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COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACE

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Abstract. The concept of occasionally weakly compatible (shortly, owc) mappings introduced by Al-Thagafi and Shahzad [2], which is more general than the concept of weakly compatible maps. In this paper, we prove a common fixed point theorem for owc mappings in Menger space using arbitrary continuous t-norm for a nonlinear case.

1 Introduction

K. Menger [11] introduced the notion of probabilistic metric space, which is a generalization of the metric space. The study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [14, 15]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications.

In 1986, Jungck [6] introduced the notion of compatible mappings in metric spaces. Mishra [12] extended the notion of compatibility to probabilistic metric spaces. And this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [7, 8]. The concept of weakly compatible mappings is most general as each pair of compatible mappings is weakly compatible but the reverse is not true. Recently, Aamri and El Moutawakil [1] introduced the (E.A) property and thus generalized the concept of non-compatible maps. The results obtained in the metric fixed point theory by using the notion of non-compatible maps or the (E.A) property, are very interesting. Lastly, Al-Thagafi and Shahzad [2] introduced the notion of occasionally weakly compatible mappings which is more general than the concept of weakly compatible maps. Several interesting and elegant results have been obtained by various authors in this direction [3, 4, 5, 9, 10, 13, 17].

In this paper, we prove a common fixed point theorem for occasionally weakly compatible mappings in Menger space using arbitrary continuous t-norm for a non-linear case.

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2 Preliminaries

Definition 1. *[15] A triangular norm $\triangle$ (shortly t-norm) is a binary operation on the unit interval $[0,1]$ and the following conditions are satisfied: for all $a,b,c,d \in [0,1]$,*

(i) $\triangle(a,1) = a$ for all $a \in [0,1]$;
(ii) $\triangle(a,b) = \triangle(b,a)$;
(iii) $\triangle(a,b) \leq \triangle(c,d)$ for $a \leq c$, $b \leq d$;
(iv) $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$;
Examples of t-norms are $\triangle(a,b) = ab$ and $\triangle(a,b) = \min\{a,b\}$.
Now t-norms are recursively defined by $\triangle^1 = \triangle$ and

$$\triangle^n(x_1,\ldots,x_{n+1}) = \triangle\left(\triangle^{n-1}(x_1,\ldots,x_n),x_{n+1}\right).$$

Definition 2. *[15] A mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$.*

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 3. *[15] A probabilistic metric space (shortly PM-space) is an ordered pair $(X,\mathcal{F})$, where X is a nonempty set of elements and $\mathcal{F}$ is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions: for all $x,y,z \in X$ and $t,s > 0$,*

(i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only $x = y$;
(ii) $F_{x,y}(0) = 0$;
(iii) $F_{x,y}(t) = F_{y,x}(t)$;
(iv) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$,

The ordered triple $(X,\mathcal{F},\triangle)$ is called a Menger space if $(X,\mathcal{F})$ is a PM-space, $\triangle$ is a t-norm and the following inequality holds:

(v) $F_{x,y}(t+s) \geq \triangle(F_{x,z}(t), F_{z,y}(s))$, for all $x,y,z \in X$ and $t,s > 0$.

Every metric space (X,d) can always be realized as a PM-space by considering $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x,y))$ for all $x,y \in X$. So PM-spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations.

Definition 4. *[15] Let $(X,\mathcal{F},\triangle)$ be a Menger space with continuous t-norm.*

(i) A sequence $\{x_n\}$ in X is said to be converge to a point x in X if and only if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$.

(iii) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 5. *[12] Two self mappings A and B of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible if and only if $F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X.*

Definition 6. *[9] Let $(X, \mathcal{F}, \triangle)$ be a Menger space and A, B be self maps of X. A point $x \in X$ is called a coincidence point of A and B if and only if $Ax = Bx$. In this case $w = Ax = Bx$ is called a point of coincidence of A and B.*

Definition 7. *[16] Two self mappings A and B of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be weakly compatible if they commute at their coincidence points, that is, if $Ax = Bx$ for some $x \in X$, then $ABx = BAx$.*

Remark 8. *[16] Two compatible self-maps are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.*

The following concept [2] is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point. The counterpart of the concept of occasionally weakly compatible maps in PM-spaces is as follows:

Definition 9. *[9] Two self mappings A and B of a Menger space $(X, \mathcal{F}, \triangle)$ are occasionally weakly compatible if and only if there is a point $x \in X$ which is a coincidence point of A and B at which A and B commute.*

Lemma 10. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space, A and B are occasionally weakly compatible self maps of X. If A and B have a unique point of coincidence, $w = Ax = Bx$, then w is the unique common fixed point of A and B.*

Proof. Since A and B are occasionally weakly compatible, there exists a point $x \in X$ such that $Ax = Bx = w$ and $ABx = BAx$. Thus, $AAx = ABx = BAx$, which says that Ax is also a point of coincidence of A and B. Since the point of coincidence $w = Ax$ is unique by hypothesis, $BAx = AAx = Ax$, and $w = Ax$ is a common fixed point of A and B. Moreover, if z is any common fixed point of A and B, then $z = Az = Bz = w$ by the unique of the point of coincidence. □

3 Result

Theorem 11. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space. Further, let (L, A) and (M, S) are occasionally weakly compatible maps in X satisfying*

$$\min\{F_{Lx,My}(kt), F_{Sy,Lx}(kt)\} + \gamma F_{Sy,My}(kt) \geq [\alpha F_{Ax,Lx}(t) + \beta F_{Ax,Sy}(t)] \tag{3.1}$$

for all $x, y \in X, k \in (0,1)$ *and* $t > 0$ *where* $0 < \alpha, \beta < 1$ *and* $0 \leq \gamma < 1$ *such that* $\alpha + \beta - \gamma = 1$;

Then L, A, M *and* S *have a unique common fixed point in* X.

Proof. Since the pairs (L, A) and (M, S) are occasionally weakly compatible, there exist points $u, v \in X$ such that $Lu = Au, LAu = ALu$ and $Mv = Sv, MSv = SMv$. Now we show that $Lu = Mv$.

By putting $x = u$ and $y = v$ in inequality (3.1), then we get

$$\begin{aligned} \min\{F_{Lu,Mv}(kt), F_{Sv,Lu}(kt)\} + \gamma F_{Sv,Mv}(kt) &\geq [\alpha F_{Au,Lu}(t) + \beta F_{Au,Sv}(t)], \\ \min\{F_{Lu,Mv}(kt), F_{Mv,Lu}(kt)\} + \gamma F_{Mv,Mv}(kt) &\geq [\alpha F_{Lu,Lu}(t) + \beta F_{Lu,Mv}(t)], \\ F_{Lu,Mv}(kt) + \gamma &\geq [\alpha + \beta F_{Lu,Mv}(t)], \\ F_{Lu,Mv}(kt) &\geq [\beta F_{Lu,Mv}(t) + (\alpha - \gamma)] \\ F_{Lu,Mv}(kt) &\geq \left(\frac{\alpha - \gamma}{1 - \beta}\right) \\ F_{Lu,Mv}(kt) &\geq 1. \end{aligned}$$

Thus, we have $Lu = Mv$. Therefore, $Lu = Au = Mv = Sv$. Moreover, if there is another point z such that $Lz = Az$. Then using the inequality (3.1) it follows that $Lz = Az = Mv = Sv$, or $Lu = Lz$. Hence $w = Lu = Au$ is the unique point of coincidence of L and A. By Lemma 10, w is the unique common fixed point of L and A. Similarly, there is a unique point $z \in X$ such that $z = Mz = Sz$. Suppose that $w \neq z$ and taking $x = u, y = z$ in inequality (3.1), then we get

$$\begin{aligned} \min\{F_{Lw,Mz}(kt), F_{Sz,Lw}(kt)\} + \gamma F_{Sz,Mz}(kt) &\geq [\alpha F_{Aw,Lw}(t) + \beta F_{Aw,Sz}(t)], \\ \min\{F_{w,z}(kt), F_{z,w}(kt)\} + \gamma F_{z,z}(kt) &\geq [\alpha F_{w,w}(t) + \beta F_{w,z}(t)], \\ F_{w,z}(kt) + \gamma &\geq [\alpha + \beta F_{w,z}(t)], \\ F_{w,z}(kt) &\geq [\beta F_{w,z}(t) + (\alpha - \gamma)] \\ F_{w,z}(kt) &\geq \left(\frac{\alpha - \gamma}{1 - \beta}\right) \\ F_{w,z}(kt) &\geq 1. \end{aligned}$$

Thus, we have $w = z$. That is w is the unique common fixed point of L, A, M and S in X. □

Now, we give an example which illustrates Theorem 11.

Example 12. *Let* $X = [0,1]$ *with the metric* d *defined by* $d(x, y) = |x - y|$ *and for each* $t \in [0,1]$ *define*

$$F_{x,y}(t) = \begin{cases} e^{\frac{-|x-y|}{t}}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$*. Clearly* $(X, \mathcal{F}, \triangle)$ *be a Menger space. Define* L, A, M *and* $S : X \to X$ *by*

$$L(x) = \begin{cases} x, & \text{if } 0 \le x \le \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} < x \le 1. \end{cases} \quad A(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le \frac{1}{2}; \\ 0, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

$$M(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} < x \le 1. \end{cases} \quad S(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{x}{4}, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Then L, A, M *and* S *satisfy all the conditions of Theorem 11 for* $k \in (0, 1)$ *with respect to the distribution function* $F_{x,y}$*.*

First, we have
$L\left(\frac{1}{2}\right) = \frac{1}{2} = A(\frac{1}{2})$ *and* $LA(\frac{1}{2}) = \frac{1}{2} = AL(\frac{1}{2})$
and
$M(\frac{1}{2}) = \frac{1}{2} = S(\frac{1}{2})$ *and* $MS(\frac{1}{2}) = \frac{1}{2} = SM(\frac{1}{2})$,
that is, L *and* A *as well as* M *and* S *are occasionally weakly compatible. Also* $\left(\frac{1}{2}\right)$ *is the unique common fixed point of* L, A, M *and* S*. On the other hand, it is clear to see that the mappings* L, A, M *and* S *are discontinuous at* $\left(\frac{1}{2}\right)$*.*

On taking $L = M$ and $A = S$ in Theorem 11 then we get the following interesting result.

Corollary 13. *Let* $(X, \mathcal{F}, \triangle)$ *be a Menger space. Further, let* (L, A) *be occasionally weakly compatible maps in* X *satisfying*

$$\min\{F_{Lx,Ly}(kt), F_{Ay,Lx}(kt)\} + \gamma F_{Ay,Ly}(kt) \ge [\alpha F_{Ax,Lx}(t) + \beta F_{Ax,Ay}(t)] \tag{3.2}$$

for all $x, y \in X, k \in (0, 1)$ *and* $t > 0$ *where* $0 < \alpha, \beta < 1$ *and* $0 \le \gamma < 1$ *such that* $\alpha + \beta - \gamma = 1$*;*

Then L *and* A *have a unique common fixed point in* X*.*

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