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ALL MAXIMAL IDEMPOTENT SUBMONOIDS OF $Hyp_G(n)$

Weerapong Wongpinit and Sorasak Leeratanavalee

Abstract. A generalized hypersubstitution of type τ is a mapping which maps any operation symbol to the set of all terms of the same type which does not necessarily preserve the arity. The set of all generalized hypersubstitutions together with a binary operation and the identity element forms a monoid. In this paper we determine all maximal idempotent submonoids of this monoid of type $\tau = (n)$.

1 Introduction

The concept of a generalized hypersubstitution is a generalization of the concept of a hypersubstitution introduced by S. Leeratanavalee and K. Denecke. It is used to study strong hyperidentities and strongly solid varieties, respectively (see [2]). A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ from the set of all n_i-ary operation symbols into the set of all terms built up by elements of the alphabet $X := \{x_1, x_2, ...\}$ and operation symbols from $\{f_i \mid i \in I\}$ which does not necessarily preserve the arity.

The set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ is necessary. It is defined by the following steps:

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

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Every generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

A binary operation $\circ_G$ on $Hyp_G(\tau)$ is defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. Then we have the following proposition.

Proposition 1. ([2]) *For arbitrary $t, t_1, t_2, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitution $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \hat{\sigma}[t_2], ..., \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, t_2, ..., t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $(Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid.

In [3], W. Wongpinit and S. Leeratanavalee determined all maximal idempotent submonoids of $Hyp_G(2)$.

Throughout this paper, let f be the binary operation symbol of type $\tau = (n)$. By σ_t we denote the generalized hypersubstitution which maps f to a term $t \in W_{(n)}(X)$. For $t \in W_{(n)}(X)$, we introduce the following notation:

(i) $leftmost(t)$:= the first variable (from the left) occurring in t,

(ii) $rightmost(t)$:= the last variable occurring in t,

(iii) $var(t)$:= the set of all variables occurring in t.

For a type $\tau = (n)$ with an n-ary operation symbol f, $t \in W_{(n)}(X)$ and $1 \leq i \leq n$. An $i - most(t)$ is defined inductively by:

(i) if t is a variable, then $i - most(t) = t$,

(ii) if $t = f(t_1, t_2, \ldots, t_n)$, then $i - most(t) := i - most(t_i)$.

Notice that $1 - most(t) = leftmost(t)$ and $n - most(t) = rightmost(t)$.

Example 2. Let $\tau = (4)$ be a type, $t = f(x_2, x_2, f(x_5, x_8, x_1, x_3), f(x_1, x_4, x_6, x_5))$. Consider $1 - most(t) = x_2$, $2 - most(t) = x_2$, $3 - most(t) = 3 - most(f(x_5, x_8, x_1, x_3))$ $= x_1$ and $4 - most(t) = 4 - most(f(x_1, x_4, x_6, x_5)) = x_5$.

2 Maximal Idempotent Submonoids of $Hyp_G(n)$

Let S be a semigroup. An element $a \in S$ is called idempotent if $aa = a$. We denote the set of all idempotent elements of a semigroup S by $E(S)$. Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} \mid x_i \in X\}$,

$R_2 := \{\sigma_t \mid var(t) \cap X_n = \emptyset\}$,

$R_3 := \{\sigma_t \mid t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}\}$.

In 2010, W. Puninagool and S. Leeratanavalee [3] proved that the set of all idempotent elements in $Hyp_G(n), E(Hyp_G(n)) = \bigcup_{i=1}^{3} R_i$.

Remark 3. *$\bigcup_{i=1}^{3} R_i$ is not closed under $\circ_G$, i.e., $\bigcup_{i=1}^{3} R_i$ is not a subsemigroup of $Hyp_G(n)$.*

Let $\sigma_t \in Hyp_G(n)$, we denote

$R'_3 := \{\sigma_t \mid t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l$ $\}$.

For each $\emptyset \neq I = \{x_{i_1}, ..., x_{i_{m'}}\} \subset X_n$, we denote

$E_I := \{\sigma_t \mid t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\} \subsetneq I$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{i_1, ..., i_{m'}\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \in \{i_1, ..., i_{m'}\} \setminus \{i_l\}\}$,

$F_I := \{\sigma_t \mid t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}, I \subseteq var(t) \cap X_n$ and if $x_{i_l} \notin I, x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l\}$,

$(MI)_{Hyp_G(n)} := R_1 \cup R_2 \cup R'_3$, and

$(MI_I)_{Hyp_G(n)} := R_1 \cup R_2 \cup E_I \cup F_I$.

Lemma 4. *Let $s, t \in W_{(n)}(X)$. If $j - most(t) = x_k \in X_n$ and $k - most(s) = x_i$, then $j - most(\hat{\sigma}_t[s]) = x_i$.*

Proof. We will prove this lemma by induction on complexity of the term s. If $s = x_i \in X$. Then $j - most(\hat{\sigma}_t[s]) = x_i$. Assume that $s = f(s_1, ..., s_n)$ and $j - most(\hat{\sigma}_t[s_k]) = x_i$. Then $j - most(\hat{\sigma}_t[s]) = j - most(S^n(t, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])) = j - most(S^n(x_k, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])) = j - most(\widehat{\sigma}_t[s_k]) = x_i$. □

Proposition 5. *$(MI)_{Hyp_G(n)}$ is an idempotent submonoid of $Hyp_G(n)$.*

Proof. We show that $(MI)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$.

Let $\sigma_t, \sigma_s \in (MI)_{Hyp_G(n)}$.

If $\sigma_t \in R_1$, then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in R_1$.

If $\sigma_t \in R_2$ and $\sigma_s \in (R_2 \cup R_3')$, then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in R_2$.

If $\sigma_t, \sigma_s \in R_3'$, then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, .., x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j{-}most(t_k) \neq x_{i_l}$ for all $j \neq i_l$, and $s = f(s_1, ..., s_n)$ where $s_{q_1} = x_{q_1}, ..., s_{q_{m'}} = x_{q_{m'}}$ for distinct $q_1, ..., q_{m'} \in \{1, ..., n\}$, $var(s) \cap X_n = \{x_{q_1}, .., x_{q_{m'}}\}$ and if $x_{q_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$ and $k' \in \{1, ..., n\} \setminus \{i_1, ..., i_{m'}\}$, then $j' - most(t_{k'}) \neq x_{q_{l'}}$ for all $j' \neq q_{l'}$. Hence

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, ..., s_n)] = S^n(f(t_1, ..., t_n), \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(t_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])$ for all $i \in \{1, ..., n\}$, and

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1, ..., t_n)] = S^n(f(s_1, ..., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = f(w_1, ..., w_n)$$

where $w_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ for all $i \in \{1, ..., n\}$.

Case 1: $var(t_k) \cap X_n = \emptyset$ for all $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$ and $var(s_{k'}) \cap X_n = \emptyset$ for all $k' \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}\}$. Consider $(\sigma_t \circ_G \sigma_s)(f) = f(u_1, ..., u_n)$.

Case 1.1: $i \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$. Then $u_i = S^n(t_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = t_i$.

Case 1.2: $i \in \{i_1, ..., i_m\}$. Then

$$u_i = S^n(t_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = S^n(x_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = \widehat{\sigma}_t[s_i].$$

If $i \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}\}$, then $var(u_i) \cap X_n = \emptyset$. If $i \in \{q_1, ..., q_{m'}\}$, then $u_i = x_i$.

By Case 1.1 and Case 1.2, we get $\sigma_t \circ_G \sigma_s \in (R_2 \cup R_3') \subset (MI)_{Hyp_G(n)}$. In the same way, we can show that $\sigma_s \circ_G \sigma_t \in (R_2 \cup R_3') \subset (MI)_{Hyp_G(n)}$.

Case 2: $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$, for some $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$ and $var(s_{k'}) \cap X_n = \emptyset$ for all $k' \in \{1, .., n\} \setminus \{q_1, ..., q_{m'}\}$.

To show that $\sigma_t \circ_G \sigma_s \in (R_2 \cup R_3')$. Consider $(\sigma_t \circ_G \sigma_s)(f) = f(u_1, ..., u_n)$.

Case 2.1: $i \in \{i_1, ..., i_m\} \setminus \{i_l\}$. Then $u_i = S^n(t_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = S^n(x_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = \widehat{\sigma}_t[s_i]$, we can prove in the same manner as in Case 1.2.

Case 2.2: $i = i_l$. Then

$$u_{i_l} = S^n(t_{i_l}, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = S^n(x_{i_l}, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = \widehat{\sigma}_t[s_{i_l}].$$

If $i_l \in \{q_1, ..., q_{m'}\}$, then $u_{i_l} = x_{i_l}$. Since $u_k = S^n(t_k, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])$ and $x_{i_l} \in var(t_k)$, we have $x_{i_l} \in var(u_k)$. Since $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l$ and $var(s_{k'}) \cap X_n = \emptyset$ for all $k' \in \{1, .., n\} \setminus \{q_1, ..., q_{m'}\}$, we have $j - most(u_k) \neq x_{i_l}$ for all $j \neq i_l$. If $i_l \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}\}$ then $var(u_{i_l}) \cap X_n = \emptyset$ and $var(u_k) \cap X_n = \emptyset$.

Case 2.3: $i \in \{1, ..., n\} \setminus \{i_1, ..., i_m, k\}$. Then $var(u_i) \cap X_n = \emptyset$.

By Case 2.1, Case 2.2 and Case 2.3, we get $\sigma_t \circ_G \sigma_s \in (R_2 \cup R_3') \subset (MI)_{Hyp_G(n)}$.

To show that $\sigma_s \circ_G \sigma_t \in (R_2 \cup R_3')$. Consider $(\sigma_s \circ_G \sigma_t)(f) = f(w_1, ..., w_n)$.

Case 2.4: $i \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}\}$. Since $var(s_i) \cap X_n = \emptyset$, then

$$w_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = s_i.$$

Case 2.5: $i \in \{q_1, ..., q_{m'}\}$. Then

$$w_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(x_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_i].$$

If $i \in \{1, ..., n\} \setminus \{i_1, .., i_m\}$, then $var(w_i) \cap X_n = \emptyset$. If $i \in \{i_1, .., i_m\}$, then $w_i = \widehat{\sigma}_s[t_i] = x_i$.

By Case 2.4 and Case 2.5, we get $\sigma_s \circ_G \sigma_t \in (R_2 \cup R'_3) \subset (MI)_{Hyp_G(n)}$.

Case 3: $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ for some $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$ and $x_{q_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$ for some $k' \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}\}$.

To show that $\sigma_t \circ_G \sigma_s \in (R_2 \cup R'_3)$. Consider $(\sigma_t \circ_G \sigma_s)(f) = f(u_1, ..., u_n)$.

Case 3.1: $i \in \{i_1, ..., i_m\} \setminus \{i_l\}$. Then $u_i = \widehat{\sigma}_t[s_i]$.
If $i \in \{q_1, ..., q_{m'}\}$, then $u_i = x_i$.
If $i = k'$, then $u_i = \widehat{\sigma}_t[s_i]$

If $i_l = q_{l'}$, then $u_{i_l} = x_{i_l}$ and $x_{i_l} \in var(s_{k'})$. Since $j - most(t) \neq x_{i_l}$ for all$j \neq i_l$ and $k - most(s_i) \neq x_{i_l}$ for all $k \neq i_l$, by Lemma 4 $j - most(u_i) \neq x_{i_l}$ for all $j \neq i_l$.

If $i_l \neq q_{l'}$, then $var(s_i) \cap \{x_{i_l}\} = \emptyset$. So $var(u_i) \cap \{x_{i_l}\} = \emptyset$.
If $i \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}, k'\}$ then $var(u_i) \cap X_n = \emptyset$.

Case 3.2: $i = i_l$. Then $u_i = \widehat{\sigma}_t[s_i]$.
If $i \in \{q_1, ..., q_{m'}\}$, then $u_i = x_i$. Since $u_k = S^n(t_k, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])$ and $x_i \in var(t_k)$, we have $x_i \in var(u_k)$.

If $j - most(t_k) \notin X_n$, then $j - most(u_k) \notin X_n$ for all $j \neq i_l$.

If $j - most(t_k) = x_{k'} \in X_n$, then $j - most(u_k) = j - most(\widehat{\sigma}_t[s_{k'}])$. Since $j - most(t) \neq x_{i_l}$ for all $j \neq i_l$ and $j - most(s_{k'}) \neq x_{i_l}$ for all $j \neq i_l$, by Lemma 4 we have $j - most(u_k) \neq x_{i_l}$ for all $j \neq i_l$.
If $i_l \in \{1, ..., n\} \setminus \{q_1, ..., q_{m'}\}$, then $var(u_{i_l}) \cap X_n = \emptyset$ and $var(u_k) \cap X_n = \emptyset$.

Case 3.3: $i \in \{1, ..., n\} \setminus \{i_1, ..., i_m, k\}$. Then $var(u_i) \cap X_n = \emptyset$.

By Case 3.1, Case 3.2 and Case 3.3, we get $\sigma_t \circ_G \sigma_s \in (R_2 \cup R'_3) \subset (MI)_{Hyp_G(n)}$. Therefore $(MI)_{Hyp_G(n)}$ is closed under $\circ_G$. □

Proposition 6. *$(MI)_{Hyp_G(n)}$ is a maximal idempotent submonoid of $Hyp_G(n)$.*

Proof. Let K be a proper idempotent submonoid of $Hyp_G(n)$ such that $(MI)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. If $\sigma_t \in K$ where $\sigma_t \in R_3 \setminus R'_3$, then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for some $i_1, ..., \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) = x_{i_l}$ for some $j \neq i_l$. We can choose $\sigma_s \in R'_3$ where $s = f(s_1, ..., s_n)$ such that $s_j = x_j, s_k = x_k$ and $var(s_{i_l}) \cap X_n = \emptyset$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma_s}[f(t_1, ..., t_n)] = S^n(f(s_1, .., s_n), \widehat{\sigma_s}[t_1], ..., \widehat{\sigma_s}[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(s_i, \widehat{\sigma_s}[t_1], ..., \widehat{\sigma_s}[t_n])$ for all $i \in \{1, ..., n\}$. Then

$$u_{i_l} = S^n(s_{i_l}, \widehat{\sigma_s}[t_1], ..., \widehat{\sigma_s}[t_n]) = s_{i_l}.$$

Consider $u_k = S^n(s_k, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(x_k, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_k]$. Since $j - most(s) = x_j$ and $j - most(t_k) = x_{i_l}$, by Lemma 4 we get $x_{i_l} \in var(u_k)$ and $var(u_{i_l}) \cap X_n = \emptyset$. Hence $\sigma_s \circ_G \sigma_t$ is not an idempotent element in $Hyp_G(n)$ which is a contradiction. So $\sigma_t \in R'_3$. Therefore $K = (MI)_{Hyp_G(n)}$. □

Proposition 7. *For each $\emptyset \neq I = \{x_{i_1}, ..., x_{i_{m'}}\} \subset X_n$, $(MI_I)_{Hyp_G(n)}$ is an idempotent submonoid of $Hyp_G(n)$.*

Proof. We show that $(MI_I)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$. Let $\sigma_t, \sigma_s \in (MI_I)_{Hyp_G(n)}$.

If $\sigma_t \in R_1$, then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in R_1 \subset (MI_I)_{Hyp_G(n)}$.

If $\sigma_t \in R_2$ and $\sigma_s \in (R_2 \cup E_I \cup F_I)$, then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in R_2 \subset (MI_I)_{Hyp_G(n)}$.

If $\sigma_t, \sigma_s \in E_I$, then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\} \subsetneq I$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{i_1, ..., i_{m'}\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \in \{i_1, ..., i_{m'}\} \setminus \{i_l\}$, and $s = f(s_1, ..., s_n)$ where $s_{q_1} = x_{q_1}, ..., s_{q_{m'}} = x_{q_{m'}}$ for distinct $q_1, ..., q_{m'} \in \{1, ..., n\}$, $var(s) \cap X_n = \{x_{q_1}, .., x_{q_{m'}}\} \subsetneq I$ and if $x_{q_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$ and $k' \in \{i_1, ..., i_{m'}\} \setminus \{q_1, ..., q_{m'}\}$, then $j - most(s_{k'}) \neq x_{q_{l'}}$ for all $j \in \{i_1, ..., i_{m'}\} \setminus \{q_{l'}\}$. To show that $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in E_I$, we can prove in the same manner as in Case 1 to Case 3 of Proposition 5.

If $\sigma_t \in E_I$ and $\sigma_s \in F_I$, then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\} \subsetneq I$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{i_1, ..., i_{m'}\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \in \{i_1, ..., i_{m'}\} \setminus \{i_l\}$ and $s = f(s_1, ..., s_n)$ where $s_{q_1} = x_{q_1}, ..., s_{q_{m'}} = x_{q_{m'}}$ for distinct $q_1, ..., q_{m'} \in \{1, ..., n\}$, $I \subseteq var(s) \cap X_n = \{x_{q_1}, .., x_{q_{m'}}\}$ and if $x_{q_{l'}} \notin I, x_{q_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$ and $k' \in \{1, ..., n\} \setminus \{i_1, ..., i_{m'}\}$, then $j' - most(t_{k'}) \neq x_{q_{l'}}$ for all $j' \neq q_{l'}$.

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, ..., s_n)] = S^n(f(t_1, ..., t_n), \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = f(t_1, ..., t_n),$$

and

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1, ..., t_n)] = S^n(f(s_1, ..., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = f(w_1, ..., w_n)$$

where $w_i = S^n(x_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ and $var(f(w_1, ..., w_n)) \cap X_n \subsetneq I$. To show that $\sigma_s \circ_G \sigma_t \in E_I$, we can prove in the same manner as in Case 1 to Case 3 of Proposition 5.

If $\sigma_t, \sigma_s \in F_I$, then $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for distinct $i_1, ..., i_m \in \{1, ..., n\}$, $I \subseteq var(t) \cap X_n = \{x_{i_1}, .., x_{i_m}\}$ and if $x_{i_l} \notin I$, $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq q_l$ and $s = f(s_1, ..., s_n)$ where $s_{q_1} = x_{q_1}, ..., s_{q_{m'}} = x_{q_{m'}}$ for distinct $q_1, ..., q_{m'} \in \{1, ..., n\}$, $I \subseteq var(s) \cap X_n = \{x_{q_1}, .., x_{q_{m'}}\}$ and if $x_{q_{l'}} \notin I$, $x_{q_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$ and $k' \in \{1, ..., n\} \setminus \{i_1, ..., i_{m'}\}$, then $j' - most(t'_k) \neq x_{q_{l'}}$ for all $j' \neq q_{l'}$.

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, ..., s_n)] = S^n(f(t_1, ..., t_n), \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(t_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])$ and then $I \subseteq var(f(u_1, ..., u_n)) \cap X_n$, and

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1, ..., t_n)] = S^n(f(s_1, ..., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = f(w_1, ..., w_n)$$

where $w_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ and then $I \subseteq var(f(w_1, ..., w_n)) \cap X_n$ and $I \subseteq var(f(u_1, ..., u_n)) \cap X_n$. To show that $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in F_I$, we can prove in the same manner as in Case 1 to Case 3 of Proposition 5. Therefore $(MI_I)_{Hyp_G(n)}$ is closed under $\circ_G$. Hence $(MI_I)_{Hyp_G(n)}$ is a submonoid of $Hyp_G(n)$. □

Proposition 8. *For each $\emptyset \neq I \subset X_n$, $(MI_I)_{Hyp_G(n)}$ is a maximal idempotent submonoid of $Hyp_G(n)$.*

Proof. Let K be a proper idempotent submonoid of $Hyp_G(n)$ such that $(MI_I)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$ where $\sigma_t \in R_3 \backslash E_I \cup F_I$. Suppose $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$, $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$. If $I \nsubseteq var(t) \cap X_n$, there exists $x_{i_j} \in I$ such that $x_{i_j} \notin var(t) \cap X_n$.

If $x_{i_k} = t_{i_j} \in var(t) \cap X_n \setminus \{x_{i_j}\}$, then choose $\sigma_s \in F_I$ where $s = f(s_1, ..., s_n)$ such that $s_{i_k} \cap X_n = \emptyset$ and $s_{i_j} = x_{i_j}$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1, ..., t_n)] = S^n(f(s_1, .., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ for all $i \in \{1, ..., n\}$. Then

$$u_{i_k} = S^n(s_{i_k}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \emptyset \text{ and } u_{i_j} = S^n(x_{i_j}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_{i_j}] = x_{i_k},$$

which is a contradiction.

If $t_{i_j} \notin var(t) \cap X_n$, then choose $\sigma_s \in E_I$ where $s = f(s_1, ..., s_n)$ such that $s_{i_j} = s_{i_l} = x_{i_j}$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, ..., s_n)] = S^n(f(t_1, .., t_n), \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(t_i, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n])$ for all $i \in \{1, ..., n\}$. Then

$$u_{i_j} = S^n(t_{i_j}, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) \neq x_{i_j} \text{ and}$$
$$u_{i_l} = S^n(x_{i_l}, \widehat{\sigma}_t[s_1], ..., \widehat{\sigma}_t[s_n]) = \widehat{\sigma}_t[s_{i_l}] = x_{i_j},.$$

which is a contradiction. If $I \subseteq var(t) \cap X_n = \{x_{i_1}, .., x_{i_m}\}$ and if $x_{i_l} \notin I$, $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) = x_{i_l}$ for some $j \neq q_l$. We can choose $\sigma_s \in E_I$ where $s = f(s_1, ..., s_n)$ such that $s_j = x_j, s_k = x_k$ and $var(s_{i_l}) \cap X_n = \emptyset$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t_1, ..., t_n)] = S^n(f(s_1, .., s_n), \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = f(u_1, ..., u_n)$$

where $u_i = S^n(s_i, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n])$ for all $i \in \{1, ..., n\}$. Then

$$u_{i_l} = S^n(s_{i_l}, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = s_{i_l}.$$

Consider

$$u_k = S^n(s_k, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = S^n(x_k, \widehat{\sigma}_s[t_1], ..., \widehat{\sigma}_s[t_n]) = \widehat{\sigma}_s[t_k].$$

Since $j - most(s) = x_j$ and $j - most(t_k) = x_{i_l}$, by Lemma 4 we get $x_{i_l} \in var(u_k)$ and $var(u_{i_l}) \cap X_n = \emptyset$ which is a contradiction. If $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\} \subsetneq I$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{i_1, ..., i_{m'}\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) = x_{i_l}$ for some $j \in \{i_1, ..., i_{m'}\} \setminus \{i_l\}$, we can prove the same manner case $I \subseteq var(t) \cap X_n = \{x_{i_1}, .., x_{i_m}\}$ and if $x_{i_l} \notin I$, $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$, then $j - most(t_k) = x_{i_l}$ for some $j \neq q_l$. Hence $\sigma_s \circ_G \sigma_t$ is not an idempotent element in $Hyp_G(n)$ which is a contradiction. So $\sigma_t \in E_I \cup F_I$. Therefore $K = (MI_I)_{Hyp_G(n)}$. □

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Weerapong Wongpinit
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai 50200, Thailand.
E-mail: sivaree_sudsanit@hotmail.com

Sorasak Leeratanavalee
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai 50200, Thailand.
E-mail: sorasak.l@cmu.ac.th

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