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NEUMANN SYSTEM AND HYPERELLIPTIC AL FUNCTIONS

Shigeki Matsutani

Abstract. This article shows that the Neumann dynamical system is described well in terms of the Weierstrass hyperelliptic al functions. The descriptions are very primitive; their proofs are provided only by residual computations but don't require any theta functions.

1 Introduction

The Neumann dynamical system is a well-known integrable nonlinear dynamical system, whose Lagrangian for $(q, \dot{q}) \in \mathbb{R}^{2g+2}$ is given by,

$$L = \frac{1}{2}\sum_{i=1}^{g+1} \dot{q}_i^2 - \frac{1}{2}\sum_{i=1}^{g+1} a_i q_i^2, \tag{1}$$

with a holonomic constraint,

$$\Phi(q) = 0, \quad \Phi(q) := \sum_{i=1}^{g+1} q_i^2 - 1, \tag{2}$$

which was proposed by C. Neumann in 1859 for the case of $g = 2$ [16]. This is studied well in frameworks of the dynamical system [14], of the symplectic geometry [8], of the algebraic geometry [15], of the representation of the infinite Lie algebra [1, 17].

D. Mumford gave explicit expressions of the Neumann system in terms of hyperelliptic functions based upon classical and modern hyperelliptic function theories [15]. This article gives more explicit expressions of the Neumann system using Weierstrass hyperelliptic al functions [19].

In the case of elliptic functions theory [20], Weierstrass $\wp$ functions and Jacobi sn, cn, dn functions play important roles in the theory even though they are expressed by

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the theta functions and all relations among them are rewritten by the theta functions. The expressions of Weierstrass $\wp$ functions and Jacobi sn, cn, dn functions make the theory of elliptic functions fruitful and reveal the essentials of elliptic functions [20].

Unfortunately in the case of higher genus case, such studies are not enough though F. Klein and K. Weierstrass discovered hyperelliptic versions of these $\wp$, sn, cn, dn functions [10, 19]. The history, especially of the al functions, a generalization of sn, cn, dn function, is well-described in Chapter "Fonctions ellptiques et intégrales abéliennes" in [6] as in [13]. The al function was discovered by Weierstrass in order to obtain his hyperelliptic θ function, Al, in 1854 [19], which is the first attempt to higher general genus version of N. H. Abel's theory of elliptic functions following the Abelian integral theory of hyperelliptic curves by C. G. J. Jacobi [11]. The name "al" and "Al" are honor to N. H. Abel. Klein sophisticated Weierstrass' "Al" to hyperelliptic σ function following the Weierstrass' elliptic σ function theory [10] and defined hyperelliptic $\wp$ functions.

These studies were basically succeeded by the modern algebraic geometry and the Abelian function theory. However their concreteness of the theories in the nineteenth century [2, 3, 4, 10, 19] faded out. Thus Mumford picked up the theory of Jacobi [11] and connected it with the modern theory [15]. For the similar purpose, several authors devote themselves to reinterpretations of the modern theory of hyperelliptic functions in terms of these functions in [2, 3, 4, 10, 19] and developing studies of these functions as special functions [5, 12, 13, and their references]. In this article, we also proceed to make the hyperelliptic function theory more fruitful and show that the Weierstrass al functions give natural descriptions of the Neumann dynamical system. As in Theorem 10, the configuration q^i of i-th particle (or coordinate) is directly given by the al function,

$$q^i(t) = \mathfrak{al}_i(t).$$

Here $\mathfrak{al}_i(t)$ is defined in Definition 4 which was originally defined by Weierstrass as a generalization of Jacobi sn, cn, and dn functions over a elliptic curve to that over a hyperelliptic curve. As Jacobi sn, cn, dn functions are associated with several nonlinear phenomena and these relations enable us to recognize the essentials of the phenomena [18], we expect that this expression also plays a role in hyperelliptic function case. (Even though it is not well-known, a differential equation which is known as the sine-Gordon equation plays the central role in the discovery of the elliptic and hyperelliptic functions [19, p.296], [6, 13].)

In fact the description in terms of the al functions makes several properties of the Neumann system rather simple. For examples, an essential property of the Neumann system $\sum_i^g (q^i(t))^2 = 1$ is interpreted as a hyperelliptic version of $\mathrm{sn}^2(u)+\mathrm{cn}^2(u) = 1$. Its hamiltonian is given as a manifestly constant quantity in Theorem 10 (3). Here we don't need any theta functions and the theory of the theta functions at all. Following [2, 3, 4, 5, 19], we give proofs in this article, which basically need only primitive residual computations. This is in contrast to the previous works, e.g., [15].

We give our plan of this article. §2 gives a short review of the Neumann system. In §3, we introduce the hyperelliptic al functions and hyperelliptic $\wp$ functions. There we also give a short review of their basic properties following [2, 3, 5, 19]. §4 is our main section, where we give our main theorem. There $\mathfrak{al}$ function naturally describes the Neumann system.

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2 Neumann System

We shortly review the Neumann system $(q, \dot{q}) \in \mathbb{R}^{2g+2}$ whose Lagrangian and constraint condition are given (1) and (2) in Introduction. The constraint (2) means $\dot{\Phi}(q) = 0$,

$$\sum_{i=1}^{g+1} \dot{q}_i q_i = 0. \tag{3}$$

The canonical momentum p_i to q_i is given as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}.$$

Purely kinematic investigations lead the following proposition [15].

Proposition 2. *The hamiltonian of this system is given by*

$$H := \frac{1}{2} \sum_{i=1}^{g+1} \dot{q}_i^2 + \frac{1}{2} \sum_{i=1}^{g+1} a_i q_i^2, \tag{4}$$

and the hamiltonian vector field is given by

$$D_H = \sum \dot{q}_i \frac{\partial}{\partial q_i} - \sum a_i q_i \frac{\partial}{\partial \dot{q}_i} + \left(\sum [a_i q_i^2 - \dot{q}_i^2] \right) \sum q_i \frac{\partial}{\partial \dot{q}_i}. \tag{5}$$

The equation of motion is given by

$$\dot{q}_i = \dot{q}_i, \quad \ddot{q}_i = -(2L + a_i) q_i. \tag{6}$$

3 Hyperelliptic Functions

In this article, we consider a hyperelliptic curve C_g given by an affine equation [15, 7],

$$y^2 = f(x), \quad f(x) = A(x)Q(x),$$

$$A(x) := (x-a_1)(x-a_2)\cdots(x-a_{g+1}),$$
$$Q(x) := (x-c_1)(x-c_2)\cdots(x-c_g),$$

where a_i's and c_i's are complex numbers. Let $b_i := a_i$ $(i = 1, \cdots, g+1)$ and $b_{g+i+1} := c_i$ $(i = 1, \cdots, g)$.

From here we deal with $(x_1, x_2, \cdots, x_g)$ belonging to g symmetric product $\mathrm{Sym}^g(C_g)$ of C_g.

Let us introduce the canonical coordinate $u := (u_1, \cdots, u_g)$ in $\mathbb{C}^g$ related to in the Jacobian $\mathcal{J}_g$ of C_g [5],

$$u_i := \sum_{a=1}^{g} \int_{\infty}^{(x_a, y_a)} \frac{x^{i-1} dx}{2y}.$$

Here $u_- := (u_1, \cdots, u_{g-1})$, $u = (u_-, u_g)$. The Jacobian $\mathcal{J}_g$ is given by $\mathbb{C}^g/\Lambda$ for a certain lattice Λ associated with the periodic matrices of C^g [3, 5].

Due to Abel's theorem [9], the following proposition holds.

Proposition 3. $(u_1, u_2, \cdots, u_g)$ *are linearly independent in* $\mathbb{C}^g$*. In other words, there are paths in* $\mathrm{Sym}^g(C_g)$ *so that* $\{u_g\}$ *is equal to* $\mathbb{C}$ *with fixing* u_-*.*

As Mumford studied the Neumann system using UVW-expression of the hyperelliptic functions [15], we give U, V and W functions [15],

$$U(x) =: (x-x_1)\cdots(x-x_g),$$

$$V(x) := \sum_{a=1}^{g} \frac{y_a U(x)}{U'(x_a)(x-x_a)}, \quad W(x) := \frac{f(x) - V(x)^2}{U(x)}.$$

In this article, we will express the system in terms of the hyperelliptic $\wp$ functions and al functions which are written only in terms of the data of the curve. Let us introduce these functions as follows.

Definition 4. *The hyperelliptic* $\wp_{gi}$ $(i = 1, 2, \cdots, g+1)$ *functions of* u*'s are defined by*

$$U(x) = x^g + \sum_{i=1}^{g} (-1)^i \wp_{gi} x^{g-i}, \tag{7}$$

e.g., $\wp_{gg} := x_1 + \cdots + x_g$.

The Weierstrass al_i *and* $\mathfrak{al}_i$ $(i = 1, 2, \cdots, g)$ *functions are defined by [2, 3, 19],*

$$\mathfrak{al}_r(u) := \gamma_r \mathrm{al}_r(u), \quad \mathrm{al}_r(u) := \sqrt{U(a_r)}(u), \tag{8}$$

where we set $\gamma_r = 1/\sqrt{A'(a_r)}$ *in this article. We write*

$$\mathrm{al}_r^{[i]}(u) := \frac{\partial}{\partial u_i}\mathrm{al}_r(u), \quad \mathfrak{al}_r^{[i]}(u) := \frac{\partial}{\partial u_i}\mathfrak{al}_r(u).$$

As the constant factor is less important, we call bath functions al-functions though the original version defined by K. Weierstrass has another factor in [19].

First we have primitive relations between differentials of al functions and UVW expressions:

Lemma 5. *1.* $\mathrm{al}_i^{[g]}(u) = -\dfrac{V(a_i)(u)}{\mathrm{al}_i(u)}, \quad \mathfrak{al}_i^{[g]}(u) = -\dfrac{V(a_i)(u)}{\mathfrak{al}_i(u)A'(a_i)}.$

2. $\dfrac{U(x)}{A(x)} = \displaystyle\sum_{i=1}^{g+1} \frac{\mathfrak{al}_i(u)^2}{x - a_i}, \qquad \dfrac{V(x)}{A(x)} = -\displaystyle\sum_{i=1}^{g+1} \frac{\mathfrak{al}_i(u)\mathfrak{al}_i^{[g]}(u)}{x - a_i},$

$$\frac{W(x)}{A(x)} = \sum_{i=1}^{g+1} \frac{\mathfrak{al}_i^{[g]}(u)^2}{x - a_i}.$$

Proof. Noting $\dfrac{\partial}{\partial u_g} = \displaystyle\sum_{a=1}^{g} \frac{2y_a}{U'(x_a)}\frac{\partial}{\partial x_a}$ [4] and $\dfrac{\partial}{\partial x_a}U(x) = -\dfrac{U(x)}{(x - x_a)}$, we find that $\dfrac{1}{2}\dfrac{\partial}{\partial u_g}U(x) = -V(x)$, which directly gives the relations in 1. The relations in 2 are obtained from the definition of $\mathfrak{al}_r$, 1 and the fact that $f(a_i)$ vanishes. □

The sn and cn functions are defined by $\mathrm{sn}(u) := \sqrt{a_1 - a_3}/\sqrt{x - a_3}$ and $\mathrm{cn}(u) := \sqrt{x - a_1}/\sqrt{x - a_3}$, which can be alternatively defined by

$$\mathrm{sn}(u + \Omega) = \frac{\sqrt{x - a_3}}{\sqrt{a_2 - a_3}}, \quad \mathrm{cn}(u + \Omega) = \frac{\sqrt{x - a_2}}{\sqrt{a_3 - a_2}}.$$

As the right hand sides of both definitions correspond to $\mathfrak{al}_1$ precisely, al functions should be recognized as an extension of sn, cn functions. As sn and cn functions have the relations,

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1, \quad k^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1,$$

the $\mathfrak{al}$ function also has similar relations as follows.

Proposition 6.

$$\sum_{i=1}^{g+1} \mathfrak{al}_i^2(u) = 1, \quad \sum_{i=1}^{g+1} \frac{1}{a_i^2} [\mathfrak{al}_i^{[g]}]^2(u) = 0.$$

Though this relation was also studied in [15] as a generalization of Frobenius identity of theta functions, we will prove it by primitive method, without any theta functions.

Proof. The left hand side is given by

$$\sum_{i=1}^{g+1} \frac{U(a_i)}{A'(a_i)} = \frac{1}{2} \sum_{i=1}^{g+1} \mathrm{res}_{(a_i,0)} \frac{U(x)}{A(x)},$$

since around the finite ramified point $(a_i, 0)$ of the curve C_g, we have a local parameter $t^2 = (x - a_i)$ and

$$\mathrm{res}_{(a_i,0)} \frac{U(x)}{A(x)} dx = \mathrm{res}_{(a_i,0)} \frac{2U(t^2 + a_i) t dt}{(t^2 + a_i - a_1) \cdots t^2 \cdots (t^2 + a_i - a_{g+1})}.$$

Let us consider an integral over a boundary of polygon expression C_0 of C_g,

$$\oint_{\partial C_0} \frac{U(x)}{A(x)} dx = 0,$$

which gives the relation,

$$\sum_{i=1}^{g+1} \mathrm{res}_{(a_i,0)} \frac{U(x)}{A(x)} dx = -\mathrm{res}_{\infty} \frac{U(x)}{A(x)} dx.$$

At ∞, a local parameter t of C_g is given by $x = 1/t^2$:

$$\mathrm{res}_{\infty} \frac{U(x)}{A(x)} dx = \mathrm{res}_{\infty} \frac{\frac{1}{t^{2g}} (1 - x_1 t^2) \cdots (1 - x_g t^2)}{\frac{1}{t^{2g+2}} (1 - a_1 t^2) \cdots (1 - a_{g+1} t^2)} \frac{-2}{t^3} dt = -2.$$

Hence it is proved. Similarly we obtain the relations for $\mathfrak{al}_r^{[g]}$ though we should evaluate $W(x)/x^2 A(x)$ using Lemma 5 2. □

There is a natural relation between al function and $\wp_{gg}$ function

Proposition 7.

$$\frac{\partial^2}{\partial u_g^2} \mathfrak{al}_i(u) = \frac{\partial}{\partial u_g} \mathfrak{al}_i^{[g]}(u) = \left(\sum_{j=1, b_i \neq a_i}^{2g+1} b_j - 2\wp_{gg}(u) \right) \mathfrak{al}_i.$$

Proof. This is directly obtained due to Lemma 5 and 8. □

Lemma 8.

$$\frac{1}{2}\frac{\partial}{\partial u_g}V(a_i) = U(a_i)\left(\sum_{i=1}^{2g+1} b_i - a_i - 2\sum_{a=1}^{g} x_a\right) - \frac{1}{U(a_i)}V(a_i)^2.$$

Proof. Here we check the left hand side $\frac{\partial}{\partial u_g}V(a_i)$:

$$\begin{aligned}
&= \sum_{a,b=1}^{g} \frac{2y_a}{U'(x_a)}\frac{\partial}{\partial x_a}\frac{y_b U(a_i)}{U'(x_b)(a_i - x_b)} \\
&= \sum_{a=1}^{g} \frac{y_a}{U'(x_a)}\frac{\partial}{\partial x_a}\frac{2y_a U(a_i)}{U'(x_a)(a_i - x_a)} + \sum_{a\neq b}^{g} \frac{y_a}{U'(x_a)}\frac{\partial}{\partial x_a}\frac{2y_b U(a_i)}{U'(x_b)(a_i - x_b)} \\
&= U(a_i)\sum_{a=1}^{g}\left[\frac{1}{U'(x)}\frac{\partial}{\partial x}\left(\frac{f(x)U(a_i)}{U'(x)(a_i - x)}\right)\right]_{x=x_a} \\
&\quad + U(a_i)\sum_{a\neq b}^{g}\frac{f(x_a)}{U'(x_a)^2(a_i - x_a)^2} \\
&\quad - U(a_i)\sum_{a\neq b}^{g}\frac{2y_a}{U'(x_a)}\frac{y_b}{U'(x_b)(a_i - x_a)}\left(\frac{1}{(a_i - x_b)} - \frac{1}{(x_a - x_b)}\right) \\
&= \sum_{i=1}^{2g+1} b_i - a_i - 2\sum_{a=1}^{g} x_a - U(a_i)\left(\sum_{a\neq b}^{g}\frac{y_a}{U'(x_a)(a_i - x_a)}\right)^2.
\end{aligned}$$

Here we used the following relations.

1. $\frac{\partial}{\partial x_a}U'(x_a) = \frac{1}{2}\frac{\partial^2}{\partial x^2}U(x)|_{x=x_a}$,

2. $$\sum_{a\neq b}^{g}\frac{2y_a}{U'(x_a)}\frac{y_b}{U'(x_b)(a_i - x_a)}\left(\frac{1}{(a_i - x_b)} - \frac{1}{(x_a - x_b)}\right) = \sum_{a\neq b}^{g}\frac{y_a}{U'(x_a)}\frac{y_b}{U'(x_b)(a_i - x_a)(a_i - x_b)},$$

3. $\left[\frac{1}{U'(x)}\frac{\partial}{\partial x}\left(\frac{f(x)}{U'(x)(a_i - x_b)}\right)\right]_{x=x_a} = \mathrm{res}_{(x_a,y_a)}\frac{f(x)}{U(x)^2(a_i - x)}dx$, and

4. $$\sum_{i=1}^{2g+1} b_i - a_i - 2\sum_{a=1}^{g} x_a = \sum_{a=1}^{g} \mathrm{res}_{(x_a,y_a)} \frac{f(x)}{U(x)^2(a_i - x)} dx.$$

The fourth relation is obtained by an evaluation of the integral $\oint_{\partial C_0} \frac{f(x)}{U(x)^2(a_i - x)} dx.$ □

Remark 9. The Klein hyperelliptic $\wp$ function obeys the KdV equations [5, 12]. On the other hand, $\frac{\partial}{\partial u_g} \log \mathrm{al}_r$ is a solution of the MKdV equation [12] and $\log \mathrm{al}_r$ obeys the sine-Gordon equations [13]. The relation in Proposition 7 means so-called *Miura transformation*,

$$\left(\frac{\partial}{\partial u_g} \log \mathrm{al}_i\right)^2 + \frac{\partial^2}{\partial u_g^2} \log al_i = (\mathcal{L} - a_i),$$

where $\mathcal{L} := \frac{1}{2}\left(2\wp_{gg} - \sum_{i=1}^{2g+1} b_i\right)$.

4 Neumann system and hyperelliptic al functions

This section gives our main theorem as follows.

Theorem 10. *Suppose that configurations of* $(x_1, \cdots, x_g) \in \mathrm{Sym}^g(C_g)$ *are given so that* (al_i) *belongs to* $\mathbb{R}^{g+1}$, $u_g \in \mathbb{R}$ *fixing* $u_- \in \mathbb{R}^{g-1}$.

1. $\mathfrak{al}_i$ *obey the Neumann system, i.e.,*

$$q_i(t) = \mathfrak{al}_i(u_-, t), \quad \dot{q}_i = -\mathfrak{al}_i^{[g]}(u_-, t), \tag{9}$$

where the time t *of the system is identified with* $-u_g$ *and thus the hamiltonian vector field is given by*

$$D_H := \frac{d}{dt} \equiv -\frac{\partial}{\partial u_g}.$$

2. *The hamiltonian (4) and the Lagrangian (1) are given by*

$$H = \frac{1}{2}\left(\sum_{i=1}^{g+1} a_i - \sum_{a=1}^{g} c_a\right), \quad L = \frac{1}{2}\left(2\wp_{gg} - \sum_{i=1}^{2g+1} b_i\right).$$

3. *The conserved quantities are* c_i $(i = 1, \cdots, g)$ *and*

$$m_i := q_i^2 + \sum_{i=1}^{g+1} \sum_{j=1,\neq i}^{g+1} \frac{(q_i \dot{q}_j - q_j \dot{q}_i)^2}{a_i - a_j}, \quad (i = 1, \cdots, g+1),$$

which obey relations,

$$m_i = \frac{Q(a_i)}{A'(a_i)}, \quad \sum_{i=1}^{g+1} m_i = 1, \quad \sum_{i=1}^{g+1} a_i m_i = H.$$

These relations were essentially proved in [15] using UVW expression and the properties of the theta functions without al functions. However by following the method [3, 4, 19], we show them directly using nature of al functions without theta functions. We use only the data of the curve C_g and simple residual computations. Our method is very primitive in contrast to [15]. Since the theta function has excess parameters for higher genus case, we believe that our method has some advantage, at least, for concrete problems of geometry and physics.

Proof. Assumptions are asserted by Proposition 3. 1: Due to Proposition 6, $\mathfrak{al}_r$'s obviously obeys the constraint condition $\Phi(\mathfrak{al}) = 0$ (2) and $\dot{\Phi}(\mathfrak{al}) = 0$ (3) by differentiating the both sides of the identity in u_g. We should check whether they obey the equation of motion (6), which are proved in Proposition 7 if we assume the form of the Lagrangian L in 2. 2 is directly obtained by using the relations in Lemma 12. Finally 3 is proved in Remark 14. □

Remark 11. 1. The equation of motion (6) is directly related to Proposition 7, which is connected with the Miura transformation. Further the constraint (2) satisfies due to the identity of $\mathfrak{al}$ function as mentioned in Proposition 6. These exhibits essentials of al functions. Hence the Neumann system should be expressed by the al functions as some dynamical systems are expressed by Jacobi sn, cn, dn functions [18].

2. We remark that the hamiltonian depends only upon a_i's and c_i's which determines the hyperelliptic curve C_g. Thus it is manifest that it is invariant for the time u_g development of the system.

3. There are $2g$ degrees of freedom as a kinematic system because the constraints Φ and $\dot{\Phi}$ reduce $(2g+2)$ ones to $2g$ ones. The independent conserved quantities m_i are $g = g + 1 - 1$; "-1" comes from $\sum m_i = 1$. Since the sum of m_i gives hamiltonian H, H is not linearly independent conserved quantities. Since there are other g conserved quantities c_i but their sum gives the hamiltonian $\sum m_i$, the dimensional of independent c_i is $g - 1$. However $\sum_{i=1}^{g+1} \dot{q}_i^2 / a_i^2 = 0$ compensates the lacking one. Hence the degrees of freedom of this system is equal to number of the conserved quantities.

4. By the definition of c_i's, c_i depends upon the initial condition of the Neumann system whereas a_i is fixed as coupling constants of the Neumann system. Thus $\mathcal{S}_g := \{C_g \ : \ y^2 = A(x)Q(x) \mid c_1, c_2, \cdots c_g \in \mathbb{C}\}$ corresponds to the solution space $\mathcal{N}_g$ of the Neumann system if $u_g \in \mathbb{R}$ and $(\mathfrak{al}, \mathfrak{al}^{[g]}) \in \mathbb{R}^{2g+2}$. The $\mathcal{S}_g$ is a subspace of the moduli $\mathcal{M}_g$ of hyperelliptic curves of genus g.

Let us give a lemma and remarks as follows, which are parts of the proofs of the theorem.

Lemma 12. *1.* $\displaystyle\sum_{i=1}^{g+1} [\mathfrak{al}_i^{[g]}(u)]^2 = \wp_{gg}(u) - \sum_{a=1}^{g} c_a.$

2. $\displaystyle\sum a_i \mathfrak{al}_i(u)^2 = \sum_{i=1}^{g+1} a_i - \wp_{gg}(u).$

Proof. (1) Due to Lemma 5, we deal with $\displaystyle\oint_{\partial C_0} \frac{V(x)^2}{U(x)A(x)} dx = 0$ giving

$$2\sum_{i=1}^{g+1} \frac{V(a_i)^2}{U(a_i)A'(a_i)} + \sum_{a=1, \epsilon=\pm}^{g} \mathrm{res}_{(x_a, \epsilon y_a)} \frac{V(x)^2}{U(x)A(x)} dx + \mathrm{res}_{\infty} \frac{V(x)^2}{U(x)A(x)} dx = 0.$$

Whereas the third term vanishes, each element in the second term is given by

$$\mathrm{res}_{(x_a, \pm y_a)} \frac{V^2(x)}{U(x)A(x)} dx = \frac{Q(x_a)}{U'(x_a)}.$$

Further we also evaluate an integral, $\displaystyle\oint_{\partial C_0} \frac{Q(x)}{U(x)} dx = 0$. The integrand has singularities at $(x_a, \pm y_a)$ and infinity. Similar consideration leads us to the identities

$$\sum_{a=1, \epsilon=\pm}^{g} \frac{Q(x_a)}{U'(x_a)} = 2(c_1 + \cdots + c_g) - 2(x_1 + \cdots + x_g).$$

Due to these relations, we have the relation 1.

2. Next we consider an integral, $\displaystyle\oint_{\partial C_0} x \frac{U(x)}{A(x)} dx = 0$. A residual computation gives $\displaystyle\sum_{i=1}^{g+1} a_i \frac{U(a_i)}{A'(a_i)} = -\mathrm{res}_{\infty} x \frac{U(x)}{A(x)} dx$. The infinity term gives $2((x_1 + \cdots x_g) - (a_1 + \cdots a_{g+1}))$. Hence we also have the relation in 2. □

Remark 13. Using the fact $\frac{\partial x_a}{\partial u_g} = \frac{2y_a}{U'(x_a)}$, we obtain anther form of Lemma 12 [7], $\sum_{i=1}^{g+1} [\mathfrak{al}_i^{[g]}]^2 = \sum_{a,b=1}^{g} g(x)_{a,b} \frac{\partial x_a}{\partial u_g}\frac{\partial x_b}{\partial u_g}$, where $g(x)_{a,b} := -\sum_{i=1}^{g+1} \frac{U(a_i)}{(a_i - x_a)(a_i - x_b)A'(a_i)}$ whose off-diagonal part does not vanish for the case genus $g > 2$ in general.

Remark 14. (Proof of Theorem 10 3). Here we give the conserved quantities of the Neumann system as a proof of Theorem 10.3. Let us consider,

$$m_i(x) = q_i^2 + \sum_{i=1}^{g+1} \sum_{j=1,\neq i}^{g+1} \frac{(q_i\dot{q}_j - q_j\dot{q}_i)^2}{x - a_j}.$$

Then we have identities

$$\frac{f(x)}{A(x)^2} \equiv \frac{U(x)W(x) + V(x)^2}{A(x)^2} = \sum_{i=1}^{g+1} \frac{m_i(x)}{x - a_i},$$

$$m_i = \mathrm{res}_{a_i} \frac{m_i(x)}{x - a_i} = q_i^2 + \sum_{i=1}^{g+1} \sum_{j=1,\neq i}^{g+1} \frac{(q_i\dot{q}_j - q_j\dot{q}_i)^2}{a_i - a_j}.$$

The direct computation gives the relations in Theorem 10 3, when we deal with the integrals of differentials $\frac{Q(x)}{A(x)}dx$, $\frac{xQ(x)}{A(x)}dx$.

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8-21-1 Higashi-Linkan, Sagamihara, 228-0811,
Japan.
e-mail: rxb01142@nifty.com