BOUNDED LINEAR OPERATORS ON FINITE DIMENSIONAL PROBABILISTIC NORMED SPACES

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Abstract. Probabilistic normed spaces were introduced by Šerstnev and have been redefined by Alsina, Schweizer, and Sklar. In this paper, we obtain some conditions under which linear operators on finite dimensional probabilistic normed spaces are bounded and continuous.

1 Introduction

A probabilistic normed space (PN space) is a natural generalization of an ordinary normed linear space. In PN spaces, the norms of the vectors are represented by probability distribution functions rather than a positive number. Such spaces were introduced by Šerstnev in [15] and have been redefined by Alsina, Schweizer, and Sklar in [2]. Linear operators in probabilistic normed spaces were first studied by Lafuerza-Guillén, Rodríguez-Lallena, and Sempi in [6] and [9]. In this paper, we state Theorem 17 which is required in the sequel. Then for a linear operator $T : (V_1, \nu, \tau_1, \tau_1^*) \to (V_2, \nu', \tau_2, \tau_2^*)$, where $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ are PN spaces, we obtain some conditions on V_1 and V_2 under which the linear operator T is continuous and bounded. After that we give some examples, which show that those conditions are necessary for boundedness and continuity of the linear operator T. We first recall some notations and definitions of the probabilistic normed spaces that will be used in the sequel.

A distribution function, briefly a d.f., is a function F defined on the extended reals $\overline{\mathbb{R}} = [-\infty, +\infty]$ that is non-decreasing, left-continuous on $\mathbb{R}$ and such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ; the subset of those d.f.'s such that $F(0) = 0$ will be denoted by Δ^+ and by $\mathcal{D}^+$ the subset of the d.f.'s in Δ^+ such that $\lim_{x \to +\infty} F(x) = 1$. For every $a \in \overline{\mathbb{R}}$, ε_a is the d.f. defined

by

$$\varepsilon_a(x) = \left\{ \begin{array}{ll} 0, & x \le a, \\ 1, & x > a. \end{array} \right.$$

Definition 1. *A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+,*

(1) $\tau(F, \varepsilon_0) = F$,
(2) $\tau(F, G) = \tau(G, F)$,
(3) $\tau(F, G) \le \tau(H, K)$ whenever $F \le H$, $G \le K$,
(4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

The set Δ^+ can be made into a metric space with the metric $d_L : \Delta^+ \times \Delta^+ \to \mathbb{R}$, known as Levy metric, defined by

$$d_L(F, G) = \inf\{h \in [0, 1] |\ F(t - h) - h \le G(t) \le F(t + h) + h \text{ for all } t \in \mathbb{R}\},$$

under which a sequence of distribution functions $\{F_n\}_{n \in \mathbb{N}}$ converges to $F \in \Delta^+$ if and only if at each point $t \in \mathbb{R}$, where F is continuous, $F_n(t) \to F(t)$ [14]. Typical continuous triangle functions are the operations τ_T, τ_{T^*} and Π_T, which are, respectively, given by

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

$$\tau_{T^*}(F, G)(x) = \inf_{s+t=x} T^*(F(s), G(t))$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x))$$

for all $F, G \in \Delta^+$ and all $x \in \mathbb{R}$. Here T is a continuous t-norm, i.e., a continuous binary operation on the interval $[0, 1]$ that is associative, commutative, non-decreasing in each variable, and has 1 as identity; T^* is a continuous t-conorm, by which means a continuous binary operation on $[0, 1]$ which is related to a continuous t-norm through

$$T^*(x, y) = 1 - T(1 - x, 1 - y).$$

The most important t-norms are the functions W, Π and M which are defined, respectively, by

$$W(a, b) = \max(a + b - 1, 0),$$

$$\Pi(a, b) = a.b,$$

$$M(a, b) = \min(a, b).$$

Their corresponding t-conorms are given, respectively, by

$$W^*(a, b) = \min(a + b, 1),$$

$$\Pi^*(a, b) = a + b - a.b,$$

$$M^*(a, b) = \max(a, b).$$

Definition 2. *(see [2]) A Šerstnev PN space or a Šerstnev space is a triple (V, ν, τ), where V is a (real or complex) linear space, ν is a mapping from V into Δ^+ and τ is a continuous triangle function and the following conditions are satisfied for all p and q in V:*

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(Š) $\forall \alpha \in \mathbb{R} \setminus \{0\}, \quad \forall x \in \overline{\mathbb{R}}_+, \quad \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$.

Notice that condition (Š) implies
(N2) $\forall p \in V, \qquad \nu_{-p} = \nu_p$.

Definition 3. *(see [2]) A PN space is a quadruple (V, ν, τ, τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions such that $\tau \leq \tau^*$, and the mapping $\nu : V \to \Delta^+$ satisfies, for all p and q in V, the conditions:*

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2) $\forall p \in V \quad \nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(N4) $\forall \alpha \in [0,1], \quad \nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$.

The function ν is called the probabilistic norm. If ν satisfies the condition, weaker than (N1),

$$\nu_\theta = \varepsilon_0,$$

then (V, ν, τ, τ^*) is called a Probabilistic Pseudo–Normed space (briefly, a PPN space). If ν satisfies the conditions (N1) and (N2), then (V, ν, τ, τ^*) is said to be a Probabilistic seminormed space (briefly, PSN space). If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t–norm T and its t–conorm T^*, then $(V, \nu, \tau_T, \tau_{T^*})$ is denoted by (V, ν, T) and is called a Menger PN space.

Definition 4. *(see [1]) A quadruple (V, ν, τ, τ^*) is said to satisfy the ϕ–Šerstnev condition if*

(ϕ-Š) $\nu_{\lambda p}(x) = \nu_p\left(\widehat{\phi}\left(\frac{\phi(x)}{|\lambda|}\right)\right)$ for every $p \in V$, for every $x > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

A PN space (V, ν, τ, τ^*) which satisfies the ϕ–Šerstnev condition is called a ϕ–Šerstnev PN space. If $\phi(x) = x^{1/\alpha}$ for a fixed positive real number α, the condition (ϕ-Š) takes the form

(α-Š) $\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|^\alpha}\right)$ for every $p \in V$, for every $x > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

PN spaces satisfying the condition (α-Š) are called α–Šerstnev PN spaces.

Definition 5. *(see [5]) A PSN space (V, ν) is said to be equilateral if there is a d.f. $F \in \Delta^+$, different from ε_0 and from ε_∞, such that, for every $p \neq \theta$, $\nu_p = F$. It is immediate that every equilateral PSN space (V, ν) is a PN space under $\tau = \Pi_M$ and $\tau^* = \Pi_M$. An equilateral PN space will be denoted by (V, F, M).*

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Definition 6. *(see [5]) Let $G \in \Delta^+$ be different from ε_0 and from ε_∞, let $(V, \|.\|)$ be a normed space and defined $\nu : V \to \Delta^+$ by $\nu_\theta = \varepsilon_0$ and, if $p \neq \theta$, by*

$$\nu_p(t) = G(\frac{t}{\|p\|}) \qquad (t > 0).$$

The pair (V, ν) is called the simple space generated by $(V, \|.\|)$ and by G. The simple space generated by $(V, \|.\|)$ and by G is a Menger PN space under M, denoted by $(V, \|.\|, G, M)$, and a Šerstnev space.

Definition 7. *(see [14]) Let (V, ν, τ, τ^*) be a PN space. For each $p \in V$ and $\lambda > 0$, the strong λ-neighborhood of p is the set*

$$N_p(\lambda) = \{q \in V : \nu_{q-p}(\lambda) > 1 - \lambda\}.$$

The system of neighborhoods $\{N_p(\lambda) : \lambda > 0, \ p \in V\}$ determines a Hausdorff topology on V, called the strong topology.

Definition 8. *(see [12]) A subset A of a PN space (V, ν, τ, τ^*) is said to be D-compact if every sequence $\{p_n\}_n$ in A has a subsequence $\{p_{n_k}\}_k$ converges to a vector $p \in A$ with respect to the strong topology.*

Let A be a subset of a PN space (V, ν, τ, τ^*). The probabilistic radius R_A of a nonempty set A in PN space (V, ν, τ, τ^*) is defined by

$$R_A(x) = \begin{cases} l^-\phi_A(x), & x \in [0, +\infty), \\ 1, & x = \infty, \end{cases}$$

where $l^- f(x)$ denotes the left limit of the function f at the point x and $\phi_A(x) = \inf\{\nu_p(x) : p \in A\}$.

Definition 9. *(see [9, Definition 2.1]) A nonempty set A in a PN space (V, ν, τ, τ^*) is said to be:*

(a) certainly bounded, if $R_A(x_0) = 1$ for some $x_0 \in (0, +\infty)$;

(b) perhaps bounded, if one has $R_A(x) < 1$ for every $x \in (0, \infty)$, and $l^- R_A(+\infty) = 1$.

Moreover, the set A will be said to be D-bounded if either (a) or (b) holds, i.e., if $R_A \in \mathcal{D}^+$.

Definition 10. *(see [9]) Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be PN spaces. A linear operator $T : V_1 \to V_2$ is said to be:*

(a) certainly bounded if, and only if, it maps every certainly bounded set A of the space $(V_1, \nu, \tau_1, \tau_1^*)$ into a certainly bounded set TA of the space $(V_2, \nu', \tau_2, \tau_2^*)$;

(b) bounded if, and only if, it maps every D-bounded set of the space $(V_1, \nu, \tau_1, \tau_1^*)$ into a D-bounded set TA of the space $(V_2, \nu', \tau_2, \tau_2^*)$;

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(c) strongly bounded if, and only if, there exists a constant $k > 0$ such that, for every $p \in V_1$ and for every $x > 0$,

$$\mu_{T(p)}(x) \geq \nu_p(\frac{x}{k}).$$

2 Bounded linear operators in finite dimensional probabilistic normed spaces

In [9] there are some examples which show that in general there is not any relation between boundedness, certainly boundedness and continuity. The following theorem gives some relations between strongly bounded operators, bounded operators, and continuous operators in probabilistic normed spaces.

Theorem 11. *(see [9] and [13]) If $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ are PN spaces and $T : V_1 \to V_2$ is a linear operator, then*

(a) If T is strongly bounded operator, then T is bounded, certainly bounded, and continuous with respect to strong topology.
(b) If $\nu(V_1) \subseteq \mathcal{D}^+$, $\nu'(V_2) \subseteq \mathcal{D}^+$ and T is continuous, then T is strongly bounded with respect to strong topology.

We first state the fundamental facts about a definition and some results which are required in the sequel.

Definition 12. *([8]) The PN space (V, ν, τ, τ^*) is said to satisfy the double infinity–condition (briefly, DI-condition) if the probabilistic norm ν is such that, for all $\lambda \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}$ and $p \in V$,*

$$\nu_{\lambda p}(x) = \nu_p(\varphi(\lambda, x)),$$

where $\varphi : \mathbb{R} \times [0, +\infty) \to [0, +\infty)$ satisfies $\lim_{x \to +\infty} \varphi(\lambda, x) = +\infty$ for all λ and $\lim_{\lambda \to 0} \varphi(\lambda, x) = +\infty$ for all x.

Definition 13. *(see [7]) A subset A of a topological vector space (briefly, TV space) E is topologically bounded, if for every sequence $\{\lambda_n\}_n$ of real numbers that converges to 0 as $n \to \infty$ and for every sequence $\{p_n\}_n$ of elements of A, one has $\lambda_n p_n \to \theta$ in the topology of E. Also by [11, Theorem 1.30], A is topologically bounded if, and only if, for every neighborhood U of θ, we have $A \subseteq tU$ for all sufficiently large t.*

Theorem 14. *([8], Theorem 2.1) Let (V, ν, τ, τ^*) be a PN space which satisfies the DI–condition. Then for a subset $A \subseteq V$, the following statements are equivalent:*

(a) A is D–bounded.

(b) A is bounded, namely, for every $n \in \mathbb{N}$ and for every $p \in A$, there is $k \in \mathbb{N}$ such that $\nu_{p/k}(1/n) > 1 - 1/n$.

(c) A is topologically bounded.

Example 15. *Let (V, ν, τ, τ^*) be an α–Šerstnev PN space. It is easy to see that (V, ν, τ, τ^*) satisfies the DI–condition, where*

$$\varphi(\lambda, x) = \frac{x}{|\lambda|^\alpha}.$$

Theorem 16. *Let (V, ν, τ, τ^*) be an α-Šerstnev PN space. Then for a subset $A \subseteq V$, the same statements as in Theorem 14 are equivalent.*

Now we state the following theorem which is used frequently in this paper.

Theorem 17. *If $dimV = n < \infty$ and (V, ν, τ, τ^*) is a PN space that is also a TV space and A is a subspace of V, then*

(a) V is normable.

(b) A is D-compact if, and only if, it is compact.

Also if (V, ν, τ, τ^*) is an α-Šerstnev PN space, then

(c) A is D-compact if, and only if, it is compact if, and only if, it is closed and D-bounded.

Proof. (a) Let $\{e_1, e_2, ..., e_n\}$ be a Hamel basis for V. Then for every p in V, there are unique $\alpha_1, \alpha_2, ..., \alpha_n$ in $\mathbb{R}$ such that $p = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$. If we define $\|p\| := \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$, then $\| \cdot \|$ is a norm on V. It is easy to check that $(V, \| \cdot \|)$ is a TV space. By [4, p. 105], if $\mathcal{T}_1$ is the strong topology and $\mathcal{T}_2$ is the normed topology on V which was defined as above, then $\mathcal{T}_1 = \mathcal{T}_2$. Thus, V is normable.

(b) Since $\mathcal{T}_1 = \mathcal{T}_2$, the identity map $I : (V, \mathcal{T}_1) \to (V, \mathcal{T}_2)$ is a homeomorphism. Hence [10, Theorem 28.2] implies the desired conclusion.

(c) Let $(\mathbb{R}^n, \| \cdot \|)$ be the Euclidean space and $\{e_1, e_2, ..., e_n\}$ be a Hamel basis for V. We define $f : (V, \mathcal{T}_2) \to (\mathbb{R}^n, \| \cdot \|)$ by $f(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n) = (\alpha_1, \alpha_2, ..., \alpha_n)$. It is clear that f is a homeomorphism. Therefore, by the Heine-Borel Theorem and Theorem 16, A is compact in the strong topology if, and only if, it is closed and D-bounded. $\square$

Corollary 18. *Let M be a subspace of V with $dimM = n < \infty$ and A is a subspace of M. If (V, ν, τ, τ^*) is a PN space that is also a TV space, then*

(a) M is normable.

(b) A is D-compact if, and only if, it is compact.

Also if (V, ν, τ, τ^*) is an α-Šerstnev PN space, then

(c) A is D-compact if, and only if, it is compact if, and only if, it is closed and D-bounded.

Proof. By [4, p. 105], M is closed and so the proof is obvious. ∎

Theorem 19. *Let $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be TV PN spaces and $dimV_1 < \infty$. If $T : V_1 \to V_2$ is a linear operator, then T is continuous with respect to strong topology.*

Proof. Since V_1 and Ran$T$ satisfy the hypotheses of Theorem 17, they are normable. Hence by [4, Proposition 3.4], $T : V_1 \to \text{Ran}(T)$ is continuous with respect to strong and normed topology. Let U be an open subset of V_2 with respect to strong topology. Then $U \bigcap \text{Ran}(T)$ is open in $\text{Ran}(T)$ and so $T^{-1}(U \bigcap \text{Ran}(T)) = T^{-1}(U)$ is open in V_1. Therefore, T is continuous with respect to strong topology. ∎

Theorem 20. *Suppose that $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ be α-Šerstnev PN spaces that are also TV spaces. Let $dimV_1 < \infty$. If $T : V_1 \to V_2$ be a linear operator, then T is bounded.*

Proof. Let A be a D-bounded subset of V_1. Then $\overline{A}$ is D-bounded and closed with respect to strong topology. Hence by Theorem 17, $\overline{A}$ is compact. If we apply Theorem 19, then $T(\overline{A})$ is compact in V_2. Thus, $T(\overline{A})$ is compact in Ran$T$. Again by using Theorem 17, we get $T(\overline{A})$ is D-bounded and so $T(A)$ is D-bounded. ∎

Corollary 21. *Suppose that $(V_1, \nu, \tau_1, \tau_1^*)$, $(V_2, \nu', \tau_2, \tau_2^*)$ and T satisfy the hypotheses of Theorem 20, then*

(a) $T : V_1 \to V_2$ is bounded.
(b) $T : V_1 \to RanT$ is bounded with respect to normed topology.
(c) $T : V_1 \to V_2$ is continuous with respect to strong topology.

Corollary 22. *Suppose that $(V_1, \nu, \tau_1, \tau_1^*)$, $(V_2, \nu', \tau_2, \tau_2^*)$ and T satisfy the hypotheses of Theorem 19. Let $dimV_1 = dimV_2 < \infty$ and T be 1-1. Then T is a homeomorphism.*

Corollary 23. *Let $(V_1, \|.\|)$ and $(V_2, \|.\|)$ be two normed spaces and $dimV_1 < \infty$. Consider the simple spaces $(V_1, \|.\|, G_1, M)$ and $(V_2, \|.\|, G_2, M)$, where G_1 and G_2 belong to $\mathcal{D}^+$. If $T : V_1 \to V_2$ is a linear operator, then T is bounded.*

Proof. See [7, Theorem 2] and Theorem 20. ∎

Example 24. *Suppose that $(V_1, \|.\|)$ is a finite dimensional normed space and define $\nu : V_1 \to \Delta^+$ via*

$$\nu_p = \varepsilon_{\|p\|}.$$

It is not hard to see that $(V_1, \nu, \tau_T, \tau_{T^*})$ is an α–Šerstnev PN space, where T is a continuous t-norm and T^* is it's t-conorm. Consider the simple PN space $(V_2, \|.\|, F, M)$, where $F \in \mathcal{D}^+$. Also suppose that $T : V_1 \to V_2$ is a linear operator. Then by [7, Theorem 2], Theorems 19 and 20, T is continuous and bounded.

The following examples show that the hypotheses of Theorem 20 are necessary.

Example 25. *Consider the PN space $(\mathbb{R}, \nu, \tau_W, \tau_M)$, where $\nu : \mathbb{R} \to \Delta^+$ is define by*

$$\nu_p = \frac{1}{|p|+2}\varepsilon_0 + \frac{|p|+1}{|p|+2}\varepsilon_\infty$$

for every $p \neq 0$ in $\mathbb{R}$ and $\nu_\theta = \epsilon_0$. By [3], $(\mathbb{R}, \nu, \tau_W, \tau_M)$ is not a TV space. Let $(\mathbb{R}, F, M)$ be an equilateral PN space, where $F \in \mathcal{D}^+$. By [14, Section 12.3], the strong topology in an equilateral PN space is discrete. Therefore, the identity map $I : (\mathbb{R}, F, M) \to (\mathbb{R}, \nu, \tau_W, \tau_M)$ is continuous. In the other hand, I is not bounded. Because for every nonzero element $p \in \mathbb{R}$, $\{p\}$ is D-bounded in $(\mathbb{R}, F, M)$ and it is not D-bounded in $(\mathbb{R}, \nu, \tau_W, \tau_M)$.

Example 26. *Let $\beta \in (0,1)$. For $p = (p_1, p_2) \in \mathbb{R}^2$, one defines the probabilistic norm ν by $\nu_\theta = \varepsilon_0$ and*

$$\nu_p(x) = \begin{cases} \varepsilon_\infty(x) & p_1 \neq 0, \\ \beta\varepsilon_0(x), & otherwise. \end{cases}$$

It is easy to see that $(\mathbb{R}^2, \nu, \Pi_M, \Pi_M)$ is an α–Šerstnev PN space that is not TV space. Also consider the Šerstnev PN space $(\mathbb{R}^2, \nu', \Pi_\Pi, \Pi_M)$, where ν', the probabilistic norm, is a map $\nu' : \mathbb{R}^2 \to \Delta^+$ defined via $\nu'_p(x) = e^{\frac{-|p|}{x}}$ for $x > 0$ and $\nu'_\theta = \varepsilon_0$. If $F : (\mathbb{R}^2, \nu', \Pi_\Pi, \Pi_M) \to (\mathbb{R}^2, \nu, \Pi_M, \Pi_M)$ is defined by $F((p_1, p_2)) = (p_2, p_1)$, then F is not D-bounded. Because for every $(p_1, p_2) \in \mathbb{R}^2$, $\{(p_1, p_2)\}$ is D-bounded in $(\mathbb{R}^2, \nu', \Pi_\Pi, \Pi_M)$, but it is not D-bounded in $(\mathbb{R}^2, \nu', \Pi_M, \Pi_M)$.

Example 27. *Let $(V, \|.\|)$ be a normed space and T be a t-norm. Suppose that f is a right continuous non-increasing function and satisfying the following two properties:*
    (a) $f(x) = 1$ if, and only, if $x = 0$;
    (b) $f(\|p + q\|) \geq T(f(\|p\|), f(\|q\|))$ for all $p, q \in V$.
If $\nu : V \to \Delta^+$ is given by

$$\nu_p(x) = \begin{cases} 0, & x \leq 0, \\ f(\|p\|), & x \in (0, +\infty), \\ 1, & x = +\infty \end{cases}$$

for every $p \in V$, then by [7], (V, ν, τ_T, τ_M) is a TV PN space which satisfies the following properties:
    (1) (V, ν, τ_T, τ_M) is normable ;
    (2) (V, ν, τ_T, τ_M) is not an α–Šerstnev space;
    (3) $\nu(V - \{0\}) \subseteq \Delta^+ - \mathcal{D}^+$.
Also we define $\nu' : V \to \Delta^+$ by $\nu'_p(x) = e^{\frac{-\|p\|}{x}}$ for $x > 0$. Then $(V, \nu', \Pi_\Pi, \Pi_M)$ is a

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Šerstnev PN space. Now if $I : (V, \nu', \Pi_\Pi, \Pi_M) \to (V, \nu, \tau_T, \tau_M)$ is the identity map, then for every nonzero element $p \in V$, $\{p\}$ is D-bounded in $(V, \nu', \Pi_\Pi, \Pi_M)$, but $\{p\}$ is not D-bounded in (V, ν, τ_T, τ_M). Hence I is not bounded.

Example 28. *Suppose that $dimV_1 = 1$ and $(V_1, \nu, \tau_T, \tau_{T^*})$ is the α–Šerstnev PN space which was defined in Example 24. Assume that $(\mathbb{R}, \nu, \tau_T, \tau_{T^*})$ is the PN space which was defined in Example 25. If $p \neq 0$ is a vector in V and we define $T : (V, \nu, \tau_T, \tau_{T^*}) \to (\mathbb{R}, \nu, \tau_W, \tau_M)$ by $T(\alpha p) = \alpha$ for every α in $\mathbb{R}$. Hence $\{p\}$ is D-bounded in $(V, \nu, \tau_T, \tau_{T^*})$ and $\{T(p)\} = \{1\}$ is not D-bounded in $(\mathbb{R}, \nu, \tau_W, \tau_M)$. Then T is not bounded.*

Example 29. *Let V be an one dimensional vector space. Consider the equilateral PN space (V, F, M), where $F \in \mathcal{D}^+$. Suppose that $(\mathbb{R}, \nu, \tau_W, \tau_M)$ is the PN space which was defined in Example 25. If $p \neq 0$ is a vector in V and we define $T : (V, F, M) \to (\mathbb{R}, \nu, \tau_W, \tau_M)$ by $T(\alpha p) = \alpha$ for every α in $\mathbb{R}$, then $\{p\}$ is D-bounded in (V, F, M) and $\{T(p)\} = \{1\}$ is not D-bounded in $(\mathbb{R}, \nu, \tau_W, \tau_M)$. Hence T is not bounded.*

Example 30. *Let $G_1, G_2 \in \mathcal{D}^+$ and*

$$V = \{(\alpha_1, \alpha_2, \alpha_3, ...) : \alpha_i \in \mathbb{R} \text{ and } \alpha_i = 0 \text{ for all but a finite number of } i\}.$$

For every $x = (\alpha_1, \alpha_2, \alpha_3, ...) \in V$, we define $\|x\| = \sum_{i=1}^\infty |\alpha_i|^2$. If $T : (V, \|.\|, G_1, M) \to (\mathbb{R}, |.|, G_2, M)$ is defined by $T(\alpha_1, \alpha_2, \alpha_3, ...) = \sum_{i=1}^\infty i\alpha_i$ for every $(\alpha_1, \alpha_2, \alpha_3, ...) \in V$, then T is not bounded. Because by [9, Example 2.3], the D-bounded sets of $(V, \|.\|, G_1, M)$ and $(\mathbb{R}, |.|, G_2, M)$ coincide with the bounded sets of $(V, \|.\|)$ and $(\mathbb{R}, |.|)$, respectively. Hence $A = \{(1, 0, 0, 0, ...), (0, 1, 0, 0, ...), (0, 0, 1, 0, ...), ...\}$ is D-bounded in $(V, \|.\|, G_1, M)$ and $T(A)$ is not D-bounded in $(\mathbb{R}, |.|, G_2, M)$. Therefore, the hypothesis of Theorem 20 and Corollary 23 that says $dimV_1 < \infty$ is necessary.

Remark 31. *One can easily see that Theorems 17, 19 and 20 and Corollaries 18, 21 and 22 hold for Šerstnev PN spaces with similar proofs to the above results and using [7, Theorems 2 and 5].*

Lemma 32. *([12]) Consider a finite dimensional PN space (V, ν, τ, τ^*), where τ^* is archimedean, $\nu_p \neq \varepsilon_\infty$, and $\nu(V) \subseteq \mathcal{D}^+$ and $\mathcal{D}^+$ is invariant under τ, for every $p \in V$ on the real field $(\mathbb{R}, \nu', \tau, \tau^*)$, where ν' has the LG-property, Every subset A of V is D-compact if and only if A is D-bounded and closed.*

Theorem 33. *Suppose that $(V_1, \nu, \tau_1, \tau_1^*)$ and $(V_2, \nu', \tau_2, \tau_2^*)$ are TV spaces which are PN spaces and satisfy the hypotheses of the previous lemma and also $dimV_2$ may be finite or infinite. Let $T : V_1 \to V_2$ be a linear operator. Then T is bounded.*

Proof. By the similar idea of the proof of Theorem 20 and using Lemma 32, the theorem follows. $\square$

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