^{\perp}U_i = J_1(\nabla_X E_{1i}) + J_1h(X, E_{1i}) - \mathbf{q}(X)E_{3i} + \mathbf{r}(X)E_{2i}.$$

Then by using (15), (2), (9) and the above equality we deduce that

$$\begin{aligned} g(A_jE_{1i}, X) &= g(h(X, E_{1i}), U_j) \\ &= -g(J_1h(X, E_{1i}), E_{1j}) \\ &= g(A_iX + J_1(\nabla_X E_{1i}), E_{1j}) \\ &= g(A_iX, E_{1j}) - g(\nabla_X E_{1i}, U_j) \\ &= g(X, A_iE_{1j}), \quad \forall X \in \Gamma(TN), \end{aligned}$$

which proves (20) for $a = 1$. In a similar way we obtain (20) for $a = 2$ and $a = 3$. □

Next, we define on $\Gamma(\mathcal{D})$ the 1-forms

$$\Omega_{aij}(X) = g(\nabla_{E_{ai}} E_{aj}, X), \tag{21}$$

for any $X \in \Gamma(\mathcal{D})$, $a \in \{1,2,3\}$ and $i,j \in \{1,...,n\}$. Then we state the following.

Proposition 6. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then we have:*

$$\Omega_{aij} = \Omega_{aji}, \quad \forall a \in \{1,2,3\}, \quad i,j \in \{1,...,n\}, \tag{22}$$

and

$$\begin{aligned} &(a) \quad g(\nabla_{E_{1i}} E_{2j}, X) = \Omega_{1ij}(J_3X), \quad g\left(\nabla_{E_{2j}} E_{1i}, X\right) = -\Omega_{2ij}(J_3X), \\ &(b) \quad g(\nabla_{E_{2i}} E_{3j}, X) = -\Omega_{2ij}(J_1X), \quad g\left(\nabla_{E_{3j}} E_{2i}, X\right) = -\Omega_{3ij}(J_1X), \\ &(c) \quad g(\nabla_{E_{3i}} E_{1j}, X) = \Omega_{3ij}(J_2X), \quad g\left(\nabla_{E_{1j}} E_{3i}, X\right) = \Omega_{1ij}(J_2X), \end{aligned} \tag{23}$$

for any $X \in \Gamma(\mathcal{D})$.

Proof. By using (21), (9), (13), (5), (3) and (14) we obtain

$$\begin{aligned} \Omega_{aij}(X) &= g\left(\tilde{\nabla}_{E_{ai}} J_aU_j, X\right) = g\left(J_a(\tilde{\nabla}_{E_{ai}} U_j), X\right) = \\ &= -g\left(\tilde{\nabla}_{E_{ai}} U_j, J_aX\right) = g(A_jE_{ai}, J_aX), \end{aligned} \tag{24}$$

for any $a \in \{1,2,3\}$ and $i,j \in \{1,...,n\}$. Then (22) follows by using (24) and (20). Next, by using (13), (2), (5), (1), (9) and (21) we deduce that

$$\begin{aligned} g(\nabla_{E_{1i}} E_{2j}, X) &= g\left(\tilde{\nabla}_{E_{1i}} E_{2j}, X\right) = g\left(J_3\left(\tilde{\nabla}_{E_{1i}} E_{2j}\right), J_3X\right) \\ &= g\left(\tilde{\nabla}_{E_{1i}} E_{1j}, J_3X\right) = \Omega_{1ij}(J_3X), \quad \forall X \in \Gamma(\mathcal{D}) \end{aligned}$$

which proves the first equality in (23a). In a similar way are obtained all the other equalities in (23). □

Theorem 7. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold $(M,\ \mathbf{V},\ g)$. Then the following assertions are equivalent:*

(i) The distribution $\mathcal{D}^{\perp}$ is integrable.

(ii) $\Omega_{aij} = 0$, $\forall a \in \{1,2,3\}$, $i,j \in \{1,...,n\}$.

(iii) For any $X \in \Gamma(\mathcal{D})$ and $Y \in \Gamma\left(\mathcal{D}^{\perp}\right)$ we have

$$h(X,Y) = 0. \tag{25}$$

Proof. Taking into account that ∇ is torsion-free, and by using (21) and (22) we deduce that

$$g([E_{ai}, E_{aj}], X) = 0, \quad \forall X \in \Gamma(\mathcal{D}), \quad a \in \{1,2,3\}, \quad i,j \in \{1,...,n\}. \tag{26}$$

On the other hand, by using (23) we obtain

$$\begin{array}{lrcl} \text{(a)} & g([E_{1i}, E_{2j}], X) & = & \Omega_{1ij}(J_3X) + \Omega_{2ij}(J_3X), \\ \text{(b)} & g([E_{2i}, E_{3j}], X) & = & \Omega_{3ij}(J_1X) - \Omega_{2ij}(J_1X), \\ \text{(c)} & g([E_{3i}, E_{1j}], X) & = & \Omega_{3ij}(J_2X) - \Omega_{1ij}(J_2X), \end{array} \tag{27}$$

for any $X \in \Gamma(\mathcal{D})$. Then from (26) and (27) we infer that (ii) implies (i), since $\{E_{ai}\}$, $a \in \{1,2,3\}$, $i,j \in \{1,...,n\}$ is an orthonormal basis of $\Gamma\left(\mathcal{D}^{\perp}\right)$. Now, we suppose that $\mathcal{D}^{\perp}$ is integrable. Then taking into account that J_a, $a \in \{1,2,3\}$, are automorphisms of $\Gamma(\mathcal{D})$ from (27) we deduce that Ω_{aij} satisfy the system

$$\Omega_{1ij} + \Omega_{2ij} = 0, \quad \Omega_{3ij} - \Omega_{2ij} = 0, \quad \Omega_{3ij} - \Omega_{1ij} = 0.$$

Hence $\Omega_{aij} = 0$, for all $a \in \{1,2,3\}$, and $i,j \in \{1,...,n\}$. Thus we proved that (i) implies (ii). Finally, by using (24) and (15) we obtain

$$\Omega_{aij}(X) = g\left(h\left(J_aX, E_{ai}\right), U_j\right),$$

for any $a \in \{1,2,3\}$, and $i,j \in \{1,...,n\}$, which implies the equivalence of (ii) and (iii). This completes the proof of the theorem. □

4 Foliations on a Normal Anti-Invariant Submanifold

Let $\mathcal{F}$ be a foliation on (N,g). Then we say that $\mathcal{F}$ is *totally geodesic* if each leaf of $\mathcal{F}$ is totally geodesic immersed in (N,g). Denote by $\mathcal{F}(\mathcal{D})$ and $\mathcal{F}(\mathcal{D}^{\perp})$ the foliations determined by $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, provided these distributions are integrable.

Theorem 8. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold $(M,\ \mathbf{V},\ g)$. Then we have the assertions:*

(i) If $\mathcal{D}$ is integrable, then the foliation $\mathcal{F}(\mathcal{D})$ is totally geodesic.

(ii) If $\mathcal{D}^{\perp}$ is integrable, then the foliation $\mathcal{F}(\mathcal{D}^{\perp})$ is totally geodesic.

Proof. Suppose $\mathcal{D}$ is integrable. Then from (19) we deduce that for any . Thus F(D) is a totally geodesic foliation. Next, we suppose that is integrable. Then by using the assertion (ii) of Theorem 7 in (21) and (23) we obtain

$$g\left(\nabla_U V, X\right) = 0, \quad \forall U, V \in \Gamma\left(\mathcal{D}^{\perp}\right), \; X \in \Gamma\left(\mathcal{D}\right),$$

since $\{E_{ai}\}$, $a \in \{1, 2, 3\}$, $i \in \{1, ..., n\}$, is an orthonormal basis in $\Gamma\left(\mathcal{D}^{\perp}\right)$. Thus $\nabla_U V \in \Gamma\left(\mathcal{D}^{\perp}\right)$ for any $U, V \in \Gamma\left(\mathcal{D}^{\perp}\right)$, which means that $\mathcal{F}(\mathcal{D}^{\perp})$ is a totally geodesic foliation. □

Next, we say that N is a *local (global) normal anti-invariant product* if both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable and N is locally (globally) a semi-Riemannian product $(S, h) \times (S^{\perp}, k)$, where S and $S^{\perp}$ are leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

Corollary 9. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$ such that $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable. Then N is a local normal anti-invariant product. If in particular, N is complete and simply connected, then it is a global normal anti-invariant product.*

Proof. From Theorem 8 we see that both foliations $\mathcal{F}(\mathcal{D})$ and $\mathcal{F}(\mathcal{D}^{\perp})$ are totally geodesic. Hence N is a local normal anti-invariant product. If moreover, N is complete and simply connected then we apply the decomposition theorem for semi-Riemannian manifolds (cf. Wu [8]) and obtain the last assertion of the corollary. □

Corollary 10. *A totally geodesic normal anti-invariant submanifold N of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$ is a local normal semi-invariant product. If moreover, N is complete and simply connected, then it is a global anti-invariant product.*

Proof. Taking into account that the second fundamental form h of N vanishes identically on N, from Theorems 4 and 7 we deduce that both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable. Then we apply Corollary 9 and obtain the assertions in this corollary. □

Foliations with bundle-like metric on Riemannian manifolds have been introduced by Reinhart[5]. The main properties of these foliations can be found in Reinhart [6], Tondeur [7] and Bejancu-Farran [2]. Here we need the following characterization of such foliations. Let $\mathcal{F}$ be a non-degenerate foliation on a semi-Riemannian manifold (N, g). Denote by $\mathcal{D}$ and $\mathcal{D}^{\perp}$ the tangent distribution and normal distribution to $\mathcal{F}$ respectively. Then g is a *bundle-like metric* for $\mathcal{F}$ if and only if (cf. Bejancu-Farran[2], p. 112)

$$g\left(\nabla_U V + \nabla_V U, X\right) = 0, \quad \forall U, V \in \Gamma\left(\mathcal{D}^{\perp}\right), \; X \in \Gamma\left(\mathcal{D}\right). \tag{28}$$

In general, the distribution $\mathcal{D}^{\perp}$ is not necessarily integrable when (28) is satisfied. However, for normal anti-invariant submanifolds we prove the following.

Theorem 11. *Let N be a normal anti-invariant submanifold of a paraquaternionic Kähler manifold $(M,\ \mathbf{V},\ g)$ such that the paraquaternionic distribution $\mathcal{D}$ is integrable. Then N is a local normal anti-invariant product if and only if g is a bundle-like metric for the foliation $\mathcal{F}(\mathcal{D})$.*

Proof. First, suppose that N is a local normal anti-invariant product. Then $\mathcal{D}^{\perp}$ is integrable and its leaves are totally geodesic immersed in (N, g). Thus $\nabla_U V \in \Gamma\left(\mathcal{D}^{\perp}\right)$ for any $U, V \in \Gamma\left(\mathcal{D}^{\perp}\right)$ and therefore (28) is satisfied. Thus g is bundle-like for $\mathcal{F}(\mathcal{D})$. Conversely, suppose that g is bundle-like for $\mathcal{F}(\mathcal{D})$. Then, by using (28), (21) and (22) we deduce that $\Omega_{aij} = 0$, for any $a \in \{1, 2, 3\}$, $i, j \in \{1, ..., n\}$. Thus by Theorem 7, $\mathcal{D}^{\perp}$ is integrable. Moreover, by assertion (ii) of Theorem 8 we infer that the foliation $\mathcal{F}(\mathcal{D}^{\perp})$ is totally geodesic. As $\mathcal{F}(\mathcal{D})$ is also totally geodesic (by (i) of Theorem 8), we conclude that N is a local normal anti-invariant product. □

Finally, taking into account Theorem 11 and Corollary 9 we obtain the following.

Corollary 12. *Let N be a complete and simply connected normal antiinvariant submanifold of a paraquaternionic Kähler manifold $(M,\ \mathbf{V},\ g)$ such that the paraquaternionic distribution $\mathcal{D}$ is integrable. Then N is a global normal anti-invariant product if and only if g is a bundle-like metric for the foliation $\mathcal{F}(\mathcal{D})$.*

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