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ALMOST-PERIODIC SOLUTION FOR BAM NEURAL NETWORKS

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Abstract. In this paper, we study the existence and uniqueness solution and investigate the conditions that make it almost-periodic solution for **BAM** neural networks with retarded delays. The existence of solution established by using Schauder fixed point theorem. The uniqueness established by using Banach fixed point theorem. Moreover we study the parametric stability of such a solution. Also we illustrate our results with an example.

1 Introduction and Preliminaries

Recently, the concept of almost periodicity solutions (see[4, 17]), for differential and integral equations is an important area of research. It naturally arises in diverse fields such as population biology, economics, neural networks and chemical processes (see[5, 6]). Our aim is to study the existence and uniqueness of almost-periodic solution for a class of two-layer associative networks, called bidirectional associative memory **BAM** neural networks (see[11, 8]) with and without delays, has been proposed and used in many fields (see[13, 12]). The study in neural dynamic systems involves a discussion of stability properties (see[10, 7]), periodic and almost-periodic oscillatory [1, 2], chaos [3] and bifurcation [16]. Moreover, we examine the parametric stability of this solution. The parametric stability(see[9, 15]) together with robust stability for nonlinear systems admits the stability of equilibrium points for such systems. The problem of robust stability is to find how much we can perturb the parameters of the systems and still retain stability of the equilibrium points. And the maximal value of parameter that retains stability of the equilibrium is called the parametric stability margin.

The main subject of this paper is to study the existence and uniqueness solution and investigate the conditions that make this solution is almost-periodic solution for

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the **BAM** neural networks

$$x_i' = -A_i x_i(t) + p_i(t)\sum_{j=1}^{m} \mu_{ji} f_j(y_j(t-js)) + I_i(t), \tag{1.1}$$

$$y_j' = -B_j y_j(t) + q_j(t)\sum_{i=1}^{n} \omega_{ij} f_i(x_i(t-is)) + J_j(t),$$

subject to the initial conditions

$$x_i(t_0) = x_{i0}, \quad y_j(t_0) = y_{j0}, \quad t_0 \in [-T, 0], \ T < \infty \ \textit{and} \ t \in J := [-T, T] \tag{1.2}$$

where s is a positive number. $i = 1, ..., n$, $j = 1, ..., m$, $x_i : J \to \mathbb{R}$, $y_j : J \to \mathbb{R}$, are the activation of the $i-th$ and $j-th$ neurons respectively. A_i, B_j, are positive constants which are denoting the connected matrices. $p_i(t)$ *and* $q_j(t)$ are continuous functions. μ_{ji} and ω_{ij} are connection weights. $I_i(t)$ and $J_j(t)$ are continuous functions and they denoted the external bias on the $i-th$ and $j-th$ units respectively.

Definition 1. *[4, 17] A function $f \in \mathcal{B}$ ($\mathcal{B}$ is a Banach space) is called almost periodic in $t \in \mathbb{R}$ uniformly in any $K \subset \mathcal{B}$ a bounded subset, if for each $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that every interval of length $\delta_\epsilon > 0$ contains a number s with the following property:*

$$\|f(t+s,u) - f(t,u)\| < \epsilon, \quad t \in \mathbb{R}, u \in K.$$

Definition 2. *[9, 15] Consider the nonlinear system of the form*

$$x^{\cdot} = f(x,\mu) = f_\mu(x), \tag{1.3}$$

which has an equilibrium at $x^* = 0$ when $\mu = \mu^*$. The equilibrium $x^* = 0$ is called parametrically stable at μ^* if there exists a small neighborhood $N(\mu^*)$ such that for any $\mu \in N(\mu^*)$, the following two conditions hold: (a) There exists an equilibrium $x^e(\mu)$ of the nonlinear system (1.3). (b) For any given $\epsilon > 0$, there exist correspondingly a $\delta = \delta(\epsilon,\mu) > 0$ such that

$$\|x_0 - x^e(\mu)\| < \delta \ \textit{implies} \ \|x(t;x_0,\mu) - x^e(\mu)\| < \epsilon \ \textit{for all} \ t \geq 0.$$

The equilibrium $x^* = 0$ is called parametrically unstable at μ^* if it is not parametrically stable.

Definition 3. *[9, 15] Consider the system (1.3) which has an equilibrium at $x^* = 0$ when $\mu = \mu^*$. The equilibrium $x^* = 0$ is called parametrically asymptotically stable at μ^* if there exists a small neighborhood $N(\mu^*)$ such that for any $\mu \in N(\mu^*)$, the following two conditions hold:*
(a) The equilibrium $x^* = 0$ is parametrically stable at μ^*.
(b) For all $\mu \in N(\mu^*)$, there exists a number $\delta(\mu) > 0$ such that

$$\|x_0 - x^e(\mu)\| < \delta \ \textit{implies} \ \|x(t;x_0,\mu) - x^e(\mu)\| \to 0 \ \textit{as} \ t \to \infty.$$

Definition 4. *[9, 15] Consider the system (1.3) which has an equilibrium at $x^* = 0$ when $\mu = \mu^*$. The equilibrium $x^* = 0$ is called parametrically exponentially stable at μ^* if there exists a small neighborhood $N(\mu^*)$ such that for any $\mu \in N(\mu^*)$, the following two conditions hold:*
(a) The equilibrium $x^* = 0$ is parametrically stable at μ^*.
(b) For all $\mu \in N(\mu^*)$, there exists a number $\delta(\mu) > 0$ such that

$$\|x_0 - x^e(\mu)\| < \delta \ \text{implies} \ \|x(t; x_0, \mu) - x^e(\mu)\| < Me^{-at}\|x(0) - x^e(\mu)\|,$$

for some positive constants M, a.

Lemma 5. *[15] The equilibrium point x^* of the nonlinear system (1.3) is parametrically exponentially stable at μ^* if the nonlinear system $x^{\cdot} = f(x, \mu^*) = f_{\mu^*}(x)$ is locally exponentially stable at $x = 0$.*

2 The existence and uniqueness solution.

In this section we give conditions for the existence and uniqueness of a solution for the system (1.1). For arbitrary vector: $(x(t), y(t)) := (x_1(t), ..., x_n(t), y_1(t), ..., y_m(t))^T$, $t \in J$, define the norm: $\|(x, y)\| = \|x\| + \|y\|$ where, $\|x\| = sup_{t\in J} max_{1\leq i\leq n}\{|x_i(t)|\}$ *and* $\|y\| = sup_{t\in J} max_{1\leq j\leq m}\{|y_j(t)|\}$. Set $\mathcal{B}^{n+m} := \{(x, y)|(x, y) = (x_1, ..., x_n, y_1, ..., y_m)^T\}$ then $\mathcal{B}^{n+m}$ is a Banach space endowed with the above norm. To facilitate our discussion, let us first state the following assumption denoted **(A)** :

1. A_i is a positive constant such that $\overline{A} := max_{1\leq i\leq n}\{|A_i|\} < \infty$.

2. B_j is a positive constant such that $\overline{B} := max_{1\leq j\leq m}\{|B_j|\} < \infty$.

3. $I_i(t)$ is a continuous function on J such that $\overline{I} := sup_{t\in J} max_{1\leq i\leq n}\{|I_i(t)|\} < \infty$.

4. $J_j(t)$ is a continuous function on J such that $\overline{J} := sup_{t\in J} max_{1\leq j\leq m}\{|J_j(t)|\} < \infty$.

5. $p_i(t)$ is a continuous function on J such that $\overline{p} := sup_{t\in J} max_{1\leq i\leq n}\{|p_i(t)|\} < \infty$.

6. $q_j(t)$ is a continuous function on J such that $\overline{q} := sup_{t\in J} max_{1\leq i\leq n}\{|q_i(t)|\} < \infty$.

7. μ_{ji} is a parameter such that $\overline{\mu} := max\{|\mu_{ji}|\} < \infty$.

8. ω_{ij} is a parameter such that $\overline{\omega} := max\{|\omega_{ij}|\} < \infty$.

9. Denotes $R := \{\overline{A} + \overline{B} + \overline{a}[m\overline{\mu}\overline{p} + n\overline{\omega}\overline{q}]\}$ such that $0 < 2TR < 1$.

10. f_k is a continuous function on $\mathbb{R}$ such that $|f_k(x)| \leq a(t)|x|$, where $a : J \to J$ is a positive continuous function with $\overline{a} = sup_{t\in J} a(t)$.

The solution of the system (1.1), subject to the initial conditions (1.2), can be written as

$$x_i(t) = x_{i0} + \int_{-T}^{t} [-A_i x_i(\tau) + p_i(\tau) \sum_{j=1}^{m} \mu_{ji} f_j(y_j(\tau - js)) + I_i(\tau)] d\tau, \tag{2.1}$$

$$y_j(t) = y_{j0} + \int_{-T}^{t} [-B_j y_j(\tau) + q_j(\tau) \sum_{i=1}^{n} \omega_{ij} f_i(x_i(\tau - is)) + J_j(\tau)] d\tau.$$

Define an operator $P : \mathcal{B}^{n+m} \to \mathcal{B}^{n+m}$ by

$$P(x, y) := (\int_{-T}^{t} [-A_1 x_1(\tau) + p_1(\tau) \sum_{j=1}^{m} \mu_{j1} f_j(y_j(\tau - js)) + I_1(\tau)] d\tau, ...,$$

$$\int_{-T}^{t} [-A_n x_n(\tau) + p_n(\tau) \sum_{j=1}^{m} \mu_{jn} f_j(y_j(\tau - js))$$

$$+ I_n(\tau)] d\tau, \int_{-T}^{t} [-B_1 y_1(\tau) + q_1(\tau) \sum_{i=1}^{n} \omega_{i1} f_i(x_i(\tau - is)) + J_1(\tau)] d\tau, ...,$$

$$\int_{-T}^{t} [-B_m y_m(\tau) + q_m(\tau) \sum_{i=1}^{n} \omega_{im} f_i(x_i(\tau - is)) + J_m(\tau)] d\tau)^T.$$

Let B_r be a convex close subset of $\mathcal{B}^{n+m}$ define by $B_r := \{(x, y)|(x, y) \in \mathcal{B}^{n+m}, \|(x, y) - (x_0, y_0)\| \leq r\}$ where $(x_0, y_0) = (\int_{-T}^{t} I_1(\tau)d\tau, ..., \int_{-T}^{t} I_n(\tau)d\tau, \int_{-T}^{t} J_1(\tau)d\tau, ..., \int_{-T}^{t} J_m(\tau)d\tau)^T$ and $r \geq \frac{4T^2 R[\overline{I} + \overline{J}]}{1 - 2TR}$.

Theorem 6. *Let assumption* **(A)** *hold. Then the modelling system* (1.1) *has a solution.*

Proof. In order to show that (1.1) has a solution we only need to prove that P has a fixed point. According to the definition of the norm of Banach space $\mathcal{B}^{n+m}$, we have $\|(x_0, y_0)\| \leq 2T[\overline{I} + \overline{J}]$. Now we prove that P has a fixed point.

$$\|P(x, y) - (x_0, y_0)\| \leq sup_{t\in J} \int_{-T}^{t} max_{1\leq i\leq n}\{| - A_i x_i(\tau)|$$

$$+ |p_i(\tau) \sum_{j=1}^{m} \mu_{ji} f_j(y_j(\tau - js))|\} d\tau + sup_{t\in J} \int_{-T}^{t} max_{1\leq j\leq m}\{| - B_j y_j(\tau)|$$

$$+ |q_j(\tau) \sum_{i=1}^{n} \omega_{ij} f_i(x_i(\tau - is))|\} d\tau \leq 2T\{\overline{A}\|x\| + m\overline{\mu p a}\|y\| + \overline{B}\|y\| + n\overline{\omega q a}\|x\|\}$$

$$\leq 2T\{\overline{A} + \overline{B} + \overline{a}[m\overline{\mu p} + n\overline{\omega q}]\}\|(x, y)\| \leq 2TR(\|(x, y) - (x_0, y_0)\| + \|(x_0, y_0)\|)$$

we obtain that

$$\|P(x,y)-(x_0,y_0)\| \leq \frac{2TR\|(x_0,y_0)\|}{1-2TR} \leq \frac{4T^2R[\overline{I}+\overline{J}]}{1-2TR},$$

that is $P : B_r \to B_r$. Then P maps B_r into itself. In fact, P maps the convex closure of $P[B_r]$ into itself. Since f are bounded on B_r, $P[B_r]$ is equicontinuous and the Schauder fixed point Theorem shows that P has a fixed point $(x,y) \in \mathcal{B}^{n+m}$ such that $P(x,y)=(x,y)$, which is corresponding to the solution of (1.1).
In the next theorem, we study the uniqueness solution of (1.1). □

For this purpose, we illustrate the following assumption denoted **(B)** :

1. There exists $\ell_i > 0$ such that $|f_i(\phi_i)-f_i(\varrho_i)| \leq \ell_i\|\phi_i-\varrho_i\|$ for all $i=1,...,n$.
2. There exists $\ell_j > 0$ such that $|f_j(\xi_j)-f_j(\nu_j)| \leq \ell_j\|\xi_j-\nu_j\|$ for all $j=1,...,m$.
3. $2T\{\overline{A}+\overline{B}+\overline{\ell}[m\overline{\mu}\overline{p}+n\overline{\omega}\overline{q}]\} < 1$, where $\overline{\ell} := max\{\ell_k\}$ such that $k=1,...,max\{n,m\}$.

Theorem 7. *Let assumptions* **(A)** *and* **(B)** *hold. Then system* (1.1) *has a unique solution.*

Proof. We only need to prove that the fixed point of P is unique. Let (x,y) and (u,v) in U. By the definition of the norm we have $\|(x,y)-(u,v)\| = \|(x-u,y-v)\| = \|x-u\|+\|y-v\|$.

$$\|P(x,y)-P(u,v)\| = sup_{t\in J}\{|\int_{-T}^{t} max_{1\leq i\leq n}[-A_i(x_i(\tau)-u_i(\tau))$$

$$+p_i(\tau)\sum_{j=1}^{m}\mu_{ji}(f_j(y_j(\tau-js))-f_j(v_j(\tau-js)))]d\tau|\}$$

$$+sup_{t\in J}\{|\int_{-T}^{t} max_{1\leq j\leq m}[-B_j(y_j(\tau)-v_j(\tau))$$

$$+q_j(\tau)\sum_{i=1}^{n}\omega_{ij}(f_i(x_i(\tau-is))-f_i(u_i(\tau-is)))]d\tau|\}$$

$$\leq sup_{t\in J}\int_{-T}^{t} max_{1\leq i\leq n}\{|-A_i||(x_i(\tau)-u_i(\tau))|$$

$$+|p_i(\tau)|\sum_{j=1}^{m}|\mu_{ji}||(f_j(y_j(\tau-js))-f_j(v_j(\tau-js)))|\}d\tau$$

$$+sup_{t\in J}\int_{-T}^{t} max_{1\leq j\leq m}\{|-B_j||(y_j(\tau)-v_j(\tau))$$

$$+|q_j(\tau)|\sum_{i=1}^{n}|\omega_{ij}||(f_i(x_i(\tau-is))-f_i(u_i(\tau-is)))|\}d\tau$$

$$\leq 2T\{\overline{A}\|x-u\| + m\overline{p}\overline{\mu}\ell\|y-v\| + \overline{B}\|y-v\| + n\overline{q}\overline{\omega}\ell\|x-u\|\}$$
$$= 2T\{\overline{A} + n\overline{q}\overline{\omega}\ell\}\|x-u\| + 2T\{\overline{B} + m\overline{p}\overline{\mu}\ell\}\|y-v\|$$
$$\leq 2T\{\overline{A} + n\overline{q}\overline{\omega}\ell\}(\|x-u\| + \|y-v\|) + 2T\{\overline{B} + m\overline{p}\overline{\mu}\ell\}(\|x-u\| + \|y-v\|)$$
$$= 2T\{\overline{A} + \overline{B} + \overline{\ell}[m\overline{\mu}\overline{p} + n\overline{\omega}\overline{q}]\}(\|x-u\| + \|y-v\|)$$

by assumption **(B)**, implies that P is a contraction mapping then by Banach fixed point theorem, P has a unique fixed point which is corresponds to the solution of system (1.1). □

3 Almost periodic solution.

In this section, we introduce the conditions that let every solution of system (1.1) be almost periodic solution. we illustrate the following assumption denoted **(C)** :

Setting $\|(x,y)\| := \kappa$. And suppose $p_i(t), q_j(t), I_i(t)$ and $J_j(t)$ are almost periodic functions of period s, such that

$$|p_i(t+s) - p_i(t)| < \frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8m\overline{\mu}T\kappa\overline{a}}, \quad |q_j(t+s) - q_j(t)| < \frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8n\overline{\omega}T\kappa\overline{a}},$$

$$|I_i(t+s) - I_i(t)| < \frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8T} \quad and \;\; |J_i(t+s) - J_i(t)| < \frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8T}.$$

Lemma 8. *Let assumption* **(C)** *hold. Then operator P is almost periodic function.*

Proof. By the proof of Theorem 2.1, P is bounded operator. By assumption **(C)**, we have $\forall\epsilon > 0$ there exist δ_ϵ such that there exist $s \in [\gamma, \gamma+\delta\epsilon]$ with the following properties:

$$|P(x(t+s), y(t+s)) - P(x(t), y(t))| \leq sup_{t\in J}\int_{-T}^{t} max_{1\leq i\leq n}|-A_i||(x(\tau+s),$$
$$y(\tau+s)) - (x(\tau), y(\tau))|d\tau + (m\overline{\mu}a(t)\|y\|)sup_{t\in J}\int_{-T}^{t} max_{1\leq i\leq n}|p_i(\tau+s) - p_i(\tau)|d\tau$$
$$+ sup_{t\in J}\int_{-T}^{t} max_{1\leq i\leq n}|I_i(\tau+s) - I_i(\tau)|d\tau$$
$$+ sup_{t\in J}\int_{-T}^{t} max_{1\leq j\leq m}|-B_j||(x(\tau+s), y(\tau+s)) - (x(\tau), y(\tau))|d\tau$$

$$+ (n\overline{\omega}a(t)\|x\|) sup_{t\in J}\int_{-T}^{t} max_{1\leq j\leq m}|q_j(\tau+s)-q_j(\tau)|d\tau$$

$$+ sup_{t\in J}\int_{-T}^{t} max_{1\leq j\leq m}|J_j(\tau+s)-J_j(\tau)|d\tau + 2T|(x(t+s),y(t+s))$$

$$-(x(t),y(t))|(\overline{A}+\overline{B}) \leq (m\overline{\mu}a(t)\|(x,y)\|) sup_{t\in J}\int_{-T}^{t} max_{1\leq i\leq n}|p_i(\tau+s)-p_i(\tau)|d\tau$$

$$+ sup_{t\in J}\int_{-T}^{t} max_{1\leq i\leq n}|I_i(\tau+s)-I_i(\tau)|d\tau$$

$$+ (n\overline{\omega}a(t)\|(x,y)\|) sup_{t\in J}\int_{-T}^{t} max_{1\leq j\leq m}|q_j(\tau+s)$$

$$-q_j(\tau)|d\tau + sup_{t\in J}\int_{-T}^{t} max_{1\leq j\leq m}|J_j(\tau+s)-J_j(\tau)|d\tau$$

then we have

$$|P(x(t+s),y(t+s))-P(x(t),y(t))| \leq \frac{2m\overline{\mu}T\kappa\overline{a}}{[1-2T(\overline{A}+\overline{B})]}\times\frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8m\overline{\mu}T\kappa\overline{a}}$$
$$+\frac{2n\overline{\omega}T\kappa\overline{a}}{[1-2T(\overline{A}+\overline{B})]}\times\frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8n\overline{\omega}T\kappa\overline{a}}+\frac{2T}{[1-2T(\overline{A}+\overline{B})]}\times\frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8T}$$
$$+\frac{2T}{[1-2T(\overline{A}+\overline{B})]}\times\frac{\epsilon[1-2T(\overline{A}+\overline{B})]}{8T}=\epsilon.$$

Implies that P is almost periodic function. □

Theorem 9. *Let assumptions* **(A)**, **(B)** *and* **(C)** *hold. Then system* (1.1) *has a unique almost periodic solution.*

Proof. By Theorems 6 , 7 and Lemma 8. □

4 Parametric stability.

In this section, we discuss the conditions of parametric stability for the almost periodic solution of modelling system (1.1). The study of stability of system (1.1) is equivalent to the study of stability of the system

$$x_i' = -A_i x_i(t) + p_i(t)\sum_{j=1}^{m}\mu_{ji} f_j(y_j(t-js)) \tag{4.1}$$

$$y_j' = -B_j y_j(t) + q_j(t)\sum_{i=1}^{n}\omega_{ij} f_i(t-is)).$$

It is easy to prove the following result

Lemma 10. *$(x^*, y^*) := (0,0)$ is an equilibrium point for system (4.1) at $(\mu^*, \omega^*) := (0,0)$.*

Theorem 11. *If for small values c_i and d_j, the system*

$$[-A_i x_i(t)+p_i(t)\sum_{j=1}^{m}\mu_{ji}f_j(y_j(t-js)) = c_i,\ -B_j y_j(t)+q_j(t)\sum_{i=1}^{n}\omega_{ij}f_i(x_i(t-js)) = d_j] \tag{4.2}$$

is solvable, then the equilibrium point for system (4.1) is parametric asymptotically stable.

Proof. (By [14] section3) or [15] . □

Lemma 12. *For the homogeneous system*

$$[x_i' = -A_i x_i(t),\quad y_j' = -B_j y_j(t)], \tag{4.3}$$

$(x, y) = (0, 0)$ is an exponentially stable point.

Proof. We can put the system in a matrix formula $\acute{X} = WX$, where W is $n + m \times n + m$ diagonal matrix. It is easily seen to be a *Hurwitz* matrix with the eigenvalues $-A_1, ..., -A_n, B_1, ..., B_m$. Thus system (4.3) is exponentially stable at $(x, y) = (0, 0)$. □

Theorem 13. *System (4.1) is parametric exponentially stable in the equilibrium point $(x^*, y^*) = (0, 0)$ at $(\mu^*, \omega^*) = (0, 0)$.*

Proof. At $(\mu^*, \omega^*) = (0, 0)$, system (4.1) reduce to the homogeneous system (4.3). Then by Lemma 4.2, $(x, y) = (0, 0)$ is exponentially stable. Thus in view of Lemma 1.1, the equilibrium point $(x^*, y^*) = (0, 0)$ for system (4.1) is parametric exponentially stable at $(\mu^*, \omega^*) = (0, 0)$. □

5 An example.

In this section, we give an example to illustrate our results. Consider the following simple **BAM** networks with almost periodic coefficients of period 2π.

$$x_i' = -A_i x_i(t) + p_i(t)\sum_{j=1}^{2}\mu_{ji}f_j(y_j(t-js)) + I_i(t), \tag{5.1}$$

$$y_j' = -B_j y_j(t) + q_j(t)\sum_{i=1}^{2}\omega_{ij}f_i(x_i(t-is)) + J_j(t),$$

subject to the initial conditions $x_i(t_0) = x_{i0} = 0.25, \quad y_i(t_0) = y_{i0} = 0.25, \quad t \in J := [-1/8, 1/8]$, where $i = j = 1, 2$, $\overline{A} = 1/2$, $\overline{B} = 1/2$, $p_i(t) = (1, \ sin(t))^T$ with $\overline{p} = 1$ and $q_j(t) = (sin(t), \ 1)^T$ with $\overline{q} = 1$ and $I_i = J_j = 1$. Setting

$$\begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.1 \end{pmatrix}$$

$$\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 \\ 0.5 & 0.5 \end{pmatrix}$$

with $\overline{\mu} = 0.5$, $\overline{\omega} = 0.5$. Define the function f as follows $f_i(x) = x$ and $f_j(y) = y$ where $a(t) = 1$. From above, we see that the functions involved in the previous example satisfy assumption **(A)**. Then in view of Theorem 7, the system has a solution in $U := \{(x, y) | \|(x, y) - (x_0, y_0)\| \leq r = 3/2\}$. Now if $\ell = 0.5$, the solution is unique (Theorem 6). Also all the parameters of the example are almost-periodic functions in t. Thus the system has a unique almost-periodic solution. It is clear that $(x^*, y^*) = (0, 0)$ is an equilibrium point for the system at $(\mu^*, \omega^*) = (0, 0)$. In order to examine the parametric stability of the system, we can easy to show that the following system is solvable:

$$-A_i x_i(t) + p_i(t) \sum_{j=1}^{m} \mu_{ji} f_j(y_j(t - js)) = c_i \tag{5.2}$$

$$-B_j y_j(t) + q_j(t) \sum_{i=1}^{n} \omega_{ij} f_i(x_i(t - js)) = d_j$$

for fixed constants c_1, c_2, d_1 and d_2. Thus we obtain that the system is parametric asymptotically stable (see Theorem 11). Now, since the system

$$x_i' = -A_i x_i(t)$$

$$y_j' = -B_j y_j(t)$$

is locally exponentially stable at $(x, y) = (0, 0)$, where

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{pmatrix} = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{pmatrix}$$

then in view of Theorem 13, the equilibrium point $(x^*, y^*) = (0, 0)$ for system (5.1) is parametric exponentially stable at $(\mu^*, \omega^*) = (0, 0)$.

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