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THE NECESSARY AND SUFFICIENT CONDITIONS OF THE SHEAF OPTIMIZATION PROBLEM

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Abstract. The purpose of this paper is to introdue the sheaf optimization problem (SOP) and find the necessary and sufficient conditionss of SOP.

1 Introduction

In this paper, we will present the sheaf optimization problem (SOP) and find the necessary and sufficient conditionss of SOP in $\mathbb{R}^n$.

Most of the results about properties and comparison of the sheaf solutions can be found in ([1]-[4]). The problems of sheaf differential equation are still open.

2 Preliminaries

In n-dimension Euclidian space $\mathbb{R}^n$, we have considered the control systems (CS):

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)) \tag{2.1}$$

where $x : [0, T] \to \mathbb{R}^n$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A solution to (2.1) is $x(t) = x(t, t_0, x_0, u)$ which as:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s))ds \tag{2.2}$$

$x_0 \in H_0 \subset \mathbb{R}^n$, $u(t) \in U$ admissible controls, U is the unbounded and closed set in $\mathbb{R}^n$, $t \in [0, T] \subset R^+$.

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Definition 1. *A state pair* (x_0, x_1) *of solutions of control systems (2.1) will be a controllable if after time* t_1 *we shall find a control* $u(t) \in U$ *such that:*

$$x(t_1) = x(t_0, x_0, t_1, u(t_1)) = x_1 \tag{2.3}$$

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Definition 2. *A control system (2.1) is said to be:*

(GC) - *global controllable if every state pair of set solution* $(x_0, x_1) \in \mathbb{R}^n$.

(GA) - *global achievable if for every* $x_1 \in \mathbb{R}^n$ *we have a state pair of solution* $(0, x_1)$ *that is GC.*

(GAZ) - *global achievable to zero if for every* $x_1 \in \mathbb{R}^n$ *we have a state pair* $(x_1, 0) \in \mathbb{R}^n$ *that will be controlable.*

In [2] the authors have study the comparison problems of sheaf solutions for set control differential equations (SCDEs).

In [4] the author has study the problems (GC), (GA) and (GAZ) for set control differential equations (SCDEs).

In [6] the authors have study the same problems (GC), (GA) and (GAZ) for fuzzy set control differential equations (FSCDEs).

Definition 3. *The sheaf solution (or sheaf trajectory)* H_t *is denoted by a number of solutions that make into sheaves (lung one on top of the other and often tied toghether):*

$$H_{t,u} = \{x(t) = x(t_0, x_0, t, u(t)) \,|\, x_0 \in H_0, t \in [t_0, T], u(t) \in U\}\,. \tag{2.4}$$

Assume that at time $t \in [t_0, T]$, $u(0) = 0$, $x(0) = x_0$ for two admissible controls $u(t)$, $\overline{u}(t) \in U$ we have two form of sheaf solutions:

$$\begin{aligned} H_{t,u} &= \{x(t) = x(t_0, x_0, t, u(t)) \,|\, x_0 \in H_0, t \in [t_0, T], u(t) \in U\} \\ H_{t,\overline{u}} &= \{\overline{x}(t) = x(t_0, x_0, t, \overline{u}(t)) | x_0 \in H_0, t \in [t_0, T], \overline{u}(t) \in U\} \end{aligned} \tag{2.5}$$

where $x(t) = x(t, x_0, t, u(t))$- solution of CS (2.1) (see Fig.1)

Definition 4. *The Hausdorff distance between set* $H_{t,u}$ *and* $H_{t,\overline{u}}$ *is denoted by:*

$$d_H\left(H_{t,u}, H_{t,\overline{u}}\right) = \max\left\{\sup_{x(t)\in H_{t,u}} d(x(t), H_{t,\overline{u}}), \sup_{\overline{x}(t)\in H_{t,\overline{u}}} d(\overline{x}(t), H_{t,u})\right\}.$$

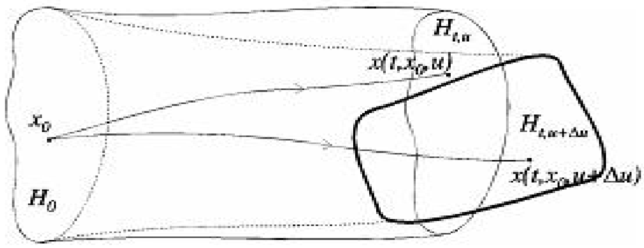

Figure 1: *The sheaf solutions of control systems (2.1) in two admissible controls* $u(t)$ and $\bar{u}(t) = u(t) + \Delta u$).

Definition 5. *The pair of sheaf solutions* H_0, $H_1 \subset \mathbb{R}^n$ *will be controllable if after time* t_1 *we shall find a control* $u(t) \in U$ *and one map* $\sigma : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ *such that:*

$$\sigma(H_0, u(t_1)) = H_1 \tag{2.6}$$

Theorem 6. *(See [5]) If every state pair* (x_0, x_1) *belongs to solutions of CS (2.1) then the sheaf solution of CS (2.1) is controllable.*

Definition 7. *The control system (2.1) is said to be:*

(SC1) sheaf controllable in type 1, if for any set $H_1 \subset \mathbb{R}^n$ *such that the pair of sheaf solutions* H_0, H_1 *is controllable.*

(SC2) sheaf controllable in type 2, if for all $\epsilon > 0$*, there exists* $\delta(\epsilon) > 0$ *such that* $x_0 \in H_0,\ \overline{x}_0 \in \overline{H}_0$ *with* $d_H(H_0, \overline{H}_0) < \delta$ *then*

$$d_H(H_{t,u},\, \overline{H}_{t,u}) < \epsilon. \tag{2.7}$$

(SC3) sheaf controllable in type 3, if for all $\epsilon > 0$*, there exists* $\delta(\epsilon) > 0$ *such that* $x_0 \in H_0,\ \overline{x}_0 \in \overline{H}_0$ *with* $d_H(H_0, \overline{H}_0) < \delta$ *then*

$$d_H(H_{t,u},\, \overline{H}_{t,\overline{u}}) < \epsilon$$

or

$$d_H(H_{t,\overline{u}},\, \overline{H}_{t,u}) < \epsilon. \tag{2.8}$$

Lemma 8. *(See [2]) If control system (2.1) with* $\|f(t, x(t), u(t))\| \leqslant c(t, \|x(t)\|)$ *for all* $\epsilon > 0$*, there exists* $\delta(\epsilon) > 0$ *such that:* $d(x(t_1),\ x(t_2)) < \delta$ *then*

$$d_H(H_{t_1,u(t_1)}, H_{t_2,u(t_2)}) < \epsilon \tag{2.9}$$

Theorem 9. *(See [5]) Assume that $f(t, x(t), u(t))$ in CS (2.1) satisfy:*

$$\|f(t, \overline{x}(t), \overline{u}(t)) - f(t, x(t), u(t))\| \leqslant L\left(1 + \|\overline{x}(t) - x(t)\| + \|\overline{u}(t) - u(t)\|\right) \quad (2.10)$$

then the control systems CS (2.1) is sheaf-controllable in type 1 (SC1).

Corollary 10. *If CS (2.1) is SC1, the right hand side $f(t, x(t), u(t))$ satisfies condition of Lemma (10) then for all $\epsilon > 0$ there exists $t_1 \in I$ such that:*

$$|t_1 - t_0| < \delta, d_H(H_0, H_1) < \epsilon$$

Theorem 11. *(See [5]) Assume that $x_0 \in H_0$, $\overline{x}_0 \in \overline{H}_0$ for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that: $d_H(H_0, \overline{H}_0) < \delta$ then*

$$d_H\left(H_{t,u}, \overline{H}_{t,u}\right) < \epsilon$$

that means CS (2.1) is sheaf controllable in type 2 (CS2).

Theorem 12. *(See [5]) Assume that for all $\epsilon > 0, \exists\, \delta(\epsilon) > 0$ we have:*

(i) $u(t), \overline{u}(t) \in U$, $\|\Delta u\| < \delta$

(ii) H_0, $\overline{H}_0 \in Q : d_H(H_0, \overline{H}_0) < \delta$

(iii) $H_{t,u(t)}$, $\overline{H}_{t,\overline{u}(t)}$ *sheaf solutions CS (2.1) is SC1 and SC2 then control system CS (2.1) is sheaf - controllable in type 3 (SC3).*

Theorem 13. *(See [5]) Assume that for CS (2.1) a right hand side $f \in C(I \times \mathbb{R}^n \times \mathbb{R}^d, \mathbb{R}^n)$ satisfies:*

$$\|f\left(t, \overline{x}(t), \overline{u}(t)\right) - f\left(t, x(t), u(t)\right)\| \leqslant c(t)\left[\|\Delta x\| + \|\Delta u\|\right] \quad (2.11)$$

where $c(t)$ infinite integrable on $I = [0, T]$. We have for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$:

(a) If $u(t), \overline{u}(t) \in U : \|\Delta u\| < \delta$ then CS (2.1) is SC1.

(b) If $x_0 \in H_0, \overline{x}_0 \in \overline{H}_0$ with $d_H(H_0, \overline{H}_0) < \delta$ then CS (2.1) is SC2.

(c) The control system (2.1) with (2.11) and (a), (b) is SC3.

3 The necessary and sufficient conditions of the sheaf optimization problem

Let's consider again the control systems (CS):

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)), \tag{3.1}$$

$x(0) = x_0 \in H_0$, where $t \in J = [0, T] \subset \mathbb{R}^+$, $x(t) \in Q$, Q is the open set in $\subset R^n$, $u(t) \in U \subset R^n$ - admissible controls. Assume that for CS (3.1) there exist solution $x(t) = x(t, t_0, x_0, u)$ and sheaf solution $H_{t,u}$.

Definition 14. *We say that for control system (3.1) is given: SOP - the Sheaf Optimization Problem, if it denotes:*

$$\begin{cases} I(u) = \int\limits_0^T \int\int\limits_{H_{t,u}} \varphi(t, x(t), u(t))dxdt + h(x(T)) \to \min \\ \frac{dx(t)}{dt} = f(t, x(t), u(t)) \\ H_{t,u} = \{x(t_0, x_0, t, u(t))\,|x_0 \in H_0, u \in U, t \in [t_0, T] \subset \mathbb{R}^+\}, \end{cases} \tag{3.2}$$

where $h : \mathbb{R}^n \to \mathbb{R}, \varphi : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *are integrable continuous functions.*

We consider the Hamilton - Jacobi - Bellman's (HJB) partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) + V(t, x) \div f(t, x, u) + \varphi(t, x, u) = 0 \tag{3.3}$$

with bundary condition $\int\int\limits_{H_{T,u}} V(T, x)dx(T) = h(x(T)), V(0, x) = V(0, x_0)$.

Lemma 15. *Assume that* $V(t, x)$ *is a solution of HJB (3.3) with the boundary conditions* $\int\int\limits_{H_{T,u}} V(T, x)dx(T) = h(x(T)), V(0, x) = V(0, x_0)$, *if function* $W(t, x, u) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) + V(t, x) \div f(t, x, u) + \varphi(t, x, u) \geq 0$ *and* $u(t)$ *is admissible control then for SOP (3.2) there exists estimate:*

$$I(u) - \iint\limits_{H_{t,u}} V(0, x_0)dx_0 = \int\limits_0^T \iint\limits_{H_{t,u}} W(t, x(t), u(t))dxdt.$$

Proof. Putting $P(t,u) = \iint\limits_{H_{t,u}} V(t,x)dx + \int\limits_0^t \iint\limits_{H_{t,u}} \varphi(t,x(t),u(t))dxdt$, (*) we have

$$\begin{aligned}
\frac{d}{dt}P(t,u) &= \iint\limits_{H_{t,u}} \left[\frac{d}{dt}V(t,x).dx + V(t,x).\frac{d}{dt}dx\right] + \iint\limits_{H_{t,u}} \varphi(t,x(t),u(t))dx \\
&= \iint\limits_{H_{t,u}} \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}.f + V. \div f + \varphi(t,x(t),u(t))\right] dx \\
&= \iint\limits_{H_{t,u}} W(t,x(t),u(t,x(t)))dx,
\end{aligned}$$

where $\div f.dx = \frac{d}{dt}dx$, then

$$\begin{aligned}
P(T,u) - P(0,u) &= \int\limits_0^T \frac{dP}{dt}dt \\
&= \int\limits_0^T \iint\limits_{H_{t,u}} W(t,x(t),u(t,x(t)))dxdt(**)
\end{aligned}$$

By (*) we have

$$\begin{aligned}
P(T,u) &= \iint\limits_{H_{T,u}} V(T,x)dx(T) + \int\limits_0^T \iint\limits_{H_{t,u}} \varphi(t,x,u)dxdt \\
&= h(x(T)) + \int\limits_0^T \iint\limits_{H_{t,u}} \varphi(t,x,u)dxdt \\
&= I(u)
\end{aligned}$$

and $P(0,u) = \iint\limits_{H_0} V(0,x_0)dx_0$ then (**) impilies that

$$I(u) - \iint\limits_{H_0} V(0,x_0)dx_0 = \int\limits_0^T \iint\limits_{H_{t,u}} W(t,x(t),u(t,x(t)))dxdt.$$

□

Theorem 16. *(Necessary conditions) If SOP (3.2) has solution, that means there exists optimal control* $u*(t)$ *such that* $I(u^*) = \min\limits_{u(t)\in U} I(u)$ *and* $V(t,x)$ *is a solution of HJB (3.3), then the necessary conditions for SOP (3.2) are:*

(i) $\iint\limits_{H_{t,u}} V(T,x)dx(T) = h(x(T))$

(ii) $W(t,x^*,u^*) = 0$,
where $W(t,x,u) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) + V(t,x) \div f(t,x,u) + \varphi(t,x,u) \geq 0.$

Proof. Suppose that the SOP (3.2) has a solution, that means $I(u^*) = \min\limits_{u(t)\in U} I(u)$. Because $V(t,x)$ - solution of HJB (3.3):

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) + V. \div f(t,x,u) + \varphi(t,x,u) = 0$$

with $\int\int\limits_{H_{T,u}} V(T,x)dx(T) = h(x(T))$, if function $W(t,x,u)$ satisfies:

$$W(t,x,u) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) + V. \div f(t,x,u) + \varphi(t,x,u) \geq 0$$

that integrable on sheaf solutions $H_{t,u}$. By Lemma 15, if $u(t)$ is admissible control then for optimization control problem SOP (3.2) there exists estimate:

$$I(u) = \iint\limits_{H_0} V(0,x_0)dx_0 + \int\limits_0^T \iint\limits_{H_{t,u}} W(t,x(t),u(t))dxdt.$$

Assume that for SOP (3.2) has optimal control $u^*(t)$ then for all $t \in [0,T]$, we have $W(t,x^*,u^*) = 0$. □

Theorem 17. *(Sufficient conditions) Assume that* u *- any admissible control for SOP (3.2) and* $V(t,x)$ *solution of HJB (3.3).*

If satisfy the followings:

(i) $\int\int\limits_{H_{T,u}} V(T,x)dx(T) = h(x(T))$

(ii) $W(t,x,u) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) + V. \div f(t,x,u) + \varphi(t,x,u) \geq 0$

(iii) there exists u^* *such that* $I(u^*) = \iint\limits_{H_0} V(0,x_0)dx_0$

then u^* *is optimal control for SOP (3.2) .*

Proof. Suppose that any admissible control $u(t) \in U \subset R^d$, and solution of HJB (3.3) with $\int\int\limits_{H_{T,u}} V(T,x)dx(T) = h(x(T))$ such that for SOP (3.2) we have

$$I(u) = \iint\limits_{H_0} V(0,x_0)dx_0 + \int\limits_0^T \iint\limits_{H_{t,u}} W(t,x(t),u(t))dxdt$$

By condition (iii) of this theorem implies that:

$$I(u^*) - \iint\limits_{H_0} V(0, x_0)dx_0 = \int\limits_0^T \iint\limits_{H_{t,u}} W\left(t, x^*\left(t\right), u^*(t)\right) dxdt = 0$$

and implies that $u^*\left(t\right)$ - optimal control for SOP (3.2). □

References

[1] A. D. Ovsyannikov, *Mathematical methods in sheaf controls* (In Russian), Leningrad University Pub., 1980.

[2] N. D. Phu and T.T. Tung, *Some properties of sheaf-solutions of sheaf fuzzy control Problems*, Electronic Journal of Differential Equations Vol (2006), N. 108, 1-8. MR2255223. Zbl 1110.93031.

[3] N. D. Phu and T.T. Tung, *Some results on sheaf solutions of sheaf set control problem*, J. Nonlinear Analysis: TMA, **9** (2007), 1309 - 1315. MR2323280. Zbl 1113.49003.

[4] N. D. Phu, *On the Global Controllable for Set Control Differential Equations*, International Journal of Evolution Equations, **4** (3) (2009), 281-292. MR2654508. Zbl 1195.93062.

[5] N. D. Phu and P. V. Tri, *Some kinds of sheaf control problems for control systems*, J. Applied Mathematics **3** (2012), 39-44. MR2904668.

[6] N. D. Phu and L. Q. Dung, *On the Stability and Controllability of Fuzzy Set Control Differential Equations*, International Journal of Reliability and Safety **5** No 3–4 (2011), 320-335.

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