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HIGHER *-DERIVATIONS BETWEEN UNITAL C*-ALGEBRAS

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Abstract. Let $\mathcal{A},\mathcal{B}$ be two unital C^*-algebras. We prove that every sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$, $H=\{h_0,h_1,...,h_m,...\}$, which satisfy $h_m(3^n uy)=\sum_{i+j=m}h_i(3^n u)h_j(y)$ for each $m\in\mathbb{N}_0$, for all $u\in U(\mathcal{A})$, all $y\in\mathcal{A}$, and all $n=0,1,2,...$, is a higher derivation. Also, for a unital C^*-algebra $\mathcal{A}$ of real rank zero, every sequence of continuous mappings from $\mathcal{A}$ into $\mathcal{B}$, $H=\{h_0,h_1,...,h_m,...\}$, is a higher derivation when $h_m(3^n uy)=\sum_{i+j=m}h_i(3^n u)h_j(y)$ holds for all $u\in I_1(\mathcal{A}_{sa})$, all $y\in\mathcal{A}$, all $n=0,1,2,...$ and for each $m\in\mathbb{N}_0$. Furthermore, by using the fixed points methods, we investigate the Hyers–Ulam–Rassias stability of higher $*-$derivations between unital C^*-algebras.

1 Introduction

The stability of functional equations was first introduced by S. M. Ulam [27] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2,d) and a positive number ϵ, does there exist a $\delta>0$ such that if a function $f:G_1\longrightarrow G_2$ satisfies the inequality $d(f(xy),f(x)f(y))<\delta$ for all $x,y\in G1$, then there exists a homomorphism $T:G_1\to G_2$ such that $d(f(x),T(x))<\epsilon$ for all $x\in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D. H. Hyers [10] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [24] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [24] is called the Hyers–Ulam–Rassias stability. According to Th. M. Rassias theorem:

Theorem 1. *Let $f:E\longrightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon(\|x\|^p+\|y\|^p)$$

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for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$, and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is linear.

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [9, 12, 15].
D.G. Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [1, 2, 3, 11, 17, 18, 20, 22, 25] and references therein.

Jun and Lee [14] proved the following: Let X and Y be Banach spaces. Denote by $\phi : X - \{0\} \times Y - \{0\} \to [0, \infty)$ a function such that $\tilde{\phi}(x, y) = \sum_{n=0}^{\infty} 3^{-n}\phi(3^n x, 3^n y) < \infty$ for all $x, y \in X - \{0\}$. Suppose that $f : X \longrightarrow Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \phi(x, y)$$

for all $x, y \in X - \{0\}$. Then there exists a unique additive mapping $T : X \longrightarrow Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3}(\tilde{\phi}(x, -x) + \tilde{\phi}(-x, 3x))$$

for all $x \in X - \{0\}$.

Recently, C. Park and W. Park [21] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra(See also [13]). Throughout this paper, let $\mathcal{A}$ be a unital C^*-algebra with unit e, and $\mathcal{B}$ a unital C^*-algebra. Let $U(\mathcal{A})$ be the set of unitary elements in $\mathcal{A}$, $\mathcal{A}_{sa} := \{x \in \mathcal{A} | x = x^*\}$, and $I_1(\mathcal{A}_{sa}) = \{v \in \mathcal{A}_{sa} | \|v\| = 1, v \in Inv(\mathcal{A})\}$.

A linear mapping $d : A \to A$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in A$.

Let $\mathbb{N}$ be the set of natural numbers. For $m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, a sequence $H = \{h_0, h_1, ..., h_m\}$ (resp. $H = \{h_0, h_1, ..., h_n, ...\}$) of linear mappings from A into B is called a higher derivation of rank m (resp. infinite rank) from A into B if

$$h_n(xy) = \sum_{i+j=n} h_i(x)h_j(y)$$

holds for each $n \in \{0, 1, ..., m\}$ (resp. $n \in \mathbb{N}_0$) and all $x, y \in A$. The higher derivation H from A into B is said to be onto if $h_0 : A \to B$ is onto. The higher derivation

H on A is called be strong if h_0 is an identity mapping on A. Of course, a higher derivation of rank 0 from A into B (resp. a strong higher derivation of rank 1 on A) is a homomorphism (resp. a derivation). So a higher derivation is a generalization of both a homomorphism and a derivation.

In this paper, we prove that every sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$, $H = \{h_0, h_1, ..., h_m, ...\}$ is a higher derivation when for each $m \in \mathbb{N}_0$, $h_m(3^n uy) = \sum_{i+j=m} h_i(3^n u)h_j(y)$ for all $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, ...$, and that for a unital C^*-algebra $\mathcal{A}$ of real rank zero (see [4]), every sequence of continuous mappings from $\mathcal{A}$ into $\mathcal{B}$, $H = \{h_0, h_1, ..., h_m, ...\}$ is a higher derivation when for each $m \in \mathbb{N}_0$, $h_m(3^n uy) = \sum_{i+j=m} h_i(3^n u)\ h_j(y)$ for all for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, ...$ Furthermore, we investigate the Hyers–Ulam–Rassias stability of higher $*-$derivations between unital C^*-algebras by using the fixed pint methods.

Note that a unital C^*-algebra is of real rank zero, if the set of invertible self-adjoint elements is dense in the set of self–adjoint elements (see [4]). We denote the algebric center of algebra $\mathcal{A}$ by $Z(\mathcal{A})$.

2 Higher *-derivations on unital C*-algebras

By a following similar way as in [19], we obtain the next theorem.

Theorem 2. *Suppose that $F = \{f_0, f_1, ..., f_m, ...\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that $f_m(0) = 0$ for each $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,*

$$f_m(3^n uy) = \sum_{i+j=m} f_i(3^n u)f_j(y) \tag{2.1}$$

for all $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$, all $n = 0, 1, 2, ...$ and for each $m \in \mathbb{N}$. If there exists a function $\phi : (\mathcal{A}-\{0\})^2 \times \mathcal{A} \to [0, \infty)$ such that $\tilde{\phi}(x, y, z) = \sum_{n=0}^{\infty} 3^{-n}\phi(3^n x, 3^n y, 3^n z) < \infty$ for all $x, y \in \mathcal{A} - \{0\}$ and all $z \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$,

$$\|2f_m(\frac{\mu x + \mu y}{2}) - \mu f_m(x) - \mu f_m(y) + f_m(u^*) - f_m(u)^*\| \le \phi(x, y, u), \tag{2.2}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in \mathcal{A}, u \in (U(\mathcal{A}) \cup \{0\})$. If $\lim_n \frac{f_m(3^n e)}{3^n} \in U(\mathcal{B}) \cap Z(\mathcal{B})$, then the sequence $F = \{f_0, f_1, ..., f_m, ...\}$ is a higher $*-$derivation.

Proof. Put $u = 0, \mu = 1$ in (2.2), it follows from Theorem 1 of [14] that there exists a unique additive mapping $h_m : \mathcal{A} \to \mathcal{B}$ such that

$$\|f_m(x) - h_m(x)\| \le \frac{1}{3}(\tilde{\phi}(x, -x, 0) + \tilde{\phi}(-x, 3x, 0)) \tag{2.3}$$

for all $x \in \mathcal{A} - \{0\}$ and for each $m \in \mathbb{N}_0$. These mappings are given by

$$h_m(x) = \lim_n \frac{f_m(3^n x)}{3^n}$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. By the same reasoning as the proof of Theorem 1 of [19], h_m is $\mathbb{C}-$linear and $*-$preserving for each $m \in \mathbb{N}_0$. It follows from (2.1) and (2.2) that

$$\begin{aligned} h_m(uy) &= \lim_n \frac{f_m(3^n uy)}{3^n} = \lim_n \sum_{i+j=m} \frac{f_i(3^n u) f_j(y)}{3^n} \\ &= \sum_{i+j=m} h_i(u) f_j(y) \end{aligned} \tag{2.4}$$

for all $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Since h_m is additive, then by (2.4), we have

$$3^n h_m(uy) = h_m(u(3^n y)) = \sum_{i+j=m} h_i(u) f_j(3^n y)$$

for all $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Hence,

$$h_m(uy) = \lim_n \sum_{i+j=m} h_i(u) \frac{f_j(3^n y)}{3^n} = \sum_{i+j=m} h_i(u) h_j(y) \tag{2.5}$$

for all $u \in U(\mathcal{A})$, all $y \in A$ and for each $m \in \mathbb{N}_0$. By the assumption, we have

$$h_m(e) = \lim_n \frac{f_m(3^n e)}{3^n} \in U(\mathcal{B}) \cap Z(\mathcal{B})$$

and for each $m \in \mathbb{N}_0$, hence, it follows by (2.4) and (2.5) that

$$\sum_{i+j=m} h_i(e) h_j(y) = h_m(ey) = \sum_{i+j=m} h_i(e) f_j(y)$$

for all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Since $h_m(e)$ is invertible, then by induction $h_m(y) = f_m(y)$ for all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. We have to show that $F = \{f_0, f_1, ..., f_m, ...\}$ is higher derivation. To this end, let $x \in \mathcal{A}$. By Theorem 4.1.7 of [16], x is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{n} c_j u_j \quad (c_j \in$

$\mathbb{C}, u_j \in U(\mathcal{A})$), it follows from (2.5) that

$$\begin{aligned} f_m(xy) = h_m(xy) = h_m(\sum_{k=1}^{n} c_k u_k y) &= \sum_{k=1}^{n} c_k h_m(u_k y) \\ &= \sum_{k=1}^{n} c_k (\sum_{i+j=m} h_i(u_k) h_j(y)) \\ &= \sum_{i+j=m} h_i(\sum_{k=1}^{n} c_k u_k) h_j(y) \\ &= \sum_{i+j=m} h_i(x) h_j(y) \\ &= \sum_{i+j=m} f_i(x) f_j(y) \end{aligned}$$

for all $y \in \mathcal{A}$. And this completes the proof of theorem. □

Corollary 3. *Let $p \in (0,1), \theta \in [0,\infty)$ be real numbers. Suppose that*

$$F = \{f_0, f_1, ..., f_m, ...\}$$

is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that $f_m(0) = 0$ for each $m \in \mathbb{N}_0$,

$$f_m(3^n uy) = \sum_{i+j=m} f_i(3^n u) f_j(y)$$

for all $u \in U(\mathcal{A})$, all $y \in \mathcal{A}$, all $n = 0, 1, 2, ...$ and for each $m \in \mathbb{N}_0$. Suppose that

$$\|2f_m(\frac{\mu x + \mu y}{2}) - \mu f_m(x) - \mu f_m(y) + f_m(z^*) - f_m(z)^*\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. If $\lim_n \frac{f_m(3^n e)}{3^n} \in U(\mathcal{B}) \cap Z(\mathcal{B})$, then the sequence $F = \{f_0, f_1, ..., f_m, ...\}$ is a higher $*-$derivation.

Proof. Setting $\phi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ all $x, y, z \in \mathcal{A}$. Then by Theorem 1 we get the desired result. □

Theorem 4. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero. Suppose that*

$$F = \{f_0, f_1, ..., f_m, ...\}$$

is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that $f_m(0) = 0$ for each $m \in \mathbb{N}_0$,

$$f_m(3^n uy) = \sum_{i+j=m} f_i(3^n u) f_j(y) \tag{2.6}$$

for all $u \in I_1(\mathcal{A}_{sa})$, *all* $y \in \mathcal{A}$, *all* $n = 0, 1, 2, ...$ *and for each* $m \in \mathbb{N}_0$. *Suppose that there exists a function* $\phi : (\mathcal{A} - \{0\})^2 \times \mathcal{A} \to [0, \infty)$ *satisfying* (2.2) *and* $\tilde{\phi}(x, y, z) < \infty$ *for all* $x, y \in \mathcal{A} - \{0\}$ *and all* $z \in \mathcal{A}$. *If* $\lim_n \frac{f_m(3^n e)}{3^n} \in U(\mathcal{B}) \cap Z(\mathcal{B})$, *then the sequence* $F = \{f_0, f_1, ..., f_m, ...\}$ *is a higher* $*-$*derivation.*

Proof. By the same reasoning as the proof of Theorem 1, there exist a unique involutive $\mathbb{C}-$linear mappings $h_m : \mathcal{A} \to \mathcal{B}$ satisfying (2.3) for each $m \in \mathbb{N}_0$. It follows from (2.6) that

$$h_m(uy) = \lim_n \frac{f_m(3^n uy)}{3^n} = \lim_n \sum_{i+j=m} \frac{f_i(3^n u) f_j(y)}{3^n} = \sum_{i+j=m} h_i(u) f_j(y) \tag{2.7}$$

for all $u \in I_1(\mathcal{A}_{sa})$, and all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. By additivity of h_m and (2.7), we obtain that

$$3^n h_m(uy) = h_m(u(3^n y)) = \sum_{i+j=m} h_i(u) f_j(3^n y)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Hence,

$$h_m(uy) = \lim_n \sum_{i+j=m} h_i(u) \frac{f_j(3^n y)}{3^n} = \sum_{i+j=m} h_i(u) h_j(y) \tag{2.8}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. By the assumption, we have

$$h_m(e) = \lim_n \frac{f_m(3^n e)}{3^n} \in U(\mathcal{B}) \cap Z(\mathcal{B})$$

and for each $m \in \mathbb{N}_0$. Similar to the proof of Theorem 1, it follows from (2.7) and (2.8) that $h_m = f_m$ on A for each $m \in \mathbb{N}_0$. So h_m is continuous for each $m \in \mathbb{N}_0$. On the other hand $\mathcal{A}$ is real rank zero. On can easily show that $I_1(\mathcal{A}_{sa})$ is dense in $\{x \in \mathcal{A}_{sa} : \|x\| = 1\}$. Let $v \in \{x \in \mathcal{A}_{sa} : \|x\| = 1\}$. Then there exists a sequence $\{z_n\}$ in $I_1(\mathcal{A}_{sa})$ such that $\lim_n z_n = v$. Since h_m is continuous for each $m \in \mathbb{N}_0$, it follows from (2.8) that

$$\begin{aligned} h_m(vy) &= h_m(\lim_n (z_n y)) = \lim_n h_m(z_n y) = \lim_n \sum_{i+j=m} h_i(z_n) h_j(y) \\ &= \sum_{i+j=m} h_i(\lim_n z_n) h_j(y) = \sum_{i+j=m} h_i(v) h_j(y) \end{aligned} \tag{2.9}$$

for all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Now, let $x \in \mathcal{A}$. Then we have $x = x_1 + ix_2$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$ are self–adjoint.

First consider $x_1 = 0, x_2 \neq 0$. Since h_m is $\mathbb{C}-$linear for each $m \in \mathbb{N}_0$, it follows from (2.9) that

$$\begin{aligned} f_m(xy) &= h_m(xy) = h_m(ix_2y) = h_m(i\|x_2\|\frac{x_2}{\|x_2\|}y) \\ &= i\|x_2\|h_m(\frac{x_2}{\|x_2\|}y) = i\|x_2\| \sum_{i+j=m} h_i(\frac{x_2}{\|x_2\|})h_j(y) \\ &= \sum_{i+j=m} h_i(i\|x_2\|\frac{x_2}{\|x_2\|})h_j(y) = \sum_{i+j=m} h_i(ix_2)h_j(y) \\ &= \sum_{i+j=m} h_i(x)h_j(y) = \sum_{i+j=m} f_i(x)f_j(y) \end{aligned}$$

for all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$.

If $x_2 = 0, x_1 \neq 0$, then by (2.9), we have

$$\begin{aligned} f_m(xy) &= h_m(xy) = h_m(x_1y) = h_m(\|x_1\|\frac{x_1}{\|x_1\|}y) \\ &= \|x_1\|h_m(\frac{x_1}{\|x_1\|}y) = \|x_1\| \sum_{i+j=m} h_i(\frac{x_1}{\|x_1\|})h_j(y) \\ &= \sum_{i+j=m} h_i(\|x_1\|\frac{x_1}{\|x_1\|})h_j(y) = \sum_{i+j=m} h_i(x_1)h_j(y) \\ &= \sum_{i+j=m} h_i(x)h_j(y) = \sum_{i+j=m} f_i(x)f_j(y) \end{aligned}$$

for all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$.

Finally, consider the case that $x_1 \neq 0, x_2 \neq 0$. Then it follows from (2.9) that

$$\begin{aligned} f(xy) &= h_m(xy) = h_m(x_1y + (ix_2)y) \\ &= h_m(\|x_1\|\frac{x_1}{\|x_1\|}y) + h_m(i\|x_2\|\frac{x_2}{\|x_2\|}y) \\ &= \|x_1\|h_m(\frac{x_1}{\|x_1\|}y) + i\|x_2\|h_m(\frac{x_2}{\|x_2\|}y) \\ &= \|x_1\| \sum_{i+j=m} h_i(\frac{x_1}{\|x_1\|})h_j(y) + i\|x_2\| \sum_{i+j=m} h_i(\frac{x_2}{\|x_2\|})h_j(y) \\ &= \sum_{i+j=m} \left[h_i(\|x_1\|\frac{x_1}{\|x_1\|})h_j(y) + h_i(i\|x_2\|\frac{x_2}{\|x_2\|})h_j(y)\right] \\ &= \sum_{i+j=m} [h_i(x_1) + h_i(ix_2)]h_j(y) = \sum_{i+j=m} h_i(x)h_j(y) = \sum_{i+j=m} f_i(x)f_j(y) \end{aligned}$$

for all $y \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Hence, $f_m(xy) = \sum_{i+j=m} f_i(x)f_j(y)$ for all $x, y \in \mathcal{A}$, for each $m \in \mathbb{N}_0$, and $F = \{f_0, f_1, ..., f_m, ...\}$ is higher $*-$derivation. $\square$

Corollary 5. *Let $\mathcal{A}$ be a C^*-algebra of rank zero. Let $p \in (0,1), \theta \in [0,\infty)$ be real numbers. Suppose that $F = \{f_0, f_1, ..., f_m, ...\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that $f_m(0) = 0$ for each $m \in \mathbb{N}_0$,*

$$f_m(3^n uy) = \sum_{i+j=m} f_i(3^n u) f_j(y)$$

for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, all $n = 0, 1, 2, ...$ and for each $m \in \mathbb{N}_0$. Suppose that

$$\|2f_m(\frac{\mu x + \mu y}{2}) - \mu f_m(x) - \mu f_m(y) + f_m(z^*) - f_m(z)^*\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. If $\lim_n \frac{f_m(3^n e)}{3^n} \in U(\mathcal{B}) \cap Z(\mathcal{B})$ for each $m \in \mathbb{N}_0$, then the sequence $F = \{f_0, f_1, ..., f_m, ...\}$ is a higher $*-$derivation.

Proof. Setting $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ all $x, y, z \in \mathcal{A}$. Then by Theorem 4 we get the desired result. □

3 Stability of higher *-derivations: a fixed point approach

We investigate the generalized Hyers–Ulam–Rassias stability of higher $*$-derivations on unital C^*-algebras by using the alternative fixed point.

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (see also [6, 7, 8, 22, 23, 26]). Before proceeding to the main result of this section, we will state the following theorem.

Theorem 6. *(The alternative of fixed point [5]). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$,*

either

$$d(T^m x, T^{m+1} x) = \infty \text{ for all } m \geq 0,$$

or there exists a natural number m_0 such that

$$d(T^m x, T^{m+1} x) < \infty \text{ for all } m \geq m_0;$$

the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;

y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$;

$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.

Theorem 7. *Suppose that* $F = \{f_0, f_1, ..., f_m, ...\}$ *is a sequence of mappings from* $\mathcal{A}$ *into* $\mathcal{B}$ *such that for each* $m \in \mathbb{N}_0$, $f_m(0) = 0$ *for which there exists a function* $\phi : \mathcal{A}^5 \to [0, \infty)$ *satisfying*

$$\|f_m(\frac{\mu x + \mu y + \mu z}{3}) + f_m(\frac{\mu x - 2\mu y + \mu z}{3}) + f_m(\frac{\mu x + \mu y - 2\mu z}{3}) - \\ -\mu f_m(x) + f_m(uv) - \sum_{i+j=m} f_i(u) f_j(v) + f_m(w^*) - f_m(w)^*\| \leq \\ \leq \phi(x, y, z, u, v, w), \tag{3.1}$$

for all $\mu \in \mathbb{T}$, *and all* $x, y, z, u, v \in \mathcal{A}, w \in U(\mathcal{A}) \cup \{0\}$ *and for each* $m \in \mathbb{N}_0$. *If there exists an* $L < 1$ *such that* $\phi(x, y, z, u, v, w) \leq 3L\phi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}, \frac{u}{3}, \frac{v}{3}, \frac{w}{3})$ *for all* $x, y, z, u, v, w \in \mathcal{A}$, *then there exists a unique higher* $*-$*derivation*

$$H = \{h_0, h_1, ..., h_m, ...\}$$

such that

$$\|f_m(x) - h_m(x)\| \leq \frac{L}{1 - L}\phi(x, 0, 0, 0, 0, 0) \tag{3.2}$$

for all $x \in \mathcal{A}$ *and for each* $m \in \mathbb{N}_0$.

Proof. It follows from $\phi(x, y, z, u, v, w) \leq 3L\phi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}, \frac{u}{3}, \frac{v}{3}, \frac{w}{3})$ that

$$lim_j 3^{-j}\phi(3^j x, 3^j y, 3^j z, 3^j u, 3^j v, 3^j w) = 0 \tag{3.3}$$

for all $x, y, z, u, v, w \in \mathcal{A}$.
Put $y = z = w = u = 0$ in (3.1) to obtain

$$\|3f_m(\frac{x}{3}) - f_m(x)\| \leq \phi(x, 0, 0, 0, 0, 0) \tag{3.4}$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Hence,

$$\|\frac{1}{3}f_m(3x) - f_m(x)\| \leq \frac{1}{3}\phi(3x, 0, 0, 0, 0, 0) \leq L\phi(x, 0, 0, 0, 0, 0) \tag{3.5}$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$.
Consider the set $X := \{g_m \mid g_m : \mathcal{A} \to \mathcal{B}, m \in \mathbb{N}_0\}$ and introduce the generalized metric on X:

$$d(h, g) := inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0, 0, 0)\forall x \in A\}.$$

It is easy to show that (X, d) is complete. Now we define the linear mapping $J : X \to X$ by

$$J(h)(x) = \frac{1}{3}h(3x)$$

for all $x \in \mathcal{A}$. By Theorem 3.1 of [5],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all $g, h \in X$.
It follows from (2.5) that

$$d(f_m, J(f_m)) \leq L.$$

By Theorem 6, J has a unique fixed point in the set $X_1 := \{h \in X : d(f_m, h) < \infty\}$. Let h_m be the fixed point of J. h_m is the unique mapping with

$$h_m(3x) = 3h_m(x)$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$ satisfying there exists $C \in (0, \infty)$ such that

$$\|h_m(x) - f_m(x)\| \leq C\phi(x, 0, 0, 0, 0, 0)$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. On the other hand we have $lim_n d(J^n(f_m), h_m) = 0$. It follows that

$$lim_n \frac{1}{3^n} f_m(3^n x) = h_m(x) \tag{3.6}$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. It follows from $d(f_m, h_m) \leq \frac{1}{1-L} d(f_m, J(f_m))$, that

$$d(f_m, h_m) \leq \frac{L}{1-L}.$$

This implies the inequality (3.2). It follows from (3.1), (3.3) and (3.6) that

$$\begin{aligned}
&\|3h_m(\frac{x+y+z}{3}) + h_m(\frac{x-2y+z}{3}) + h_m(\frac{x+y-2z}{3}) - h_m(x)\| \\
&= lim_n \frac{1}{3^n} \|f_m(3^{n-1}(x+y+z)) + f_m(3^{n-1}(x-2y+z)) + \\
&\quad + f_m(3^{n-1}(x+y-2z)) - f_m(3^n x)\| \\
&\leq lim_n \frac{1}{3^n} \phi(3^n x, 3^n y, , 3^n z, 0, 0, 0) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. So

$$h_m(\frac{x+y+z}{3}) + h_m(\frac{x-2y+z}{3}) + h_m(\frac{x+y-2z}{3}) = h_m(x)$$

for all $x, y, z \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Put $w = \frac{x+y+z}{3}, t = \frac{x-2y+z}{3}$ and $s = \frac{x+y-2z}{3}$ in above equation, we get $h_m(w+t+s) = h_m(w) + h_m(t) + h_m(s)$ for all $w, t, s \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Hence, h_m for each $m \in \mathbb{N}_0$ is Cauchy additive. By putting $y = z = x$, $v = w = 0$ in (2.1), we have

$$\|\mu f_m(\frac{3\mu x}{3}) - \mu f_m(x)\| \leq \phi(x, x, , x, 0, 0, 0)$$

for all $x \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. It follows that

$$\|h_m(\mu x) - \mu h_m(x)\| = \\ = lim_m \frac{1}{3^m}\|f_m(\mu 3^m x) - \mu f_m(3^m x)\| \le lim_m \frac{1}{3^m}\phi(3^m x, 3^m x, 3^m x, 0, 0, 0) = 0$$

for all $\mu \in \mathbb{T}$, and all $x \in \mathcal{A}$. One can show that the mapping $h : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}-$linear. By putting $x = y = z = u = v = 0$ in (2.1) it follows that

$$\begin{aligned} &\|h_m(w^*) - (h_m(w))^*\| \\ &= lim_m \|\frac{1}{3^m} f_m((3^m w)^*) - \frac{1}{3^m}(f_m(3^m w))^*\| \\ &\le lim_m \frac{1}{3^m}\phi(0,0,0,0,0,3^m w) \\ &= 0 \end{aligned}$$

for all $w \in U(\mathcal{A})$ and for each $m \in \mathbb{N}_0$. By the same reasoning as the proof of Theorem 1, we can show that $h_m : \mathcal{A} \to \mathcal{B}$ is $*-$preserving for each $m \in \mathbb{N}_0$.

Since h_m is $\mathbb{C}-$linear, by putting $x = y = z = w = 0$ in (2.1) it follows that

$$\begin{aligned} \|h_m(uv) - \sum_{i+j=m} h_i(u)h_j(v)\| &= lim_m \|\frac{1}{9^m} f_m(9^m(uv)) - \frac{1}{9^m}\sum_{i+j=m} f_i(3^m u) f_j(3^m v)\| \\ &\le lim_m \frac{1}{9^m}\phi(0,0,0,3^m u, 3^m v, 0) \\ &\le lim_m \frac{1}{3^m}\phi(0,0,0,3^m u, 3^m v, 0) \\ &= 0 \end{aligned}$$

for all $u, v \in \mathcal{A}$ and for each $m \in \mathbb{N}_0$. Thus $H = \{h_0, h_1, ..., h_m, ...\}$ is higher $*-$derivation satisfying (3.2), as desired. □

We prove the following Hyers–Ulam–Rassias stability problem for higher $*$-derivations on unital C^*-algebras:

Corollary 8. *Let $p \in (0,1), \theta \in [0,\infty)$ be real numbers. Suppose that*

$$F = \{f_0, f_1, ..., f_m, ...\}$$

is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $m \in \mathbb{N}_0$, $f_m(0) = 0$ and

$$\|f_m(\frac{\mu x + \mu y + \mu z}{3}) + f_m(\frac{\mu x - 2\mu y + \mu z}{3}) + f_m(\frac{\mu x + \mu y - 2\mu z}{3}) - \mu f_m(x) + f_m(uv)$$

$$- \sum_{i+j=m} f_i(u)f_j(v) + f_m(w^*) - f_m(w)^*\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p),$$

for all $\mu \in \mathbb{T}$ *and all* $x, y, z, u, v \in \mathcal{A}, w \in U(\mathcal{A}) \cup \{0\}$. *Then there exists a unique higher* $*-$*derivation* $H = \{h_0, h_1, ..., h_m, ...\}$ *such that*

$$\|f_m(x) - h_m(x)\| \leq \frac{3^p\theta}{3 - 3^p}\|x\|^p$$

for all $x \in \mathcal{A}$ *and for each* $m \in \mathbb{N}_0$.

Proof. Setting$\phi(x, y, z, u, v, w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$ all $x, y, z, u, v, w \in \mathcal{A}$. Then by $L = 3^{p-1}$ in Theorem 7, one can prove the result. □

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