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ON THE REGULARITY OF MILD SOLUTIONS TO COMPLETE HIGHER ORDER DIFFERENTIAL EQUATIONS ON BANACH SPACES.

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Abstract. For the complete higher order differential equation $u^{(n)}(t) = \Sigma_{k=0}^{n-1} A_k u^{(k)}(t) + f(t)$, $t \in R$ (*) on a Banach space E, we give a new definition of mild solutions of (*). We then characterize the regular admissibility of a translation invariant subspace $\mathcal{M}$ of $BUC(R, E)$ with respect to (*) in terms of solvability of the operator equation $\Sigma_{j=0}^{n-1} A_j X \mathcal{D}^j - X\mathcal{D}^n = C$. As application, almost periodicity of mild solutions of (*) is proved.

1 Introduction

The qualitative theory of mild solutions on the whole line of the higher order differential equation of the type

$$u^{(n)}(t) = Au(t) + f(t), \qquad t \in R, \tag{1.1}$$

where A is a closed operator on a Banach space E, has been of increasing interest in the last years. When $n = 1$ and A generates a C_0-semigroup $(T(t))_{t\geq 0}$, the mild solution of (1.1) is defined by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau, \quad t \geq s. \tag{1.2}$$

The qualitative behavior of mild solution (1.2) has been intensively investigated by many authors (see [7], [10], [13], [15], [19] and references therein). For second order differential equation $u''(t) = Au(t) + f(t)$ with A generating a cosine family $(C(t))$, the mild solution is then defined by

$$u(t) = C(t-s)u(s) + S(t-s)u'(s) + \int_s^t S(t-\tau)f(\tau)d\tau, \tag{1.3}$$

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where $S(t)$ is the associate sine family. The qualitative properties of mild solution (1.3) have also been studied in [8] and [14].

Recently, Schweiker [17] and Vu Quoc Phong and Schuler [16] studied the first and second order differential equation, in which A is not the generator of a C_0-semigroup or of a cosine family (respectively). Although their definitions of mild solutions are different, they all showed that the existence and uniqueness of mild solutions, which belong to a subspace $\mathcal{M}$ of $BUC(R,E)$, are closely related to the solvability of the operator equation of the form

$$AX - X\mathcal{D} = -\delta_0.$$

Here $\mathcal{D}$ is the differential operator in $\mathcal{M}$ and δ_0 is the Dirac operator defined by $\delta_0(f) := f(0)$. On the other hand, in [2], Arendt and Batty showed the existence of almost periodic mild solution to second order differential equation by using a different way. In [9], the author extended those results to higher order differential equations.

Unfortunately, for the complete higher differential equations, we have had little consideration about the regularity of their solutions, mainly because of the complexity of the structure of the equation. In this paper, we consider the complete higher order differential equation

$$u^{(n)}(t) = \sum_{j=0}^{n-1} A_j u^{(j)}(t) + f(t) \quad t \in R, \tag{1.4}$$

where A_j $(j = 0, 1, 2, ..., n-1)$ are closed linear operator on E and f is a continuous function from R to E. First, we give a general definition of mild solutions to Equation (1.4). Several properties of mild solutions are then shown in Section 2.

In Section 3, we consider the conditions for the solvability of operator equation of the form $B(\sum_{j=0}^{n-1} B_j X D^j) - XD^n = C$, in particular, when $D = \mathcal{D}$, the differential operator on a function space, and $C = -\delta_0$.

Assume that $\mathcal{M}$ is a closed, translation-invariant subspace of $BUC(R,E)$. $\mathcal{M}$ is said to be *regularly admissible* with respect to Equation (1.4), if for every $f \in \mathcal{M}$ Equation (1.4) has a unique mild solution $u \in \mathcal{M}$. In Section 4 we characterize the regular admissibility of $\mathcal{M}$ in terms of solvability of an operator equation. Namely, we show that the subspace $\mathcal{M}$ is regularly admissible if and only if the operator equation of the form

$$B(\sum_{j=0}^{n-1} B_j X\mathcal{D}^j) - X\mathcal{D}^n = C \tag{1.5}$$

has a unique bounded solution. As applications, in Section 5, we prove that if $\sigma(S) \cap iR$ is countable and $\mathcal{F}$ is a certain subspace of $BUC(R)$, then each bounded mild solution of the complete higher order equation is in $\mathcal{F}$ whenever f is in $\mathcal{F}$. The results in this paper extend some well-known results on the regularity of mild solutions of the first and higher order differential equations to the complete higher order differential equations.

Throughout this paper, if not otherwise indicated, we assume that A_i, $i = 0, 1, ..., n-1$, are linear, closed operators on E with the domains $Dom(A_i)$ that satisfy the following condition:

Condition F: *There exists a linear, closed operator B on E with $Dom(B) \subset \cap_{j=0}^{n-1} Dom\ (A_j)$ and $0 \in \varrho(B)$ such that B commutes with A_i and $B^{-1}A_j$ can be extended to bounded operators $B_j = \overline{B^{-1}A_j} = A_jB^{-1}$ for all $j = 0, 1, ..., n-1$.*

Examples:

1) Consider the higher order differential equation

$$u^{(n)}(t) = Au(t) + f(t) \quad t \in R,$$

where A is a closed operator on E with $\varrho(A) \neq \emptyset$. Then A satisfies **Condition F** with $B = (\lambda - A)$, where $\lambda \in \varrho(A)$.

2) Consider the complete, higher order differential equation:

$$\prod_{j=1}^{n}(\frac{d}{dt} - A_j)u(t) = f(t), \quad t \in R,$$

where A_j are closed, commuting operators on E with $\varrho(A_j) \neq \emptyset$. Then A_j satisfy **Condition F** with $B = \prod_{j=1}^{n}(\lambda_j - A_j)$, where $\lambda_j \in \varrho(A_j)$.

For a number $\lambda \in C$, define the operator

$$S(\lambda) = \lambda^n - B(\sum_{j=0}^{n-1} \lambda^j B_j) \tag{1.6}$$

with $Dom(S(\lambda)) = \{x \in E : \sum_{j=0}^{n-1} \lambda^j B_j x \in Dom(B)\}$. It is not hard to see that $\cap_{j=0}^{n-1} Dom(A_j) \subseteq Dom(S(\lambda))$. Moreover, since $B^{-1}S(\lambda)$ is bounded, $S(\lambda)$ is a closed operator. Finally, we define the resolvent $\varrho(S)$ and spectrum $\sigma(S)$ by

$$\varrho(S) := \{\lambda \in C : S(\lambda) \text{ is injective and surjective}\}$$

and

$$\sigma(S) = C \backslash \varrho(S).$$

Since $S(\lambda)$ is a closed operator, if $\lambda \in \varrho(S)$, then $S(\lambda)^{-1}$ is a bounded operator on E.

2 Mild Solutions of Higher Order Differential Equations

Let us fix some notations. By $C^{(n)}(R,E)$ we denote the space of continuous functions with continuous derivatives $u', u'', ...u^{(n)}$, and by $BUC(R,E)$ the space of bounded, uniformly continuous functions with values in E. The operator $I : C(R,E) \to C(R,E)$ is defined by $If(t) := \int_0^t f(s)ds$ and $I^n f := I(I^{n-1}f)$.

(1) A continuous function u is called a mild solution of (1.4), if $\sum_{j=0}^{n-1} B_j I^{n-j}u(t) \in Dom(B)$ and there exist n vectors x_0, x_1, ..., x_{n-1} in E such that

$$u(t) = \sum_{j=0}^{n-1} \frac{t^j}{j!}x_j + B\left(\sum_{j=0}^{n-1} B_j I^{n-j}u(t)\right) + I^n f(t) \tag{2.1}$$

for all $t \in R$.

(2) A function u is a classical solution of (1.4), if u is n-times continuously differentiable, $\sum_{j=0}^{n-1} B_j u^{(j)}u(t) \in Dom(B)$ and

$$u^{(n)}(t) = B\left(\sum_{j=0}^{n-1} B_j u^{(j)}(t)\right) + f(t)$$

holds for $t \in R$.

Remark. Using the standard arguments, we can prove the following.

(i) If a mild solution u is m times differentiable, $0 \le m < n$, then x_j, $j = 0, 1, ..., m$, are the initial values, i.e. $u(0) = x_0$, $u'(0) = x_1$, ..., and $u^{(m)}(0) = x_m$.

(ii) If u is a bounded mild solution of (1.4) corresponding to a bounded inhomogeneity f and $\phi \in L^1(R,E)$ then $u * \phi$ is a mild solution of (1.4) corresponding to $f * \phi$.

The mild solution to (1.4) defined by (2.1) is really an extension of classical solution in the sense that every classical solution is a mild solution and conversely, if a mild solution is n-times continuously differentiable, then it is a classical solution. That statement is actually contained in the following lemma. For the sake of simplicity, for $j < 0$, we denote $I^j u(t) := u^{(j)}(t)$, the j^{th} derivative of $u(t)$.

Lemma 1. *Suppose m is an integer with $0 \le m \le n$ and u is a mild solution of (1.4), which is m-times continuously differentiable. Then $\sum_{j=0}^{n-1} B_j I^{n-m-j}u(t) \in D(B)$ and*

$$u^{(m)}(t) = \sum_{j=m}^{n-1} \frac{t^{j-m}}{(j-m)!}x_j + B\left(\sum_{j=0}^{n-1} B_j I^{n-m-j}u(t)\right) + I^{n-m}f(t). \tag{2.2}$$

Proof. If $m=0$, then (2.2) coincides with (2.1). We prove for $m=1$: Let

$$v(t) := B\Big(\sum_{j=0}^{n-1} B_j I^{n-j}u(t)\Big) = u(t) - \sum_{j=0}^{n-1}\frac{t^j}{j!}x_j - I^n f(t).$$

By the assumptions, v is continuously differentiable and

$$v'(t) = u'(t) - \sum_{j=1}^{n-1}\frac{t^{j-1}}{(j-1)!}x_j - I^{n-1}f(t).$$

Let $h>0$ and put

$$v_h := \sum_{j=0}^{n-1} B_j \frac{1}{h}\int_t^{t+h} I^{n-j-1}u(s)ds.$$

Then $v_h \to \sum_{j=0}^{n-1} B_j(I^{n-j-1}u)(t)$ for $h\to 0$ and

$$\begin{aligned} Bv_h &= B\sum_{j=0}^{n-1}\frac{1}{h}\Big(B_j\int_0^{t+h} I^{n-j-1}u(s)ds - B_j\int_0^{t} I^{n-j-1}u(s)ds\Big) \\ &= \frac{1}{h}\Big(B\sum_{j=0}^{n-1}B_j\int_0^{t+h} I^{n-j-1}u(s)ds - B\sum_{j=0}^{n-1}B_j\int_0^{t} I^{n-j-1}u(s)ds\Big) \\ &= \frac{1}{h}\Big(B\sum_{j=0}^{n-1}B_jI^{n-j}u(t+h) - B\sum_{j=0}^{n-1}B_jI^{n-j}u(t)\Big) \\ &= \frac{1}{h}(v(t+h)-v(t)) \\ &\to v'(t) \text{ for } h\to 0. \end{aligned}$$

Since B is a closed operator, we obtain that $\sum_{j=0}^{n-1} B_j(I^{n-j-1}u)(t)\in Dom(B)$ and

$$B\sum_{j=0}^{n-1}B_j(I^{n-j-1}u)(t) = u'(t) - \sum_{j=1}^{n-1}\frac{t^{j-1}}{(j-1)!}x_j - I^{n-1}f(t),$$

from which (2.2) with $m=1$ follows. If $m>1$, we obtain (2.2) by repeating the above process $(m-1)$ times. $\square$

In particular, if f is continuous and the mild solution u is n-times continuously differentiable, i.e. $m=n$, then (2.2) becomes $u^{(n)}(t) = B\sum_{j=0}^{n-1}B_jI^{-j}u(t)+f(t) = B\sum_{j=0}^{n-1}B_ju^{(j)}(t)+f(t)$, which means u is a classical solution of (1.4).

In the following we consider the spectrum of mild solutions of (1.4). For a bounded function $u \in L^\infty(R,E)$, *the Carleman transform* $\hat{u}$ of u is defined by

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t}u(t)dt \text{ for } Re(\lambda) > 0, \\ -\int_{-\infty}^0 e^{-\lambda t}u(t)dt \text{ for } Re(\lambda) < 0. \end{cases} \tag{2.3}$$

It is clear that $\hat{u}$ is holomorphic on $C \setminus iR$. A point $\mu \in R$ is called a *regular point* if $\hat{u}$ has a holomorphic extension in a neighborhood of $i\mu$. The spectrum of u is defined as follows

$$sp(u) = \{\mu \in R : \mu \text{ is not regular }\}$$

The following lemma, whose proof can be found in [5] and [11], will be needed later.

Lemma 2. *Let f, g be in $BUC(R,E)$ and $\phi \in L^1(R,E)$. Then*

- *(i) $sp(f)$ is closed and $sp(f) = \emptyset$ if and only if $f = 0$.*
- *(ii) $sp(f+g) \subset sp(f) \cup sp(g)$.*
- *(iii) $sp(f * \phi) \subset sp(f) \cap supp\mathcal{F}\phi$, where $\mathcal{F}\phi$ is the Fourier transform of ϕ.*

The following lemma is the first result about the spectrum of mild solutions of Equation (1.4).

Lemma 3. *Let f be a bounded continuous function and u be a bounded mild solution of (1.4). Then*

$$sp(u) \subseteq \{\mu \in R : i\mu \in \sigma(S)\} \cup sp(f).$$

Proof. It is easy to see that $\widehat{Iu}(\lambda) = \frac{1}{\lambda}\hat{u}(\lambda)$, hence $\widehat{I^n u}(\lambda) = \frac{1}{\lambda^n}\hat{u}(\lambda)$. Taking the Carleman transform on both sides of Equation (2.1) we have

$$\hat{u}(\lambda) = Q(\lambda) + B\sum_{j=0}^{n-1} B_j \frac{\hat{u}(\lambda)}{\lambda^{n-j}} + \frac{1}{\lambda^n}\hat{f}(\lambda), \tag{2.4}$$

where $Q(\lambda) = \int_0^\infty e^{-\lambda t}(\sum\limits_{i=0}^{n-1} \frac{t^i}{i!}x_i)dt = \sum\limits_{i=0}^{n-1} x_i/\lambda^{i+1}$. From Equation (2.4) we obtain

$$S(\lambda)\hat{u}(\lambda) = \lambda^n Q(\lambda) + \hat{f}(\lambda)$$

for $\lambda \notin iR$. Hence, for $\lambda \in \varrho(S)$ we have

$$\hat{u}(\lambda) = S(\lambda)^{-1}(\lambda^n Q(\lambda) + \hat{f}(\lambda)).$$

Note that $\lambda^n Q(\lambda)$ is a holomorphic function in terms of λ. It implies that if $\mu \in R$ is a regular point of f and $i\mu \in \varrho(S)$, then $\hat{u}$ has holomorphic extension in a neighborhood of $i\mu$, i.e. μ is a regular point of u. Hence we have the inclusive relation. □

From Lemma 3, it directly follows.

Corollary 4. *If u is a bounded mild solution of (1.4) corresponding to $f \equiv 0$, then $sp(u) \subseteq \{\mu \in R : i\mu \in \sigma(S)\}$*

Corollary 5. *If $iR \cap \sigma(S) = \emptyset$, then (1.4) has at most one bounded mild solution.*

3 The Equation $B(\sum_{j=0}^{n-1} B_j X D^j) - XD^n = C$

Let A and D be closed, generally unbounded, linear operators on Banach spaces E and F, respectively, and let C be a bounded linear operator from E to F. A bounded operator $X : F \to E$ is called a *solution* of the operator equation

$$AX - XD = C \tag{3.1}$$

if for every $f \in Dom(D)$ we have $Xf \in Dom(A)$ and $AXf - XDf = Cf$. Equation (3.1) has been considered by many authors. It was first studied intensively for bounded operators by Daleckii and Krein [3], Rosenblum [12]. For unbounded case, (3.1) was studied in [1], [18], [15] and [19] when A and D are generators of C_0-semigroups, and in [13], [16] when A and D are closed operators.

In this paper, we consider operator equation of the form:

$$B(\sum_{j=0}^{n-1} B_j X D^j) - XD^n = C, \tag{3.2}$$

where B and B_j, $j = 0, 1, ..., n-1$, are defined as in Section 1, D is a closed operator on F and C is a bounded operator from F to E. A bounded operator $X : F \to E$ is called a *solution* of (3.2) if for each $f \in Dom(D^n)$, $\sum_{j=0}^{n-1} B_j X D^j f \in Dom(B)$ and

$$B(\sum_{j=0}^{n-1} B_j X D^j f) - XD^n f = Cf.$$

We have the following results:

Theorem 6. *(i) If Equation (3.2) has a unique bounded solution for every bounded operator C, then $\sigma(S) \cap \sigma(D) = \emptyset$;*

(ii) Suppose D is a bounded operator such that $\sigma(S) \cap \sigma(D) = \emptyset$. Then for every bounded operator C, Equation (3.2) has a unique bounded solution X, which has the following integral form

$$X = -\frac{1}{2\pi i} \int_\Gamma S(\lambda)^{-1} C (\lambda - D)^{-1} d\lambda, \tag{3.3}$$

where Γ is a closed Cauchy contour around $\sigma(D)$ and separated from $\sigma(S)$.

Proof. The proof of (i) is almost the same as the one of ([1, Theorem 2.1]) with little modification, and is omitted.

To prove (ii), let X be as in (3.3). We will show that X is a solution of (3.2). Let j be a positive integer and suppose $f \in Dom(D^j)$, then by a straightforward calculation we have

$$(\lambda - D)^{-1}D^j f = \lambda^j(\lambda - D)^{-1}f - \sum_{k=0}^{j-1}\lambda^k D^{j-k-1}f. \tag{3.4}$$

Using definition (3.3) and identity (3.4) we obtain

$$\begin{aligned}
\sum_{j=0}^{n-1} B_j X D^j f &= -\frac{1}{2\pi i}\sum_{j=0}^{n-1}\int_\Gamma B_j S(\lambda)^{-1}C(\lambda - D)^{-1}D^j f d\lambda) \\
&= -\frac{1}{2\pi i}\sum_{j=0}^{n-1}\int_\Gamma B_j S(\lambda)^{-1}C\Big(\lambda^j(\lambda - D)^{-1}f - \sum_{k=0}^{j-1}\lambda^k D^{j-k-1}f\Big)d\lambda \\
&= -\frac{1}{2\pi i}\sum_{j=0}^{n-1}\Big(\int_\Gamma \lambda^j B_j S(\lambda)^{-1}C(\lambda - D)^{-1}f d\lambda - \sum_{k=0}^{j-1}\int_\Gamma \lambda^k B_j S(\lambda)^{-1}CD^{j-k-1}f d\lambda\Big) \\
&= -\frac{1}{2\pi i}\int_\Gamma\Big(\sum_{j=0}^{n-1}\lambda^j B_j S(\lambda)^{-1}C(\lambda - D)^{-1}f\Big)d\lambda.
\end{aligned}$$

Here we used the fact that $\int_\Gamma \lambda^k S(\lambda)^{-1}CD^{j-k-1}f d\lambda = 0$ for all $k = 0, 1, ..., j$. Note that

$$S(\lambda)^{-1}C(\lambda - D)^{-1}f \in Dom(S(\lambda)).$$

Thus,

$$\sum_{j=0}^{n-1}\lambda^j B_j S(\lambda)^{-1}C(\lambda - D)^{-1}f \in Dom(B)$$

and

$$\begin{aligned}
B(\sum_{j=0}^{n-1}\lambda^j B_j S(\lambda)^{-1}C(\lambda - D)^{-1}f) = \\
&= \lambda^n S(\lambda)^{-1}C(\lambda - D)^{-1}f - S(\lambda)S(\lambda)^{-1}C(\lambda - D)^{-1}f \\
&= \lambda^n S(\lambda)^{-1}C(\lambda - D)^{-1}f - C(\lambda - D)^{-1}f
\end{aligned}$$

is a holomorphic function on $C\backslash(\sigma(D) \cup \sigma(S))$. Hence,

$$\sum_{j=0}^{n-1} B_j X D^j f = -\frac{1}{2\pi i}\int_\Gamma\Big(\sum_{j=0}^{n-1}\lambda^j B_j S(\lambda)^{-1}C(\lambda - D)^{-1}f\Big)d\lambda \in Dom(B)$$

and

$$
\begin{aligned}
&B(\sum_{j=0}^{n-1} B_j X D^j f) - XD^n f = \\
&= -\frac{1}{2\pi i} B \int_\Gamma \Big(\sum_{j=0}^{n-1} \lambda^j B_j S(\lambda)^{-1} C(\lambda - D)^{-1} f \Big) d\lambda + \\
&\qquad\qquad + \frac{1}{2\pi i} \int_\Gamma S(\lambda)^{-1} C(\lambda - D)^{-1} D^n f d\lambda \\
&= -\frac{1}{2\pi i} \int_\Gamma \Big(\lambda^n S(\lambda)^{-1} C(\lambda - D)^{-1} f - C(\lambda - D)^{-1} f \Big) d\lambda + \\
&\quad + \Big(\frac{1}{2\pi i} \int_\Gamma \lambda^n S(\lambda)^{-1} C(\lambda - D)^{-1} f d\lambda - \frac{1}{2\pi i} \sum_{k=0}^{n-1} \int_\Gamma \lambda^k S(\lambda)^{-1} C D^{n-k-1} f d\lambda \Big) \\
&= C \frac{1}{2\pi i} \int_\Gamma (\lambda - D)^{-1} f d\lambda - \frac{1}{2\pi i} \sum_{k=0}^{n-1} \int_\Gamma \lambda^k S(\lambda)^{-1} C D^{n-k-1} f d\lambda \Big) \\
&= Cf,
\end{aligned}
$$

which shows X is an operator solution to (3.2). Here we used again identity (3.4) and the fact that $\int_\Gamma \lambda^k S(\lambda)^{-1} C D^{n-k-1} f d\lambda = 0$ for all $k = 0, 1, ..., n$.

To show the uniqueness of the solution of (3.2), it suffices to show $X = 0$, where X is a solution of

$$B(\sum_{j=0}^{n-1} B_j X D^j) - XD^n = 0. \tag{3.5}$$

Let X be a solution of (3.5). Then for each $f \in Dom(D^n)$ we have

$$XD^n(\lambda - D)^{-1} f = B(\sum_{j=0}^{n-1} B_j X D^j (\lambda - D)^{-1} f). \tag{3.6}$$

Thus,

$$
\begin{aligned}
S(\lambda)^{-1} X D^n (\lambda - D)^{-1} f &= S(\lambda)^{-1} B(\sum_{j=0}^{n-1} B_j X D^j (\lambda - D)^{-1} f) \\
&= B(\sum_{j=0}^{n-1} B_j S(\lambda)^{-1} X D^j (\lambda - D)^{-1} f). \qquad (3.7)
\end{aligned}
$$

Using Identity (3.4) for the left side and the definition of operator B on the right side of (3.7) we have

$$S(\lambda)^{-1} X D^n (\lambda - D)^{-1} f = \lambda^n S(\lambda)^{-1} X (\lambda - D)^{-1} f - \sum_{k=0}^{n-1} \lambda^k S(\lambda)^{-1} X D^{n-k-1} f \tag{3.8}$$

and

$$B\left(\sum_{j=0}^{n-1} \lambda^j B_j S(\lambda)^{-1} X(\lambda - D)^{-1} f\right) = \lambda^n S(\lambda)^{-1} X(\lambda - D)^{-1} f - X(\lambda - D)^{-1} f. \quad (3.9)$$

Comparing (3.8) and (3.9) we have

$$X(\lambda - D)^{-1} f = \sum_{k=0}^{n-1} \lambda^k S(\lambda)^{-1} X D^{n-k-1} f,$$

which implies

$$Xf = \frac{1}{2\pi i} \int_\Gamma X(\lambda - D)^{-1} f d\lambda = \frac{1}{2\pi i} \sum_{k=0}^{n-1} \int_\Gamma \lambda^k S(\lambda)^{-1} X D^{n-k-1} f = 0$$

and hence, $X = 0$. ☐

We now consider the situation when $F = \mathcal{M}$, a translation-invariant subspace of $BUC(R, E)$ and $D = \mathcal{D}_\mathcal{M}$, the restriction of $\mathcal{D}$ to $\mathcal{M}$, where $\mathcal{D} := \frac{d}{dt}$ on $BUC(R, E)$. It is well-known that $\sigma(\mathcal{D}) = iR$ and $\sigma(\mathcal{D}^n) = (\sigma(\mathcal{D}))^n$.

Let now $\mathcal{M}_k := \{f \in \mathcal{M} : sp(f) \subset [-ik, ik]\}$, $k \geq 1$. Then the following properties hold (See [4, 16]).

i) $\mathcal{M}_k$ are translation invariant subspaces,
ii) $\mathcal{M}_k \subset \mathcal{M}_{k+1}$ and
iii) $\mathcal{D}_{\mathcal{M}_k}$ is bounded.

We first need the following Lemma, which was proved in [16].

Lemma 7. $\sigma(\mathcal{D}_\mathcal{M}) = \cup_{k=1}^{\infty} \sigma(\mathcal{D}_{\mathcal{M}_k})$.

We now return to the operator equation

$$B(\sum_{j=0}^{n-1} B_j X \mathcal{D}^j_\mathcal{M}) - X\mathcal{D}^n_\mathcal{M} = \delta_0^\mathcal{M}, \quad (3.10)$$

where $\delta_0^\mathcal{M}$ is the restriction of the Dirac operator to $\mathcal{M}$. Assume that

$$\sigma(S) \cap \sigma(\mathcal{D}_\mathcal{M}) = \emptyset. \quad (3.11)$$

Then it implies $\sigma(S) \cap \sigma(\mathcal{D}_{\mathcal{M}_k}) = \emptyset$. By Theorem 6, the operator equation

$$B(\sum_{j=0}^{n-1} B_j X \mathcal{D}^j_{\mathcal{M}_k}) - X\mathcal{D}^n_{\mathcal{M}_k} = \delta_0^{\mathcal{M}_k}$$

has a unique bounded solution X_k, which is of the form

$$X_k = -\frac{1}{2\pi i}\int_{\Gamma_k} S(\lambda)^{-1}\delta_0^{\mathcal{M}_k}(\lambda - \mathcal{D}_{\mathcal{M}_k})^{-1} d\lambda, \tag{3.12}$$

where Γ_k is a contour around $\sigma(\mathcal{D}_{\mathcal{M}_k})$ and separated from $\sigma(S)$. Moreover, the uniqueness of X_k implies

$$X_k|\mathcal{M}_l = X_l \text{ for } l < k.$$

We state a result about the existence and uniqueness of bounded solutions of Equation (3.10), whose proof is similar to that of [16, Theorem 7], and is omitted.

Theorem 8. *Assume that condition* (3.11) *holds. Then the operator equation* (3.10) *has a unique bounded solution if and only if*

$$sup_{k\geq 1}\|X_k\| < \infty, \tag{3.13}$$

where X_k *are defined by* (3.12).

4 Regularly Admissible Subspaces

Let $\mathcal{M}$ be a closed, translation-invariant subspace of $BUC(R, E)$, which is regularly admissible with respect to Equation (1.4). Define the linear operator G on $\mathcal{M}$ such that for each $f \in \mathcal{M}$, Gf is the unique mild solution of (1.4) in $\mathcal{M}$, we have the following.

Lemma 9. *G is a linear, bounded operator on $\mathcal{M}$.*

Proof. We define operator $\tilde{G} : \mathcal{M} \to \mathcal{M} \otimes E^n$ by

$$\tilde{G}f := (u, x_0, x_1, ..., x_{n-1}),$$

where u is the unique mild solution of (1.4) corresponding to f and $x_0, x_1, ..., x_{n-1}$ are contained in the mild solution

$$u(t) = \sum_{j=0}^{n-1} \frac{t^i}{j!}x_i + B\left(\sum_{j=0}^{n-1} B_j I^{n-j}u(t)\right) + I^n f(t). \tag{4.1}$$

We will show that $\tilde{G}$ is closed. Let $(f_k)_{k\in N} \subseteq \mathcal{M}$ with $\lim_{k\to\infty} f_k = f$ and $\tilde{G}f_k = (u_k, x_{0,k}, ..., x_{n-1,k})$ with $\lim_{k\to\infty} \tilde{G}f_k = (u, x_0, ..., x_{n-1})$, i.e. $\lim_{k\to\infty} u_k = u$ and $\lim_{k\to\infty} x_{j,k} = x_j$ for $j = 0, 1, ..., n-1$. Then we have

$$\lim_{k\to\infty}\sum_{j=0}^{n-1} B_j I^{n-j}u_k(t) = \sum_{j=0}^{n-1} B_j I^{n-j}u(t)$$

and, by Equation (4.1),

$$\begin{aligned}B\left(\sum_{j=0}^{n-1} B_j I^{n-j} u_k(t)\right) &= u_k(t) - \sum_{0}^{n-1} \frac{t^i}{i!} x_{i,k} - I^n f_k(t)\\ &\to u(t) - \sum_{0}^{n-1} \frac{t^i}{i!} x_k - I^n f(t) \text{ as } k \to \infty.\end{aligned}$$

Since B is closed we obtain that $\sum_{j=0}^{n-1} B_j I^{n-j} u(t) \in Dom(B)$ and

$$B\left(\sum_{j=0}^{n-1} B_j I^{n-j} u(t)\right) = u(t) - \sum_{0}^{n-1} \frac{t^i}{i!} x_i - I^n f(t).$$

That means $\tilde{G} f = (u, x_0, x_1, ..., x_{n-1})$. Hence, $\tilde{G}$ is closed and thus bounded. Since $G = \tilde{G} \circ P$, where $P : \mathcal{M} \otimes E^n \to \mathcal{M}$ is the projection on the first coordinate and thus a bounded operator, we obtain that G is bounded. ☐

In the next lemma, we show that G, which is called the solution operator of (1.4), commutes with the translation operator and hence, commutes with the differential operator.

Lemma 10. *Let $\mathcal{M}$ be a regularly admissible subspace of $BUC(R, E)$. Then the following statements hold.*

i) $S_h \cdot G = G \cdot S_h$, where S_h is the translation operator on $\mathcal{M}$.

ii) $\mathcal{D}_{\mathcal{M}} \cdot G = G \cdot \mathcal{D}_{\mathcal{M}}$

Proof. *i)* Let $u = Gf$ be the unique mild solution of equation (1.4). We show that $S_h u$ is the unique mild solution to (1.4) corresponding to $S_h f$. By a short calculation we can show that

$$(I^m S_h u)(t) = (I^m u)(t + h) + \sum_{j=0}^{m-1} t^j v_j,$$

where v_j are certain vectors in E depending only on h. Hence,

$$\begin{aligned}&B\left(\sum_{j=0}^{n-1} B_j I^{n-j} (S_h u)(t)\right) + I^n S_h f(t)\\ &= B(\sum_{j=0}^{n-1} B_j I^{n-j} u(t+h)) + I^n f(t+h) + \sum_{j=0}^{m-1} t^j w_j\\ &= u(t+h) - \sum_{j=0}^{m-1} \frac{(t+h)^j}{j!} x_j + \sum_{j=0}^{m-1} t^j w_j\\ &= S_h u(t) - \sum_{j=0}^{m-1} \frac{t^j}{j!} y_j.\end{aligned}$$

Hence,

$$S_h u(t) = \sum_{j=0}^{m-1} \frac{t^j}{j!} y_j + B\Big(\sum_{j=0}^{n-1} B_j I^{n-j}(S_h u)(t)\Big) + I^n S_h f(t),$$

where w_i and y_j, $i = 0, 1, ..., n-1$ are certain vectors in E. This means $S_h u$ is the mild solution to (1.4) corresponding to $S_h f$. Part *ii)* is a direct consequence of *i)*, and the lemma is proved □

Corollary 11. *Let $\mathcal{M}$ be a regularly admissible subspace of $BUC(R, E)$ and u be the unique mild solution corresponding to f in $\mathcal{M}$. If $f \in C^n(R, E)$ such that $f', f'', ..., f^{(n)}$ belong to $\mathcal{M}$, then u is a classical solution.*

In what follows, we assume that $\mathcal{M}$ satisfies the following additional assumption:

$$\textit{For all } C \in \mathcal{L}(\mathcal{M}, E) \textit{ and } f \in \mathcal{M}, \tag{4.2}$$

$$\textit{the function } \Phi(t) = CS(t)f \textit{ belongs to } \mathcal{M}.$$

The regular admissibility of a space is closely related to the solvability of operator equation (3.1). That relation was shown for higher order differential equations (see [15] when $n = 1$, [17] and [16] when $n = 2$ and [9] for any n). The following theorem is a generalization of those results to complete, higher order differential equations.

Theorem 12. *Let $\mathcal{M}$ be a translation invariant subspace in $BUC(R, E)$, which satisfies the assumption (4.2). Then the following are equivalent.*

(i) $\mathcal{M}$ is a regularly admissible.

(ii) The operator equation

$$B(\sum_{j=0}^{n-1} B_j X \mathcal{D}^j_{\mathcal{M}}) - X\mathcal{D}^n_{\mathcal{M}} = -\delta_0 \tag{4.3}$$

has a unique solution.

(iii) For every bounded operator $C : \mathcal{M} \to E$, the operator equation

$$B(\sum_{j=0}^{n-1} B_j X \mathcal{D}^j_{\mathcal{M}}) - X\mathcal{D}^n_{\mathcal{M}} = C \tag{4.4}$$

has a unique solution.

Proof $(i) \Rightarrow (ii)$. Let $G : \mathcal{M} \to \mathcal{M}$ be the bounded operator defined by $Gf = u$ where u is the unique mild solution in $\mathcal{M}$. We define the operator $X : \mathcal{M} \mapsto E$ by

$$Xf := (Gf)(0).$$

Then X is a bounded operator. Let now $f \in \mathcal{D}^n_{\mathcal{M}}$. By Corollary 11, $u = Gf$ is a classical solution of (1.4), i.e.,

$$(Gf)^{(n)}(t) = B\left(\sum_{j=0}^{n-1} B_j (Gf)^{(j)}(t)\right) + f(t). \tag{4.5}$$

By Lemma 10, $(Gf)^{(j)} = Gf^{(j)}$ for $j = 1, 2, ..., n$. Using that fact when we put $t = 0$ in (4.5), we have

$$B\left(\sum_{j=0}^{n-1} B_j X \mathcal{D}^j_{\mathcal{M}}\right) f - X\mathcal{D}^n_{\mathcal{M}} f = -\delta_0 f$$

for any $f \in \mathcal{D}^n_{\mathcal{M}}$, i.e. X is a bounded solution of (4.3).

To show the uniqueness, we assume that X_0 is a solution of Equation (4.3). Then for every $f \in \mathcal{D}^n_{\mathcal{M}}$, the function $u \in \mathcal{M}$, defined by $u(t) = X_0 S(t) f$, is a classical solution of Equation (1.4). Indeed,

$$\begin{aligned} u^{(n)}(t) &= X_0 \mathcal{D}^n S(t) f \\ &= B\left(\sum_{j=0}^{n-1} B_j X_0 \mathcal{D}^j_{\mathcal{M}} S(t) f\right) + \delta_0 S(t) f \\ &= B\left(\sum_{j=0}^{n-1} B_j u^{(j)}(t)\right) + f(t) \end{aligned}$$

for all $t \in R$. We will show that $u(\cdot) = X_0 S(\cdot) f$ is a mild solution of (1.4) for every $f \in \mathcal{M}$. To this end, let $f \in \mathcal{M}$. Then there exists a sequence $(f_k)_{k\in N} \subset D(\mathcal{D}^n_{\mathcal{M}})$ with $lim_{k\to\infty} f_k = f$. Using the boundedness of operator G we have

$$Gf = lim_{k\to\infty} G f_k = lim_{k\to\infty} X_0 S(\cdot) f_k = X_0 S(\cdot) f,$$

i.e., $u(\cdot) = X_0 S(\cdot) f$ is a mild solution of (1.4).

Assume now that X_1 and X_2 are two solutions of (4.3). Then, for every $f \in \mathcal{M}$, $u = (X_1 - X_2) S(\cdot) f$ is a mild solution of the higher order equation $u^{(n)}(t) = B(\sum_{j=0}^{n-1} B_j u^{(j)}(t))$. By the uniqueness of the mild solution we have $u \equiv 0$, which implies $X_1 = X_2$.

(ii) $\Rightarrow$ *(iii)* Let X be the unique solution of (4.3). Define the bounded operator $Y : \mathcal{M} \to E$ by $Yf := X\overline{f}$, where $\overline{f}(\cdot) := -CS(\cdot) f$. Let $f \in Dom(\mathcal{D}^n_{\mathcal{M}})$, then $\overline{(\mathcal{D}^n_{\mathcal{M}} f)}(t) = -CS(t)\mathcal{D}^n_{\mathcal{M}} f = \mathcal{D}^n_{\mathcal{M}} \overline{f}(t)$. Hence, we have

$$\begin{aligned} B(\sum_{j=0}^{n-1} B_j Y \mathcal{D}^j_{\mathcal{M}} f) &= B(\sum_{j=0}^{n-1} B_j X \mathcal{D}^j_{\mathcal{M}} \tilde{f}) \\ &= X\mathcal{D}^n_{\mathcal{M}} \tilde{f} + \delta_0 \tilde{f} \\ &= Y\mathcal{D}^n_{\mathcal{M}} f + Cf, \end{aligned}$$

i.e. Y is a bounded solution of (4.4).

The uniqueness of the solution of (4.4) follows directly from the uniqueness of the solution of the homogeneous equation $B(\sum_{j=0}^{n-1} B_j X\mathcal{D}^j_{\mathcal{M}}) - X\mathcal{D}^n_{\mathcal{M}} = 0$, which, again, follows from the uniqueness of the solution of (4.3).

$(iii) \Rightarrow (i)$ We have shown above that, if X is a bounded solution of (4.3), then $u(t) := XS(t)f$ is a mild solution of the higher order equation (1.4). It remains to show that this solution is unique. In order to do it, assume that u is a mild solution of the homogeneous equation $u^{(n)}(t) = \sum_{j=0}^{n-1} A_j u^{(j)}(t)$, $t \in R$. By Corollary 4, $isp(u) \subseteq \sigma(S)$. On the other hand, since $u \in \mathcal{M}$, $isp(u) \subseteq \sigma(\mathcal{D}_{\mathcal{M}})$. By Theorem 6(i), it follows from (iii) that $\sigma(S) \cap \sigma(\mathcal{D}_{\mathcal{M}}) = \emptyset$. Hence, $sp(u) = \emptyset$, so $u \equiv 0$ and the theorem is proved. ☐

5 Applications

We now apply the results in Chapter 4 to some function spaces. Let $\mathcal{G}$ be a closed, translation-invariant subspace of $BUC(R, E)$. We define the *reduced spectrum* of a function $u \in \mathcal{G}$ by

$$\begin{aligned} sp_{\mathcal{G}}(u) : \;&=\; \{\lambda \in R : \forall \epsilon > 0 \;\; \exists g \in L^1(R) \text{ such that} \\ &\quad\; \operatorname{supp}\mathcal{F}g \subset (\lambda - \epsilon, \lambda + \epsilon) \text{ and } g * u \notin \mathcal{G}\}. \end{aligned}$$

Theorem 13. *Assume that $f \in \mathcal{G}$ and u is a mild solution of (1.4). Then we have*

$$sp_{\mathcal{G}}(u) \subset iR \cap \sigma(S).$$

Proof. Let λ be any point in R such that $i\lambda \in \varrho(S)$, we will show that $\lambda \notin sp_{\mathcal{G}}(u)$, i.e., there is $\epsilon > 0$ such that for every $\phi \in L^1(R)$ with $\operatorname{supp}\mathcal{F}\phi \subset (\lambda - \epsilon, \lambda + \epsilon)$, the function $\phi * u$ is in $\mathcal{G}$.

Since $\varrho(S)$ is an open set, there exists $\epsilon > 0$ such that $i\Gamma \subset \varrho(S)$, where $\Gamma = [\lambda - \epsilon, \lambda + \epsilon]$. Let $\mathcal{M} = X(\Gamma)$ be the subspace of $BUC(R, E)$ consisting of all functions f with $sp(f) \subset \Gamma$. It is easy to see that $\mathcal{M}$ satisfies condition (4.2). Moreover, $\mathcal{D}_{\mathcal{M}}$ is bounded, $\sigma(\mathcal{D}_{\mathcal{M}}) = i\Gamma$ and $\sigma(S) \cap (i\Gamma) = \emptyset$. Hence, by Theorem 6$(ii)$, the equation $B(\sum_{j=0}^{n-1} B_j X\mathcal{D}^j_{\mathcal{M}}) - X\mathcal{D}^n_{\mathcal{M}} = -\delta_0$ has a unique solution. By Theorem 12, $\mathcal{M}$ is regularly admissible and for any function $\tilde{f} \in \mathcal{M}$, if $\tilde{f} \in \mathcal{G}$, then the mild solution $\tilde{u}(t) = XS(t)\tilde{f}$ is also in $\mathcal{G}$.

Let ϕ be a function in $L^1(R)$ with $\operatorname{supp}\mathcal{F}\phi \subset \Gamma$. Put $\tilde{u} := u * \phi$ and $\tilde{f} := f * \phi$. Then $\tilde{u}$ and $\tilde{f}$ are in $X(\Gamma)$ (due to Lemma 2(iii)) and $\tilde{f}$ is a function in $\mathcal{G}$. Moreover, $\tilde{u}$ is the unique mild solution of (1.4) corresponding to $\tilde{f}$ in $X(\Gamma)$ (due to Remark (iv) in Section 2). Hence, $\tilde{u}$ is also in $\mathcal{G}$, and the theorem is proved. ☐

We apply the above theorem with $\mathcal{G} = AP(R, E)$, the space of all continuous, almost periodic function from R to E. We know that if u is almost periodic, then $sp_{AP}(u)$

is countable, but we do not have the converse implication. The following theorem, which can be found in [6] (part (a) and (b)) and [13] (part (c)), gives conditions for the almost periodicity of a function, if its reduced spectrum is countable.

Theorem 14. *Let $u \in BUC(R, E)$ such that $sp_{AP}(u)$ is countable. Assume that*

(a) $E \not\supseteq c_0$*; or*

(b) The range of $u(t)$ *is weakly relatively compact; or*

(c) u *is totally ergodic.*

Then u *is almost periodic.*

Combining Theorem 13 and Theorem 14 we have

Theorem 15. *For the equation*

$$u^{(n)}(t) = \sum_{j=0}^{n-1} A_j u^{(j)}(t) + f(t), \quad t \in R, \tag{5.1}$$

we assume that f *is almost periodic and* $\sigma(S) \cap (iR)$ *is countable. Let* $u \in BUC(R, E)$ *be a mild solution of Equation (5.1). Then* u *is almost periodic if one of the following conditions is satisfied.*

(a) $E \not\supseteq c_0$*; or*

(b) The range of $u(t)$ *is weakly relatively compact; or*

(c) u *is totally ergodic.*

We can extend the above results to a class of subspaces in $BUC(R, E)$ using a result from [2].

Theorem 16. (c.f.[2, Theorem 3.4]) *Suppose* $\mathcal{F}$ *is a closed, translation-invariant subspace of* $BUC(R, E)$ *satisfying the following conditions:*

(i) $\mathcal{F}$ *contains a constant functions;*

(ii) $\mathcal{F}$ *is invariant by multiplication by* $e^{i\lambda \cdot}$ *for all* $\lambda \in R$*;*

(iii) whenever $f \in \mathcal{F}$ *and* $F(t) = \int_0^t f(s)ds \in BUC(R, E)$*, then* $F \in \mathcal{F}$*;*

Then for each function $u \in BUC(R, E)$ *with* $\sigma_{\mathcal{F}}(u)$ *being countable, we have* $u \in \mathcal{F}$*.*

Combining Theorem 13 and Theorem 16, we have

Theorem 17. *Let* F *be a subspace of* $BUC(R, E)$ *satisfying conditions in Theorem 16. Suppose* $f \in \mathcal{F}$ *and* $iR \cap \sigma(S)$ *is countable. Let* $u \in BUC(R, E)$ *be a mild solution of Equation (5.1). Then* u *is in* $\mathcal{F}$*.*

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