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EULER'S CONSTANT, SEQUENCES AND SOME ESTIMATES

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Abstract. We give a class of sequences with the argument of the logarithmic term modified and that converge quickly to a generalization of Euler's constant denoted by $\gamma(a)$, i.e. the limit of the sequence $\left(\sum_{k=1}^{n} \frac{1}{a+k-1} - \ln \frac{a+n-1}{a}\right)_{n\in\mathbb{N}}$, where $a \in (0, +\infty)$.

Also, we obtain estimates for $\gamma - \left(\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24(n+1/2)}\right)\right)$, where $\gamma = \gamma(1)$ is the Euler's constant.

1 Introduction

Let $D_n = H_n - \ln n$, where H_n denotes the nth harmonic number, i.e. $H_n = \sum_{k=1}^{n} \frac{1}{k}$. The limit $\gamma = \lim_{n\to\infty} D_n$ is the Euler's constant and, as he said, it is "worthy of serious consideration" ([10, pp. xx, 51]). Unfortunately, the definition sequence of the Euler's constant, the sequence $(D_n)_{n\in\mathbb{N}}$, converges slowly to γ. We have the following estimates for $D_n - \gamma$ (see [22], [23], [1], [4]):

$$\frac{1}{2n + \frac{2\gamma-1}{1-\gamma}} \le D_n - \gamma < \frac{1}{2n + \frac{1}{3}}, \quad n \in \mathbb{N}.$$

The numbers $\frac{2\gamma-1}{1-\gamma}$ and $\frac{1}{3}$ are the best constants with this property, i.e. $\frac{2\gamma-1}{1-\gamma}$ cannot be replaced by a smaller one and $\frac{1}{3}$ cannot be replaced by a larger one, so that the above-mentioned inequalities to hold for all $n \in \mathbb{N}$.

Quicker convergences to γ, as well as estimates related to γ have been given in the literature. We remind some of them. In [6], D. W. DeTemple considered the sequence $(R_n)_{n\in\mathbb{N}}$, with $R_n = H_n - \ln\left(n + \frac{1}{2}\right)$, and he obtained that $\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}$, $n \in \mathbb{N}$. A while later C.-P. Chen proved in [2] that

$$\frac{1}{24(n+a)^2} \le R_n - \gamma < \frac{1}{24(n+b)^2}, \quad n \in \mathbb{N},$$

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where $a = 1/\sqrt{24\left(-\gamma+1-\ln(3/2)\right)} - 1$ and $b = 1/2$ are the best constants with this property. B.-N. Guo and F. Qi obtained in [9, Theorem 1] that

$$\frac{1}{12n^2+\frac{6}{5}} < \gamma - \left(D_n - \frac{1}{2n}\right) \leq \frac{1}{12n^2+\frac{2(7-12\gamma)}{2\gamma-1}}, \quad n \in \mathbb{N},$$

the numbers $6/5$ and $2(7-12\gamma)/(2\gamma-1)$ being the best constants with this property. Estimates for $\gamma - \left(H_n - \ln\left(n+\frac{1}{2}\right) - \frac{1}{24(n+1/2)^2}\right)$ were given by C.-P. Chen in [3], and for $H_n - \ln\left(n+\frac{1}{2}+\frac{1}{24n}-\frac{1}{48n^2}+\frac{23}{5760n^3}\right) - \gamma$ by C.-P. Chen and C. Mortici in [5]. Interesting type of sequences that converge to γ have been given by C. Mortici in [15], [16].

Let $a \in (0,+\infty)$. We consider a generalization of Euler's constant as the limit $\gamma(a)$ of the sequence $(y_n(a))_{n\in\mathbb{N}}$ defined by (see, for example, [11, p. 453], [18], [19], [20])

$$y_n(a) = \sum_{k=1}^{n} \frac{1}{a+k-1} - \ln\frac{a+n-1}{a}.$$

Clearly, $\gamma(1) = \gamma$. Numerous results regarding the generalization of Euler's constant $\gamma(a)$ can be found in [18], [19], [20], [21]. See also [14] and [12].

In Section 2 we give a class of sequences with the argument of the logarithmic term modified and that converge quickly to $\gamma(a)$, and in Section 3 we prove some estimates for $\gamma - \left(H_n - \ln\left(n+\frac{1}{2}+\frac{1}{24(n+1/2)}\right)\right)$.

We remind the following lemma (C. Mortici [13, Lemma]), which is a consequence of the the Stolz-Cesàro Theorem, the 0/0 case [7, Theorem B.2, p. 265].

Lemma 1. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers and $x^* = \lim_{n\to\infty} x_n$. We suppose that there exists $\alpha \in \mathbb{R}$, $\alpha > 1$, such that*

$$\lim_{n\to\infty} n^{\alpha}(x_n - x_{n+1}) = l \in \overline{\mathbb{R}}.$$

Then there exists the limit

$$\lim_{n\to\infty} n^{\alpha-1}(x_n - x^*) = \frac{l}{\alpha-1}.$$

Also, recall that the digamma function ψ is the logarithmic derivative of the gamma function, i.e.

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x \in (0,+\infty).$$

It is known that ([8, Section 8.365, Entry 4, p. 904], [17, Section 5.4, Entry 5.4.14, p. 137])

$$\psi(n+1) = -\gamma + H_n, \quad n \in \mathbb{N}. \tag{1.1}$$

From the recurrence formula ([8, Section 8.365, Entry 1, p. 904], [17, Section 5.5, Entry 5.5.2, p. 138])

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad x \in (0, +\infty),$$

and the asymptotic formula ([17, Section 5.11, Entry 5.11.2, p. 140], see also [8, Section 8.367, Entry 13, p. 906])

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty),$$

one obtains

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty). \tag{1.2}$$

2 Sequences that converge to $\gamma(a)$

Theorem 2. *Let $a \in (0, +\infty)$. We specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n\in\mathbb{N}}$ from Introduction.*

(*i*) *We consider the sequence $(\alpha_n(a))_{n\in\mathbb{N}}$ defined by*

$$\alpha_n(a) = \sum_{k=1}^{n} \frac{1}{a+k-1} - \ln\left(\frac{a+n-\frac{1}{2}}{a} + \frac{1}{24a\left(a+n-\frac{1}{2}\right)}\right).$$

Then

$$\lim_{n\to\infty} n^4(\gamma(a) - \alpha_n(a)) = \frac{37}{5760}.$$

(*ii*) *We consider the sequence $(\beta_n(a))_{n\in\mathbb{N}}$ defined by*

$$\beta_n(a) = \alpha_n(a) + \frac{37}{5760\left(a+n-\frac{1}{2}\right)^4}.$$

Then

$$\lim_{n\to\infty} n^6(\beta_n(a) - \gamma(a)) = \frac{1109}{290304}.$$

(*iii*) *We consider the sequence $(\delta_n(a))_{n\in\mathbb{N}}$ defined by*

$$\delta_n(a) = \beta_n(a) - \frac{1109}{290304\left(a+n-\frac{1}{2}\right)^6}.$$

Then

$$\lim_{n\to\infty} n^8(\gamma(a) - \delta_n(a)) = \frac{27427}{6635520}.$$

Proof. (i) We have

$$\alpha_{n+1}(a)-\alpha_n(a)=\frac{1}{a+n}-\ln\left(a+n+\frac{1}{2}+\frac{1}{24\left(a+n+\frac{1}{2}\right)}\right)$$
$$+\ln\left(a+n-\frac{1}{2}+\frac{1}{24\left(a+n-\frac{1}{2}\right)}\right).$$

Set $\varepsilon_n:=\frac{1}{a+n}$, $n\in\mathbb{N}$. Since $\frac{1}{2}\,\varepsilon_n+\frac{1}{24}\cdot\frac{\varepsilon_n^2}{1+\frac{1}{2}\,\varepsilon_n}\in(-1,1]$ and $-\frac{1}{2}\,\varepsilon_n+\frac{1}{24}\cdot\frac{\varepsilon_n^2}{1-\frac{1}{2}\,\varepsilon_n}\in(-1,1]$, for every $n\in\mathbb{N}$, using the series expansion ([11, pp. 171–179]) we obtain

$$\begin{aligned}
&\alpha_{n+1}(a)-\alpha_n(a)\\
&=\varepsilon_n-\ln\left(1+\frac{1}{2}\,\varepsilon_n+\frac{1}{24}\cdot\frac{\varepsilon_n^2}{1+\frac{1}{2}\,\varepsilon_n}\right)+\ln\left(1-\frac{1}{2}\,\varepsilon_n+\frac{1}{24}\cdot\frac{\varepsilon_n^2}{1-\frac{1}{2}\,\varepsilon_n}\right)\\
&=\frac{37}{1440}\varepsilon_n^5+\frac{445}{48384}\varepsilon_n^7+\frac{343}{165888}\varepsilon_n^9+\frac{11765}{43794432}\varepsilon_n^{11}\\
&\quad-\frac{17413}{1242169344}\varepsilon_n^{13}-\frac{43561}{2293235712}\varepsilon_n^{15}+O(\varepsilon_n^{17}).
\end{aligned}$$

It follows that

$$\lim_{n\to\infty}n^5(\alpha_{n+1}(a)-\alpha_n(a))=\frac{37}{1440}.$$

Now, according to Lemma 1, we get

$$\lim_{n\to\infty}n^4(\gamma(a)-\alpha_n(a))=\frac{37}{5760}.$$

(ii) We are able to write that

$$\begin{aligned}
&\beta_n(a)-\beta_{n+1}(a)\\
&=\alpha_n(a)-\alpha_{n+1}(a)+\frac{37}{5760\left(a+n-\frac{1}{2}\right)^4}-\frac{37}{5760\left(a+n+\frac{1}{2}\right)^4}\\
&=\alpha_n(a)-\alpha_{n+1}(a)+\frac{37}{5760}\cdot\frac{\varepsilon_n^4}{(1-\frac{1}{2}\,\varepsilon_n)^4}-\frac{37}{5760}\cdot\frac{\varepsilon_n^4}{(1+\frac{1}{2}\,\varepsilon_n)^4}\\
&=\frac{1109}{48384}\,\varepsilon_n^7+\frac{16933}{829440}\,\varepsilon_n^9+\frac{515707}{43794432}\varepsilon_n^{11}+\frac{6874549}{1242169344}\varepsilon_n^{13}+O(\varepsilon_n^{15}).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}n^7(\beta_n(a)-\beta_{n+1}(a))=\frac{1109}{48384},$$

and applying Lemma 1, we obtain

$$\lim_{n\to\infty}n^6(\beta_n(a)-\gamma(a))=\frac{1109}{290304}.$$

(*iii*) We have

$$\begin{aligned}
&\delta_{n+1}(a)-\delta_n(a)\\
&=\beta_{n+1}(a)-\beta_n(a)-\frac{1109}{290304\left(a+n+\frac{1}{2}\right)^6}+\frac{1109}{290304\left(a+n-\frac{1}{2}\right)^6}\\
&=\beta_{n+1}(a)-\beta_n(a)-\frac{1109}{290304}\cdot\frac{\varepsilon_n^6}{(1+\frac{1}{2}\,\varepsilon_n)^6}+\frac{1109}{290304}\cdot\frac{\varepsilon_n^6}{(1-\frac{1}{2}\,\varepsilon_n)^6}\\
&=\frac{27427}{829440}\varepsilon_n^9+\frac{2119277}{43794432}\varepsilon_n^{11}+O(\varepsilon_n^{13}).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} n^9(\delta_{n+1}(a)-\delta_n(a))=\frac{27427}{829440},$$

and using Lemma 1, this yields

$$\lim_{n\to\infty} n^8(\gamma(a)-\delta_n(a))=\frac{27427}{6635520}.$$

□

In the same manner as in the proof of Theorem 2, considering the sequence in each of the following parts, we get the indicated limit:

$$\eta_n(a)=\delta_n(a)+\frac{27427}{6635520\left(a+n-\frac{1}{2}\right)^8},\ \text{for every } n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{10}(\eta_n(a)-\gamma(a))=\frac{3311269}{437944320};$$

$$\theta_n(a)=\eta_n(a)-\frac{3311269}{437944320\left(a+n-\frac{1}{2}\right)^{10}},\ \text{for every } n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{12}(\gamma(a)-\theta_n(a))=\frac{10998972697}{521711124480};$$

$$\lambda_n(a)=\theta_n(a)+\frac{10998972697}{521711124480\left(a+n-\frac{1}{2}\right)^{12}},\ \text{for every } n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{14}(\lambda_n(a)-\gamma(a))=\frac{2675115071}{32105299968};$$

$$\mu_n(a)=\lambda_n(a)-\frac{2675115071}{32105299968\left(a+n-\frac{1}{2}\right)^{14}},\ \text{for every } n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{16}(\gamma(a)-\mu_n(a))=\frac{33177521752619}{74851213639680}.$$

We remark the pattern in forming the sequences from Theorem 2 and those mentioned above. For example, the general term of the sequence $(\mu_n(a))_{n\in\mathbb{N}}$ can be written in the form

$$\mu_n(a) = \sum_{k=1}^{n} \frac{1}{a+k-1} - \ln\left(\frac{a+n-\frac{1}{2}}{a} + \frac{1}{2}\cdot\frac{B_2}{2}\cdot\frac{1}{a\left(a+n-\frac{1}{2}\right)}\right) - \sum_{k=2}^{7}\left(\frac{2^{2k-1}-1}{2^{2k-1}}\cdot\frac{B_{2k}}{2k} + \frac{1}{k}\left(-\frac{1}{2}\cdot\frac{B_2}{2}\right)^k\right)\frac{1}{\left(a+n-\frac{1}{2}\right)^{2k}},$$

where B_{2k} is the Bernoulli number of index $2k$. Related to this remark, see also [18, Remark 3.4], [20, Remark 2.1.3, p. 71; Remark 3.1.6, pp. 100, 101], [21].

For Euler's constant $\gamma = 0.5772156649\ldots$ we obtain, for example:

$\alpha_2(1) = 0.5770647254\ldots$; $\alpha_3(1) = 0.5771747758\ldots$;
$\beta_2(1) = 0.5772291698\ldots$; $\beta_3(1) = 0.5772175820\ldots$;
$\delta_2(1) = 0.5772135225\ldots$; $\delta_3(1) = 0.5772155039\ldots$;
$\eta_2(1) = 0.5772162314\ldots$; $\eta_3(1) = 0.5772156875\ldots$;
$\theta_2(1) = 0.5772154386\ldots$; $\theta_3(1) = 0.5772156600\ldots$;
$\lambda_2(1) = 0.5772157923\ldots$; $\lambda_3(1) = 0.5772156663\ldots$;
$\mu_2(1) = 0.5772155686\ldots$; $\mu_3(1) = 0.5772156643\ldots$.

As can be seen, $\mu_3(1)$ is accurate to nine decimal places in approximating γ.

In this section we have obtained that the sequence $(\alpha_n(a))_{n\in\mathbb{N}}$ converges to $\gamma(a)$ as n^{-4}, $(\beta_n(a))_{n\in\mathbb{N}}$ as n^{-6}, $(\delta_n(a))_{n\in\mathbb{N}}$ as n^{-8}, $(\eta_n(a))_{n\in\mathbb{N}}$ as n^-10, $(\theta_n(a))_{n\in\mathbb{N}}$ as n^{-12}, $(\lambda_n(a))_{n\in\mathbb{N}}$ as n^{-14}, and $(\mu_n(a))_{n\in\mathbb{N}}$ as n^{-16}.

3 Best bounds

Let $(\alpha_n)_{n\in\mathbb{N}}$ be the sequence defined by $\alpha_n = \alpha_n(1)$. In Theorem 2, part (i), we have proved that

$$\lim_{n\to\infty} n^4(\gamma - \alpha_n) = \frac{37}{5760}. \tag{3.1}$$

Proposition 3. *We have*

$$\alpha_n < \alpha_{n+1} < \gamma,$$

for every $n \in \mathbb{N}$.

Proof. We have

$$\alpha_{n+1} - \alpha_n = \frac{1}{n+1} - \ln\frac{24\left(n+\frac{3}{2}\right)^2+1}{n+\frac{3}{2}} + \ln\frac{24\left(n+\frac{1}{2}\right)^2+1}{n+\frac{1}{2}}.$$

Considering the function $h : [1, +\infty) \to \mathbb{R}$, defined by

$$h(x) = \frac{1}{x+1} - \ln\frac{24\left(x+\frac{3}{2}\right)^2+1}{x+\frac{3}{2}} + \ln\frac{24\left(x+\frac{1}{2}\right)^2+1}{x+\frac{1}{2}},$$

and differentiating it, we obtain that

$$\begin{aligned} h'(x) &= -\frac{1}{(x+1)^2} - \frac{48\left(x+\frac{3}{2}\right)}{24\left(x+\frac{3}{2}\right)^2+1} + \frac{1}{x+\frac{3}{2}} + \frac{48\left(x+\frac{1}{2}\right)}{24\left(x+\frac{1}{2}\right)^2+1} - \frac{1}{x+\frac{1}{2}} \\ &= -\frac{296x^2+592x+247}{(x+1)^2(2x+1)(2x+3)(24x^2+24x+7)(24x^2+72x+55)} < 0, \end{aligned}$$

for every $x \in [1, +\infty)$. It follows that the function h is strictly decreasing on $[1, +\infty)$. Also, one can observe that $\lim_{x\to\infty} h(x) = 0$. These imply that $h(x) > 0$, for every $x \in [1, +\infty)$. Therefore $\alpha_{n+1} - \alpha_n > 0$, for every $n \in \mathbb{N}$, i.e. the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is strictly increasing. Because $\lim_{n\to\infty} \alpha_n = \gamma$, we conclude that $\alpha_n < \alpha_{n+1} < \gamma$, for every $n \in \mathbb{N}$. □

Now we give our main result of this section.

Theorem 4. *Let* $c = \sqrt[4]{\frac{37}{5760\left(\gamma - 1 + \ln\frac{55}{36}\right)}}$. *We have*

$$\frac{37}{5760(n+c-1)^4} \le \gamma - \alpha_n < \frac{37}{5760\left(n+\frac{1}{2}\right)^4},$$

for every $n \in \mathbb{N}$. *Moreover, the constants* $c-1$ *and* $\frac{1}{2}$ *are the best possible with this property.*

Proof. Note that h is the function from the proof of Proposition 3. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$u_n = \alpha_n + \frac{37}{5760(n+c-1)^4}.$$

We have

$$u_{n+1} - u_n = \alpha_{n+1} - \alpha_n + \frac{37}{5760(n+c)^4} - \frac{37}{5760(n+c-1)^4}.$$

We consider the function $f : [1, +\infty) \to \mathbb{R}$ defined by

$$f(x) = h(x) + \frac{37}{5760(x+c)^4} - \frac{37}{5760(x+c-1)^4}.$$

Differentiating, we get that

$$\begin{aligned} f'(x) &= h'(x) - \frac{37}{1440(x+c)^5} + \frac{37}{1440(x+c-1)^5} \\ &= -[(x-2)\sum_{k=0}^{10} a_k x^k + a]/[1440(x+1)^2(2x+1)(2x+3) \\ &\quad \times (24x^2+24x+7)(24x^2+72x+55)(x+c)^5(x+c-1)^5], \end{aligned}$$

where

$$\begin{aligned}
a_0 &= 1704960c^{10} + 29131200c^9 + 186501600c^8 + 736070400c^7 + 1853668800c^6 \\
&\quad +3168930240c^5 + 2411292900c^4 - 4850625300c^3 - 16664565450c^2 \\
&\quad -19017183900c - 7986894645, \\
a_1 &= 426240c^{10} + 14918400c^9 + 96991200c^8 + 358070400c^7 + 937152000c^6 \\
&\quad +1579556160c^5 + 1207540385c^4 - 2426894770c^3 - 8330914280c^2 \\
&\quad -9509169335c - 3993353213, \\
a_2 &= 4262400c^9 + 57542400c^8 + 179078400c^7 + 447552000c^6 + 814550400c^5 \\
&\quad +596637000c^4 - 1215797260c^3 - 4162697310c^2 - 4755280360c - 1996652945, \\
a_3 &= 19180800c^8 + 127872000c^7 + 194040000c^6 + 390801600c^5 + 338914540c^4 \\
&\quad -623634680c^3 - 2079474610c^2 - 2376201250c - 998753397, \\
a_4 &= 51148800c^7 + 179020800c^6 + 113500800c^5 + 190996800c^4 \\
&\quad -278703440c^3 - 1054242840c^2 - 1183443900c - 499821050, \\
a_5 &= 89510400c^6 + 161118720c^5 - 5156160c^4 - 81924480c^3 \\
&\quad -518885880c^2 - 593930280c - 248977367, \\
a_6 &= 107412480c^5 + 85248000c^4 - 97693440c^3 \\
&\quad -213589440c^2 - 297681680c - 123649040, \\
a_7 &= 89084160c^4 + 9377280c^3 - 114423840c^2 - 133212000c - 61684628, \\
a_8 &= 49443840c^3 - 23016960c^2 - 61800000c - 28772640, \\
a_9 &= 16623360c^2 - 16623360c - 12999840, \\
a_{10} &= 2557440c - 3836160, \\
a &= 3765600c^{10} + 56484000c^9 + 376560000c^8 + 1468584000c^7 + 3709116000c^6 \\
&\quad +6337504800c^5 + 4822372125c^4 - 9700823250c^3 - 33329558250c^2 \\
&\quad -38034154125c - 15973832025.
\end{aligned}$$

One can verify that $a_i > 0$, $i \in \{0, 1, \ldots, 10\}$ and $a > 0$. It follows that $f'(x) < 0$, for every $x \in [2, +\infty)$. Hence, the function f is strictly decreasing on $[2, +\infty)$. Also, one can see that $\lim_{x\to\infty} f(x) = 0$. From these we obtain that $f(x) > 0$, for every $x \in [2, +\infty)$. So, $u_{n+1} - u_n > 0$, for every $n \geq 2$, i.e. the sequence $(u_n)_{n\geq 2}$ is strictly increasing. Having in view that $\lim_{n\to\infty} u_n = \gamma$, we are able to write that $u_n < \gamma$, for every $n \geq 2$. Consequently,

$$\frac{37}{5760(n+c-1)^4} \leq \gamma - \alpha_n,$$

for every $n \in \mathbb{N}$, and the constant $c - 1$ is the best possible with this property (the equality holds only when $n = 1$).

Let $(v_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$v_n = \alpha_n + \frac{37}{5760\left(n+\frac{1}{2}\right)^4}.$$

Then

$$v_{n+1} - v_n = \alpha_{n+1} - \alpha_n + \frac{37}{5760\left(n+\frac{3}{2}\right)^4} - \frac{37}{5760\left(n+\frac{1}{2}\right)^4}.$$

Differentiating the function $g:[1,+\infty)\to\mathbb{R}$, defined by

$$g(x) = h(x) + \frac{37}{5760\left(x+\frac{3}{2}\right)^4} - \frac{37}{5760\left(x+\frac{1}{2}\right)^4},$$

we obtain that

$$\begin{aligned} g'(x) &= h'(x) - \frac{37}{1440\left(x+\frac{3}{2}\right)^5} + \frac{37}{1440\left(x+\frac{1}{2}\right)^5} \\ &= (4258560x^8 + 34068480x^7 + 117018784x^6 + 225153984x^5 \\ &\quad +265337840x^4 + 196269760x^3 + 89193746x^2 + 22861476x + 2546975) \\ &\quad /[45(x+1)^2(2x+1)^5(2x+3)^5(24x^2+24x+7)(24x^2+72x+55)]. \end{aligned}$$

Thus $g'(x)>0$, for every $x\in[1,+\infty)$. Hereby, the function g is strictly increasing on $[1,+\infty)$. Clearly, $\lim_{x\to\infty} g(x)=0$. These yield $g(x)<0$, for every $x\in[1,+\infty)$. Therefore $v_{n+1}-v_n<0$, for every $n\in\mathbb{N}$, which means that the sequence $(v_n)_{n\in\mathbb{N}}$ is strictly decreasing. Since $\lim_{n\to\infty} v_n=\gamma$, it follows that $\gamma<v_n$, for every $n\in\mathbb{N}$. We can then write that

$$\gamma - \alpha_n < \frac{37}{5760\left(n+\frac{1}{2}\right)^4}, \tag{3.2}$$

for every $n\in\mathbb{N}$. It remains to prove that the constant $\frac{1}{2}$ is the best possible with the property that the above inequality (3.2) holds for every $n\in\mathbb{N}$, and this can be achieved as follows. We have just proved that

$$\sqrt[4]{\frac{37}{5760(\gamma-\alpha_n)}} - n > \frac{1}{2}, \quad n\in\mathbb{N}. \tag{3.3}$$

Using (1.1) and (1.2), we get that

$$\begin{aligned}\gamma-\alpha_n &= \gamma-H_n+\ln\left(n+\frac{1}{2}+\frac{1}{24\left(n+\frac{1}{2}\right)}\right)\\ &= -\psi(n+1)+\ln\left(n+\frac{1}{2}+\frac{1}{24\left(n+\frac{1}{2}\right)}\right)\\ &= -\frac{1}{2n}+\frac{1}{12n^2}-\frac{1}{120n^4}+\frac{1}{252n^6}+O\left(\frac{1}{n^8}\right)+\ln\left(1+\frac{1}{2n}+\frac{1}{24n^2\left(1+\frac{1}{2n}\right)}\right)\\ &= \frac{37}{5760n^4}-\frac{37}{2880n^5}+O\left(\frac{1}{n^6}\right). \end{aligned} \tag{3.4}$$

Let $A_n=\sqrt[4]{\frac{37}{5760n^4(\gamma-\alpha_n)}}$, $n\in\mathbb{N}$. Clearly, $\lim\limits_{n\to\infty}A_n=1$, having in view (3.1). Then, based on (3.4), we have

$$\begin{aligned}\sqrt[4]{\frac{37}{5760(\gamma-\alpha_n)}}-n &= n(A_n-1)\\ &= \frac{n}{A_n^3+A_n^2+A_n+1}\left(\frac{1}{\frac{5760}{37}n^4(\gamma-\alpha_n)}-1\right)\\ &= \frac{n}{A_n^3+A_n^2+A_n+1}\left(\frac{1}{1-\frac{2}{n}+O\left(\frac{1}{n^2}\right)}-1\right)\\ &= \frac{1}{A_n^3+A_n^2+A_n+1}\cdot\frac{2+O\left(\frac{1}{n}\right)}{1-\frac{2}{n}+O\left(\frac{1}{n^2}\right)}\to\frac{1}{4}\cdot 2=\frac{1}{2}\quad (n\to\infty). \end{aligned} \tag{3.5}$$

Indeed, from (3.3) and (3.5) we obtain that $\frac{1}{2}$ is the best constant with the property that the inequality (3.2) holds for every $n\in\mathbb{N}$, and now the proof is complete. □

References

[1] H. Alzer, *Inequalities for the gamma and polygamma functions*, Abh. Math. Semin. Univ. Hamb. **68** (1998), 363–372. MR1658358(99k:33002). Zbl 0945.33003.

[2] C.-P. Chen, *Inequalities for the Euler–Mascheroni constant*, Appl. Math. Lett. **23** (2) (2010), 161–164. MR2559461(2010k:11193). Zbl 1203.26025.

[3] C.-P. Chen, *Monotonicity properties of functions related to the psi function*, Appl. Math. Comput. **217** (7) (2010), 2905–2911. MR2733736. Zbl 1213.33003.

[4] C.-P. Chen, F. Qi, *The best lower and upper bounds of harmonic sequence*, RGMIA **6** (2) (2003), 303–308.

[5] C.-P. Chen, C. Mortici, *New sequence converging towards the Euler–Mascheroni constant*, Comput. Math. Appl. **64** (4) (2012), 391–398. MR2948588. Zbl 1252.33002.

[6] D. W. DeTemple, *A quicker convergence to Euler's constant*, Amer. Math. Monthly **100** (5) (1993), 468–470. MR1215533(94e:11146). Zbl 0858.11068.

[7] O. Furdui, *Limits, Series, and Fractional Part Integrals. Problems in Mathematical Analysis*, Springer, New York, 2013. MR3097674. Zbl 06135851.

[8] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products* (7th ed.), Elsevier/Academic Press, Amsterdam, 2007. MR2360010(2008g:00005). Zbl 1208.65001.

[9] B.-N. Guo, F. Qi, *Sharp bounds for harmonic numbers*, Appl. Math. Comput. **218** (3) (2011), 991–995. MR2831344. Zbl 1229.11027.

[10] J. Havil, *Gamma. Exploring Euler's Constant*, Princeton University Press, Princeton and Oxford, 2003. MR1968276(2004k:11195). Zbl 1023.11001.

[11] K. Knopp, *Theory and Application of Infinite Series*, Blackie & Son Limited, London and Glasgow, 1951. Zbl 0042.29203.

[12] V. Lampret, *A double inequality for a generalized-Euler-constant function*, J. Math. Anal. Appl. **381** (1) (2011), 155–165. MR2796199(2012c:11277). Zbl 1225.65008.

[13] C. Mortici, *New approximations of the gamma function in terms of the digamma function*, Appl. Math. Lett. **23** (1) (2010), 97–100. MR2566111(2010k:33003). Zbl 1183.33003.

[14] C. Mortici, *Improved convergence towards generalized Euler-Mascheroni constant*, Appl. Math. Comput. **215** (9) (2010), 3443–3448. MR2576834. Zbl 1186.11076.

[15] C. Mortici, *On new sequences converging towards the Euler-Mascheroni constant*, Comput. Math. Appl. **59** (8) (2010), 2610–2614. MR2607965(2011b:33005). Zbl 1193.33003.

[16] C. Mortici, *On some Euler-Mascheroni type sequences*, Comput. Math. Appl. **60** (7) (2010), 2009–2014. MR2719721(2011h:40002). Zbl 1205.40006.

[17] F. W. J. Olver (ed.), D. W. Lozier (ed.), R. F. Boisvert (ed.), C. W. Clark (ed.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010. MR2723248(2012a:33001). Zbl 1198.00002.

[18] A. Sîntămărian, *A generalization of Euler's constant*, Numer. Algorithms **46** (2) (2007), 141–151. MR2358246(2009a:11260). Zbl 1130.11075.

[19] A. Sîntămărian, *Some inequalities regarding a generalization of Euler's constant*, J. Inequal. Pure Appl. Math. **9** (2) (2008), 7 pp., Article 46. MR2417328(2009i:11149). Zbl 1195.11168.

[20] A. Sîntămărian, *A Generalization of Euler's Constant*, Editura Mediamira, Cluj-Napoca, 2008. Zbl 1163.26300.

[21] A. Sîntămărian, *Some new sequences that converge to a generalization of Euler's constant*, Creat. Math. Inform. **20** (2) (2011), 191–196. MR2918517(2012m:11181). Zbl 06160785.

[22] L. Tóth, *Problem* E 3432, Amer. Math. Monthly **98** (3) (1991), 264.

[23] L. Tóth, *Problem* E 3432 (*Solution*), Amer. Math. Monthly **99** (7) (1992), 684–685.

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