Surveys in Mathematics and its Applications

ISSN 1842-6298
Volume **2** (2007), 29 – 41

PERTURBATION ANALYSIS FOR THE COMPLEX MATRIX EQUATION $Q \pm A^{\mathrm{H}} X^p A - X = 0$

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Abstract. We study the sensitivity of the solution of a general type matrix equation $Q \pm A^{\mathrm{H}} X^p A - X = 0$. Local and nonlocal perturbation bounds are derived. The results are obtained using the technique of Lyapunov majorants and fixed point principles. A numerical example is given.

1 Introduction

In this paper we shall use the following notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ – the sets of natural, real and complex numbers, respectively; $\mathbb{K}^{m\times n}$ – the space of $m \times n$ matrices over $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$; $A^\top$, $\overline{A}$ and A^{H} – the transpose, complex conjugate and complex conjugate transpose of the matrix A; I_n – the unit $n \times n$ matrix; $\|A\|_{\mathrm{F}}$ – the Frobenius norm of a matrix A; $\mathrm{vec}(A) = [a_1^\top, a_2^\top, \ldots, a_n^\top]^\top \in \mathbb{K}^{mn}$ – the column–wise vector representation of the matrix $A = [a_1, a_2, \ldots, a_n] \in \mathbb{K}^{m\times n}$, $a_j \in \mathbb{K}^m$; $P_{m,n} \in \mathbb{R}^{mn\times mn}$ – the vec-permutation matrix, such that $\mathrm{vec}(M^\top) = P_{m,n}\mathrm{vec}(M)$ for $M \in \mathbb{K}^{m\times n}$; $A \otimes B = [a_{ij}B] \in \mathbb{K}^{mp\times nq}$ – the Kronecker product of the matrices $A = [a_{i,j}] \in \mathbb{K}^{m\times n}$ and $B \in \mathbb{K}^{p\times q}$; $\mathrm{spect}(A)\{\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\}$ – the full spectrum (the collection of eigenvalues counted according to their algebraic multiplicities) of $A \in \mathbb{K}^{n\times n}$; $\mathbb{S}_+^{n\times n} \subset \mathbb{K}^{n\times n}$ – the set of Hermitian positive definite matrices; $\lambda_{\min}(A)$ – the minimum eigenvalue of the matrix $A \in \mathbb{S}_+^{n\times n}$; $\mathcal{L}(n)$ – the space of linear matrix operators $\mathbb{K}^{n\times n} \to \mathbb{K}^{n\times n}$. The abbreviation ":=" stands for "equal by definition".

In what follows we present a complete local perturbation analysis for the matrix complex equation

$$Q \pm A^{\mathrm{H}} X^p A - X = 0, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$, $Q \in \mathbb{S}_+^{n\times n}$ and $X \in \mathbb{S}_+^{n\times n}$ is the unknown matrix and $p \in \mathbb{R}$. We derive nonlocal perturbation bounds for the case $p = 1/s$, $s \in \mathbb{N}$. The computation of the principal s–th root $A^{1/s}$ in this case may be done the algorithms considered in [4].

Nonlinear matrix equations of the form (1) arise in many areas of theory and practice. More general matrix equations $X + A^{\mathrm{H}}\mathcal{G}(X)A = Q$ have been studied in [19, 18]. Most of the existing results in this area are connected with particular classes of such equations. Various cases for $\mathcal{G}(X)$ were studied in [8, 7, 1] for $\mathcal{G}(X) = \pm X^{-1}$, in [2, 13] for $\mathcal{G}(X) = \pm X^{-2}$ and in [12, 9, 10, 17, 11] for $\mathcal{G}(X) = \pm X^{-n}$, $n = 1, 2, \ldots$.

2000 Mathematics Subject Classification: 15A24
Keywords: Perturbation bounds, sensitivity analysis, nonlinear matrix equations

We recall that the Frechét derivative $\mathcal{F}(p,X)$ of the function $X \to X^p$, $p \in \mathbb{Q}$ at the point $X \in \mathbb{S}_+^{n\times n}$ is the linear operator $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that

$$(X+E)^p = X^p + \mathcal{F}(p,X)(E) + \mathrm{O}(\|E\|^2),\ E \to 0,$$

where the Hermitian matrix E is a given increment of X and $\|E\|_2 < \lambda_{\min}(X)$. More general increments E may also be considered under a modified definition for X^p.

Note that Frechét derivatives of first and higher order for general operator functions f have been considered in [3]. In particular the cases $f(X) = X^r$ and $f(X) = X^{1/r}$ are studied, where $r \in \mathbb{N}$, $X \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ is the space of bounded linear operators on the Hilbert space $\mathcal{H}$. The authors investigate the norms of the first and higher order Frechét derivatives of these and other operator valued functions.

We shall need the following theorem, which has been proved in [5]

Theorem 1. *For rational powers p the operator $\mathcal{F}(p,X)$ is defined by*

1. *for $p = r$; $r \in \mathbb{N}$, $\mathcal{F}(r,X)(E)\sum_{k=0}^{r-1} X^{r-1-k}EX^k$;*
2. *for $p = -r$; $r \in \mathbb{N}$, $\mathcal{F}(-r,X) = -\sum_{k=0}^{r-1} X^{-1-k}EX^{k-r}$;*
3. *for $p = 1/s$; $s \in \mathbb{N}$, $\mathcal{F}(1/s,X) = (\mathcal{F}(s,X^{1/s}))^{-1}$;*
4. *for $p = r/s$; $r,s \in \mathbb{N}$, $\mathcal{F}(r/s,X) = \mathcal{F}(r,X^{1/s}) \circ (\mathcal{F}(s,X))$;*
5. *for $p = -r/s$; $r,s \in \mathbb{N}$, $\mathcal{F}(-r/s,X)\mathcal{F}(-1,X^{r/s}) \circ \mathcal{F}(r/s,X)$.*

When p is irrational, explicit expressions for $\mathcal{F}(p,X)$ may be found (after reduction of X into diagonal Schur form) for some particular classes of arguments X, namely in the most non-generic case and in the generic case. The complete description of $\mathcal{F}(p,X)$ for irrational p involves the analysis of some unsolved intermediate cases.

In the first case X shall be a scalar matrix, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ and $X = \mathrm{diag}(\lambda,\lambda,\ldots,\lambda)$. This is the most "non-generic" case since here X belongs to a line (one-dimensional variety). In this case we have the following result, which has been proved in [14].

Theorem 2. *The Frechét derivative at $X = \lambda I_n$ is given from $\mathcal{F}(p,X)(E) = p\lambda^{p-1}E$, or*

$$\mathcal{F}(p,X) = p\lambda^{p-1}\mathcal{I}, \tag{2}$$

where $\mathcal{I}$ is the identity operator in $\mathbb{K}^{n\times n}$.

In the second (generic) case the positive definite matrix X has pairwise distinct eigenvalues, namely $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. This is the generic case when X belongs to a part of an open variety in the Zariski topology.

Next we describe in explicit form the action of the operator $\mathcal{F}(p,X)$ in the generic case (see [14]). Let $U \in \mathbb{K}^{n\times n}$ be an unitary matrix such that $\Lambda = U^{\mathrm{H}}XU = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ and $G = U^{\mathrm{H}}EU$. Then the following result is valid.

Theorem 3. *The action of the Frechét derivative $\mathcal{F}(p,X)$ in the generic case is given by*

$$\mathcal{F}(p,X)(E) = U(\mathcal{F}(p,\Lambda)(U^{\mathrm{H}}EU))U^{\mathrm{H}}, \tag{3}$$

where the elements of $\mathcal{F}(p,\Lambda)(G)$ are

$$\begin{aligned} (\mathcal{F}(p,\Lambda)(G))_{ii} &= p\lambda_i^{p-1}g_{ii}, \\ (\mathcal{F}(p,\Lambda)(G))_{ij} &= \frac{\lambda_i^p - \lambda_j^p}{\lambda_i - \lambda_j}g_{ij},\ i \neq j. \end{aligned} \tag{4}$$

2 Perturbed problem

Denote by $\Sigma := (Q, A)$ and X_0 the collection of matrix coefficients and the solution for the equation (1), respectively [16]. Suppose that the matrices Q and A are perturbed as

$$Q \to Q + \Delta Q,\ A \to A + \Delta A.$$

Then the problem is to estimate the perturbation in the solution X_0 as a function of the norms of the perturbations ΔQ and ΔA in the data matrices Q and A. The perturbation in the solution ΔX is a hermitian matrix, such that $X_0 + \Delta X \in \mathbb{S}_+^{n\times n}$. This shall be fulfilled if $\|\Delta X\| < \lambda_{\min}(X)$ (see [5]).

Let the perturbed collection of matrix coefficients be $\Delta\Sigma := (\Delta Q, \Delta A)$. Hence the perturbed equation is

$$F(X_0 + \Delta X, \Sigma + \Delta\Sigma) := Q + \Delta Q \pm (A + \Delta A)^{\mathrm{H}}(X_0 + \Delta X)^p(A + \Delta A) - X_0 - \Delta X = 0. \qquad (5)$$

Denote by

$$\delta = [\delta_1, \delta_2]^\top := [\Delta_Q, \Delta_A]^\top \in \mathbb{R}_+^2$$

the vector, whose elements are the Frobenius norms of the perturbations in the data matrices, i.e.

$$\Delta_Q = \|\Delta Q\|_{\mathrm{F}},\ \Delta_A = \|\Delta A\|_{\mathrm{F}}.$$

The perturbation problem is to find a bound

$$\Delta_X \le f(\delta),\ \delta \in \Omega \subset \mathbb{R}_+^2,$$

for the perturbation $\Delta_X := \|\Delta X\|_{\mathrm{F}}$, where Ω is a given set and f is a continuous function, nondecreasing in all of its arguments and satisfying $f(0) = 0$.

3 Equivalent operator equation

In this section we rewrite the perturbation problem for equation (1) as an equivalent operator equation. This equation is used to obtain local and non–local perturbation bounds. Applying the technique of Lyapunov majorants and the Schauder fixed point principle to the operator equation, we may find conditions for the existence of a small solution to this equation. For this purpose we use the technique of Frechét derivatives. Recall that $F : \mathcal{X} \to \mathbb{K}^{n\times n}$, where $\mathcal{X}$ is an open set of $\mathbb{K}^{n\times n}$, is Frechét differntiable in X_0, if there exists a linear operator $F_X(X_0) : \mathbb{K}^{n\times n} \to \mathbb{K}^{n\times n}$ such that

$$F(X_0 + Z) = F(X_0) + F_X(X_0)(Z) + \mathrm{o}(\|Z\|),\ Z \to 0.$$

In our case the function depends on several matrix arguments and we shall use partial Frechét derivatives in Q and X and the partial (pseudo) derivative in A. We rewrite the perturbed equation (5) as

$$F(X_0 + \Delta X, \Sigma + \Delta\Sigma) = F_1(\Delta X, \Delta\Sigma) + F_2(\Delta X, \Delta\Sigma) = 0,$$

where

$$F_1(\Delta X, \Delta\Sigma) := \sum_{Z\in\{Q,A\}} F_Z(X_0, \Sigma)(\Delta Z) + F_X(X_0, \Sigma)(\Delta X) + F_{12}(\Delta\Sigma)$$

and $F_2(\Delta X, \Delta\Sigma)$ contains second and higher order terms in ΔX and $\Delta\Sigma$. Here $F_Z(X_0, \Sigma)$ are the partial Frechét derivatives of Z in (X_0, Σ) and Z stands for Q or A. The derivative $F_A(X_0, \Sigma)$ is the partial Frechét (pseudo) derivative in A, and for $Z = X$ and $Z = Q$ we have $F_Z(X_0, \Sigma)(Z) \in \mathcal{L}(n)$.

The derivatives in Q, A and X are

$$\begin{aligned} F_Q(X_0,\Sigma)(Z) &= Z, \\ F_A(X_0,\Sigma)(Z) &= \pm Z^{\mathrm{H}} X_0^p A \pm A^{\mathrm{H}} X_0^p Z, \\ F_X(X_0,\Sigma)(Z) &= \pm A^{\mathrm{H}} \mathcal{F}(p,X)(Z)A - Z. \end{aligned}$$

Now we represent $F_1(\Delta X, \Delta\Sigma)$ as

$$\begin{aligned} F_1(\Delta X,\Delta\Sigma) &= F_X(X_0,\Sigma)(\Delta X) + F_Q(X_0,\Sigma)(\Delta Q) \\ &\quad + F_A(X_0,\Sigma)(\Delta A) \pm \Delta A^{\mathrm{H}} X^p \Delta A. \end{aligned}$$

Also we have

$$F_2(\Delta\Sigma,\Delta X) = F_{21}(\Delta\Sigma,\Delta X) + F_{22}(\Delta\Sigma,\Delta X),$$

where with $F_{21}(\Delta\Sigma,\Delta X)$ and $F_{22}(\Delta\Sigma,\Delta X)$ we denote respectively

$$\begin{aligned} F_{21}(\Delta\Sigma,\Delta X) &= \pm\left[A^{\mathrm{H}}\mathcal{F}(p,X)(\Delta X)\Delta A + \Delta A^{\mathrm{H}}\mathcal{F}(p,X)(\Delta X)\Delta A + \Delta A^{\mathrm{H}}\mathcal{F}(p,X)A\right], \\ F_{22}(\Delta\Sigma,\Delta X) &= \pm(A^{\mathrm{H}} + \Delta A^{\mathrm{H}})(\mathrm{O}(\|\Delta X\|^2))(A+\Delta A). \end{aligned}$$

Now we rewrite (5) as

$$\begin{aligned} F_X(X_0,\Sigma)(\Delta X) = -F_Q(X_0,\Sigma)(\Delta Q) - F_A(X_0,\Sigma)(\Delta A) \\ \mp \Delta A^{\mathrm{H}} X^p \Delta A - F_{21}(\Delta\Sigma,\Delta X) - F_{22}(\Delta\Sigma,\Delta X). \end{aligned} \tag{6}$$

We shall consider various cases for the power p as described above.

1) The case $p = 1/s$.

Here the Frechét derivative of the function $X \to X^{1/s}$ is

$$\mathcal{F}(1/s,X)(\Delta X) = \mathcal{F}^{-1}(s, X^{1/s})(\Delta X)$$

and in this case the matrix of $F_X(X_0,\Sigma)$ is

$$L_p = \pm(A^\top \otimes A^{\mathrm{H}})L_1^{-1} - I_{n^2}, \tag{7}$$

where

$$L_1 := \sum_{k=0}^{s-1} (X^{k/s})^\top \otimes X^{(s-1-k)/s}$$

is the matrix of the operator $\mathcal{F}(1/s,X)$.

2) The case $p = r/s$, r and s are coprime.

Now the operator representing the Frechét derivative for $X^{r/s}$ is

$$\mathcal{F}(r/s,X)(\Delta X) = \mathcal{F}(r, X^{1/s}) \circ \mathcal{F}^{-1}(s, X^{1/s}).$$

Then the matrix of $F_X(X_0,\Sigma)$ is

$$L_p = \pm(A^\top \otimes A^{\mathrm{H}})L_2 - I_{n^2}, \tag{8}$$

where

$$L_2 = L_1^{-1} \sum_{k=0}^{r-1} (X^{k/s})^\top \otimes X^{(r-1-k)/s}$$

is the matrix of the operator $\mathcal{F}(r/s,X)$.

3) The case $p = -r/s$.

Here we have a negative rational power and

$$\mathcal{F}(-r/s, X)(\Delta X) = \mathcal{F}(-1, X^{r/s}) \circ \mathcal{F}(r/s, X).$$

The matrix of $F_X(X_0, \Sigma)$ in this case is

$$L_p = \mp(A^\top \otimes A^{\mathrm{H}})L_3 - I_{n^2}, \tag{9}$$

where

$$L_3 = L_2\left((X^{-r/s})^\top \otimes X^{-r/s}\right)$$

is the matrix of the operator $\mathcal{F}(-r/s, X)$. Thus all cases of rational powers have been considered.

4) The non-generic case for real p.

According to (2) we have

$$L_p = \mp(A^\top \otimes A^{\mathrm{H}})L_4 - I_{n^2}, \tag{10}$$

where

$$L_4 = p\lambda^{p-1}I_{n^2}$$

is the matrix of the operator $\mathcal{F}(p, X)$.

5) The generic case for real p.

Having in mind (3) and (4) we obtain the matrix of the operator F_X as

$$L_p = \mp(A^\top \otimes A^{\mathrm{H}})L_5 - I_{n^2}. \tag{11}$$

Here L_5 is the matrix of the operator $\mathcal{F}(p, X)$ with elements

$$\begin{aligned} L_{5(ii)} &= (U \otimes U)(U^{\mathrm{H}} \otimes U^{\mathrm{H}})p\lambda_i^{p-1} \\ L_{5(ij)} &= (U \otimes U)(U^{\mathrm{H}} \otimes U^{\mathrm{H}})\frac{\lambda_i^p - \lambda_g^p}{\lambda_i - \lambda_j},\ i \neq j. \end{aligned}$$

We suppose that the operator $F_X = F_X(X_0, \Sigma)$ is invertible. Then from equation (6) it follows that

$$\begin{aligned} \Delta X &= F_X^{-1}(-\Delta Q \mp \Delta A^{\mathrm{H}} X_0^p A \mp A^{\mathrm{H}} X_0^p \Delta A) \mp F_X^{-1}(\Delta A^{\mathrm{H}} X_0^p \Delta A) \\ &\quad -F_X^{-1}(F_{21}) - F_X^{-1}(F_{22}). \end{aligned}$$

We also have

$$\begin{aligned} \mathrm{vec}(\Delta X) &= -L_p^{-1}\mathrm{vec}(\Delta Q) \mp L_p^{-1}((X_0^p A)^\top \otimes I_n)\mathrm{vec}(\Delta A^{\mathrm{H}}) \\ &\quad \mp(I_n \otimes (A^{\mathrm{H}} X_0^p))\mathrm{vec}(\Delta A) + \mathrm{O}(\Delta\Sigma)^2. \end{aligned} \tag{12}$$

Set

$$\begin{aligned} x &:= \mathrm{vec}(\Delta X), \\ y_1 &:= \mathrm{vec}(\Delta Q), \\ y_2 &:= \mathrm{vec}(\Delta A) \end{aligned}$$

and

$$\begin{aligned} M_1 &:= -L_p^{-1}, \\ M_2 &:= \mp L_p^{-1}\left[I_n \otimes (A^{\mathrm{H}} X_0^p)\right], \\ M_3 &:= \mp L_p^{-1}\left[(X_0^p A)^\top \otimes I_n\right] P_{n^2}, \end{aligned}$$

With these notations the equivalent vector equation may be written as

$$x = \Psi(y, x) := \Psi_1(y) + \Psi_2(y, x), \tag{13}$$

$$\Psi_1(x) := \Psi_{11}(y) + \Psi_{12}(y),$$

$$\Psi_2(x) := \Psi_{21}(y, x) + \Psi_{22}(y, x),$$

where

$$\begin{aligned}
\Psi_{11}(y) &:= -L_p^{-1}\mathrm{vec}(\Delta Q \pm \Delta A^{\mathrm{H}} X_0^p A \pm A^{\mathrm{H}} X_0^p \Delta A)), \\
\Psi_{12}(y) &:= \mp L_p^{-1}\mathrm{vec}(\Delta A^{\mathrm{H}} X_0^p \Delta A), \\
\Psi_{21}(y, x) &:= -L_p^{-1}\mathrm{vec}(F_{21}(\Delta\Sigma, \Delta X)), \\
\Psi_{22}(y, x) &:= -L_p^{-1}\mathrm{vec}(F_{22}(\Delta\Sigma, \Delta X)).
\end{aligned}$$

Moreover, it is fulfilled that

$$\begin{aligned}
\Psi_{11}(y) &= M_1 y_1 + M_2 y_2 + M_3 \overline{y_2}, \\
\|\Psi_{12}(y)\|_2 &\leq \delta_2^2 \|L_p^{-1}\|_2 \|X_0\|_2^p.
\end{aligned}$$

4 Local perturbation analysis

In this section we use the results from Section 3 in order to derive condition numbers and local first order bounds for the perturbation $\Delta_X = \|\Delta X\|_{\mathrm{F}}$ in the solution X_0 of equation (1). In calculating condition numbers and first order estimates in the complex case a special technique [15] must be used based on the theory of additive complex operators. The reason is that the function $A \to A^{\mathrm{H}}$ is not linear (it is additive but not homogeneous). In the Frobenius norm the absolute condition numbers K_Z for the solution of the equation (1) relative to the matrix coefficients, $Z = Q, A$ are

$$\begin{aligned}
K_Q &= \|L_p^{-1}\|_2, \\
K_A &= \|\Theta(M_2, M_3)\|_2,
\end{aligned}$$

where

$$\Theta(M_2, M_3) := \left[\begin{array}{cc} M_{20} + M_{30} & M_{21} - M_{31} \\ M_{21} + M_{31} & M_{20} - M_{30}, \end{array} \right] \in \mathbb{R}^{n^2 \times n^2}$$

and

$$M_2 = M_{20} + \imath M_{21}, \; M_3 = M_{30} + \imath M_{31}$$

are complex $n \times n$ matrices with $M_{20}, M_{21}, M_{30}, M_{31}$ real. Here L_p is the matrix given in (7), (8), (9), (10) and (11) corresponding to various cases for p.

Let

$$\xi := \|\Delta X\|_{\mathrm{F}} = \|\mathrm{vec}(\Delta X)\|_2 = \|x\|_2.$$

Then the first local estimate, based on the condition numbers, is

$$\xi \leq \omega_1(\delta) + \mathrm{O}(\|\delta\|^2), \; \delta \to 0,$$

where

$$\omega_1(\delta) := K_Q\delta_1 + K_A\delta_2.$$

We shall give two more local bounds including an improved first order homogeneous bound. These bounds are not formulated in terms of condition numbers. The reason is that linear local bounds, based on condition numbers, may be more conservative than other first order homogeneous bounds. The second bound is

$$\xi \le \omega_2(\delta) + \mathrm{O}(\|\delta\|^2),\ \ \delta \to 0,$$

where

$$\omega_2(\delta) := \left\| \left[\begin{array}{cc} M_1^{\mathbb{R}} & \Theta(M_2, M_3) \end{array} \right] \right\|_2 \|\delta\|_2$$

and $M_1^{\mathbb{R}}$ is a real representation of matrix M_1, i.e.

$$M_1^{\mathbb{R}} := \left[\begin{array}{cc} M_{10} & -M_{11} \\ M_{11} & M_{10} \end{array} \right] \in \mathbb{R}^{2n\times 2n}$$

and

$$M_1 = M_{10} + \imath M_{11}.$$

The third perturbation bound is

$$\xi \le \omega_3(\delta) + \mathrm{O}(\|\delta\|^2),\ \ \delta \to 0,$$

where

$$\omega_3(\delta) := \sqrt{\delta^\top M \delta}$$

and

$$M := [m_{ij}] \in \mathbb{R}_+^{2\times 2}$$

is a symmetric matrix with non–negative elements

$$M = \left[\begin{array}{cc} (M_1^{\mathbb{R}})^{\mathrm{H}} M_1^{\mathbb{R}} & (M_1^{\mathbb{R}})^{\mathrm{H}} \Theta(M_2, M_3) \\ \Theta(M_2, M_3)^{\mathrm{H}} M_1^{\mathbb{R}} & \Theta(M_2, M_3)^{\mathrm{H}} \Theta(M_2, M_3) \end{array} \right].$$

We have $\omega_3(\delta) \le \omega_1(\delta)$ for all δ, while both inequalities $\omega_2(\delta) < \omega_3(\delta)$ and $\omega_2(\delta) > \omega_3(\delta)$ are possible for some non–negative 2–vectors δ. Hence for small δ we obtain the improved bound

$$\xi \le \omega(\delta) + \mathrm{O}(\|\delta\|^2),\ \ \delta \to 0,$$

where

$$\omega(\delta) := \min\{\omega_2(\delta), \omega_3(\delta)\}.$$

As a corollary of the above considerations we may formulate the following result giving a local perturbation bound for the solution of the equation.

Theorem 4. *For small $\|\delta\|$ the norm $\|\Delta X\|_{\mathrm{F}}$ of the perturbation ΔX in the solution X_0 of equation* (1) *satisfies the local perturbation estimate*

$$\xi \le \omega(\delta) + \mathrm{O}(\|\delta\|^2),\ \ \delta \to 0,$$

where

$$\omega(\delta) := \min\{\omega_2(\delta), \omega_3(\delta)\}.$$

5 Nonlocal perturbation analysis

In this section we present a nonlocal perturbation analysis for the equation (1) for some particular cases for the power p. For the nonlocal perturbation analysis we show that if δ belongs to a certain small set Ω then the equivalent operator Ψ from (13) maps a closed convex set $\mathcal{B} \subset \mathbb{R}^{n^2}$ into itself.

Moreover, the set $\mathcal{B}$ is small being of diameter $f(\delta) = \mathrm{O}(\|\delta\|)$. According to the Schauder fixed point principle there exists a solution $\xi \in \mathcal{B}$ of (13) and $\Delta_X = \|\xi\|_2 \leq f(\delta)$.

Consider the operator equation

$$x = \Psi(y,x),$$

$$\Psi(y,x) = \Psi_1(y) + \Psi_2(y,x).$$

Then the next estimates are valid

$$\begin{aligned} \|\Psi(y,x)\|_2 &\leq \|\Psi_1(y)\|_2 + \|\Psi_2(y,x)\|_2, \\ \|\Psi_1(y)\|_2 &\leq b_0(\delta) : \omega(\delta) + \delta_2^2\|L_p^{-1}\|_2\|X_0\|_2^p, \\ \|\Psi_2(y,x)\|_2 &\leq \|\Psi_{21}(y,x)\|_2 + \|\Psi_{22}(y,x)\|_2, \end{aligned} \tag{14}$$

where

$$\|\Psi_{21}(y,x)\| \leq \beta_1\delta_2 + \beta_2\delta_2^2.$$

Here we have set

$$\beta_1 = \|L_p^{-1}\| \left\|(I_n \otimes A^{\mathrm{H}})\mathrm{Mat}(\mathcal{F}(p,X)) + (A^{\mathrm{H}} \otimes I_n)\mathrm{Mat}(\mathcal{F}(p,X))\right\|_2,$$

$$\beta_2 = \|L_p^{-1}\|\|\mathrm{Mat}(\mathcal{F}(p,X))\|_2,$$

where $\mathrm{Mat}(\mathcal{F}(p,X))$ are the matrices L_1, L_2 and L_3 from(7), (8), (9) for the different cases of p.

Next we shall we need some results for the accuracy of the affine approximation of matrix power functions. We have

$$(X + \Delta X)^p = X^p + \mathcal{F}(p,X)(\Delta X) + \mathrm{O}(\|\Delta X\|)^2.$$

Denote

$$G_s = (X + \Delta X)^p - X^p - \mathcal{F}(p,X)(\Delta X) = \mathrm{O}(\|\Delta X\|^2),\ X \to 0.$$

Here we shall consider the cases $p = 1/2$, $p = 1/3$ and the general case $p = 1/s$. Set $\varepsilon := \|\Delta X\|_{\mathrm{F}}$ and suppose that $\|\Delta X\|_2 < \lambda_{\min}(X)$.

5.1 The case $p = 1/2$.

For G_2 we have the next estimate (see [6])

$$\|G_2\|_{\mathrm{F}} \leq \frac{2l_2^3\varepsilon^2}{1 - 2l_2^2\varepsilon + \sqrt{1 - 4l_2^2\varepsilon}},$$

where

$$l_2 = \|\mathrm{Mat}(\mathcal{F}(1/2,X))\|_2.$$

Then for Ψ_{22} we have

$$\|\Psi_{22}(y,x)\|_2 \quad \leq \quad \|L_{1/2}^{-1}\|(\|A\|_2 + \delta_2)^2 \frac{2l_2^3\varepsilon^2}{1 - 2l_2^2\varepsilon + \sqrt{1 - 4l_2^2\varepsilon}}$$

and

$$\begin{aligned} \|\Psi_2(y,x)\|_2 &\leq b_1(\delta)\varepsilon + b_2(\delta)\varepsilon^2, \\ b_1(\delta) &= \beta_1\delta_2 + \beta_2\delta_2^2, \\ b_2(\delta) &= \|L_{1/2}^{-1}\|(\|A\|_2 + \delta_2)^2 l_2^3. \end{aligned} \tag{15}$$

5.2 The case $p = 1/3$.

Here we have (see [6])

$$\|G_3\|_{\mathrm{F}} \leq \frac{2a_0(\varepsilon)}{1 - a_1(\varepsilon) + \sqrt{(1 - a_1(\varepsilon))^2 - 4a_0(\varepsilon)\hat{a_2}(\varepsilon)}},$$

where

$$\begin{aligned} b &:= \|X_0\|_2^{1/3}, \\ l_3 &= \|\mathrm{Mat}(\mathcal{F}(1/3, X))\|_2, \\ a_0 &= l_3^3\varepsilon^2(3b + l_3\varepsilon), \\ a_1 &= 3l_3^2(2b + l_3\varepsilon), \\ \hat{a_2} &= 2l_3(b + l_3\varepsilon) + \sqrt{b^2l_3^2 + l_3/3}. \end{aligned}$$

For small ε it is fulfilled

$$\|G_3\|_{\mathrm{F}} \leq 3bl_3^3\varepsilon^2 + \mathrm{O}(\varepsilon^3).$$

Finally we have the next estimate

$$\begin{aligned} \|\Psi_2(y, x)\|_2 &\leq b_1(\delta)\varepsilon + b_2(\delta)\varepsilon^2, \\ b_1(\delta) &= \beta_1\delta_2 + \beta_2\delta_2^2, \\ b_2(\delta) &= 3\|L_{1/3}^{-1}\|(\|A\|_2 + \delta_2)^2 l_3^3 b. \end{aligned} \tag{16}$$

5.3 The case $p = 1/s$, $s \in \mathbb{N}$

In this case we have the next estimate for $\|G_s\|_{\mathrm{F}}$ (see [6])

$$\|G_s\|_{\mathrm{F}} \leq \frac{2a_0(\varepsilon)}{1 - a_1(\varepsilon) + \sqrt{(1 - a_1(\varepsilon)^2 - 4a_0(\varepsilon)(a_2(\varepsilon) + \hat{b}(\varepsilon))}},$$

where

$$\hat{b}(\varepsilon) := \sum_{j=2}^{s-1} \alpha_{j+1}(\varepsilon),$$

$$\alpha_{j+1} := a_{j+1}^{1/j}\left(\frac{1 - a_1}{j + 1}\right)^{1-1/j}, \ j = 2, 3, \ldots, s - 1.$$

The coefficients are

$$\begin{aligned} a_0 &= l_s\left[(b + l_s\varepsilon)^s - sb^{s-1}(l_s\varepsilon) - b^s\right], \\ a_1 &= l_s s\left[(b + l_s\varepsilon)^{s-1} - b^{s-1}\right], \\ a_i &= l_s\binom{s}{i}(b + l_s\varepsilon)^{s-i}, \ i = 2, 3, \ldots, s. \end{aligned}$$

Here

$$b := \|X_0\|_2^{1/s}, \ l_s := \|\mathrm{Mat}(\mathcal{F}(1/s, X))\|_2.$$

For small ε it is fulfilled

$$\|G_s\|_{\mathrm{F}} \leq \frac{l_s^3 b^{s-2}\varepsilon^2 s(s - 1)}{2} + \mathrm{O}(\varepsilon^3).$$

Now it may be shown that

$$\begin{aligned} \|\Psi_2(y,x)\|_2 &\leq b_1(\delta)\varepsilon + b_2(\delta)\varepsilon^2, \\ b_1(\delta) &= \beta_1\delta_2 + \beta_2\delta_2^2, \\ b_2(\delta) &= \|L_{1/s}^{-1}\|(\|A\|_2+\delta_2)^2 \frac{l_s^3 b^{s-2}\varepsilon^2 s(s-1)}{2}. \end{aligned} \tag{17}$$

The next step is to construct the majorant equation, whose solution will give the desired nonlocal perturbation bounds

$$\xi \leq h(\delta,\varepsilon) := b_0(\delta) + b_1(\delta)\varepsilon + b_2(\delta)\varepsilon^2.$$

Here the coefficients b_0, b_1 and b_2 are given by (14), (15) for $p = 1/2$, by (14), (16) for $p = 1/3$ and by (14), (17) for $p = 1/s$. The function h is Lyapunov majorant (see [15]) of second degree for the vector operator equation

$$x = \Psi(y,x).$$

Consider the domain

$$\Omega := \left\{\delta \in \mathbb{R}_+^2 : b_1(\delta) + 2\sqrt{b_0(\delta)b_2(\delta)} \leq 1\right\} \tag{18}$$

in $\mathbb{R}_+^2$. If $\delta \in \mathcal{B}$, then the majorant equation

$$\varepsilon = h(\delta,\varepsilon),$$

or, equivalently,

$$b_2(\delta)\varepsilon^2 - (1-b_1(\delta))\varepsilon + b_0(\delta) = 0,$$

has a solution. Let $f(\delta)$ be the smaller solution of the majorant equation. Then

$$\varepsilon_0 = f(\delta) = \frac{2b_0(\delta)}{1-b_1(\delta)+\sqrt{(1-b_1(\delta))^2-4b_0(\delta)b_2(\delta)}}, \tag{19}$$

where the coefficients b_k are determined from (14), (15) for $p = 1/2$, (14), from (16) for $p = 1/3$ and from (14), (17) for $p = 1/s$. Hence, for $\delta \in \Omega$ the operator $\Psi(x,.)$ maps the set $\mathcal{B}_{f(\delta)}$ into itself, where $\mathcal{B}_r$ is the closed central ball of radius $r \geq 0$. According to the Schauder fixed point principles there exists a solution $\xi \in \mathcal{B}_{f(\delta)}$ of equation (13) and we have the following result.

Theorem 5. *Let $\delta \in \Omega$, where Ω is given by (18). Then the perturbed equation (5) has a solution $Y = X_0 + \Delta X$ in a neighborhood of X_0 such that*

$$\|\Delta X\|_{\mathrm{F}} \leq f(\delta),$$

where f is defined from (19).

Next we will give some remarks about the case, when p is a real number.

5.4 The case $p \in \mathbb{R}$

Supposing that we have an estimate

$$\|G(\Delta X)\| \leq g(\varepsilon)$$

we may rewrite (12) as

$$\begin{aligned} \mathrm{vec}(\Delta X) &= -L_p^{-1}\mathrm{vec}(\Delta Q) \mp L_p^{-1}((X_0^p A)^\top \otimes I_n)\mathrm{vec}(\Delta A^{\mathrm{H}}) \\ &\quad \mp L_p^{-1}(I_n \otimes (A^{\mathrm{H}} X_0^p))\mathrm{vec}(\Delta A) + R(\Delta X), \end{aligned}$$

where

$$\begin{aligned}\|R\|_2 &\leq 2\|A\|(\|\mathcal{F}(p,X)\| + g(\varepsilon))\delta_2 + \delta_2^2\|(X+\Delta X)^p\| \\ &= 2\|A\|(\|\mathcal{F}(p,X)\| + g(\varepsilon))\delta_2 + \delta_2^2(\|X_0\|_2^p + \|\mathcal{F}(p,X)\| + g(\varepsilon)) \\ &:= r(\delta,\varepsilon).\end{aligned}$$

Then

$$\xi \leq h(\delta,\varepsilon),$$

where $h(\delta,\varepsilon)$ is a Lyapunov majorant for the operator equation $x = \Psi(y,x)$ and

$$h(\delta,\varepsilon) = \omega(\delta) + r(\delta,\varepsilon)\left\|L_p^{-1}\right\|.$$

We stress that this majorant is not in general of polynomial type.

6 Numerical example

Example 6. Consider the complex matrix equation

$$Q + A^{\mathrm{H}}X^{1/3}A - X = 0$$

with coefficient matrices

$$A = \begin{bmatrix} 0.2 & 0 & 0.1 \\ -0.1 & -0.1\imath & 1 \\ -0.1 & 0.2 & 0 \end{bmatrix}; \qquad Q = \begin{bmatrix} 1,92 & 0.03-0.1\imath & 0.07 \\ 0.3+0.1\imath & 0.93 & -0.1\imath \\ 0.07 & 0.1\imath & 0.98 \end{bmatrix}.$$

The solution of the equation is

$$X_0 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The condition numbers for the matrix coefficients A and Q are

$$K_A = 2.0385,\ K_Q = 1,1927,$$

which shows that the equation is very well conditioned.

Let the perturbation in the matrix A be

$$\Delta A = \varepsilon_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\varepsilon_1 > 0$ is a small parameter. Here we use two parameters, namely ε_1 and the Frobenius norm of the perturbation in the matrix Q. Then

$$\|\Delta A\|_{\mathrm{F}} = 1.7321\,\varepsilon_1.$$

The estimate est3 $:= \omega_3$ give better results than the estimates est1 $:= \omega_1$ and est2 $=: \omega_2$. The perturbation $\|\Delta X\|_{\mathrm{F}}$ in the solution of the equation is estimated by the local bound est from Section (4) and by the nonlinear nonlocal bound from Section (5). The results derived for local and nonlocal bounds are presented in Table 1 for $\varepsilon_1 = 10^{-k}$ and $k = 8,7,\ldots,1$.

Table 1

k	$\|\Delta Q\|_{\mathrm{F}}$	$\|\Delta X\|_{\mathrm{F}}$	local bounds	nonlocal bounds
8	2.1466×10^{-7}	2.2405×10^{-7}	2.8818×10^{-7}	2.8818×10^{-7}
7	2.1466×10^{-6}	2.2405×10^{-6}	2.8818×10^{-6}	2.8818×10^{-6}
6	2.1466×10^{-5}	2.2405×10^{-5}	2.8818×10^{-5}	2.8818×10^{-5}
5	2.1466×10^{-4}	2.2405×10^{-4}	2.8818×10^{-4}	2.8823×10^{-4}
4	2.1465×10^{-3}	2.2405×10^{-3}	2.8817×10^{-3}	2.8869×10^{-3}
3	2.1464×10^{-2}	2.2405×10^{-2}	2.8815×10^{-2}	2.9355×10^{-2}
2	2.1450×10^{-1}	2.2405×10^{-1}	2.8880×10^{-1}	3.6116×10^{-1}
1	2.1252×10^{0}	2.2405×10^{0}	2.8562×10^{0}	$*$

For $k = 1$ the nonlocal perturbation bound does not exist because of break of the condition $\delta \in \Omega$.

For this example both the local and nonlocal bounds are very tight.

Acknowledgment. The authors would like to thank the anonymous referee for the helpful remarks.

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