Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 1 – 16

NEW RESULT OF EXISTENCE OF PERIODIC SOLUTION FOR A HOPFIELD NEURAL NETWORKS WITH NEUTRAL TIME-VARYING DELAYS

Chuanzhi Bai and Chunhong Li

Abstract. In this paper, a Hopfield neural network with neutral time-varying delays is investigated by using the continuation theorem of Mawhin's coincidence degree theory and some analysis technique. Without assuming the continuous differentiability of time-varying delays, sufficient conditions for the existence of the periodic solutions are given. The result of this paper is new and extends previous known result.

1 Introduction

In recent years, the cellular neural networks have been extensively studied and applied in many different fields such as signal and image processing, pattern recognition and optimization. In implementation of networks, time delays are inevitably encountered because of the finite switching speed of amplifiers. Thus, it is very important to investigate the dynamics of delay neural networks. From the view of theory and application, a number of the existence of periodic solutions of neural networks model can be found in the papers [1, 2, 3, 4, 5, 6].

Due to the complicated dynamic properties of the neural cells in the real world, the existing neural network models in many cases cannot characterize the properties of the neural reaction process precisely. It is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions. Therefore, it is important and, in effect, necessary to introduce a new type of networks - neural networks of neutral-type. Such networks arise in high speed computers where nearly lossless transmission lines are used to interconnect switching circuits. Also, the neutral systems often appear in the study of automatic control, population dynamics,

2000 Mathematics Subject Classification: 34K13; 92B20
Keywords: Hopfield neural networks; Neutral delay; Coincidence degree theory; Periodic solution
This work was supported by the National Natural Science Foundation of China (10771212) and the Natural Science Foundation of Jiangsu Education Office (06KJB110010).

and vibrating masses attached to an elastic bar. Recently, the study of the neural networks with neutral delays has received much attention, see, for instance, Refs, [7, 8, 9, 10] and the references cited therein.

Recently, Gui, Ge and Yang [11] have investigated the following Hopfield networks model with neutral delays

$$x_i'(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} b_{ij} g_j(x_j'(t-\sigma_{ij}(t))) + J_i(t),\ i=1,2,...,n. \tag{1.1}$$

By means of an abstract continuous theorem of k-set contractive operator and some analysis technique, the existence of periodic solution of system (1) is obtained. But the condition that the time-varying delays $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ are continuously differentiable is required. Furthermore, the criterion for the existence of periodic solutions of Hopfield neural networks model in [11] depends on the τ_{ij}' and σ_{ij}'.

In this paper, we consider the Hopfield neural networks with neutral time-varying delays

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t) g_j(x_j'(t-\sigma_{ij}(t))) + I_i(t), \tag{1.2}$$

where $i = 1,2,...,n$, $x_i(t)$ denotes the potential (or voltage) of cell i at time t; $c_i(t) > 0$ denotes the neuron firing rate of cell i at time t; $a_{ij}(t)$ and $b_{ij}(t)$ represent the delayed strengths of connectivity and neutral delayed strengths of connectivity between cell i and j at time t, respectively; f_j and g_j are the activation functions in system (1.1); $I_i(t)$ is an external input on the ith unit at time t, in which $I_i : R \to R$, $i = 1,...,n$, are continuous periodic functions with period ω; $\tau_{ij}(t)$ and $\sigma_{ij}(t) \geq 0$ are the transmission delays.

By using the continuation theorem of coincidence degree theory and some analysis technique, we obtain some new sufficient conditions for the existence of the periodic solutions of system (1.2). The conditions imposed on the time-varying delays $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ do not need the assumptions of continuously differentiable. Our work in this paper is new and an extension of previous result in [11].

The paper is organized as follows. In Section 2, the basic notations, assumptions and some preliminaries are given. In Section 3, we present some new criteria to guarantee the existence of the periodic solutions of system (1.2). In Section 4, an illustrative example is given to demonstrate the effectiveness of the obtained results. Conclusions are drawn in Section 5.

2 Preliminaries

In this section, we state some notations, definitions and some Lemmas.

Let $A = (a_{ij})_{n\times n}$ be a real $n \times n$ matrix. $A > 0$ ($A \geq 0$) denotes each element a_{ij} is positive (nonnegative, respectively). Let $x = (x_1, x_2, ..., x_n)^T \in R^n$ be a vector. $x > 0$ ($x \geq 0$) denotes each element x_i is positive (nonnegative, respectively). For matrices or vectors A and B, $A \geq B$ ($A > B$) means that $A - B \geq 0$ ($A - B > 0$).

Definition 1. *[12]. Matrix $A = (a_{ij})_{n\times n}$ is said to be a nonsingular M-matrix, if*

(i) $a_{ii} > 0,\ i = 1, 2, ..., n;$

(ii) $a_{ij} \leq 0,$ for $i \neq j,\ i, j = 1, 2, ..., n;$

(iii) $A^{-1} \geq 0.$

Let X and Y be normed vector spaces, $L : \mathrm{dom}L \subset X \to Y$ be a linear mapping. L will be called a Fredholm mapping of index zero if $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$ and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that mapping $L|_{\mathrm{dom}L\cap\mathrm{Ker}P} : (I - P)X \to \mathrm{Im}L$ is invertible. We denote the inverse of the mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Now, we introduce Mawhin's continuation theorem ([13], p.40) as follows.

Lemma 2. *Let X and Y be two Banach spaces, $L : \mathrm{dom}L \to Y$ be a Fredholm operator with index zero. Assume that Ω is a open bounded set in X, and N is L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(a) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{Dom}L$, $Lx \neq \lambda Nx$;

(b) $QNx \neq 0$ for each $x \in \partial\Omega \cap \mathrm{Ker}L$, and $\deg(JNQ,\ \Omega \cap \mathrm{Ker}L,\ 0) \neq 0$,

where J is an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$. Then equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{Dom}L$.

The following lemmas will be useful to prove our main result in Section 3.

Lemma 3. *[12]. Assume that A is a nonsingular M-matrix and $Aw \leq d$, then $w \leq A^{-1}d$.*

Lemma 4. *[14]. Let $A = (a_{ij})$ with $A_{ij} \leq 0,\ i, j = 1, 2, ..., n,\ i \neq j$. Then the following statements are equivalent.*

(1) A is an M-matrix.

(2) There exists a row vector $\eta = (\eta_1, \eta_2, ..., \eta_n) > (0, 0, ..., 0)$ such that $\eta A > 0$.

(3) There exists a column vector $\xi = (\xi_1, \xi_2, ..., \xi_n)^T > (0, 0, ..., 0)^T$ such that $A\xi > 0$.

Throughout this paper, we assume that

(H_1) $a_{ij}, b_{ij}, I_i \in C(\mathbb{R}, \mathbb{R})$, $\tau_{ij}, c_i(> 0), \sigma_{ij} \in C(\mathbb{R}, \mathbb{R}^+)$ ($\mathbb{R}^+ = [0, \infty)$) are periodic functions with a common period $\omega(> 0)$, $i, j = 1, 2, ..., n$.

(H_2) $f_j, g_j \in C(\mathbb{R}, \mathbb{R})$ are Lipschitzian with Lipschitz constants L_j and l_j respectively, i.e.,

$$|f_j(x) - f_j(y)| \leq L_j|x - y|, \ \ |g_j(x) - g_j(y)| \leq l_j|x - y|,$$

for all $x, y \in \mathbb{R}$, $j = 1, 2, ..., n$.

3 Existence of periodic solution

In this section, we will use the continuation theorem of coincidence degree theory to obtain the existence of an ω-periodic solution to system (1.2).

For convenience, we introduce the following notations:

$$c_{i*} := \min_{t\in[0,\omega]} c_i(t)(> 0), \quad c_i^+ := \max_{t\in[0,\omega]} c_i(t),$$

$$a_{ij}^+ := \max_{t\in[0,\omega]} |a_{ij}(t)|, \quad b_{ij}^+ := \max_{t\in[0,\omega]} |b_{ij}(t)|, \quad I_i^+ := \max_{t\in[0,\omega]} |I_i(t)|, \ \ i, j = 1, 2, ..., n.$$

Theorem 5. *Let (H_1) and (H_2) hold. Suppose that C and $A - B(C^{-1}D)$ are two nonsingular M-matrix, where*

$$A = (\bar{a}_{ij})_{n\times n}, \quad \bar{a}_{ij} = c_{i*}\delta_{ij} - a_{ij}^+ L_j, \quad B = (\bar{b}_{ij})_{n\times n}, \quad \bar{b}_{ij} = b_{ij}^+ l_j,$$

$$C = (\bar{c}_{ij})_{n\times n}, \quad \bar{c}_{ij} = \delta_{ij} - b_{ij}^+ l_j, \quad D = (\bar{d}_{ij})_{n\times n}, \quad \bar{d}_{ij} = c_i^+\delta_{ij} + a_{ij}^+ L_j,$$

$$\delta_{ij} = \begin{cases} 1, & for\ i = j, \\ 0, & for\ i \neq j. \end{cases}$$

then system (1.2) has at least one ω-periodic solution.

Proof. Take

$$C_\omega = \{x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C(R, R^n)$$
$$: x_i(t + \omega) \equiv x_i(t), i = 1, ..., n\},$$

$$C_\omega^1 = \{x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C^1(R, R^n)$$
$$: x_i(t+\omega) \equiv x_i(t), i = 1, ..., n\}.$$

Then C_ω is a Banach space with the norm

$$\|x\|_0 = \max_{1 \le i \le n} \{|x_i|_0\}, \quad |x_i|_0 = \max_{t \in [0,\omega]} |x_i(t)|,$$

and C_ω^1 is also a Banach space with the norm $\|x\| = \max\{\|x\|_0, \|x'\|_0\}$.

For each $x(t) = (x_1(t), x_2(t), ..., x_n(t)) \in C_\omega^1$, $L : C_\omega^1 \to C_\omega$ and $N : C_\omega^1 \to C_\omega$ are defined by

$$(Lx)(t) = \frac{dx}{dt} = (x_1'(t), x_2'(t)..., x_n'(t))^T, \quad \text{and } (Nx)(t) = \begin{bmatrix} (Nx)_1(t) \\ \vdots \\ (Nx)_n(t) \end{bmatrix},$$

where

$$(Nx)_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t) f_j(x_j(t - \tau_{ij}(t)))$$
$$+ \sum_{j=1}^n b_{ij}(t) g_j(x_j'(t - \sigma_{ij}(t))) + I_i(t).$$

It is easy to see that $\text{Ker}L = \mathbb{R}^n$,

$$\text{Im}L = \left\{(x_1(t), ..., x_n(t))^T \in C_\omega : \int_0^\omega x_i(s)ds = 0, \ i = 1, ..., n\right\}$$

is closed in C_ω, and

$$\text{dimKer}L = \text{codimIm}L = n.$$

So, L is a Fredholm mapping of index zero. Let

$$Px = \frac{1}{\omega}\int_0^\omega x(t)dt = \left(\frac{1}{\omega}\int_0^\omega x_1(t)dt, ..., \frac{1}{\omega}\int_0^\omega x_n(t)dt\right)^T,$$
$$Qy = \frac{1}{\omega}\int_0^\omega y(t)dt = \left(\frac{1}{\omega}\int_0^\omega y_1(t)dt, ..., \frac{1}{\omega}\int_0^\omega y_n(t)dt\right)^T,$$

where

$$x(t) = (x_1(t), ..., x_n(t))^T \in C_\omega^1, \quad y(t) = (y_1(t), ..., y_n(t))^T \in C_\omega.$$

Obviously, P, Q are continuous projectors such that

$$\text{Im}P = \text{Ker}L, \quad \text{Ker}Q = \text{Im}L = \text{Im}(I - Q).$$

Moreover, the generalized inverse (to L) $K_P : \mathrm{Im}L \to \mathrm{KerP} \cap \mathrm{Dom}L$ is given by

$$(K_P z)(t) = \begin{bmatrix} \int_0^t z_1(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^s z_1(u)duds \\ \vdots \\ \int_0^t z_n(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^s z_n(u)duds \end{bmatrix}.$$

Thus,

$$(QNx)(t) = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega A_1(x,s)ds \\ \vdots \\ \frac{1}{\omega}\int_0^\omega A_n(x,s)ds \end{bmatrix},$$

and

$$K_P(I-Q)Nx)(t)$$
$$= \begin{bmatrix} \int_0^t A_1(x,s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t A_1(x,s)dsdt + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega A_1(x,s)ds \\ \vdots \\ \int_0^t A_n(x,s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t A_n(x,s)dsdt + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega A_n(x,s)ds \end{bmatrix},$$

where

$$A_i(x,t) = -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))$$
$$+ \sum_{j=1}^n b_{ij}(t)g_j(x_j'(t-\sigma_{ij}(t))) + I_i(t).$$

Clearly, QN and $K_P(I-Q)N$ are continuous. For any bounded open subset $\Omega \subset C_\omega^1$, $QN(\overline{\Omega})$ is obviously bounded. Moreover, applying the Arzela- Ascoli theorem, one can easily show that $K_P(I-Q)N(\overline{\Omega})$ is compact. Thus, N is L-compact on $\overline{\Omega}$ for any bounded open subset $\Omega \subset C_\omega^1$.

Now we are in a position to show that there exists an appropriate open, bounded subset Ω, which satisfies all the requirements given in the continuation theorem. According to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned} x_i'(t) =& \lambda \Bigg[-c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^n b_{ij}(t)g_j(x_j'(t-\sigma_{ij}(t))) + I_i(t) \Bigg], \end{aligned} \tag{3.1}$$

where $i = 1, ..., n$. Suppose that $x(t) = (x_1(t), ..., x_n(t))^T \in X$ is a solution of system (3.1) for some $\lambda \in (0,1)$. Hence, there exist $\xi_i \in [0,\omega]$ $(i = 1, ..., n)$ such that

$|x_i(\xi_i)| = \max_{t\in[0,\omega]}|x_i(t)| = |x_i|_0$. Thus, $x_i'(\xi_i) = 0$ for $i = 1, ..., n$. By (3.1), we have

$$\begin{aligned} c_i(\xi)x_i(\xi_i) = &\sum_{j=1}^{n} a_{ij}(\xi_i)f_j(x_j(\xi_i - \tau_{ij}(\xi_i))) \\ &+ \sum_{j=1}^{n} b_{ij}(\xi_i)g_j(x_j'(\xi_i - \sigma_{ij}(\xi_i))) + I_i(\xi_i). \end{aligned} \tag{3.2}$$

In view of (H_2) and (3.1), we have

$$\begin{aligned} c_{i*}|x_i|_0 = c_{i*}|x_i(\xi_i)| \le c_i(\xi)|x_i(\xi_i)| \le &\sum_{j=1}^{n} |a_{ij}(\xi_i)||f_j(x_j(\xi_i - \tau_{ij}(\xi_i)))| \\ &+ \sum_{j=1}^{n} |b_{ij}(\xi_i)||g_j(x_j'(\xi_i - \sigma_{ij}(\xi_i)))| + |I_i(\xi_i)| \\ \le &\sum_{j=1}^{n} a_{ij}^{+}(L_j|x_j(\xi_i - \tau_{ij}(\xi_i))| + |f_j(0)|) \\ &+ \sum_{j=1}^{n} b_{ij}^{+}(l_j|x_j'(\xi_i - \sigma_{ij}(\xi_i))| + |g_j(0)|) + I_i^{+} \\ \le &\sum_{j=1}^{n}(a_{ij}^{+}L_j|x_j|_0 + b_{ij}^{+}l_j|x_j'|_0) + \sum_{j=1}^{n}(a_{ij}^{+}|f_j(0)| + b_{ij}^{+}|g_j(0)|) + I_i^{+},\ i = 1, ..., n, \end{aligned}$$

which implies that

$$\sum_{j=1}^{n}(c_{i*}\delta_{ij} - a_{ij}^{+}L_j)|x_j|_0 \le \sum_{j=1}^{n} b_{ij}^{+}l_j|x_j'|_0 + \sum_{j=1}^{n}(a_{ij}^{+}|f_j(0)| + b_{ij}^{+}|g_j(0)|) + I_i^{+}, \tag{3.3}$$

where $i = 1, ..., n$. The formulas (3.3) may be rewritten in the form

$$AX \le BY + h, \tag{3.4}$$

where $X = (|x_1|_0, |x_2|_0, ..., |x_n|_0)^T$, $Y = (|x_1'|_0, |x_2'|_0, ..., |x_n'|_0)^T$, $h = (h_i)_{n\times 1}$, and

$$h_i = \sum_{j=1}^{n}(a_{ij}^{+}|f_j(0)| + b_{ij}^{+}|g_j(0)|) + I_i^{+}.$$

Let $\eta_i \in [0, \omega]$ $(i = 1, ..., n)$ such that $|x_i'(\eta_i)| = \max_{t\in[0,\omega]}|x_i'(t)| = |x_i'|_0$. From (3.1), ($H_1$) and ($H_2$), we get

$$|x_i'|_0 = |x_i'(\eta_i)| \le |c_i(\eta_i)||x_i(\eta_i)| + \sum_{j=1}^{n} |a_{ij}(\eta_i)||f_j(x_j(\eta_i - \tau_{ij}(\eta_i)))|$$
$$+ \sum_{j=1}^{n} |b_{ij}(\eta_i)||g_j(x_j'(\eta_i - \sigma_{ij}(\eta_i)))| + |I_i(\eta_i)|$$
$$\le c_i^+|x_i|_0 + \sum_{j=1}^{n} a_{ij}^+(L_j|x_j(\eta_i - \tau_{ij}(\eta_i)| + |f_j(0)|)$$
$$+ \sum_{j=1}^{n} b_{ij}^+(l_j|x_j'(\eta_i - \sigma_{ij}(\eta_i))| + |g_j(0)|) + I_i^+$$
$$\le c_i^+|x_i|_0 + \sum_{j=1}^{n}(a_{ij}^+L_j|x_j|_0 + b_{ij}^+l_j|x_j'|_0) + \sum_{j=1}^{n}(a_{ij}^+|f_j(0)| + b_{ij}^+|g_j(0)|) + I_i^+,$$

where $i = 1, ..., n$, that is

$$\sum_{j=1}^{n}(\delta_{ij} - b_{ij}^+l_j)|x_j'|_0 \le \sum_{j=1}^{n}(c_i^+\delta_{ij} + a_{ij}^+L_j)|x_j|_0 + \sum_{j=1}^{n} a_{ij}^+|f_j(0)| + b_{ij}^+|g_j(0)|) + I_i^+, \quad (3.5)$$

where $i = 1, ..., n$. It is easy to know that formula (3.5) may be rewritten as

$$CY \le DX + h. \quad (3.6)$$

Since C is a nonsingular M-matrix, we have by (3.6) and Lemma 3 that

$$Y \le C^{-1}DX + C^{-1}k. \quad (3.7)$$

Substituting (3.7) into (3.4), we get

$$(A - B(C^{-1}D))X \le BC^{-1}h + h := w = (w_1, w_2, ..., w_n)^T. \quad (3.8)$$

Since $A - B(C^{-1}D)$ is a nonsingular M-matrix, it follows from (3.8) and Lemma 3 that

$$X \le (A - B(C^{-1}D))^{-1}w := (R_1, R_2, ..., R_n)^T. \quad (3.9)$$

Substituting (3.9) into (3.7), we obtain

$$Y \le C^{-1}D(R_1, ..., R_n)^T + C^{-1}h := (r_1, r_2, ..., r_n)^T. \quad (3.10)$$

Since $A - B(C^{-1}D)$ is an M-matrix, we have from Lemma 4 that there exists a vector $\varsigma = (\varsigma_1, \varsigma_2, ..., \varsigma_n)^T > (0, 0, ..., 0)^T$ such that

$$(A - B(C^{-1}D))\varsigma > (0, 0, ..., 0)^T,$$

which implies that we can choose a constant $p > 1$ such that

$$p(A - B(C^{-1}D))\varsigma > w, \quad \text{and} \quad p\varsigma_i > R_i, \ i = 1, 2, ..., n. \tag{3.11}$$

Combining (3.9) with (3.6), we get

$$CY \leq D(R_1, ..., R_n)^T + h := v = (v_1, v_2, ..., v_n)^T. \tag{3.12}$$

Noticing that C is an M-matrix, we have from Lemma 4 that there exists a vector $\gamma = (\gamma_1, ..., \gamma_n)^T > (0, 0, ..., 0)^T$ such that $C\gamma > (0, 0, ..., 0)^T$, which implies that we can choose a constant $q > 1$ such that

$$qC\gamma > v, \quad \text{and} \quad q\gamma_i > r_i, \ i = 1, 2, ..., n. \tag{3.13}$$

Set

$$\bar{\varsigma} = (\bar{\varsigma}_1, \bar{\varsigma}_2, ..., \bar{\varsigma}_n)^T := p\varsigma, \quad \bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2, ..., \bar{\gamma}_n)^T := q\gamma.$$

Then, we have by (3.11) and (3.13) that

$$\bar{\varsigma}_i > R_i, \quad (A - B(C^{-1}D))\bar{\varsigma} > w, \quad \bar{\gamma}_i > r_i, \ \text{and} \ C\bar{\gamma} > v, \ i = 1, 2, ...n. \tag{3.14}$$

Now we take

$$\Omega = \left\{ x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C^1_\omega : |x_i|_0 < \bar{\varsigma}, \ |\dot{x}_i|_0 < \bar{\gamma}, \ i = 1, 2, ..., n \right\}.$$

Obviously, the condition (a) of Lemma 2 is satisfied. If $x \in \partial\Omega \cap \text{Ker}L = \partial\Omega \cap \mathbb{R}^n$, then $x(t)$ is a constant vector in R^n, and there exists some $i \in \{1, 2, ..., n\}$ such that $|x_i| = \bar{\varsigma}_i$. It follows that

$$(QNx)_i = \frac{1}{\omega} \int_0^\omega \left[-c_i(t)x_i + \sum_{j=1}^n a_{ij}(t)f_j(x_j) + \sum_{j=1}^n b_{ij}(t)g_j(0) + I_i(t) \right] dt. \tag{3.15}$$

We claim that

$$|(QNx)_i| > 0. \tag{3.16}$$

In fact, if $|(QNx)_i| = 0$, i.e.,

$$\int_0^\omega \left[c_i(t)x_i - \sum_{j=1}^n f_j(x_j)a_{ij}(t) - \sum_{j=1}^n g_j(0)b_{ij}(t) - I_i(t) \right] dt = 0.$$

Then, there exists some $t_* \in [0, \omega]$ such that

$$c_i(t_*)x_i - \sum_{j=1}^n f_j(x_j)a_{ij}(t_*) - \sum_{j=1}^n g_j(0)b_{ij}(t_*) - I_i(t_*) = 0,$$

which implies that

$$c_{i*}\bar{\varsigma}_i = c_{i*}|x_i| \le |c_i(t_*)x_i| \le \sum_{j=1}^{n}|f_j(x_j)|a_{ij}^+ + \sum_{j=1}^{n}|g_j(0)|b_{ij}^+ + I_i^+$$
$$\le \sum_{j=1}^{n}(|f_j(x_j) - f_j(0)| + |f_j(0)|)a_{ij}^+ + \sum_{j=1}^{n}|g_j(0)|b_{ij}^+ + I_i^+$$
$$\le \sum_{j=1}^{n} a_{ij}^+ L_j\bar{\varsigma}_j + \sum_{j=1}^{n}(a_{ij}^+|f_j(0)| + b_{ij}^+|g_j(0)|) + I_i^+.$$

This means that

$$(A\bar{\varsigma})_i \le h_i. \tag{3.17}$$

It is easy to know that $D\bar{\varsigma} \ge (0, 0, ..., 0)^T$. Since C is a nonsingular M-matrix, we have from $C^{-1} \ge 0$ (Lemma 3) that $C^{-1}D\bar{\varsigma} \ge (0, 0, ..., 0)^T$. Thus, we obtain

$$B(C^{-1}D)\bar{\varsigma} \ge (0, ..., 0)^T. \tag{3.18}$$

Similarly, we have

$$BC^{-1}h \ge (0, ..., 0)^T. \tag{3.19}$$

From (3.17), (3.18) and (3.19), we get

$$(A\bar{\varsigma})_i \le h_i + (BC^{-1}h)_i + (BC^{-1}D)\bar{\varsigma})_i = w_i + (BC^{-1}D\bar{\varsigma})_i.$$

This implies that

$$((A - BC^{-1}D)\bar{\varsigma})_i \le w_i.$$

which contradicts (3.14). Hence, (3.16) holds. Consequently, $QNx \ne 0$ for each $x \in \partial\Omega \cap \text{Ker}L$.

Furthermore, let

$$\Psi(x, \mu) = \mu(-x) + (1 - \mu)JQNx \quad \mu \in [0, 1].$$

Then for any $x = (x_1, x_2, ..., x_n)^T \in \partial\Omega \cap \text{Ker}L$, $(x_1, x_2, ..., x_n)^T$ is a constant vector in $\mathbb{R}^n$ with $|x_i| = \bar{\varsigma}$ for some $i \in \{1, ..., n\}$. It follows that

$$\begin{aligned}(\Psi(x, \mu))_i = & -\mu x_i + (1 - \mu)\frac{1}{\omega}\int_0^\omega \Bigg[-c_i(t)x_i + \sum_{j=1}^{n} f_j(x_j)a_{ij}(t) \\ & + \sum_{j=1}^{n} g_j(0)b_{ij}(t) + I_i(t)\Bigg] dt.\end{aligned} \tag{3.20}$$

We claim that

$$|(\Psi(x,\mu))_i| > 0. \tag{3.21}$$

If this is not true, then $|(\Psi(x,\mu))_i| = 0$, i.e.,

$$\mu x_i + (1-\mu)\frac{1}{\omega}\int_0^\omega \left[c_i(t)x_i - \sum_{j=1}^n f_j(x_j)a_{ij}(t) - \sum_{j=1}^n g_j(0)b_{ij}(t) - I_i(t)\right] dt = 0.$$

Therefore, there exists some $t^* \in [0,\omega]$ such that

$$\mu x_i + (1-\mu)\frac{1}{\omega}\left[c_i(t^*)x_i - \sum_{j=1}^n f_j(x_j)a_{ij}(t^*) - \sum_{j=1}^n g_j(0)b_{ij}(t^*) - I_i(t^*)\right] = 0, \tag{3.22}$$

which implies that

$$x_i\left[c_i(t^*)x_i - \sum_{j=1}^n f_j(x_j)a_{ij}(t^*) - \sum_{j=1}^n g_j(0)b_{ij}(t^*) - I_i(t^*)\right] \le 0.$$

Thus, we get

$$\begin{aligned}
c_{i*}|x_i|^2 \le |c_i(t^*)|x_i^2 \le x_i &\left[\sum_{j=1}^n f_j(x_j)a_{ij}(t^*) + \sum_{j=1}^n g_j(0)b_{ij}(t^*) + I_i(t^*)\right] \\
&\le |x_i|\left[\sum_{j=1}^n (|f_j(x_j) - f_j(0)| + |f_j(0)|)a_{ij}^+ + \sum_{j=1}^n |g_j(0)|b_{ij}^+ + I_i^+\right] \\
&\le |x_i|\left[\sum_{j=1}^n a_{ij}^+ L_j|x_j| + \sum_{j=1}^n (a_{ij}^+|f_j(0)| + b_{ij}^+|g_j(0)|) + I_i^+\right] \\
&\le |x_i|\left[\sum_{j=1}^n a_{ij}^+ L_j\bar{\varsigma}_j + \sum_{j=1}^n (a_{ij}^+|f_j(0)| + b_{ij}^+|g_j(0)|) + I_i^+\right],
\end{aligned}$$

this means that $(A\bar{\varsigma})_i \le h_i$. By (3.18) and (3.19), we obtain

$$((A - BC^{-1}D)\bar{\varsigma})_i \le w_i.$$

which contradicts (3.14). Hence, (3.21) holds. By the homotopy invariance theorem, we get

$$\deg\{JQN, \Omega\cap \mathrm{Ker}L, 0\} = \deg\{-x, \Omega\cap\mathrm{Ker}L, 0\} \ne 0.$$

So, condition (b) of Lemma 2 is also satisfied. Therefore, from Lemma 2 we conclude that system (1.2) has at least one ω-periodic solution. The proof is complete. □

4 Illustrative examples

In this section, we give two examples to illustrate the effectiveness of our result.

Example 6. *Consider the following Hopfield neural networks with neutral time-varying delays*

$$\begin{aligned} x_i'(t) =& - c_i(t)x_i(t) + \sum_{j=1}^{2} a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ & + \sum_{j=1}^{2} b_{ij}(t)g_j(x_j'(t-\sigma_{ij}(t))) + I_i(t), \quad i = 1, 2, \end{aligned} \tag{4.1}$$

where

$$c_i(t) = 4 - \frac{1}{4}\cos t, \quad f_i(u) = \frac{1}{8}\sin u, \quad g_i(u) = \frac{1}{6}u,$$

$$a_{11}(t) = \frac{1}{2}\cos t, \quad a_{12}(t) = \frac{1}{3}\sin t, \quad a_{21}(t) = \frac{1}{4}\sin^2 t, \quad a_{22}(t) = \frac{1}{5}\cos^2 t,$$

$$b_{11}(t) = \frac{1}{3}\sin t, \quad b_{12}(t) = \frac{1}{4}\cos 2t, \quad b_{21}(t) = \frac{1}{2}\sin 2t, \quad b_{22}(t) = \frac{1}{7}\cos t,$$

$$I_1(t) = 2 + \sin t, \quad I_2(t) = 3 - \cos t,$$

$$\tau_{ij}(u) = \sigma_{ij}(u) = \frac{1}{2}(|\sin u + 1| - |\sin u - 1|), \quad i, j = 1, 2.$$

Obviously, delays $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ are not differentiable. $f_i(u)$ and $g_i(u)$ $(i = 1, 2)$ satisfy the Lipschitz condition (H_2) with constants $L_i = \frac{1}{8}$ and $l_i = \frac{1}{6}$, respectively. $a_{ij}(t)$, $b_{ij}(t)$, $I_i(t)$, $c_i(t)$, $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ satisfy the condition (H_1) with a common period 2π. Moreover, we can easily get that $a_{11}^+ = \frac{1}{2}$, $a_{12}^+ = \frac{1}{3}$, $a_{21}^+ = \frac{1}{4}$, $a_{22}^+ = \frac{1}{5}$, $b_{11}^+ = \frac{1}{3}$, $b_{12}^+ = \frac{1}{4}$, $b_{21}^+ = \frac{1}{2}$, $b_{22}^+ = \frac{1}{7}$, $I_1^+ = 3$, $I_2^+ = 4$, with $c_{i*} = \frac{15}{4}$ and $c_i^+ = \frac{17}{4}$, for $i = 1, 2$.

Thus, we have

$$A = \begin{pmatrix} \frac{59}{16} & -\frac{1}{24} \\ -\frac{1}{32} & \frac{149}{40} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{18} & \frac{1}{24} \\ \frac{1}{12} & \frac{1}{42} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{17}{18} & -\frac{1}{24} \\ -\frac{1}{12} & \frac{41}{42} \end{pmatrix},,$$

$$C^{-1} = \begin{pmatrix} 1.0628 & 0.0454 \\ 0.0907 & 1.0283 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{69}{16} & \frac{1}{24} \\ \frac{1}{32} & \frac{171}{40} \end{pmatrix},,$$

$$A - B(C^{-1}D) = \begin{pmatrix} 3.4151 & -0.2382 \\ -0.4234 & 3.6004 \end{pmatrix},$$

and

$$(A - B(C^{-1}D))^{-1} = \begin{pmatrix} 0.2952 & 0.0195 \\ 0.0347 & 0.2800 \end{pmatrix},$$

which implies that C and $A - B(C^{-1}D)$ are two nonsingular M-matrix. Hence, all conditions of Theorem 5 are satisfied. So, by means of Theorem 5, system (4.1) has at least one 2π-periodic solution.

Example 7. *Consider the following Hopfield neural networks with constant coefficients and neutral time-varying delays*

$$\begin{aligned} x_i'(t) = &- b_i x_i(t) + \sum_{j=1}^{2} a_{ij} g_j(x_j(t - \tau_{ij}(t))) \\ &+ \sum_{j=1}^{2} b_{ij} g_j(x_j'(t - \sigma_{ij}(t))) + J_i(t), \quad i = 1, 2, \end{aligned} \tag{4.2}$$

where

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad (a_{ij}) = \begin{pmatrix} \frac{6}{\sqrt{5}} & \frac{6}{\sqrt{10}} \\ -\frac{9}{\sqrt{10}} & \frac{4}{\sqrt{5}} \end{pmatrix}, \qquad (b_{ij}) = \begin{pmatrix} \frac{4}{\sqrt{5}} & -\frac{5}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} & \frac{6}{\sqrt{5}} \end{pmatrix},$$

and

$$\tau_{11}(t) = \tau_{22}(t) = 2\pi + \frac{4}{5}\sin t, \quad \tau_{12}(t) = \tau_{21}(t) = \frac{1}{10}\cos t,$$

$$\sigma_{11}(t) = \sigma_{22}(t) = 2\pi + \frac{4}{5}\sin t, \quad \sigma_{12}(t) = \sigma_{21}(t) = \frac{1}{17}\sin t,$$

$$J_1(t) = 1 + \sin t, \quad J_2(t) = 1 - \cos t.$$

It is easy to obtain that $p_{ii} = q_{ii} = \sqrt{5}$, $i = 1, 2$, $p_{12} = p_{21} = \frac{\sqrt{10}}{3}$, $q_{12} = q_{21} = \frac{\sqrt{17}}{4}$, where p_{ij} and q_{ij} are as in [11]. Take

$$g_i(u) = \frac{1}{8}u, \qquad |g_i(u)| < \frac{1}{8}|u| + \frac{1}{8}, \quad i = 1, 2.$$

Obviously, $g_i(u)$ satisfies the conditions (H_1) and (H_3) of [11] with $\alpha_i = \frac{1}{8}$. Thus, we have

$$A_1 = (\widetilde{a}_{ij}) = \begin{pmatrix} \frac{9}{4} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{7}{2} \end{pmatrix}, \quad C_1 = (c_{ij}) = \begin{pmatrix} \frac{9}{4} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{7}{2} \end{pmatrix},$$

$$B_1 = (\widetilde{b}_{ij}) = \begin{pmatrix} \frac{1}{2} & \frac{5}{32} \\ \frac{3}{32} & \frac{3}{4} \end{pmatrix}, \quad D_1 = (d_{ij}) = \begin{pmatrix} \frac{1}{2} & -\frac{5}{32} \\ -\frac{3}{32} & \frac{1}{4} \end{pmatrix},$$

where

$$\widetilde{a}_{ij} = c_{ij} = b_i\delta_{ij} - |a_{ij}|\alpha_j p_{ij}, \quad \widetilde{b}_{ij} = |b_{ij}|\alpha_j q_{ij},$$

$$d_{ij} = \delta_{ij} - |b_{ij}|\alpha_j q_{ij}, \quad \delta_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

We easily check that

$$(A_1 - B_1(D_1^{-1}C_1))^{-1} = \begin{pmatrix} -0.0664 & -4.8894 \\ -0.9624 & -8.6460 \end{pmatrix},$$

which implies that $A_1 - B_1(D_1^{-1}C_1)$ is not a nonsingular M-matrix, that is, the condition (H_4) of [11] for system (4.2) is fails. Thus we can not apply the main result (Theorem 3.1) in [11] to judge that the system (4.2) has at least one periodic solution. But here Theorem 5 guarantees the existence of periodic solution of system (4.2). In fact, we have

$$A = \begin{pmatrix} \frac{12\sqrt{5}-3}{4\sqrt{5}} & -\frac{3}{4\sqrt{10}} \\ -\frac{9}{8\sqrt{10}} & \frac{8\sqrt{5}-1}{2\sqrt{5}} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2\sqrt{5}} & \frac{5}{8\sqrt{17}} \\ \frac{3}{8\sqrt{17}} & \frac{3}{4\sqrt{5}} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{2\sqrt{5}-1}{2\sqrt{5}} & -\frac{5}{8\sqrt{17}} \\ -\frac{3}{8\sqrt{17}} & \frac{4\sqrt{5}-3}{4\sqrt{5}} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{12\sqrt{5}+3}{4\sqrt{5}} & \frac{3}{4\sqrt{10}} \\ \frac{9}{8\sqrt{10}} & \frac{8\sqrt{5}+1}{2\sqrt{5}} \end{pmatrix}.$$

So, we get

$$C^{-1} = \begin{pmatrix} 1.3234 & 0.3018 \\ 0.1811 & 1.5460 \end{pmatrix}, \quad (A - B(C^{-1}D))^{-1} = \begin{pmatrix} 5.1515 & 5.7339 \\ 4.1651 & 5.3365 \end{pmatrix},$$

which implies that C and $A - B(C^{-1}D)$ are two nonsingular M-matrix. Therefore, all conditions of Theorem 5 are satisfied, then system (4.2) has at least one 2π-periodic solution.

5 Conclusion

In this paper, we use the continuation theorem of coincidence degree theory to study the existence of periodic solution for a Hopfield neural network with neutral time-varying delays. Without assuming the continuous differentiability of time-varying delays, sufficient conditions are obtained for the existence of the periodic solution. Moreover, two examples are given to illustrate the effectiveness of the new result.

References

[1] C. Bai, *Global exponential stability and existence of periodic solution of Cohen-Grossberg type neural networks with delays and impulses*, Nonlinear Analysis: Real World Applications **9**(2008), 747-761. MR2392372 (2009a:34122). Zbl 1151.34062.

[2] J. Cao, *New results concerning exponential stability and periodic solutions of delayed cellular neural networks*, Phys. Lett. A **307**(2003), 136-147. MR1974596(2004a:62176). Zbl 1006.68107.

[3] S. Guo, L. Huang, *Periodic oscillation for a class of neural networks with variable coefficients*, Nonlinear Anal. Real World Appl. **6** (2005), 545-561. MR2129564(2006h:34139). Zbl 1080.34051.

[4] Y. Li, *Existence and stability of periodic solutions for Cohen-Grossberg neural networks with multiple delays*, Chaos, Solotons & Fractals, **20**(2004), 459-466. MR2024869(2004i:34184). Zbl 1048.34118.

[5] B. Liu, L. Huang, *Existence and exponential stability of periodic solutions for cellular neural networks with time-varying delays*, Phys. Lett. A **349**(2006), 474-483. Zbl 1171.82329.

[6] Z. Liu, L. Liao, *Existence and global exponential stability of periodic solution of cellular neural networks with time-varying delays*, J. Math. Anal. Appl. **290** (2004), 247-262. MR2032238(2004j:34161). Zbl 1055.34135.

[7] J.H. Park, O.M. Kwon, Lee, S. M, *LMI optimization approach on stability for delayed neural networks of neutral-type*, Appl. Math. Comput. **196** (2008), 236-244. MR2382607. Zbl 1157.34056.

[8] S. Xu, J. Lam, D.W.C. Ho, Y. Zou, *Delay-dependent exponential stability for a class of neural networks with time delays*, J. Comput. Appl. Math. **183** (2005), 16-28. MR2156097(2006d:34171). Zbl 1097.34057.

[9] C. Bai, *Global stability of almost periodic solutions of Hopfield neural networks with neutral time-varying delays*, Appl. Math. Comput. **203** (2008), 72-79. MR2451540. Zbl 1173.34344.

[10] O.M. Kwon, J.H. Park, S.M. Lee, *On stability criteria for uncertain delay-differential systems of neutral type with time-varying delays*, Appl. Math. Comput. **197** (2008), 864-873. MR2400710. Zbl 1144.34052.

[11] Z. Gui, W. Ge, X. Yang, *Periodic oscillation for a Hopfield neural networks with neutral delays*, Phys. Lett. A **364** (2007), 267-273.

[12] J. Chen, X. Chen, *Special matrices*, Tsinghua Univ. Press, Beijing, 2001.

[13] R.E. Gaines, J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Mathematics. 568. Berlin-Heidelberg-New York: Springer-Verlag, 1977. MR0637067 (58 #30551). Zbl 0339.47031.

****

[14] A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Computer Science and Applied Mathematics. New York, San Francisco, London: Academic Press. XVIII, 1979. MR0544666 (82b:15013). Zbl 0484.15016.

Chuanzhi Bai
Department of Mathematics,
Huaiyin Normal University,
Huaian, Jiangsu 223300, P. R. China.
e-mail: czbai@hytc.edu.cn

Chunhong Li
Department of Mathematics,
Huaiyin Normal University,
Huaian, Jiangsu 223300, P. R.China.
e-mail: lichshy2006@126.com