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MODIFIED ADOMIAN DECOMPOSITION METHOD FOR SINGULAR INITIAL VALUE PROBLEMS IN THE SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper an efficient modification of Adomian decomposition method is introduced for solving singular initial value problem in the second-order ordinary differential equations. The scheme is tested for some examples and the obtained results demonstrate efficiency of the proposed method.

1 Introduction

In the recent years, the studies of singular initial value problems in the second order ordinary differential equations (ODEs) have attracted the attention of many mathematicians and physicists. A large amount of literature developed concerning Adomian decomposition method [1, 2, 3, 4, 6, 7], and the related modification [5, 8, 9, 11] to investigate various scientific models. It is the aim of this paper to introduce a new reliable modification of Adomian decomposition method. Our next aim consists in testing the proposed method in handling a generalization of this type of problems. For this reason a new differential operator is proposed which can be used for singular ODEs. In addition, the proposed method is tested for some examples and the obtained results show the advantage of using this method.

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2 Modified Adomian decomposition method for singular initial value problems

2.1 Modified Adomian decomposition method

Algorithm 1. *Consider the singular initial value problem in the second order ordinary differential equation in the form*

$$y^{''}+\frac{2}{x}y^{'}+f(x,y)=g(x), \tag{2.1}$$

$$y(0)=A, y(0)^{'}=B,$$

where $f(x,y)$ is a real function, $g(x)$ is given function and A and B are constants. Here, we propose the new differential operator, as below

$$L=x^{-1}\frac{d^2}{dx^2}xy, \tag{2.2}$$

so, the problem (2.1) can be written as,

$$Ly=g(x)-f(x,y). \tag{2.3}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below,

$$L^{-1}(.)=x^{-1}\int_0^x\int_0^x x(.)dxdx. \tag{2.4}$$

Applying L^{-1} of (2.4) to the first two terms $y^{''}+\frac{2}{x}y^{'}$ of Equation (2.1) we find

$$L^{-1}(y^{''}+\frac{2}{x}y^{'})=x^{-1}\int_0^x\int_0^x x(y^{''}+\frac{2}{x}y^{'})dxdx$$

$$=x^{-1}\int_0^x(xy^{'}+y-y(0))dx=y-y(0).$$

By operating L^{-1} on (2.3), we have

$$y(x)=A+L^{-1}g(x)-L^{-1}f(x,y). \tag{2.5}$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $f(x,y)$ by infinity series

$$y(x)=\sum_{n=0}^{\infty}y_n(x), \tag{2.6}$$

and

$$f(x,y)=\sum_{n=0}^{\infty}A_n. \tag{2.7}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [7, 10] to formulate Adomian polynomials. The following algorithm:

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1F^{'}(u_0),\\
A_2 &= u_2F^{'}(u_0)+\frac{u_1^2}{2!}F^{''}(u_0),\\
A_3 &= u_3F^{'}(u_0)+u_1u_2F^{''}(u_0)+\frac{u_1^3}{3!}F^{'''}(u_0),\\
&\cdots
\end{aligned} \tag{2.8}$$

can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting (2.6) and (2.7) into (2.5),

$$\sum_{n=0}^{\infty}y_n=A+L^{-1}g(x)-L^{-1}\sum_{n=0}^{\infty}A_n. \tag{2.9}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$\begin{aligned}
y_0(x) &= A+L^{-1}g(x),\\
y_{k+1}(x) &= -L^{-1}(A_k), \qquad k\geq 0,
\end{aligned} \tag{2.10}$$

which gives

$$\begin{aligned}
y_0(x) &= A+L^{-1}g(x),\\
y_1(x) &= -L^{-1}(A_0),\\
y_2(x) &= -L^{-1}(A_1),\\
y_3(x) &= -L^{-1}(A_2),\\
&\cdots
\end{aligned} \tag{2.11}$$

From (2.8) and (2.11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (2.6) can be immediately obtained.
For numerical purposes, the n-term approximate

$$\Psi_n=\sum_{n=0}^{n-1}y_k,$$

can be used to approximate the exact solution.

Example 2. *We consider the nonlinear singular initial value problem :*

$$y^{''} + \frac{2}{x}y^{'} + y^3 = 6 + x^6, \tag{2.12}$$

$$y(0) = 0, y^{'}(0) = 0.$$

In an operator form, Equation (2.12) becomes

$$Ly = 6 + x^6 - y^3. \tag{2.13}$$

Applying L^{-1} on both sides of (2.13) we find

$$y = L^{-1}(6 + x^6) - L^{-1}y^3,$$

therefore

$$y = x^2 + \frac{x^8}{72} - L^{-1}y^3.$$

By Adomian decomposition method [8] we divided $x^2 + \frac{x^8}{72}$ into two parts and we using the polynomial series for the nonlinear term, we obtain the recursive relationship

$$y_0 = x^2,$$

$$y_{k+1} = \frac{x^8}{72} - L^{-1}(A_k). \tag{2.14}$$

This in turn gives

$$y_0 = x^2,$$

$$y_1 = \frac{x^8}{72} - \frac{1}{x}\int_0^x \int_0^x x(x^2)^3 dx dx = 0$$

$$y_{k+1} = 0, k \geq 0.$$

In view of (2.14), the exact solution is given by

$$y = x^2.$$

Example 3. *Consider the linear singular initial value problem:*

$$y^{''} + \frac{2}{x}y^{'} + y = 6 + 12x + x^2 + x^3, \tag{2.15}$$

$$y(0) = y^{'}(0) = 0.$$

Proceeding as before we obtain the relation

$$y(x) = L^{-1}(6 + 12x + x^2 + x^3) - L^{-1}(y),$$

therefore

$$y_0(x) = x^2 + x^3 + \frac{x^4}{20} + \frac{x^5}{30},$$

we divided $x^2 + x^3 + \frac{x^4}{20} + \frac{x^5}{30}$ *in two parts*

$$y_0 = x^2 + x^3,$$

$$y_{k+1} = \frac{x^4}{20} + \frac{x^5}{30} - L^{-1}(y_k). \tag{2.16}$$

This in turn gives

$$y_0 = x^2 + x^3,$$

$$y_1 = \frac{x^4}{20} + \frac{x^5}{30} - L^{-1}(x^2 + x^3) = 0,$$

$$y_{k+1} = 0, k \geq 0.$$

In view of (2.16), *the exact solution is given by*

$$y = x^2 + x^3.$$

2.2 Generalization

Algorithm 4. *A generalization of Equation* (2.1) *has been studied by Wazwaz [11]. In a parallel manner, we replace the standard coefficients of* y' *and* y *by* $\frac{2n}{x}$ *and* $\frac{n(n-1)}{x^2}$ *respectively, for real* $n, n \geq 0$.
In other words, a general equation

$$y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y + f(x,y) = g(x), n \geq 0, \tag{2.17}$$

with initial conditions

$$y(0) = A, y'(0) = B,$$

we propose the new differential operator, as below

$$L = x^{-n}\frac{d^2}{dx^2}x^n y,$$

so, the problem (2.17) *can be written as,*

$$Ly = g(x) - f(x,y). \tag{2.18}$$

The inverse operator L^{-1} *is therefore considered a two-fold integral operator, as below.*

$$L^{-1} = x^{-n}\int_0^x\int_0^x x^n(.)dxdx. \tag{2.19}$$

Applying L^{-1} *of* (2.19) *to the first three terms* $y^{''}+\frac{2n}{x}y^{'}+\frac{n(n-1)}{x^2}y$ *of Equation* (2.17) *we find*

$$L^{-1}(y^{''}+\frac{2n}{x}y^{'}+\frac{n(n-1)}{x^2}y)=x^{-n}\int_0^x\int_0^x x^n(y^{''}+\frac{2n}{x}y^{'}+\frac{n(n-1)}{x^2}y)dxdx$$

$$x^{-n}\int_0^x(x^ny^{'}+nx^{n-1}y)dx=y.$$

By operating L^{-1} *on* (2.18), *we have*

$$y(x)=A+L^{-1}g(x)-L^{-1}f(x,y),$$

proceeding as before we obtain

$$y_0(x)=A+L_n^{-1}g(x),$$

$$y_{k+1}=-L_n^{-1}A_k, k\geq 0,$$

where A_k *are Adomian polynomials that represent the nonlinear term* $f(x,y)$.

Algorithm 5. *In this section, two singular initial ordinary differential equations are considered and then are solved by standard [11] and modified Adomian decomposition methods.*

Example 6. *Consider the linear singular initial value problem:*

$$y^{''}+\frac{4}{x}y^{'}+\frac{2}{x^2}y=12, \tag{2.20}$$

$$y(0)=0, y^{'}(0)=0,$$

standard Adomian decomposition method, we put

$$L^{-1}(.)=\int_0^x x^{-4}\int_0^x x^4(.)dxdx.$$

In an operator form, Equation (2.20) *becomes*

$$Ly=12-\frac{2}{x^2}y. \tag{2.21}$$

By applying L^{-1} *to both sides of* (2.21) *we have*

$$y=L^{-1}(12)-L^{-1}(\frac{2}{x^2}y),$$

proceeding as before we obtained the recursive relationship

$$y_0=\frac{6x^2}{5},$$

$$y_{k+1} = -L^{-1}(\frac{2}{x^2}y_k),$$

and the first few components are as follows:

$$y_0 = \frac{6x^2}{5},$$

$$y_1 = \frac{6x^2}{25},$$

$$y_2 = \frac{6x^2}{125},$$

$$y_3 = \frac{6x^2}{625},$$

$$\ldots$$

We can easily see that the sum of the above expressions can not give the exact solution to the problem (2.20), i.e. in this case the Adomian decomposition method diverges.

Modified Adomian decomposition method: According to (2.19) we put

$$L(.) = x^{-2}\frac{d^2}{dx^2}x^2(.),$$

so

$$L^{-1}(.) = x^{-2}\int_0^x\int_0^x x^2(.).$$

In an operator form, Equation (2.20) becomes

$$Ly = 12.$$

Now, by applying L^{-1} to both sides we have

$$L^{-1}Ly = x^{-2}\int_0^x\int_0^x 12x^2dxdx,$$

and it implies,

$$y(x) = x^2.$$

So, the exact solution is easily obtained by proposed Adomian method.

Example 7. *Consider the nonlinear singular initial value problems*

$$y'' + \frac{6}{x}y' + \frac{6}{x^2}y + y^2 = 20 + x^4, \tag{2.22}$$

$$y(0) = 0, y'(0) = 0,$$

standard Adomian decomposition method: we put

$$L(.) = x^{-6}\frac{d}{dx}x^6\frac{d}{dx}(.),$$

so

$$L^{-1}(.) = \int_o^x x^{-6}\int_0^x x^6(.)dxdx.$$

In an operator form Equation (2.22) *becomes*

$$Ly = 20 + x^4 - \frac{6}{x^2}y - y^2. \tag{2.23}$$

By applying L^{-1} *to both sides of* (2.23) *we have*

$$y = L^{-1}(20 + x^4) - L^{-1}(\frac{6}{x^2}y) - L^{-1}(y^2).$$

Proceeding as before we obtained the recursive relationship

$$y_0 = \frac{10}{7}x^2 + \frac{x^6}{66},$$

$$y_{k+1} = L^{-1}(-\frac{1}{x^2}y_k) - L^{-1}(A_k). \tag{2.24}$$

The Adomian polynomials for the nonlinear term $F(y) = y^2$ *are computed as follows*

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0 + \frac{y_1^2}{2}, \\ A_3 &= 2y_3y_0 + y_1y_2, \\ &\dots \end{aligned} \tag{2.25}$$

which are obtained by using formal algorithms in [1, 10]. Substituting (2.25) *into* (2.24) *gives the components*

$$y_0 = \frac{10}{7}x^2 + \frac{x^6}{66},$$

$$y_1 = -L^{-1}(\frac{1}{x^2}y_0) - L^{-1}(A_0) = -\frac{5}{126}x^2 - \frac{3483}{112112}x^6 - \frac{x^{10}}{3465} - \frac{x^{14}}{1158696},$$

$$y_2 = -L^{-1}(\frac{1}{x^2}y_1) + L^{-1}(A_1)$$

$$= \frac{5}{4536}x^5 + \frac{71169}{2833294464}x^6 + \frac{18528773}{2528715309120}x^{10} + \frac{811073}{15659303692032}x^{14}$$

$$+\frac{793}{6538749053760}x^{18}+\frac{1}{7631487021312}x^{22},$$

$$\dots$$

It is to see that the standard Adomian decomposition method is divergent to solve this problem.
Modified Adomian decomposition: According to (2.19). We put

$$L(.)=x^{-3}\frac{d^2}{dx^2}x^3(.),$$

so

$$L^{-1}(.)=x^{-3}\int_0^x\int_0^x x^3(.)dxdx.$$

In an operator form, Equation (2.22) becomes

$$Ly=20+x^4-y^2. \tag{2.26}$$

Now, by applying L^{-1} to both sides of (2.26) we have

$$y=L^{-1}(20+x^4)-L^{-1}(y^2),$$

therefore

$$y=x^2+\frac{x^6}{72}-L^{-1}y^2,$$

by divided $x^2+\frac{x^6}{72}$ into two parts and we obtain the recursive relationship

$$y_0=x^2,$$

$$y_{k+1}=\frac{x^6}{72}-L^{-1}(A_{k)}. \tag{2.27}$$

$$y_{k+1}=0,k\geq 0.$$

In view of (2.26) the exact solution is given by

$$y(x)=x^2.$$

so, the exact solution is easing obtained by proposed Adomian method.

The comparison between the results mentioned in Examples 6 and 7 shows the power of the proposed method of this paper for these singular initial value problems in the second (ODEs)

3 Conclusion

In this paper, we proposed an efficient modification of the standard Adomian decomposition method for solving singular initial value problem in the second-order ordinary differential equation. The study showed that the decomposition method is simple and easy to use and produces reliable results with few iterations used.

The obtained results show that the rate of convergence of modified Adomian decomposition method is higher than standard Adomian decomposition method for these problems.

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