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THE LEVI PROBLEM IN $\mathbb{C}^n$: A SURVEY

Harry J. Slatyer

Abstract. We discuss domains of holomorphy and several notions of pseudoconvexity (drawing parallels with the corresponding concepts from geometric convexity), and present a mostly self-contained solution to the Levi problem. We restrict our attention to domains of $\mathbb{C}^n$.

1 Introduction

1.1 Content

The purpose of this paper is to provide a clear, thorough and reasonably self-contained exposition of a solution to the so-called Levi problem, together with a comprehensive and detailed survey of some related classical concepts. We motivate the definitions of holomorphic convexity and pseudoconvexity by first discussing the analogous notions from geometric convexity, with the intention of providing the reader with a sound intuitive understanding of these concepts. We include almost all of the interesting proofs in full generality, without any additional smoothness or boundedness assumptions. The only major exception is the proof of Hörmander's theorem on solvability of the inhomogeneous Cauchy-Riemann equations [10], which we omit in view of length restrictions and the fact that many self-contained proofs may be found elsewhere (see [12, chapter 4], for example). This paper is targeted towards both professional mathematicians wishing to broaden their knowledge of complex analysis and geometry, and also students specialising in these areas.

Given a holomorphic function on some domain (a connected open set), it is natural to ask whether there exists a holomorphic function defined on a larger domain which agrees with the original function on its domain – that is, we seek a *holomorphic extension* of the original function. In some cases the original domain can be such that *any* holomorphic function necessarily admits a holomorphic extension to a strictly larger domain. For instance, in 1906 Hartogs showed that any function holomorphic on a domain of $\mathbb{C}^n$ (with $n \geq 2$) obtained by removing a compact set from another domain extends to a function holomorphic on the larger domain [9] –

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note the contrast to the 1-dimensional case, where there exist holomorphic functions with compactly contained singularities ($z \mapsto z^{-1}$, for example). This type of result begs the following question: which domains have the property that any holomorphic function defined on these domains necessarily admits a holomorphic extension to a larger domain? Traditionally we actually pose the inverse problem, and ask on which domains there exists a holomorphic function that *does not* extend holomorphically to points outside the domain. Such a domain is known as a *domain of holomorphy*, because it is the most natural domain of existence of some holomorphic function. One may verify that every domain of the complex plane is a domain of holomorphy – that is, given any domain in $\mathbb{C}$ there exists a function holomorphic on that domain which does not extend to a holomorphic function on any larger domain. Upon passing to multiple complex variables however, this is no longer the case (as shown by Hartogs' theorem on removal of compact singularities, for example).

Unfortunately it is typically difficult to verify directly from the definition whether a given domain is a domain of holomorphy, so it is desirable to obtain a more easily verified equivalent condition. Such a condition is provided by the solution to the so-called Levi problem. Named for Levi's pioneering work in his 1911 paper [15], the Levi problem is to show that the domains of holomorphy are precisely the *pseudoconvex* domains. Pseudoconvexity, a local property of domains which generalises the notion of convexity, is typically more easy to directly verify than whether a domain is a domain of holomorphy. For domains with twice-differentiable boundaries there is an equivalent notion of pseudoconvexity, known as *Levi pseudoconvexity*, which is particularly simple to verify for many domains [14, 15]. It is relatively easy to show that domains of holomorphy are pseudoconvex, and that for domains with twice-differentiable boundaries Levi pseudoconvexity is equivalent to pseudoconvexity. The problem of showing that the (Levi) pseudoconvex domains are domains of holomorphy, which completes the solution to the Levi problem, is much more difficult – the condition of Levi pseudoconvexity was introduced in 1910–1911 [14, 15], and was not proved to be sufficient for a domain to be a domain of holomorphy until 1942 when Oka demonstrated the fact for 2-dimensional space [20]. For arbitrary dimensions the result was not obtained until 1953–1954, when Bremermann [5], Norguet [19] and Oka [21] all presented independent proofs.

In this paper we discuss the above concepts in detail and present a solution to the Levi problem. We use definitions and methods from Boas [3], Hörmander [11], Krantz [12], Range [22], Shabat [23] and Vladimirov [26] to show that domains of holomorphy are pseudoconvex, and that the various types of pseudoconvex domains are identical. To show that pseudoconvex domains are domains of holomorphy we follow the general method of Oka [21], which is to show first that strictly pseudoconvex domains are domains of holomorphy and then use the Behnke-Stein theorem [1] together with the fact that pseudoconvex domains may be approximated by an increasing sequence of strictly pseudoconvex domains [13, 21]. While Oka shows the first step using his principle for joining two domains of holomorphy, we instead follow the argument of Boas [3] and apply Hörmander's theorem on solvability of the inhomogeneous Cauchy-Riemann equations [10].

1.2 Notation

Before commencing in earnest our discussion of the Levi problem we list some notation and terminology that will occur frequently throughout the paper:

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$ – the sets of real numbers, complex numbers, positive integers and non-negative integers

$\Re(z)$, $\Im(z)$ – the real and imaginary parts of $z \in \mathbb{C}$

$|x|$, $d(x, y)$, $x \cdot y$ – the Euclidean norm, metric and dot product for $x, y \in \mathbb{R}^n$

$|z|$, $\|z\|$, $d(z, w)$, $\rho(z, w)$, $\langle x, y\rangle$ – the Euclidean norm, L^∞ norm, Euclidean metric, L^∞ metric and scalar product for $z, w \in \mathbb{C}^n$

$B(x, r)$, $B(z, r)$, $\Delta(z, s)$ – the ball of radius $r > 0$ about $x \in \mathbb{R}^n$, about $z \in \mathbb{C}^n$ (with respect to the Euclidean metric), and the polydisc of (vector) radius s about z

$S^c, \overline{S}, \partial S, \mathring{S}, S_{(r)}$ – for a subset S of some metric space, the complement, closure, boundary, interior and r-enlargement (where $r > 0$) of S (note that for subsets of $\mathbb{C}^n$, enlargements are always with respect to the L^∞ metric)

$d(x, S)$, $d(T, S)$ – for a point x and subsets S and T of some space with metric $d(\cdot, \cdot)$, the distance from x to S ($\inf_{s\in S}\{d(x, s)\}$) and the distance from T to S ($\inf_{t\in T}\{d(t, S)\}$)

Neighbourhood of a point or set – an open set containing that point or set

$\|f\|_B$ – for a function $f : A \to \mathbb{K}$ (where $\mathbb{K}$ is a normed space) and subset $B \subset A$, the supremum of the norm of f on B (note that for functions mapping into $\mathbb{R}$ or $\mathbb{C}$, the Euclidean norm is used)

$H(C)$ – the set of functions holomorphic on a set $C \subset \mathbb{C}^n$ (note that for a set which is not open, a function is holomorphic on that set if it is holomorphic on some neighbourhood of the set)

C^k, $C^k_{\mathbb{C}}$ – the classes of k-times continuously differentiable real-valued functions and k-times continuously differentiable complex valued functions (when regarded as maps between real spaces)

f', ∇g – the derivative (gradient) of $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : V \subset \mathbb{C}^n \to \mathbb{C}$

$\Delta_\delta f(z)$ – the Levi form of $f : U \subset \mathbb{C}^n \to \mathbb{R}$ (see Proposition 55)

2 Convexity

As described in Section 1, the domains of holomorphy are precisely the domains which satisfy any of several more general analogues of geometric convexity. In this preliminary section we present several definitions of geometric convexity in the interests of motivating the generalised definitions which will appear later in the exposition. We omit the proofs in this section since they are neither particularly difficult nor relevant.

2.1 Convex sets

We first give the usual definition:

Definition 1. *A set $S \subset \mathbb{R}^n$ is **convex** if for every pair of points $x, y \in S$ the segment $\{x + t(y - x) : t \in [0, 1]\}$ is a subset of S.*

For example, open and closed balls in $\mathbb{R}^n$ are convex. It is immediate from the definition that the intersection of a collection of convex sets is convex, and that the union of an increasing sequence of convex sets (that is, a sequence $\{S_j\}_{j \geq 1}$ such that $S_j \subset S_{j+1}$ for all $j \geq 1$) is convex. A less trivial property which will be useful later is the following:

Proposition 2. *Let $U \subset \mathbb{R}^n$ be open and convex. Then for all $a \in \partial U$ there exists a supporting hyperplane P for U containing a (that is, P is a hyperplane such that $a \in P$ and $P \cap U = \emptyset$).*

Another intuitive fact is that convexity of a domain is determined locally at the boundary:

Proposition 3. *Let $U \subset \mathbb{R}^n$ be a domain. Then U is convex if and only if for each $a \in \partial U$ there is a neighbourhood $V \subset \mathbb{R}^n$ of a such that $U \cap V$ is convex.*

2.2 Convex hulls

We will find that convexity of a domain may be determined by considering the convex hulls of its compact subsets, and this fact will allow us to generalise convexity in Section 3.

Definition 4. *Let $C \subset \mathbb{R}^n$ be closed. The **convex hull** of C is the intersection of all closed convex subsets of $\mathbb{R}^n$ containing C, and is denoted $\hat{C}_L$.*

Proposition 5. *Let $U \subset \mathbb{R}^n$ be open. Then U is convex if and only if for all compact $K \subset U$ the convex hull $\hat{K}_L$ is a subset of U.*

Remark 6. *Basic topology, together with the observation that convex hulls of compact sets are compact and connected, implies that the condition $\hat{K}_L \subset U$ in the above proposition is equivalent to requiring that $\hat{K}_L \cap U$ be compact.*

To use this fact to generalise convexity we will need a different description of the convex hull of a compact set. Notice that the level sets of affine functions $\mathbb{R}^n \to \mathbb{R}$ are the hyperplanes, and the pullbacks of intervals of the form $(-\infty, t]$ and $[t, \infty)$ (where $t \in \mathbb{R}$) are the closed half-spaces. Along with the observation that the convex hull of a closed set is the intersection of all closed half-spaces containing the set, this yields:

Proposition 7. *Let $K \subset \mathbb{R}^n$ be compact. Then the convex hull $\hat{K}_L$ is given by*

$$\hat{K}_L = \{x \in \mathbb{R}^n \colon |A(x)| \leq \|A\|_K \text{ for all affine } A \colon \mathbb{R}^n \to \mathbb{R}\}.$$

With the remark following Proposition 5 in mind this implies the following, which will motivate the definition of holomorphic convexity in Section 3:

Corollary 8. *Let $U \subset \mathbb{R}^n$ be open. Then U is convex if and only if for all compact $K \subset U$,*

$$\{x \in U \colon |A(x)| \leq \|A\|_K \text{ for all affine } A \colon \mathbb{R}^n \to \mathbb{R}\} = \hat{K}_L \cap U$$

is a compact set.

2.3 The continuity principle

In this subsection we introduce the "continuity principle" which, roughly speaking, states that when a sequence of line segments converges to some limit set, and the limits of the sequences of endpoints are in the domain, then the entire limit set is also in the domain.

To state the continuity principle precisely we require some notation and a definition. If $L \subset \mathbb{R}^n$ is a closed line segment we let ∂L denote the set consisting of its two endpoints (rather than the topological boundary).

Definition 9. *Let $\{S_j\}_{j\geq 1}$ be a sequence of subsets of a metric space $\mathbb{K}$ and let $S \subset \mathbb{K}$. Then $\{S_j\}_{j\geq 1}$ **converges to** S if for all $\epsilon > 0$ there exists $J \geq 1$ such that whenever $j \geq J$ we have $S_j \subset S_{(\epsilon)}$ and $S \subset (S_j)_{(\epsilon)}$ (where the subscript ϵ is for the ϵ-dilation). In this case we write $S_j \to S$ or $\lim_{j\to\infty} S_j = S$.*

Now we can define the continuity principle formally:

Definition 10. *Let $U \subset \mathbb{R}^n$ be a domain. Then U **satisfies the continuity principle** if, for every sequence of closed line segments $\{L_j\}_{j\geq 1}$ satisfying $L_j \subset U$ for all $j \geq 1$, $\partial L_j \to B$ and $L_j \to L$ where $B \subset U$ and $L \subset \mathbb{R}^n$ are compact, we have $L \subset U$.*

Proposition 11. *A domain $U \subset \mathbb{R}^n$ is convex if and only if it satisfies the continuity principle.*

2.4 Convex exhaustion functions

In this subsection we observe that the convex domains are those on which there is a convex function (to be defined shortly) whose sublevel sets are all compact.

Definition 12. *Let $S \subset \mathbb{K}$ (where $\mathbb{K}$ is a normed space) be a set. A function $f\colon S \to \mathbb{R}$ is an **exhaustion function** for S if for all $r \in \mathbb{R}$, $f^{-1}((-\infty, r])$ is compact.*

It is obvious from the definition that continuous exhaustion functions for domains are those which tend to ∞ at each boundary point:

Proposition 13. *Let $U \subset \mathbb{K}$ (where $\mathbb{K}$ is a normed space) be a domain and $f\colon U \to \mathbb{R}$ a continuous function. Then f is an exhaustion function for U if and only if whenever $\{x_j\}_{j\geq 1} \subset U$ is a sequence with $x_j \to a \in \partial U$ or $x_j \to \infty$ (that is, $|x_j| \to +\infty$) we have $f(x_j) \to +\infty$.*

Example 14. *Let $U \subset \mathbb{C}^n$ be a domain. Let $f\colon U \to \mathbb{R}$ be given by $f(z) := |z|^2$ if $\partial U = \emptyset$ and $f(z) := |z|^2 - \ln d(z, \partial U)$ otherwise (where $d(\cdot,\cdot)$ denotes the Euclidean metric). We claim that f is an exhaustion function for U. If $\partial U = \emptyset$ this is trivial, so consider the case when $\partial U \neq \emptyset$. If $z_j \to a \in \partial U$ then $\{|z_j|^2\}_{j\geq 1}$ is bounded and $\ln d(z_j, \partial U) \to -\infty$, so $f(z_j) \to +\infty$. If $z_j \to \infty$ then for sufficiently large j we have $d(z_j, \partial U) \leq 2|z_j|$ and thus $f(z_j) \geq |z_j|^2 - \ln|z_j| - \ln 2$, which implies $f(z_j) \to +\infty$.*

Thus every domain admits an exhaustion function. To characterise convexity in terms of the existence of such functions we introduce a particular class of functions:

Definition 15. *A function $f\colon U \subset \mathbb{R} \to \mathbb{R}$ (where U is an open interval) is **convex** if for all $a, b \in U$ with $a < b$ and $x \in [a, b]$ we have $f(x) \leq f(a) + (f(b) - f(a))\frac{x-a}{b-a}$. A function $g\colon V \subset \mathbb{R}^n \to \mathbb{R}$ (where V is a domain) is **convex** if for all $a \in V$ and $\delta \in \mathbb{R}^n$ with $|\delta| = 1$ the function $x \mapsto g(a + \delta x)$ is convex on each component of $\{x \in \mathbb{R}\colon a + \delta x \in V\}$.*

The convex functions of one variable are those which satisfy the property that for any two points on the graph of the function, the graph lies below the straight line between those points. The convex functions of several variables are those whose restriction to any line segment is convex. It may be verified that convex functions are continuous (see [26, page 85]).

For twice-differentiable functions we have another condition for convexity:

Proposition 16. *Let $U \subset \mathbb{R}^n$ be a domain and $f\colon U \to \mathbb{R}$ a C^2 function. Then f is convex if and only if for all $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ and $x \in U$ we have*

$$\Delta_\delta f(x) := \sum_{j,k=1}^{n} \left.\frac{\partial^2 f}{\partial x_j \partial x_k}\right|_x \delta_j \delta_k \geq 0.$$

This result implies the following, which motivates the definition of pseudoconvexity in Section 4:

Proposition 17. *A domain $U \subset \mathbb{R}^n$ is convex if and only if there exists a convex exhaustion function for U.*

2.5 Convexity of domains with twice-differentiable boundaries

Until now we have considered arbitrary domains of $\mathbb{R}^n$, but for those domains with twice-differentiable boundaries we have an additional local characterisation of convexity.

Definition 18. *Let $U \subset \mathbb{R}^n$ be a domain and $a \in \partial U$. If $V \subset \mathbb{R}^n$ is a neighbourhood of a and $f : V \to \mathbb{R}$ is such that $U \cap V = f^{-1}((-\infty, 0))$ then f is a **local defining function** for U at a.*

*Let $k \geq 1$ (we allow $k = \infty$). The domain U is said to have k**-times differentiable boundary** or C^k **boundary** if at each boundary point there is a C^k local defining function whose gradient is non-zero at the point. If $a \in \partial U$ and $f : V \to \mathbb{R}$ is a C^k local defining function for U at a then the **tangent space** to ∂U at a with respect to f is $T_a(f) := \{\delta \in \mathbb{R}^n : \delta \cdot f'(a) = 0\}$.*

Remark 19. *By the implicit function theorem a k-times differentiable boundary is locally given by the graph of a real-valued C^k function. More precisely, consider $a \in \mathbb{R}^n$ and a C^k function $f : V \to \mathbb{R}$ (where V is a neighbourhood of a) which is zero at a and non-degenerate, and assume (without loss of generality) that $\frac{\partial f}{\partial x_n}\big|_a > 0$. By the implicit function theorem there are connected open sets $W \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$ such that $a \in W \times I \subset V$ and a C^k function $g : W \to I$ such that $f(x) = 0$ if and only if $x_n = g(x_1, \ldots, x_{n-1})$ for $x \in W \times I$. Moreover, using the fact that f is differentiable, if $x_n > g(x_1, \ldots, x_{n-1})$ then $f(x) > 0$ and if $x_n < g(x_1, \ldots, x_{n-1})$ then $f(x) < 0$. In particular, if f is a defining function for a domain U at $a \in \partial U$ then after replacing V with the subset $W \times I$ we have $U \cap V = f^{-1}((-\infty, 0)) = \{x \in V : x_n < g(x_1, \ldots, x_{n-1})\}$, which implies*

$$\partial U \cap V = \{x \in V : x_n = g(x_1, \ldots, x_{n-1})\} = f^{-1}(\{0\}). \tag{2.1}$$

It is also clear that $x \mapsto x_n - g(x_1, \ldots, x_{n-1})$ is a C^k local defining function for U at a. We will use these observations in Section 4.

One may verify that convexity of a domain with twice-differentiable boundary is determined by the curvatures of its local defining functions:

Proposition 20. *Let $U \subset \mathbb{R}^n$ be a domain with C^2 boundary. Then U is convex if and only if for all $a \in \partial U$ there is a C^2 local defining function $f : V \to \mathbb{R}$ (where V is a neighbourhood of a) such that $f'(a) \neq 0$ and $\Delta_\delta f(a) \geq 0$ for all $\delta \in T_a(f)$.*

3 Domains of holomorphy

In this section we define domains of holomorphy, discuss some examples and conditions, and then introduce the notion of holomorphic convexity and show that the domains of holomorphy are precisely the holomorphically convex domains. Using this we will prove several properties of domains of holomorphy. The material in this section is drawn from [12], [22] and [23].

3.1 Domains of holomorphy

We must first make precise the notion of holomorphic extension:

Definition 21. *Let $U \subset \mathbb{C}^n$ be open, $f \in H(U)$ and $a \in U^c$ (where $H(U)$ denotes the set of holomorphic functions on U). Then f* ***extends holomorphically*** *to a if there is a connected open neighbourhood V of a and a holomorphic function $g \in H(V)$ such that $f \equiv g$ on a non-empty open subset of $U \cap V$.*

Note that we do *not* require that the functions f and g agree on all of $U \cap V$, so it may not be the case that f extends to a function holomorphic on $U \cup V$. That is, if $f \in H(U)$ extends holomorphically to $a \in U^c$ then there is not necessarily a domain $W \subset \mathbb{C}^n$ and function $g \in H(W)$ such that $U \subset W$, $a \in W$ and $f \equiv g$ on U. We use this definition of holomorphic extension to avoid the need for multi-valued extensions. For example, the function $z \mapsto \sqrt{z}$ defined on the domain $U := \mathbb{C} \setminus \{x \colon x \in \mathbb{R}, x \geq 0\}$ extends holomorphically to the point $z = 1$, but there is no way to define this function holomorphically on a domain of $\mathbb{C}$ which contains both U and the point $z = 1$ unless we allow the function to take multiple values.

Definition 22. *Let $U \subset \mathbb{C}^n$ be a domain. Then U is a* ***domain of holomorphy*** *if there exists a function $f \in H(U)$ which does not extend holomorphically to any point of U^c. An open set $V \subset \mathbb{C}^n$ is an* ***open set of holomorphy*** *if each component of V is a domain of holomorphy.*

We introduce a particular class of functions:

Definition 23. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in H(U)$ a function. If for every domain V intersecting ∂U and every component W of $U \cap V$ there is a sequence $\{z_j\}_{j \geq 1} \subset W$ with $f(z_j) \to \infty$ then we say that f is* ***essentially unbounded*** *on ∂U.*

The following demonstrates the importance of these functions:

Proposition 24. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in H(U)$ a function essentially unbounded on ∂U. Then U is a domain of holomorphy.*

Proof. Suppose, for a contradiction, that f extends holomorphically to $b \in U^c$, so there is a domain $V \subset \mathbb{C}^n$ containing b and a function $g \in H(V)$ such that $f \equiv g$ on a non-empty open set $X \subset U \cap V$. Let W' be a component of $U \cap V$ containing a non-empty open subset of X, and let $a \in \partial W' \cap V$. Note that $a \in \partial U$ because $\partial W' \subset \partial U \cup \partial V$ and obviously $a \notin \partial V$. Now let $r > 0$ such that $\overline{B(a,r)} \subset V$ (where $B(a,r)$ denotes the open ball of radius r about a) and let W be a component of $U \cap B(a,r)$ contained in W'. Since f is essentially unbounded on ∂U there is a sequence $\{z_j\}_{j\geq 1} \subset W \subset W'$ with $f(z_j) \to \infty$. By the uniqueness theorem it follows that $f(z_j) = g(z_j)$ for each $j \geq 1$ and thus $g(z_j) \to \infty$. But $\overline{W}$ is a compact subset of V, so g attains a maximum modulus on $\overline{W}$, which is a contradiction. Therefore f does not extend holomorphically to any point of U^c, so U is a domain of holomorphy. □

In the next subsection we will prove the following, which strengthens Proposition 24 and is extremely useful in practice:

Theorem 25. *Let $U \subset \mathbb{C}^n$ be a domain and suppose for each $a \in \partial U$ there is a function $f_a \in H(U)$ tending to ∞ at a (that is, for every sequence $\{z_j\}_{j\geq 1} \subset U$ with $z_j \to a$ we have $f_a(z_j) \to \infty$). Then U is a domain of holomorphy.*

We consider some consequences of this theorem.

Corollary 26. *Let $U \subset \mathbb{C}^n$ be a domain and suppose for each $a \in \partial U$ there is a non-vanishing function $g_a \in H(U)$ which tends to zero at a. Then U is a domain of holomorphy.*

Proof. Apply Theorem 25 to the reciprocals of the functions g_a. □

Example 27. *Let $U \subset \mathbb{C}$ be a domain. If for each $a \in \partial U$ we consider the function $z \mapsto z - a$ then the hypothesis of Corollary 26 is satisfied, so U is a domain of holomorphy. Thus all domains of $\mathbb{C}$ are domains of holomorphy.*

Example 28. *Let $U \subset \mathbb{C}^n$ be a convex domain (that is, U is convex when regarded as a subset of $\mathbb{R}^{2n}$). Let $a \in \partial U$, so there exists a supporting hyperplane for U at a. That is, there exists $\delta \in \mathbb{C}^n$ such that whenever $\Re \langle z - a, \delta \rangle = 0$ we have $z \notin U$ (where we have recalled that the real part of the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ gives the dot product when the vectors are regarded as elements of $\mathbb{R}^{2n}$). Thus the function $z \mapsto \langle z - a, \delta \rangle$ is holomorphic and non-vanishing on U and tends to zero at a. By Corollary 26 it follows that U is a domain of holomorphy. Therefore the convex domains (in particular open balls) are domains of holomorphy.*

Next we construct a domain which is not a domain of holomorphy.

Proposition 29. *Let $U \subset \mathbb{C}^n$ be a complete Reinhardt domain with center 0 which is not logarithmically convex. Then U is a not domain of holomorphy.*

We recall that a complete Reinhardt domain with center 0 is a domain which can be written as a union of polydiscs centered at 0, and a domain $V \subset \mathbb{C}^n$ is logarithmically convex if its logarithmic image $\{(\ln|z_1|, \dots, \ln|z_n|) \colon z \in V, z_1 \dots z_n \neq 0\}$ is convex. One may verify that a function holomorphic in a complete Reinhardt domain with center 0 has a power series representation about 0 in the entire domain, and that the domain of convergence of any power series about 0 is logarithmically convex (see [23, subsection 7]).

Proof of Proposition 29. Let $f \in H(U)$. We know that there is a power series representation for f about 0 in U, and that the domain of convergence V of this series is a logarithmically convex domain containing U. Since U is not logarithmically convex we have that $U \neq V$, and f extends holomorphically to each point of $V \setminus U$. □

Example 30. *With e denoting Euler's number, let* $U := \Delta(0, (e, e^2)) \cup \Delta(0, (e^2, e)) \subset \mathbb{C}^2$ *(where* $\Delta(z, r)$ *denotes the polydisc of (vector) radius* r *about* $z \in \mathbb{C}^n$*), so clearly* U *is a complete Reinhardt domain with center* 0*. The logarithmic image of* U *is*

$$\{(\ln|z_1|, \ln|z_2|) \colon (z_1, z_2) \in U,\ z_1 z_2 \neq 0\} = [(-\infty, 1) \times (-\infty, 2)] \cup [(-\infty, 2) \times (-\infty, 1)],$$

which is not convex. Thus U *is not logarithmically convex. It follows from Proposition 29 that* U *is not a domain of holomorphy.*

We have defined domains of holomorphy and given some examples and a non-example, but currently we have no way to describe domains of holomorphy in simple geometric terms, and we have yet to prove the important Theorem 25.

3.2 Holomorphic convexity

We showed above that convex domains of $\mathbb{C}^n$ are domains of holomorphy, but clearly the converse is not true (take any non-convex domain of $\mathbb{C}$, for instance). Thus if we wish to describe domains of holomorphy with some notion of "convexity" we must use a more general definition. With Corollary 8 in mind we introduce the following definition, where we simply replace the affine functions with holomorphic functions:

Definition 31. *Let* $U \subset \mathbb{C}^n$ *be a domain and* $K \subset U$ *compact. The* ***holomorphically convex hull*** *of* K*, denoted* $\hat{K}$*, is*

$$\hat{K} := \{z \in U \colon |f(z)| \leq \|f\|_K \text{ for all } f \in H(U)\} = \bigcap_{f \in H(U)} |f|^{-1}([0, \|f\|_K]).$$

The domain U *is said to be* ***holomorphically convex*** *if for all compact* $K \subset U$ *the holomorphically convex hull* $\hat{K}$ *is compact.*

Note that the holomorphically convex hull of a particular compact set depends on the domain of which the compact set is a subset, so if this is not clear from the context we may write $\hat{K}_{H(U)}$ to explicitly denote the holomorphically convex hull of K as a subset of U. We now briefly note some properties of holomorphically convex hulls. Obviously the holomorphically convex hull of a compact set always contains the original compact set. Notice also that since the identity map $f(z) = z$ is holomorphic, if K is some compact subset of a domain and $\hat{K}$ is the holomorphically convex hull then for any $z \in \hat{K}$ we have $|z| = |f(z)| \leq \|f\|_K$, so holomorphically convex hulls are always bounded. Furthermore, since holomorphically convex hulls are closed in the subspace topology (this is obvious from the definition) we see that a holomorphically convex hull is compact if and only if it is a positive distance from the boundary of the domain, and this is true if and only if it is contained inside a compact subset of the domain. It is also evident from the definition that for any function $f \in H(U)$ and compact $K \subset U$ we have $\|f\|_K = \|f\|_{\hat{K}}$. This immediately implies that for a compact $K \subset U$ we have $\hat{\hat{K}} = \hat{K}$ (provided $\hat{K}$ is compact).

We have the following relationship between domains of holomorphy and holomorphically convex domains:

Theorem 32. *If $U \subset \mathbb{C}^n$ is holomorphically convex then U is a domain of holomorphy.*

To prove this we will require an intermediate result:

Lemma 33. *Let $U \subset \mathbb{C}^n$ be holomorphically convex. Then there is a sequence $\{K_j\}_{j\geq 1}$ of compact subsets of U such that $K_j \subset \mathring{K}_{j+1}$ and $K_j = \hat{K}_j$ for all $j \geq 1$, and $U = \bigcup_{j\geq 1} K_j$.*

Proof. For $j \geq 1$ define the compact set $L_j := \{z \in U : d(z, \partial U) \geq 1/j \text{ and } |z| \leq j\}$, so certainly $L_j \subset \mathring{L}_{j+1}$ for all $j \geq 1$ and $U = \bigcup_{j\geq 1} \mathring{L}_j$. Let $K_1 := \hat{L}_1$, and observe that K_1 is a compact subset of U with $K_1 = \hat{K}_1$. Let $j_2 > 1$ be sufficiently large that $K_1 \subset \mathring{L}_{j_2}$ (this is possible because $U = \bigcup_{j\geq 1} \mathring{L}_j$), and let $K_2 := \hat{L}_{j_2}$, so K_2 is a compact subset of U with $K_2 = \hat{K}_2$, and $K_1 \subset \mathring{L}_{j_2} \subset \mathring{K}_2$. Since $j_2 \geq 2$ we also have $\mathring{L}_2 \subset \mathring{K}_2$. Repeating this argument we obtain a sequence $\{K_j\}_{j\geq 1}$ of compact subsets with $K_j \subset \mathring{K}_{j+1}$, $K_j = \hat{K}_j$ and $\mathring{L}_j \subset \mathring{K}_j$ for each $j \geq 1$. From the last property and the fact that $U = \bigcup_{j\geq 1} \mathring{L}_j$ it follows that $U = \bigcup_{j\geq 1} \mathring{K}_j$, so $\{K_j\}_{j\geq 1}$ is the required sequence. □

Proof of Theorem 32. Note that if $U = \emptyset$ or $U = \mathbb{C}^n$ the assertion is trivial, so assume this is not the case.

In view of Proposition 24, to prove U is a domain of holomorphy it is enough to find a function $f \in H(U)$ which is essentially unbounded on ∂U. Let $\mathcal{A} := \{a_k\}_{k\geq 1} \subset U$ be the countable set consisting of all points in U with rational coordinates (that is, the real and imaginary parts of every component of every a_k are rational), and for each $k \geq 1$ let $B_k := B(a_k, d(a_k, \partial U))$ (note that $B_k \subset U$). Now let $\{Q_j\}_{j\geq 1}$ be a sequence of elements of $\{B_k\}_{k\geq 1}$ such that every B_k is given by Q_j for infinitely many indices j (for example, let $Q_1 := B_1$, $Q_2 := B_1$, $Q_3 := B_2$, $Q_4 := B_1$, $Q_5 := B_2$, $Q_6 := B_3$, and so on). We will find a function $f \in H(U)$ and a sequence $\{z_j\}_{j\geq 1}$ with $z_j \in Q_j$ for each $j \geq 1$ such that $f(z_j) \to \infty$ as $j \to \infty$. Suppose, for a moment, that we have found such a function and sequence. Let V be a domain intersecting ∂U and suppose W is a component of $U \cap V$. Let $a \in \partial W \cap V$ and note that we also have $a \in \partial U$. Let $r := d(a, \partial V)/2$, so because $\mathcal{A}$ is dense in U we have $a_k \in W \cap B(a, r)$ for some $k \geq 1$. Clearly $d(a_k, \partial U) < r < d(a_k, \partial V)$, meaning $B_k \subset U \cap V$ and thus $B_k \subset W$ (because B_k is connected). We have $B_k = Q_{j_l}$ for infinitely many indices $j_1 < j_2 < \dots$, so $\{z_{j_l}\}_{l\geq 1} \subset B_k \subset W$ and $f(z_{j_l}) \to \infty$. This argument applies for each component W of each domain V intersecting ∂U, so f is essentially unbounded on ∂U.

It remains to find the function $f \in H(U)$ and sequence $\{z_j\}_{j\geq 1}$. Let $\{K_j\}_{j\geq 1}$ be the sequence of compact subsets of U whose existence is asserted by the lemma. Passing to a subsequence of $\{K_j\}_{j\geq 1}$ if necessary we may assume $Q_j \cap (K_{j+1} \setminus K_j) \neq \emptyset$ for all $j \geq 1$. Thus for all $j \geq 1$ there exists $z_j \in Q_j \cap (K_{j+1} \setminus K_j)$, and since $z_j \notin K_j = \hat{K}_j$ there exists $f_j \in H(U)$ such that $|f_j(z_j)| > \|f_j\|_{K_j}$, and scaling f_j if necessary we may assume $|f_j(z_j)| > 1 \geq \|f_j\|_{K_j}$.

Let $p_1 := 1$, and inductively choose $p_j \in \mathbb{N}$ sufficiently large that for all $j \geq 1$ we have

$$\frac{1}{j^2}|f_j(z_j)|^{p_j} - \sum_{k=1}^{j-1} \frac{1}{k^2}|f_k(z_j)|^{p_k} \geq j \tag{3.1}$$

(this is possible because $|f_j(z_j)| > 1$). Now set, for all $z \in U$,

$$f(z) := \sum_{k=1}^{\infty} \frac{1}{k^2} f_k(z)^{p_k}.$$

Let $j \geq 1$, so if $z \in K_j$ then $z \in K_k$ for all $k \geq j$ and in particular $|f_k(z)| \leq 1$ for all $k \geq j$. By the Weierstrass M-test the series for f converges uniformly on K_j, so f is holomorphic on $\mathring{K}_j$. But $U = \bigcup_{j\geq 1} \mathring{K}_j$, so f is holomorphic on U. For any $j \geq 1$ we have

$$|f(z_j)| \geq \frac{1}{j^2}|f_j(z_j)|^{p_j} - \sum_{k=1}^{j-1} \frac{1}{k^2}|f_k(z_j)|^{p_k} - \sum_{k=j+1}^{\infty} \frac{1}{k^2}|f_k(z_j)|^{p_k} \geq j - \sum_{k=j+1}^{\infty} \frac{1}{k^2} \geq j - \frac{\pi^2}{6},$$

where for the second inequality we have used (3.1) and the fact that when $k > j$ we have $z_j \in K_k$ and thus $|f_k(z_j)| \leq 1$. Therefore $f(z_j) \to \infty$ as $j \to \infty$. From the earlier argument it follows that f is essentially unbounded on ∂U and thus U is a domain of holomorphy. □

As a consequence of this we may prove Theorem 25:

Proof of Theorem 25. To show U is a domain of holomorphy it suffices, by Theorem 32, to show it is holomorphically convex. Let $K \subset U$ be compact, so we must show $\hat{K}$ is compact, and it suffices to show $d(\hat{K}, \partial U) > 0$. Suppose this is not the case, so there is a sequence $\{z_j\}_{j\geq 1} \subset \hat{K}$ with $d(z_j, \partial U) \to 0$. Since $\hat{K}$ is bounded its closure is compact, so passing to a subsequence if necessary we may assume the sequence converges to a point a in this closure, and because $d(z_j, \partial U) \to 0$ we have $a \in \partial U$. By hypothesis there is a function $f_a \in H(U)$ such that $f_a(z_j) \to \infty$. But $\|f_a\|_{\hat{K}} = \|f_a\|_K < \infty$ and $\{z_j\}_{j\geq 1} \subset \hat{K}$, yielding a contradiction. Therefore $d(\hat{K}, \partial U) > 0$, so $\hat{K}$ is compact, meaning U is holomorphically convex and is hence a domain of holomorphy. □

We will find that the converse to Theorem 32 holds. We will need the following result on holomorphic extension to neighbourhoods of holomorphically convex hulls:

Proposition 34. *Let $U \subset \mathbb{C}^n$ be a domain, let $K \subset U$ be a compact subset, let $r := \rho(K, \partial U)$ (where $\rho(\cdot,\cdot)$ is the L^∞ metric) and let $\hat{K}$ be the holomorphically convex hull of K. Then every $f \in H(U)$ extends holomorphically to each point of the r-dilation $\hat{K}_{(r)} = \bigcup_{z\in\hat{K}} \Delta(z, r)$.*

Proof. Let $f \in H(U)$ and $a \in \hat{K}_{(r)}$, so $a \in \Delta(b, r)$ for some $b \in \hat{K} \subset U$. In a neighbourhood W of b we have the power series

$$f(z) = \sum_{|J|=0}^{\infty} c_J (z-b)^J, \qquad c_J := \frac{1}{J!} \frac{\partial^{|J|} f}{\partial z^J}\bigg|_b.$$

All partial derivatives of f are holomorphic on U, and $b \in \hat{K}$, so for $J \in \mathbb{N}_0^n$ (where $\mathbb{N}_0$ is the set of non-negative integers) we have

$$|c_J| = \frac{1}{J!} \left| \frac{\partial^{|J|} f}{\partial z^J}\bigg|_b \right| \leq \frac{1}{J!} \left\| \frac{\partial^{|J|} f}{\partial z^J} \right\|_K.$$

Let $r' < r$, so $S := \overline{K_{(r')}}$ is a compact subset of U. For any $p \in K$ we have $\overline{\Delta(p, r')} \subset S \subset U$ and thus $f \in H(\overline{\Delta(p, r')})$, and if we set $S_p := \{z \in U : |z_k - p_k| = r', 1 \leq k \leq n\}$ then, since $S_p \subset S$, we have $\|f\|_{S_p} \leq \|f\|_S$. Expanding f in a power series about any $p \in K$ and applying the Cauchy estimate:

$$\frac{1}{J!} \left| \frac{\partial^{|J|} f}{\partial z^J}\bigg|_p \right| \leq \frac{\|f\|_{S_p}}{r'^{|J|}} \leq \frac{\|f\|_S}{r'^{|J|}} \implies \frac{1}{J!} \left\| \frac{\partial^{|J|} f}{\partial z^J} \right\|_K \leq \frac{\|f\|_S}{r'^{|J|}}.$$

Therefore, for all $J \in \mathbb{N}_0^n$,

$$|c_J| \leq \frac{\|f\|_S}{r'^{|J|}} \implies |c_J r'^{|J|}| \leq \|f\|_S,$$

so the power series for f about b converges in $\Delta(b, r')$ (the terms of the series are bounded at the point $b + (r', \dots, r')$). This is true for all $r' < r$, so the series converges in $\Delta(b, r)$. Since a convergent power series is a holomorphic function it follows that this series yields a function holomorphic on $\Delta(b, r) \ni a$ which agrees with f on W. That is, f extends holomorphically to a. □

This immediately implies the following:

Corollary 35. *If $U \subset \mathbb{C}^n$ is a domain of holomorphy then U is holomorphically convex.*

In particular, the domains of holomorphy and the holomorphically convex domains are identical – this result is attributed to Cartan and Thullen [6], and is usually known as the Cartan-Thullen theorem. With this fact in mind we might hope that the domains of holomorphy obey some of the closure properties of convex domains.

Proposition 36. *Let $\{U_\lambda\}_{\lambda \in \Lambda}$, where Λ is some indexing set, be a set of domains of holomorphy in $\mathbb{C}^n$, and let U be a connected component of the interior of $\bigcap_{\lambda \in \Lambda} U_\lambda$. Then U is a domain of holomorphy.*

To prove this we require a lemma:

Lemma 37. *A domain $U \subset \mathbb{C}^n$ is holomorphically convex if and only if for all compact $K \subset U$ the hull $\hat{K}$ satisfies $\rho(\hat{K}, \partial U) = \rho(K, \partial U)$.*

Proof. If for every compact $K \subset U$ the hull $\hat{K}$ satisfies $\rho(\hat{K}, \partial U) = \rho(K, \partial U)$, then $\rho(\hat{K}, \partial U) > 0$, so $\hat{K}$ is compact and thus U is holomorphically convex.

Conversely, suppose U is holomorphically convex and let $K \subset U$ be compact, so $\hat{K}$ is compact. Clearly $K \subset \hat{K}$ so $\rho(\hat{K}, \partial U) \leq \rho(K, \partial U)$. Let $r := \rho(K, \partial U)$. By Proposition 34 every function holomorphic on U extends holomorphically to each point of the r-dilation $\hat{K}_{(r)}$, and since U is a domain of holomorphy it follows that $\hat{K}_{(r)} \subset U$. Thus $\rho(\hat{K}, \partial U) \geq r = \rho(K, \partial U)$, so $\rho(\hat{K}, \partial U) = \rho(K, \partial U)$ as required. □

Proof of Proposition 36. Let $K \subset U$ be compact. It suffices to show the holomorphically convex hull $\hat{K}$ is compact. Let $z \in \hat{K}$ and $\lambda \in \Lambda$, so for any $g \in H(U_\lambda)$ we have $|g(z)| \leq \|g\|_K$ (since $g\big|_U \in H(U)$), which implies $z \in \hat{K}_\lambda := \hat{K}_{H(U_\lambda)}$. Therefore $\hat{K} \subset \hat{K}_\lambda$ for all $\lambda \in \Lambda$, so $\rho(\hat{K}, \partial U_\lambda) \geq \rho(\hat{K}_\lambda, \partial U_\lambda) = \rho(K, \partial U_\lambda) \geq \rho(K, \partial U)$ (using the lemma for the equality). Therefore with $r := \rho(K, \partial U) > 0$ we have $\hat{K}_{(r)} \subset U_\lambda$ for all $\lambda \in \Lambda$, so $\hat{K}_{(r)} \subset \bigcap_{\lambda \in \Lambda} U_\lambda$. Since $\hat{K}_{(r)}$ is open it is actually contained in the interior of this intersection, and since $\hat{K}$ is in the single component U of this interior we must have $\hat{K}_{(r)} \subset U$. That is, $\rho(\hat{K}, \partial U) \geq r > 0$, so $\hat{K}$ is compact and thus U is holomorphically convex. □

Proposition 38. *Let $U_1 \subset \mathbb{C}^n$ and $U_2 \subset \mathbb{C}^m$ be domains of holomorphy. Then the Cartesian product $U := U_1 \times U_2 \subset \mathbb{C}^{n+m}$ is a domain of holomorphy.*

Proof. Let $K \subset U_1 \times U_2$ be compact, so it suffices to show the holomorphically convex hull $\hat{K}$ is compact. Let $\pi_1 \colon \mathbb{C}^{n+m} \to \mathbb{C}^n$ be defined by $\pi_1(z_1, z_2) = z_1$ (where $z_1 \in \mathbb{C}^n$, $z_2 \in \mathbb{C}^m$) and set $K_1 := \pi_1(K)$, and define $\pi_2 \colon \mathbb{C}^{n+m} \to \mathbb{C}^m$ and $K_2 := \pi_2(K)$ analogously. Then $K_j \subset U_j$ is compact for $j = 1, 2$, so the hulls $\hat{K}_j$ are compact since the domains U_j are holomorphically convex. It is easily seen that $\hat{K} \subset \hat{K}_1 \times \hat{K}_2$, and since the latter is compact it follows that $\hat{K}$ is compact, as required. □

Note this implies polydiscs $\Delta(z, r) \subset \mathbb{C}^n$ are domains of holomorphy.

As with convex domains, the union of even two domains of holomorphy need not be a domain of holomorphy – recall that $\Delta(0, (e, e^2)) \cup \Delta(0, (e^2, e)) \subset \mathbb{C}^2$ is a not a domain of holomorphy, but the polydiscs $\Delta(0, (e, e^2))$ and $\Delta(0, (e^2, e))$ are domains of holomorphy. However, we will find that the union of an increasing sequence of domains of holomorphy is a domain of holomorphy:

Theorem 39 (Behnke-Stein theorem)**.** *Let $\{U_j\}_{j \geq 1}$ be a sequence of domains of holomorphy such that $U_j \subset U_{j+1}$ for all $j \geq 1$. Then $U := \bigcup_{j \geq 1} U_j$ is a domain of holomorphy.*

We delay the proof of this fact until Section 5.

The following is immediate from the equivalence of domains of holomorphy and holomorphically convex domains:

Proposition 40. *Let $U \subset \mathbb{C}^n$ be a domain of holomorphy and $\phi \colon U \to \mathbb{C}^n$ a biholomorphic mapping. Then $\phi(U)$ is a domain of holomorphy.*

We conclude this section by introducing a particular type of open set of holomorphy which will be extremely useful later:

Definition 41. *Let $U \subset \mathbb{C}^n$ be an open set and $f_1, \ldots, f_m \in H(U)$. If $V := \{z \in U \colon |f_j(z)| < 1, 1 \leq j \leq m\}$ satisfies $\overline{V} \subset U$ then V is an **analytic polyhedron**. If $K := \{z \in U \colon |f_j(z)| \leq 1,\ 1 \leq j \leq m\}$ is compact then K is a **compact analytic polyhedron**. In either case the set $\{f_1, \ldots, f_m\}$ is a **frame** for the (compact) analytic polyhedron.*

Note that (compact) analytic polyhedra need not be connected.

Proposition 42. *If $V \subset \mathbb{C}^n$ is an analytic polyhedron then V is an open set of holomorphy.*

Proof. By definition there exists an open set $U \subset \mathbb{C}^n$ and functions $f_1, \ldots, f_m \in H(U)$ such that $V = \{z \in U \colon |f_j(z)| < 1,\ 1 \leq j \leq m\}$ and $\overline{V} \subset U$. Let W be a component of V and let $K \subset W$ be compact. We will show the holomorphically convex hull $\hat{K} := \hat{K}_{H(W)}$ is compact, and it is enough to show $\rho(\hat{K}, \partial W) > 0$. Suppose, for a contradiction, that this is not the case, so as in the proof of Theorem 25 there is a sequence $\{z_k\}_{k\geq 1} \subset \hat{K}$ with $z_k \to a \in \partial W$, and note $a \in \partial V$ because W is a component of V. Let $s := \max_{1\leq j\leq m}\{\|f_j\|_K\} < 1$. If $z \in \hat{K}$ then $|g(z)| \leq \|g\|_K$ for all $g \in H(W)$ and in particular for $g := f_j\big|_W$ $(1 \leq j \leq m)$, so $|f_j(z)| \leq \|f_j\|_K \leq s$. Thus $|f_j(z_k)| \leq s$ for $1 \leq j \leq m$ and $k \geq 1$, so $|f_j(a)| \leq s < 1$ by continuity, meaning $a \in V$, which contradicts the fact that $a \in \partial V$. Therefore $\hat{K}$ is compact for all compact $K \subset W$, so W is a domain of holomorphy. This is true for each component W of V, so V is an open set of holomorphy. □

Next we have some approximation results which will be useful in Section 5:

Proposition 43. *Let $K \subset \mathbb{C}^n$ be a compact analytic polyhedron and let V be a neighbourhood of K. Then there exists an open set of holomorphy X such that $K \subset X \subset V$.*

Proof. By definition there exists an open set $U \subset \mathbb{C}^n$ and functions $f_1, \ldots, f_m \in H(U)$ such that $K = \{z \in U \colon |f_j(z)| \leq 1,\, 1 \leq j \leq m\}$. Passing to a subset if necessary we may assume V is bounded and that $\overline{V} \subset U$, so there is a bounded neighbourhood W of $\overline{V}$ such that $\overline{W} \subset U$. Therefore $\overline{W} \setminus V$ is compact, and because for each $z \in \overline{W} \setminus V$ we have $z \notin K$ and thus $|f_j(z)| > 1$ for some $1 \leq j \leq m$ (and this inequality holds in a neighbourhood of z), there exists $s > 1$ such that whenever $z \in \overline{W} \setminus V$ we have $|f_j(z)| \geq s$ for some $1 \leq j \leq m$. Therefore for $z \in W$, if $|f_j(z)| < s$ for each $1 \leq j \leq m$ then $z \in V$. That is, the analytic polyhedron $X := \{z \in W \colon |f_j(z)/s| < 1,\, 1 \leq j \leq m\}$ satisfies $K \subset X \subset V$ (note that X is indeed an analytic polyhedron because $\overline{X} \subset \overline{V} \subset W$). By Proposition 42, X is an open set of holomorphy. □

Proposition 44. *Let $U \subset \mathbb{C}^n$ be a domain, $K \subset U$ a compact set such that $K = \hat{K}_{H(U)}$ and $V \subset U$ a neighbourhood of K. Then there exists a compact analytic polyhedron L with frame in $H(U)$ such that $K \subset L \subset V$.*

Proof. Passing to a subset of V if necessary we may assume that ∂V is compact and that $\overline{V} \subset U$. Since $\hat{K}_{H(U)} = K$, for any $z \in U \setminus K$ there is a function $f \in H(U)$ such that $\|f\|_K < |f(z)|$ (and this inequality also holds for points in a neighbourhood of z), and scaling f if necessary we may assume $\|f\|_K \leq 1 < |f(z)|$. By the compactness of ∂V there are finitely many functions $f_1, \ldots, f_m \in H(U)$ with $\|f_j\|_K \leq 1$ for each $1 \leq j \leq m$ and such that if $z \in \partial V$ then $|f_j(z)| > 1$ for some $1 \leq j \leq m$. Let $L := \{z \in V \colon |f_j(z)| \leq 1, 1 \leq j \leq m\}$, so clearly $K \subset L \subset V$. Certainly L is bounded and closed in the subspace topology on V, and $d(L, \partial V) > 0$ (otherwise there would be a sequence $\{z_j\}_{j\geq 1} \subset L$ with $z_j \to a \in \partial V$, and by continuity we would have $|f_j(a)| \leq 1$ for $1 \leq j \leq m$, which is a contradiction), so L is compact. Therefore L is the required compact analytic polyhedron. □

4 Pseudoconvexity

In the previous section the definition of a holomorphically convex domain was motivated by one of the several equivalent conditions for geometric convexity. In this section we generalise the other characterisations of convexity given in Section 2 to define various types of pseudoconvexity, and explore the consequences of these definitions. Most importantly we demonstrate that the definitions of pseudoconvexity are in fact equivalent, and that every domain of holomorphy is pseudoconvex. The material for this section is synthesised from [3], [22], [23] and [26].

4.1 The continuity principle

First we generalise the continuity principle introduced in Section 2. In that section we considered convergent sequences of line segments, which can be viewed as the images of the interval $[-1, 1]$ under affine maps. In Section 3 the definition of a convex hull was generalised by passing from affine functions to holomorphic functions. We apply a similar method here and thus obtain our first type of pseudoconvex domain.

We first need some preliminary definitions:

Definition 45. *A **holomorphic disc** is a continuous map $S\colon \overline{B(0,1)} \subset \mathbb{C} \to \mathbb{C}^n$ whose restriction to $B(0,1)$ is holomorphic. The **boundary** of the holomorphic disc S, denoted ∂S, is $S(\partial B(0,1))$.*

A holomorphic disc $S\colon \overline{B(0,1)} \to \mathbb{C}^n$ and the image $S(B(0,1))$ will often both be denoted by S.

Definition 46. *Let $U \subset \mathbb{C}^n$ be a domain. We say that U **satisfies the continuity principle** if, for every sequence of holomorphic discs $\{S_j\}_{j\geq 1}$ satisfying $S_j \cup \partial S_j \subset U$ for all $j \geq 1$, $\partial S_j \to B$ and $S_j \to T$ where $B \subset U$ and $T \subset \mathbb{C}^n$ are compact, we have $T \subset U$ (where we use the L^∞ metric to define convergence of sets).*

Note that with B and T defined as above we always have that T is connected and $B \subset T$. This observation implies the following:

Proposition 47. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^n$ be domains satisfying the continuity principle. Then each component of $U \cap V$ satisfies the continuity principle.*

Next we can show that domains of holomorphy satisfy the continuity principle. First we have a maximum modulus principle which follows directly from the maximum modulus principle of single variable complex analysis:

Lemma 48. *Let $S\colon \overline{B(0,1)} \to \mathbb{C}^n$ be a holomorphic disc with $S \cup \partial S \subset U$ where $U \subset \mathbb{C}^n$ is a domain, and let $f \in H(U)$. Then $\|f\|_S \leq \|f\|_{\partial S}$.*

Theorem 49. *If $U \subset \mathbb{C}^n$ is a domain of holomorphy then U satisfies the continuity principle.*

Proof. We show the contrapositive, so suppose U does not satisfy the continuity principle. Thus there is a sequence of holomorphic discs $\{S_j\}_{j\geq 1}$ satisfying $S_j \cup \partial S_j \subset U$ for all $j \geq 1$, $\partial S_j \to B$ and $S_j \to T$ where $B \subset U$ and $T \subset \mathbb{C}^n$ are compact, and $T \not\subset U$. Since $B \subset U$ is compact we have that the $2r$-dilation $B_{(2r)} \subset U$ for some $r > 0$. Let $K := \overline{B_{(r)}}$, and note that $K \subset U$ is compact. We will show that $\hat{K}$ is not compact, which will show that U is not holomorphically convex and hence not a domain of holomorphy. By convergence $\partial S_j \to B$ there exists $J > 0$ such that $j \geq J$ implies $\partial S_j \subset B_{(r)} \subset K$. By the above maximum modulus principle, for any $j \geq J$ and $f \in H(U)$ we have $\|f\|_{S_j} \leq \|f\|_{\partial S_j} \leq \|f\|_K$, meaning $S_j \subset \hat{K}$ for all $j \geq J$. Let $a \in T \setminus U$, so by convergence $S_j \to T$ there is a sequence $\{z_j\}_{j\geq J}$ with $z_j \to a$ and $z_j \in S_j$ for each j. Thus $\{z_j\}_{j\geq J} \subset \hat{K}$ but $a \notin \hat{K}$ (because $\hat{K} \subset U$ by definition), so $\hat{K}$ is not compact. Therefore U is not holomorphically convex and is hence not a domain of holomorphy. □

This result is useful for finding domains which are not domains of holomorphy. For instance:

Proposition 50. *Suppose $U \subset \mathbb{C}^n$ (with $n \geq 2$) is a domain such that U^c is non-empty and compact. Then U is not a domain of holomorphy.*

Proof. Since U^c is compact, the function $z \mapsto |z|$ takes its maximal value r on U^c at a point $a \in U^c$. We will assume that $a = (r, 0, \ldots, 0)$ (if this is not the case we may apply a complex rotation and appeal to the result of Proposition 40). For each $j \geq 1$ let $S_j \colon \overline{B(0,1)} \to U$ be given by $S_j(z) := (r + 1/j, z, 0, \ldots, 0)$, so clearly each S_j is a holomorphic disc satisfying $S_j \cup \partial S_j \subset U$. Furthermore, $S_j \to T$ where $T := \{(r, z, 0, \ldots, 0) \colon z \in \overline{B(0,1)}\}$ is compact and $\partial S_j \to B$ where $B := \{(r, z, 0, \ldots, 0) \colon z \in \partial B(0,1)\} \subset U$ is also compact. But $T \not\subset U$ because $a \in T \cap U^c$, so U does not satisfy the continuity principle and hence is not a domain of holomorphy. □

We remark that in fact functions holomorphic on such domains extend to functions holomorphic on all of $\mathbb{C}^n$, but this does not follow from the above result and instead can be shown using an integral representation formula such as that of Martinelli [16] and Bochner [4] to explicitly realise the holomorphic extension (see [23, page 172] for the details).

4.2 Plurisubharmonic functions

Next we define pseudoconvexity by generalising Proposition 17, which states that the convex domains are those which admit convex exhaustion functions. Recall that a convex function is a continuous function with the property that for any two points in the graph of the function, the graph lies below the straight line between those points. We will generalise this definition to functions of a complex variable to define (pluri)subharmonic functions, which satisfy the property that on each disc in the domain, the graph of the function lies below the graph of any harmonic function which dominates the function on the boundary of the disc. Here we formally define (pluri)subharmonic functions and give several important properties. We omit the proofs since they can be quite technical and may be found in many introductory complex analysis textbooks (for example [23, subsection 38] or [26, section 10]).

It will be convenient to introduce a weak notion of continuity for functions taking extended real values:

Definition 51. *A function $f\colon S \subset \mathbb{C}^n \to [-\infty, \infty]$ is* ***upper-semicontinuous*** *at $s \in S$ if, for all $\alpha > f(s)$, there exists a neighbourhood U of s such that $f(z) < \alpha$ for all $z \in U$. If f is upper-semicontinuous at each point of S then it is said to be* ***upper-semicontinuous*** *on S.*

The condition of upper-semicontinuity is clearly weaker than continuity and describes functions which do not increase by more than arbitrarily small amounts in neighbourhoods of points, but may decrease by any amount. We have a simple consequence of the definition which we will need later:

Proposition 52. *Let $f\colon K \subset \mathbb{C}^n \to [-\infty, \infty)$ (where K is compact) be upper-semicontinuous. Then f is bounded above on K and attains its maximum.*

Now we may define subharmonic functions:

Definition 53. *Let $f\colon U \subset \mathbb{C} \to [-\infty, \infty)$ (where U is open) be upper-semicontinuous. Then f is* ***subharmonic*** *on U if, for every $a \in U$ and $r > 0$ such that $\overline{B(a, r)} \subset U$ and for every continuous function $\phi\colon \overline{B(a, r)} \to \mathbb{R}$ which is harmonic on $B(a, r)$ and satisfies $\phi(z) \geq f(z)$ for all $z \in \partial B(a, r)$, we have $\phi(z) \geq f(z)$ for all $z \in B(a, r)$.*

Let $g\colon V \subset \mathbb{C}^n \to [-\infty, \infty)$ (where V is open) be an upper-semicontinuous function. Then f is ***plurisubharmonic*** *on V if, for every $a \in V$ and $\delta \in \mathbb{C}^n$ with $|\delta| = 1$, the function $z \mapsto f(a + \delta z)$ is subharmonic on $\{z \in \mathbb{C} \colon a + \delta z \in V\}$.*

It may be verified that the sum of two plurisubharmonic functions is again plurisubharmonic, and that plurisubharmonicity is a local property, which immediately implies the following:

Proposition 54. *Let $f\colon U \subset \mathbb{C}^n \to [-\infty, \infty)$ (where U is open) be upper-semicontinuous. Suppose that, for every $a \in U$ and $\delta \in \mathbb{C}^n$ with $|\delta| = 1$, the function $z \mapsto f(a + \delta z)$ is subharmonic on an open set V with $0 \in V \subset \{z \in \mathbb{C} \colon a + \delta z \in U\}$. Then f is plurisubharmonic on U.*

For twice-differentiable functions there is an equivalent condition for plurisubharmonicity that further emphasises the connection with convexity:

Proposition 55. *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{R}$ a C^2 function (that is, f is C^2 when regarded as a function of $2n$ real variables). Then f is plurisubharmonic if and only if for all $\delta \in \mathbb{C}^n$ and $z \in U$ we have*

$$\Delta_\delta f(z) := \sum_{j,k=1}^{n} \left.\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}\right|_z \delta_j \bar{\delta}_k \geq 0.$$

This result motivates the following definition which will be important later:

Definition 56. *Let $f : U \subset \mathbb{C}^n \to \mathbb{R}$ (where U is open) be a C^2 function. Then f is **strictly plurisubharmonic** if $\Delta_\delta f(z) > 0$ for all $z \in U$ and $\delta \in \mathbb{C}^n \setminus \{0\}$.*

For example, the function $z \mapsto |z|^2$ is strictly plurisubharmonic, as $\Delta_\delta |z|^2 = |\delta|^2$. Proposition 55 implies that strictly plurisubharmonic functions are plurisubharmonic.

Next we have three results which will be used to show that a domain satisfies the continuity principle if and only if it is pseudoconvex:

Proposition 57. *Let $U \subset \mathbb{C}^n$ be open and let $g \in H(U)$ be non-vanishing. Then $f : U \to \mathbb{R}$ given by $f(z) := -\ln |g(z)|$ is plurisubharmonic on U.*

Proposition 58. *Let $U \subset \mathbb{C}^n$ be open, and suppose that for each $\lambda \in \Lambda$ (where Λ is some indexing set) there is a plurisubharmonic function $f_\lambda : U \to [-\infty, \infty)$, and that the function f defined by $f(z) := \sup_{\lambda \in \Lambda}\{f_\lambda(z)\}$ maps into $[-\infty, \infty)$ and is upper-semicontinuous. Then f is plurisubharmonic.*

Proposition 59. *Let $U \subset \mathbb{C}^n$ be a domain, $S : \overline{B(0,1)} \to U$ a holomorphic disc and $f : U \to [-\infty, \infty)$ a plurisubharmonic function. Then $\sup_{z \in S}\{f(z)\} \leq \sup_{z \in \partial S}\{f(z)\}$.*

We conclude this subsection with an approximation result and its converse which will be vital in the following sections:

Proposition 60. *Let $U \subset \mathbb{C}^n$ be a domain and $f : U \to [-\infty, \infty)$ a plurisubharmonic function. Then f is the pointwise limit of a non-increasing sequence of C^∞ strictly plurisubharmonic functions $f_j : U_j \to \mathbb{R}$, where $\{U_j\}_{j \geq 1}$ is a sequence of bounded domains satisfying $U_j \subset U_{j+1}$ for each $j \geq 1$ and $\bigcup_{j \geq 1} U_j = U$.*

Proposition 61. *Let $U \subset \mathbb{C}^n$ be open and $f_j : U \to [-\infty, \infty)$ a non-increasing sequence of plurisubharmonic functions which converges pointwise to $f : U \to [-\infty, \infty)$. Then f is plurisubharmonic.*

4.3 Global pseudoconvexity

As discussed in the previous subsection, the plurisubharmonic functions of complex variables are a natural generalisation of the convex functions of real variables. Together with the characterisation of convexity in terms of convex exhaustion functions (Proposition 17), this suggests the following definition:

Definition 62. *A domain $U \subset \mathbb{C}^n$ is **pseudoconvex** if it admits a continuous plurisubharmonic exhaustion function.*

Example 63. *Let $U := B(a,r)$ for some $a \in \mathbb{C}^n$ and $r > 0$. Define $f\colon U \to \mathbb{R}$ by $f(z) := -\ln d(z, \partial U)$, so clearly f is continuous and an exhaustion function for U (it tends to $+\infty$ as the boundary points are approached). One may verify that $f(z) = -\ln \inf_{w \in \partial U}\{r^{-1}|\langle z-w, a-w\rangle|\} = \sup_{w\in\partial U}\{-\ln|r^{-1}\langle z-w,a-w\rangle|\}$, and since each $z \mapsto r^{-1}\langle z-w, a-w\rangle$ is holomorphic and non-vanishing on U we see from Propositions 57 and 58 that f is plurisubharmonic. Thus U is pseudoconvex. A similar argument shows that polydiscs are pseudoconvex.*

Proposition 64. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^n$ be pseudoconvex domains. Then each component of $U \cap V$ is pseudoconvex.*

Proof. It is easily verified that the pointwise maximum of two continuous plurisubharmonic exhaustion functions for U and V yields such an exhaustion function for each component of $U \cap V$. □

We will prove that the domains satisfying the continuity principle are precisely the pseudoconvex domains and thus demonstrate the first part of the connection between domains of holomorphy and pseudoconvexity.

Theorem 65. *If a domain $U \subset \mathbb{C}^n$ is pseudoconvex then it satisfies the continuity principle.*

Proof. By pseudoconvexity, U admits a continuous plurisubharmonic exhaustion function $f\colon U \to \mathbb{R}$. Let $\{S_j\}_{j\geq 1}$ be a sequence of holomorphic discs satisfying $S_j \cup \partial S_j \subset U$ for all $j \geq 1$, $\partial S_j \to B$ and $S_j \to T$ where $B \subset U$ and $T \subset \mathbb{C}^n$ are compact. Suppose, for a contradiction, that $T \not\subset U$, so there exists $a \in T \setminus U$, and by convergence $S_j \to T$ we have a sequence $\{z_j\}_{j\geq 1} \subset U$ with $z_j \in S_j$ for each $j \geq 1$ and $z_j \to a$. By convergence $\partial S_j \to B$ and compactness of B there exists a compact $K \subset U$ and an integer $J \geq 1$ such that $\partial S_j \subset K$ for $j \geq J$. Let $M := \|f\|_K < \infty$, so for $j \geq J$ we have $f(z_j) \leq M$ (by Proposition 59). Therefore $z_j \in f^{-1}((-\infty, M])$ for all $j \geq J$. But $z_j \to a$ and $f^{-1}((-\infty, M])$ is compact (since f is an exhaustion function), so $a \in f^{-1}((-\infty, M])$, which contradicts the fact that $a \notin U$. Thus $T \subset U$, so U satisfies the continuity principle. □

The converse is a consequence of the following intermediate results:

Lemma 66. *If $U \subset \mathbb{C}^n$ is a bounded non-empty domain which satisfies the continuity principle then $z \mapsto -\ln d(z, \partial U)$ is plurisubharmonic (and hence U is pseudoconvex).*

Proof. First we note that $z \mapsto |z|^2 - \ln d(z, \partial U)$ is a continuous exhaustion function for U (see Example 14) and that $z \mapsto |z|^2$ is plurisubharmonic, so plurisubharmonicity of $z \mapsto -\ln d(z, \partial U)$ implies pseudoconvexity of U. It remains to demonstrate plurisubharmonicity of $z \mapsto -\ln d(z, \partial U)$.

For $\delta \in \mathbb{C}^n$ with $|\delta| = 1$ and $z \in U$ define $d_\delta(z) := \inf_{w\in\mathbb{C}}\{|w|: z + \delta w \in \partial U\}$, and notice that $-\ln d(z, \partial U) = -\ln \inf_{|\delta|=1}\{d_\delta(z)\} = \sup_{|\delta|=1}\{-\ln d_\delta(z)\}$, so by Proposition 58 it is enough to show that d_δ is plurisubharmonic for each δ.

Let $\delta \in \mathbb{C}^n$ with $|\delta| = 1$ be fixed and for brevity set $f := d_\delta$ and $L_z := \{z + \delta w: w \in \mathbb{C}\}$ (for $z \in U$). It is easily verified that $f(z) = d(z, \partial U \cap L_z)$ and that f is upper-semicontinuous. Let $a \in U$ and $\delta' \in \mathbb{C}^n$ with $|\delta'| = 1$ be arbitrary. We will show that $z \mapsto f(a + \delta' z)$ is subharmonic in a neighbourhood of 0, and it will follow from Proposition 54 that f is plurisubharmonic on U.

Suppose, for a contradiction, that $z \mapsto f(a + \delta' z)$ is not subharmonic in a neighbourhood of 0. Then there exists $R > 0$ and a continuous function $\phi: \overline{B(0, R)} \subset \mathbb{C} \to \mathbb{R}$ which is harmonic on $B(0, R)$ and satisfies $\phi(z) \geq f(a+\delta' z)$ for $z \in \partial B(0, R)$ but $f(a + \delta' z) > \phi(z)$ at a point of $B(0, R)$. Clearly $z \mapsto f(a + \delta' z) - \phi(z)$ is upper-semicontinuous on $\overline{B(0, R)}$ so it attains its maximum value $\epsilon > 0$ at a point $z_0 \in B(0, R)$. Set $\psi(z) := -\phi(z) - \epsilon$, so ψ is continuous on $\overline{B(0, R)}$ and harmonic on $B(0, R)$. For $z \in \partial B(0, R)$ we have $\psi(z) + f(a+\delta' z) < 0$, so by upper-semicontinuity this inequality also holds for $z \in \partial B(0, r)$ when $r < R$ is sufficiently close to R. Thus there exists r with $|z_0| < r < R$ such that whenever $z \in \partial B(0, r)$ we have $\psi(z) < -f(a+\delta' z)$. It is also clear that when $z \in B(0, r)$ we have $\psi(z) \leq -f(a+\delta' z)$, and that $\psi(z_0) = -f(a + \delta' z_0)$.

By definition $f(a + \delta' z_0) = -\ln d_\delta(a + \delta' z_0) = -\ln d(a + \delta' z_0, \partial U \cap L_{a+\delta' z_0})$, and $d(a+\delta' z_0, \partial U \cap L_{a+\delta' z_0}) = |a+\delta' z_0 - b|$ for some $b \in \partial U \cap L_{a+\delta' z_0}$, so $b = a+\delta' z_0 + \delta w_0$ for some $w_0 \in \mathbb{C}$ and $f(a + \delta' z_0) = -\ln |w_0|$. Let $g \in H(B(0, R))$ be a holomorphic function with real part ψ. Since $|w_0| = e^{-f(a+\delta' z_0)} = e^{\psi(z_0)}$, by adding a constant imaginary number to g if necessary we may assume that $e^{g(z_0)} = w_0$.

For $t \in (0, 1]$ let $S_t: \overline{B(0, 1)} \to \mathbb{C}^n$ be given by $S_t(z) := a+\delta' rz+\delta t e^{g(rz)}$, so each S_t is a holomorphic disc. Furthermore, as $j \to \infty$ we have $S_{1-1/j} \to S_1 \cup \partial S_1$ and $\partial S_{1-1/j} \to \partial S_1$. Notice that $S_t(z) \in L_{a+\delta' rz}$ for all $t \in (0, 1]$ and $z \in \overline{B(0, 1)}$, so when $|te^{g(rz)}| < d_\delta(a+\delta' rz)$ we have $S_t(z) \in U$. But $|te^{g(rz)}| = te^{\psi(rz)}$, when $z \in \overline{B(0, 1)}$ we have $\psi(rz) \leq -f(a+\delta' rz) = \ln d_\delta(a+\delta' rz)$, and when $z \in \partial B(0, 1)$ this inequality is strict. Therefore when $t < 1$ and $z \in \overline{B(0, 1)}$ we have $|te^{g(rz)}| < e^{\psi(rz)} \leq d_\delta(a+\delta' rz)$, which implies that $S_t \cup \partial S_t \subset U$ for $t < 1$. Similarly, when $t = 1$ and $z \in \partial B(0, 1)$ we have $|te^{g(rz)}| = e^{\psi(rz)} < d_\delta(a + \delta' rz)$, meaning $\partial S_1 \subset U$. That is, we have a sequence $\{S_{1-1/j}\}_{j\geq 1}$ of holomorphic discs satisfying $S_{1-1/j} \cup \partial S_{1-1/j} \subset U$ for each $j \geq 1$, $S_{1-1/j} \to S_1 \cup \partial S_1$ and $\partial S_{1-1/j} \to \partial S_1$ where $S_1 \cup \partial S_1$ and $\partial S_1 \subset U$ are compact, so by the continuity principle we have $S_1 \cup \partial S_1 \subset U$. But $S_1(z_0/r) = a + \delta' z_0 + \delta e^{g(z_0)} = a + \delta' z_0 + \delta w_0 = b \in \partial U$, which is a contradiction. Therefore $z \mapsto f(a + \delta' z)$ is subharmonic in a neighbourhood of 0, as required. □

Lemma 67. *Let $\{U_j\}_{j\geq 1}$ be a sequence of pseudoconvex domains of $\mathbb{C}^n$ with $U_j \subset U_{j+1}$ for each $j \geq 1$. Then $U := \bigcup_{j\geq 1} U_j$ is pseudoconvex.*

Proof. If $U = \emptyset$ the result is trivial ($z \mapsto |z|^2$ is a continuous plurisubharmonic exhaustion function), so we assume this is not the case. By fixing $z_0 \in U_1$ and replacing each U_j by the component of $U_j \cap B(z_0, j)$ containing z_0 we may assume that each U_j is bounded and non-empty (note that each such domain is pseudoconvex by Example 63 and Proposition 64).

As in the proof of Lemma 66, pseudoconvexity of U will follow from plurisubharmonicity of $z \mapsto -\ln d(z, \partial U)$. Let $J \geq 1$ and $z \in U_J$. An elementary argument shows that the sequence $\{d(z, \partial U_j)\}_{j \geq J}$ is non-decreasing and converges to $d(z, \partial U)$, so by continuity of ln we see that $\{-\ln d(z, \partial U_j)\}_{j \geq J}$ is non-increasing and converges to $-\ln d(z, \partial U)$. By Theorem 65 and Lemma 66 we know that each $z \mapsto -\ln d(z, \partial U_j)$ is plurisubharmonic, so from Proposition 61 it follows that $z \mapsto -\ln d(z, \partial U)$ is plurisubharmonic on U_J. Since plurisubharmonicity is local and $U = \bigcup_{J \geq 1} U_J$ it follows that $z \mapsto -\ln d(z, \partial U)$ is plurisubharmonic on U. □

Theorem 68. *If a domain $U \subset \mathbb{C}^n$ satisfies the continuity principle then it is pseudoconvex.*

Proof. If $U = \emptyset$ the result is trivial, so suppose $U \neq \emptyset$. Let $z_0 \in U$ and for $j \geq 1$ let U_j be the component of $U \cap B(z_0, j)$ containing z_0, so U_j satisfies the continuity principle by Proposition 47, $U_j \subset U_{j+1}$ for each j and $U = \bigcup_{j \geq 1} U_j$. The result now follows from Lemmas 66 and 67. □

Along with Theorem 49, this result shows that domains of holomorphy are pseudoconvex. We will show later that the converse is also true and thus arrive at the solution to the Levi problem.

Theorem 65, together with the proofs of Lemma 66 and Theorem 68, immediately implies the following useful fact:

Corollary 69. *A domain $U \subset \mathbb{C}^n$ (with $\partial U \neq \emptyset$) is pseudoconvex if and only if $z \mapsto -\ln d(z, \partial U)$ is plurisubharmonic.*

4.4 Local pseudoconvexity

Recall that convexity of a domain is a local property of the boundary. We may introduce such a local notion of pseudoconvexity:

Definition 70. *Let $U \subset \mathbb{C}^n$ be a domain and $a \in \partial U$. Then U is* ***locally pseudoconvex*** *at a if there is a neighbourhood V of a such that every component of $U \cap V$ is pseudoconvex. If U is locally pseudoconvex at each $a \in \partial U$ it is* ***locally pseudoconvex****.*

We will find that pseudoconvexity and local pseudoconvexity are equivalent, thus indicating that pseudoconvexity itself is a local property:

Theorem 71. *Let $U \subset \mathbb{C}^n$ be a domain. Then U is pseudoconvex if and only if U is locally pseudoconvex.*

Proof. If $U = \emptyset$ or $U = \mathbb{C}^n$ the result is trivial, so suppose this is not the case.

If U is pseudoconvex then for any $a \in \partial U$ each component of $U \cap B(a,1)$ is pseudoconvex by Corollary 64, so U is locally pseudoconvex.

Now assume U is locally pseudoconvex. First suppose U is bounded. Let $a \in \partial U$ and let $V \subset \mathbb{C}^n$ be a neighbourhood of a such that each component of $U \cap V$ is pseudoconvex. Let $r_a > 0$ so that $B(a, r_a) \subset V$, so each component of $U \cap V \cap B(a, r_a) = U \cap B(a, r_a)$ is pseudoconvex (by Corollary 64). Consider such a component W, so the function $z \mapsto -\ln d(z, \partial W)$ is plurisubharmonic on W. Clearly $d(z, \partial W) = \min(d(z, \partial U), d(z, \partial B(a, r_a)))$, so when $z \in B(a, r_a/2) \cap W$ we have $d(z, \partial W) = d(z, \partial U)$ by virtue of the fact that $d(z, \partial U) \leq d(z, a) < r_a/2 < d(z, \partial B(a, r_a))$. Thus $z \mapsto -\ln d(z, \partial U)$ is plurisubharmonic on $B(a, r_a/2) \cap W$. This is true for each component W and each boundary point a, so $z \mapsto -\ln d(z, \partial U)$ is plurisubharmonic on $X := U \cap \bigcup_{a \in \partial U} B(a, r_a/2)$. Let $Y := U \cap \bigcup_{a \in \partial U} B(a, r_a/4)$, so since $Y \subset U$ is open and U is bounded, $K := U \setminus Y$ is compact and hence $z \mapsto -\ln d(z, \partial U)$ attains a maximum value M on K. Define $f : U \to \mathbb{R}$ by $f(z) := \max(-\ln d(z, \partial U), M)$, so f is continuous, and it is plurisubharmonic on X by Proposition 58 and plurisubharmonic on the interior of K because it is constant there. Clearly f is also an exhaustion function for U (it tends to $+\infty$ as boundary points are approached). Therefore U is pseudoconvex.

If U is unbounded, fix $z_0 \in U$ and for each $j \geq 1$ let U_j be the connected component of $U \cap B(z_0, j)$ containing z_0. We have $U = \bigcup_{j \geq 1} U_j$ and $U_j \subset U_{j+1}$ for all $j \geq 1$, so by Lemma 67 it suffices to show each U_j is pseudoconvex, and by the above argument it is enough to show each U_j is locally pseudoconvex. Let $a \in \partial U_j$, so either $a \in \partial U$ or $a \in \partial B(z_0, j)$. If $a \in \partial U$ there is a neighbourhood V such that each component of $U \cap V$ is pseudoconvex, so by Corollary 64 each component of $U \cap V \cap B(z_0, j) = U_j \cap V$ is pseudoconvex. Now suppose $a \notin \partial U$, so $a \in \partial B(z_0, j)$. Let $r := d(a, \partial U) > 0$, $V := B(a, r/2)$ and W a component of $U_j \cap V$. Then for $z \in W$ we have $d(z, \partial W) = \min(d(z, \partial V), d(z, \partial U_j))$, and $d(z, \partial U_j) = d(z, \partial B(z_0, j))$ (because $d(z, \partial U_j) = \min(d(z, \partial U), d(z, \partial B(z_0, j)))$, and when $z \in W$ we have $d(z, \partial B(z_0, j)) < r/2 < d(z, \partial U)$). Therefore $-\ln d(z, \partial W) = \max(-\ln d(z, \partial V), -\ln d(z, \partial B(z_0, j)))$ for $z \in W$, and the functions $z \mapsto -\ln d(z, \partial V)$ and $z \mapsto -\ln d(z, \partial B(z_0, j))$ are plurisubharmonic on W by pseudoconvexity of $V = B(a, r/2)$ and $B(z_0, j)$, so $z \mapsto -\ln d(z, \partial W)$ is plurisubharmonic by Proposition 58. Thus W is pseudoconvex, so each component of $U_j \cap V$ is pseudoconvex. Therefore each U_j is bounded and locally pseudoconvex and is hence pseudoconvex, so it follows that U is pseudoconvex. □

4.5 Levi pseudoconvexity

As with convex domains, domains with twice-differentiable boundaries admit an additional local characterisation of pseudoconvexity.

Definition 72. *Let $U \subset \mathbb{C}^n$ be a domain and $a \in \partial U$. If $V \subset \mathbb{C}^n$ is a neighbourhood* *of a and $f\colon V \to \mathbb{R}$ is such that $U \cap V = f^{-1}((-\infty, 0))$ then f is a **local defining function** for U at a. If $\overline{U} \subset V$ then f is a **global defining function** for U.*

*Let $k \geq 1$ (we allow $k = \infty$). The domain U is said to have **k-times differentiable boundary** or **C^k boundary** at a boundary point if there is a C^k local defining function whose gradient is non-zero at the point. If a domain has C^k boundary at each boundary point it is said to have **k-times differentiable boundary** or **C^k boundary**. If $a \in \partial U$ and $f\colon V \to \mathbb{R}$ is a C^k local defining function for U at a then the **(complex) tangent space** to ∂U at a with respect to f is $T^c_a(f) := \{\delta \in \mathbb{C}^n\colon \left\langle \delta, \overline{\nabla f(a)} \right\rangle = 0\}$ (where $\nabla := (\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is the complex gradient).*

Note that a global defining function immediately induces local defining functions at each boundary point. We remark that when working with a domain known to have k-times differentiable boundary at some point, we require that *all* defining functions be C^k and have non-zero gradient at the point. In view of equation (2.1) of Remark 19 (which applies in this complex setting by simply regarding subsets of $\mathbb{C}^n$ as subsets of $\mathbb{R}^{2n}$), by passing to subsets if necessary we will always assume local defining functions $f\colon V \to \mathbb{R}$ for domains U with C^k boundaries satisfy the property that $\partial U \cap V = f^{-1}(\{0\})$ (which implies $V \setminus \overline{U} = f^{-1}((0, \infty))$).

With the results of Section 2 in mind we give the following definitions:

Definition 73. *Let $U \subset \mathbb{C}^n$ be a domain with C^2 boundary. Then U is **Levi pseudoconvex** at $a \in \partial U$ if there is a C^2 local defining function $f\colon V \to \mathbb{R}$ (where V is a neighbourhood of a) such that $\Delta_\delta f(a) \geq 0$ for all $\delta \in T^c_a(f)$. If this inequality is strict for all $\delta \in T^c_a(f) \setminus \{0\}$ then U is **strictly pseudoconvex** at a. If U is Levi pseudoconvex at each boundary point it is said to be **Levi pseudoconvex**. If U is bounded and strictly pseudoconvex at each boundary point it is said to be **strictly pseudoconvex**.*

First note that a strictly pseudoconvex domain is Levi pseudoconvex. Also note the connection with plurisubharmonicity – if at each boundary point of a domain there is a non-degenerate C^2 plurisubharmonic (respectively, strictly plurisubharmonic) local defining function then the domain is Levi pseudoconvex (respectively, strictly pseudoconvex).

Definition 74. *Let $U \subset \mathbb{C}^n$ be a bounded domain that admits a C^∞ strictly plurisubharmonic global defining function which is non-degenerate on ∂U. We say that U is a **smoothly bounded strictly pseudoconvex** domain.*

It is clear that smoothly bounded strictly pseudoconvex domains have C^∞ boundaries, and in fact the converse also holds – strictly pseudoconvex domains with C^∞ boundaries are smoothly bounded in the sense of the above definition (see [22, page 59]). Regardless of this fact, we will always show explicitly that a C^∞ strictly plurisubharmonic global defining function exists when verifying that a strictly pseudoconvex domain is smoothly bounded, rather than simply showing that the boundary is C^∞.

It may be verified through a technical argument that non-negativity (and positivity) of the quantity $\Delta_\delta f(a)$ is independent of the chosen local defining function (see [22, page 56]), which means that the Levi or strict pseudoconvexity of a domain at a boundary point can be determined by considering a single defining function at that point (rather than all possible defining functions).

Example 75. *Let $U := B(c,r) \subset \mathbb{C}^n$ and let $f\colon \mathbb{C}^n \to \mathbb{R}$ be given by $f(z) := |z-c|^2 - r^2$, so f is a global defining function for U. We have $\frac{\partial f}{\partial z_j}\big|_z = \overline{z}_j - \overline{c}_j$, so $\nabla f(z) = \overline{z} - \overline{c}$, and because this is non-zero for $z \in \partial U$ it follows that U has C^2 boundary. Furthermore, $\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}\big|_z$ is 1 if $j = k$ and 0 otherwise. Therefore, for any $a \in \partial U$ and $\delta \in \mathbb{C}^n \setminus \{0\}$, $\Delta_\delta f(a) = |\delta|^2 > 0$. It follows that U is strictly pseudoconvex.*

Example 76. *Let $U := \Delta(c,r) \subset \mathbb{C}^n$ ($n \geq 2$) where r is a positive vector radius, and let $a \in \partial U$ such that $|a_1 - c_1| = r_1$ and $|a_j - c_j| < r_j$ for $j = 2, \dots, n$. Then for a neighbourhood V of a the function $f\colon V \to \mathbb{R}$ given by $f(z) := |z_1 - c_1|^2 - r_1^2$ is a local defining function for U. Clearly f is C^2 and $\nabla f(a) = (\overline{a}_1 - \overline{c}_1, 0, \dots, 0) \neq 0$, so U has C^2 boundary at a. But $\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}\big|_a$ is 1 if $j = k = 1$ and 0 otherwise, so $\Delta_\delta f(a) = |\delta_1|^2$ for all $\delta \in \mathbb{C}^n$. But if $\delta \in T_a^c(f)$ then $\delta_1 = 0$ and thus $\Delta_\delta f(a) = 0$. Hence U is Levi pseudoconvex at a (and it is also pseudoconvex by earlier results) but not strictly pseudoconvex.*

Thus there are pseudoconvex domains which are not strictly pseudoconvex. However, we have the following approximation result:

Proposition 77. *Let $U \subset \mathbb{C}^n$ be a pseudoconvex domain. Then there exists a sequence $\{U_j\}_{j\geq 1}$ of smoothly bounded strictly pseudoconvex domains such that $\overline{U_j} \subset U_{j+1}$ for all $j \geq 1$ and $U = \bigcup_{j\geq 1} U_j$.*

To prove this we will require a technical lemma of real analysis which depends on a theorem of Morse (see [17] for a proof):

Theorem 78 (Morse's theorem)**.** *Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ (where U is open) be C^∞. Then the Lebesgue measure of $\{f(x)\colon x \in U, f'(x) = 0\}$ is zero.*

This immediately implies the following:

Lemma 79. *Let $f\colon U \subset \mathbb{C}^n \to \mathbb{R}$ be C^∞ and $V \subset \mathbb{R}$ a non-empty open set. Then there exists $y \in V$ such that $\nabla f(z) \neq 0$ for all $z \in f^{-1}(\{y\})$.*

We need one more lemma, which shows that C^1 functions yield defining functions for components of their sublevel sets.

Lemma 80. *Let $f\colon U \subset \mathbb{C}^n \to \mathbb{R}$ be C^1, let $\alpha \in f(U)$ such that $\nabla f(z) \neq 0$ for all $z \in f^{-1}(\{\alpha\})$ and let V be a component of $f^{-1}((-\infty, \alpha))$. Then $z \mapsto f(z) - \alpha$ is a global defining function for V on some neighbourhood of $\overline{V}$.*

We omit the proof (which proceeds via the implicit function theorem as described in Remark 19) since it is a simple argument of real analysis and not especially instructive.

Proof of Proposition 77. If $U = \emptyset$ the assertion is trivial, so assume this is not the case. Let $f\colon U \to \mathbb{R}$ be a continuous plurisubharmonic exhaustion function for U, so by Proposition 60 there is a sequence of bounded domains $\{V_j\}_{j\geq 1}$ with $V_j \subset V_{j+1}$ for all $j \geq 1$ and $U = \bigcup_{j\geq 1} V_j$, and a non-increasing sequence of C^∞ strictly plurisubharmonic functions $f_j\colon V_j \to \mathbb{R}$ which converges pointwise to f. Let $z_0 \in V_1$, and suppose without loss of generality that $f_1(z_0) \leq 0$ (if this is not the case subtract $f_1(z_0)$ from f and each f_j). Note $[0,\infty) \subset f(U)$ by the intermediate value theorem, since $f(z_0) \leq f_1(z_0) \leq 0$ and f tends to $+\infty$ as boundary points are approached. For each $j \geq 1$ the non-empty set $K_j := f^{-1}((-\infty, j+1/2])$ is compact, so there is some $k_j \geq 1$ with $K_j \subset V_{k_j}$. By passing to a subsequence of $\{V_j\}_{j\geq 1}$ if necessary we may assume $K_j \subset V_j$ for all $j \geq 1$. For each $j \geq 1$ let $r_j \in (j-1/2, j+1/2)$ be such that $\nabla f_j(z) \neq 0$ whenever $f_j(z) = r_j$ (the possibility of this is assured by Lemma 79), and note $r_j \in f_j(V_j)$ by the intermediate value theorem, since $f_j(z_0) \leq 0 < r_j$ and for some $z \in K_j \subset V_j$ we have $f(z) = j + 1/2$ and thus $f_j(z) \geq j + 1/2 > r_j$.

For each $j \geq 1$ set $U_j := f_j^{-1}((-\infty, r_j)) \subset K_j$. If $z \in \overline{U_j}$ for some $j \geq 1$ then $f_j(z) \leq r_j$, so $f_{j+1}(z) \leq f_j(z) \leq r_j < r_{j+1}$ and thus $z \in U_{j+1}$, so $\overline{U_j} \subset U_{j+1}$. If $z \in U$ there exists $K \geq 1$ such that $r_K > f(z)$, so by convergence $f_j(z) \to f(z)$ there exists $J \geq 1$ such that $j \geq J$ implies $f_j(z) < r_K$, and in particular if we take $j \geq \max\{J, K\}$ we have $f_j(z) < r_K \leq r_j$, so $z \in U_j$. Therefore $U = \bigcup_{j\geq 1} U_j$. Now replace each U_j with the component of U_j containing z_0, so the properties $\overline{U_j} \subset U_{j+1}$ for $j \geq 1$ and $U = \bigcup_{j\geq 1} U_j$ are retained, and now $z \mapsto f_j(z) - r_j$ is a global defining function for each U_j by Lemma 80, meaning each U_j is a smoothly bounded strictly pseudoconvex domain. □

Next we show that pseudoconvex domains with C^2 boundary are Levi pseudoconvex. We will need a fact from geometry on smoothness of the so-called *signed distance function*, which is a particularly convenient defining function:

Proposition 81. *Let $U \subset \mathbb{C}^n$ be a domain with C^2 boundary and define $\eta\colon \mathbb{C}^n \to \mathbb{R}$ by*

$$\eta(z) := \begin{cases} -d(z, \partial U), & z \in \overline{U} \\ d(z, \partial U), & z \in U^c. \end{cases}$$

Then η is a local defining function for U at each boundary point (in particular η is C^2 in a neighbourhood of ∂U and is non-degenerate on ∂U).

This is a consequence of the implicit function theorem, but the details are rather technical and not particularly relevant so we omit the proof and refer the interested reader to [12, page 136]. We will also need a lemma:

Lemma 82. *Suppose* $f\colon U \to \mathbb{R}$ *is* C^2 *and has non-vanishing gradient, and that for some bounded set* S *with* $\overline{S} \subset U$ *we have* $\Delta_\delta f(z) \geq 0$ *for all* $z \in S$ *and* $\delta \in \mathbb{C}^n$ *with* $\left\langle \delta, \overline{\nabla f(z)} \right\rangle = 0$*. Then there exists* $c \geq 0$ *such that* $\Delta_\delta f(z) \geq -c|\delta| \left| \left\langle \delta, \overline{\nabla f(z)} \right\rangle \right|$ *for all* $z \in S$ *and* $\delta \in \mathbb{C}^n$*.*

Proof. If $S = \emptyset$ the assertion is trivial, so assume this is not the case. Define

$$d := 2 \sum_{j,k=1}^{n} \left\| \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} \right\|_S < \infty, \qquad c := 2d \left\| \frac{1}{|\nabla f|} \right\|_S < \infty.$$

Let $z \in S$ and $\delta \in \mathbb{C}^n$ be arbitrary. We will show the required inequality is satisfied for c as above, which does not depend on z or δ. Set $\nu := \nabla f(z)/|\nabla f(z)|$. Let $\delta' := \langle \delta, \overline{\nu} \rangle \overline{\nu}$ and $\delta'' := \delta - \delta'$, so $\langle \delta'', \overline{\nu} \rangle = 0$. Therefore, since $\Delta_{\delta''} f(z) \geq 0$,

$$\Delta_\delta f(z) = \Delta_{\delta''} f(z) + \Delta_{\delta'} f(z) + \sum_{j,k=1}^{n} \left. \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} \right|_z \left(\delta''_j \overline{\delta'}_k + \delta'_j \overline{\delta''}_k \right) \geq 0 + s + t,$$

where s is the second term of the middle expression and t is the third. Clearly $|s|/|\delta'|^2$ and $|t|/|\delta'\delta''|$ are at most d, and since $|\delta'|, |\delta''| \leq |\delta|$ it follows that $|s| \leq d|\delta'||\delta|$ and $|t| \leq d|\delta'||\delta|$. This implies that

$$\Delta_\delta f(z) \geq -2d|\delta'||\delta| = -2d|\delta| \frac{1}{|\nabla f(z)|} \left| \left\langle \delta, \overline{\nabla f(z)} \right\rangle \right| \geq -c|\delta| \left| \left\langle \delta, \overline{\nabla f(z)} \right\rangle \right|,$$

as required. □

Theorem 83. *Let* $U \subset \mathbb{C}^n$ *be a pseudoconvex domain with* C^2 *boundary. Then* U *is Levi pseudoconvex.*

Proof. Let $\eta\colon \mathbb{C}^n \to \mathbb{R}$ be the signed distance function. By pseudoconvexity of U the function $f(z) := -\ln(-\eta(z)) = -\ln d(z, \partial U)$ is plurisubharmonic on U. Thus for $z \in U$ and $\delta \in \mathbb{C}^n$ with $\left\langle \delta, \overline{\nabla \eta(z)} \right\rangle = 0$ we have, by the chain rule,

$$0 \leq \Delta_\delta f(z) = \sum_{j,k=1}^{n} \left(- \left. \frac{\partial^2 \eta}{\partial z_j \partial \overline{z}_k} \right|_z \frac{\delta_j \overline{\delta}_k}{\eta(z)} + \left. \frac{\partial \eta}{\partial z_j} \right|_z \left. \frac{\partial \eta}{\partial \overline{z}_k} \right|_z \frac{\delta_j \overline{\delta}_k}{\eta(z)^2} \right) = -\frac{1}{\eta(z)} \Delta_\delta \eta(z),$$

and since $\eta(z) < 0$ it follows that $\Delta_\delta \eta(z) \geq 0$. Let $a \in \partial U$, and let V be a neighbourhood of a on which η is C^2 and has non-vanishing gradient (the possibility of this is assured by Proposition 81 together with continuity of the gradient), and let $W \subset U \cap V$ be a bounded open set with $a \in \overline{W} \subset V$. By Lemma 82 there exists $c \geq 0$ such that $\Delta_\delta \eta(z) \geq -c|\delta| \left| \left\langle \delta, \overline{\nabla \eta(z)} \right\rangle \right|$ for all $z \in W$ and $\delta \in \mathbb{C}^n$. Fix $\delta \in \mathbb{C}^n$, so because the first two derivatives of η are continuous on V this inequality holds as $z \to a$, meaning $\Delta_\delta \eta(a) \geq -c|\delta| \left|\left\langle \delta, \overline{\nabla \eta(a)} \right\rangle\right|$. This holds for all $\delta \in \mathbb{C}^n$, so if $\delta \in T^c_a(\eta)$ then $\Delta_\delta \eta(a) \geq 0$ (as $\left\langle \delta, \overline{\nabla \eta(a)} \right\rangle = 0$). This is true for all $a \in \partial U$, so U is Levi pseudoconvex. □

We conclude the section by demonstrating the converse, thus proving that for domains with twice-differentiable boundaries the concepts of Levi pseudoconvexity and pseudoconvexity are equivalent.

Theorem 84. *Let $U \subset \mathbb{C}^n$ be Levi pseudoconvex. Then U is pseudoconvex.*

Proof. By Theorem 71 it is enough to show U is locally pseudoconvex at each boundary point. Let $a \in \partial U$ and let $f\colon V \to \mathbb{R}$ be a local defining function for U at a. Suppose, without loss of generality, that $\frac{\partial f}{\partial x_{2n}}\big|_a > 0$ (regarding f as a function of real variables), so as in Remark 19 there are open sets $W \subset \mathbb{R}^{2n-1}$ and $I \subset \mathbb{R}$ such that $a \in W \times I \subset V$ and a C^2 function $g\colon W \to I$ such that $f(x) = 0$ if and only if $x_{2n} = g(x_1, \ldots, x_{2n-1})$. Moreover, $h\colon W \times I \subset \mathbb{C}^n \to \mathbb{R}$ defined by $h(z) := \Im(z_n) - g(\Re(z_1), \ldots, \Re(z_n), \Im(z_1), \ldots, \Im(z_{n-1}))$ is a local defining function for U at each point $b \in \partial U \cap (W \times I)$. In particular, since U is Levi pseudoconvex, for each such b we have $\Delta_\delta h(b) \geq 0$ for all $\delta \in T^c_b(h)$. Let $W' \times I'$ be a neighbourhood of a such that $g(W') \subset I'$, $\overline{W'} \subset W$ and $\overline{I'} \subset I$, so by Lemma 82 there exists $c \geq 0$ such that $\Delta_\delta h(b) \geq -c|\delta| \left|\left\langle \delta, \overline{\nabla h(b)} \right\rangle\right|$ for all $b \in \partial U \cap (W' \times I')$ and $\delta \in \mathbb{C}^n$. But the derivatives of h are independent of $\Im(z_n)$, and any point $z \in W' \times I' \subset \mathbb{C}^n$ can be written in the form $z = b + (0, \ldots, 0, i\alpha)$ for some $b \in \partial U \cap (W' \times I')$ and $\alpha \in \mathbb{R}$, so this inequality holds not only on ∂U but at every point of $W' \times I'$. Thus for $z \in W' \times I'$ and $\delta \in \mathbb{C}^n$ with $|\delta| = 1$ we have

$$\Delta_\delta h(z) \geq -c \left|\left\langle \delta, \overline{\nabla h(z)} \right\rangle\right| \geq -2c \left|\left\langle \delta, \overline{\nabla h(z)} \right\rangle\right|. \tag{4.1}$$

Let $r > 0$ so that $B(a,r) \subset W' \times I'$, and let $\mu\colon B(a,r) \cap U \to \mathbb{R}$ be given by $\mu(z) := -\ln(-h(z)) + c^2|z|^2$, so clearly μ is continuous and $\mu(z) \to +\infty$ as $z \to b$ for all $b \in B(a,r) \cap \partial U$. We will show that μ is plurisubharmonic on $B(a,r) \cap U$, and since $z \mapsto -\ln d(z, \partial B(a,r))$ is plurisubharmonic by pseudoconvexity of $B(a,r)$ it will follow that $z \mapsto \mu(z) - \ln d(z, \partial B(a,r))$ is a plurisubharmonic exhaustion function for each component of $B(a,r) \cap U$ (since it tends to $+\infty$ as the boundary points are approached), which will show that U is locally pseudoconvex at a. By the chain rule we have, for any $\delta \in \mathbb{C}^n$ with $|\delta| = 1$ and $z \in B(a,r) \cap U$,

$$\Delta_\delta \mu(z) = -\frac{1}{h(z)} \Delta_\delta h(z) + \frac{1}{h(z)^2} \left|\left\langle \delta, \overline{\nabla h(z)} \right\rangle\right|^2 + c^2 \geq -2c \left|\frac{\left\langle \delta, \overline{\nabla h(z)} \right\rangle}{h(z)}\right| + \left|\frac{\left\langle \delta, \overline{\nabla h(z)} \right\rangle}{h(z)}\right|^2 + c^2 \geq 0$$

where for the first $\geq$ we have used (4.1) and the fact that $|h(z)| = -h(z)$ since $h(z) < 0$, and for the second we have observed that the left hand side is a perfect square. Therefore μ is plurisubharmonic on $B(a,r) \cap U$, so as described above it follows that U is locally pseudoconvex at a. This is true for each $a \in \partial U$, meaning U is pseudoconvex. □

5 The Levi problem

We have now demonstrated that the domains of holomorphy and the holomorphically convex domains are identical, that all of these domains are pseudoconvex, and that the pseudoconvex domains are precisely those which are locally pseudoconvex, satisfy the continuity principle, and, in the case of domains with twice-differentiable boundaries, are Levi pseudoconvex. In this section we demonstrate that pseudoconvex domains are necessarily domains of holomorphy, thus completing the solution to the Levi problem. We apply methods based on those from [2], [3], [12], [22] and [23].

5.1 The inhomogeneous Cauchy-Riemann equations

It is an elementary fact that for a real-differentiable function $f\colon U \subset \mathbb{C}^n \to \mathbb{C}$ to be complex-differentiable at a point $a \in U$ it is necessary and sufficient that the partial derivatives $\frac{\partial f}{\partial \bar{z}_j}\big|_a$ be 0 for $1 \leq j \leq n$. We may reinterpret this condition in terms of differential forms.

Recall that a real differential form is a smooth alternating covariant tensor field. That is, a real m-form on an open $U \subset \mathbb{R}^n$ is a smooth map $\tau\colon U \times (\mathbb{R}^n)^m \to \mathbb{R}$ such that, for each fixed $p \in U$, the map $\tau(p,\cdot)\colon (\mathbb{R}^n)^m \to \mathbb{R}$ is linear in each argument and alternating. There are important 1-forms dx_j (for $1 \leq j \leq n$) defined by $dx_j(p, y_1, \dots, y_n) := y_j$. An m-form τ and a p-form σ can be combined via the wedge product to form an $(m+p)$-form $\tau \wedge \sigma$, and we recall that $\wedge$ is associative and that $\tau \wedge \sigma = (-1)^{mp}\sigma \wedge \tau$. We may define complex forms in terms of these real forms:

Definition 85. *Let $U \subset \mathbb{C}^n$ be a domain and $m \geq 0$ an integer. A* ***smooth (complex)*** m***-form*** *on U is a map $\omega\colon U \times (\mathbb{C}^n)^m \to \mathbb{C}$ given by $\omega = \tau + i\sigma$, where τ and σ are smooth m-forms on U when it is regarded as a subset of $\mathbb{R}^{2n}$. The space of such smooth complex m-forms is denoted $\Omega^m(U)$. For $1 \leq j \leq n$ we write $dz_j := dx_j + idx_{n+j} \in \Omega^1(U)$ and $d\bar{z}_j := dx_j - idx_{n+j} \in \Omega^1(U)$. Let $p \geq 0$. The* ***wedge product*** *$\wedge\colon \Omega^m(U) \times \Omega^p(U) \to \Omega^{m+p}(U)$ extends the usual wedge product according to*

$$(\tau_1 + i\sigma_1) \wedge (\tau_2 + i\sigma_2) := (\tau_1 \wedge \tau_2 - \sigma_1 \wedge \sigma_2) + i(\tau_1 \wedge \sigma_2 + \sigma_1 \wedge \tau_2)$$

for any $\tau_1 + i\sigma_1 \in \Omega^m(U)$, $\tau_2 + i\sigma_2 \in \Omega^p(U)$. A ***smooth*** $(0,m)$***-form*** *on U is a smooth complex m-form ω which can be written in the form*

$$\omega = \sum_{j_1=1,\dots,j_m=1}^{n} \omega_{j_1,\dots,j_m} d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_m}, \qquad \omega_{j_1,\dots,j_m} \in C^\infty_{\mathbb{C}}(U) \tag{5.1}$$

(where $C^\infty_{\mathbb{C}}(U)$ denotes the set of functions $U \to \mathbb{C}$ which are C^∞ when regarded as functions of $2n$ real variables mapping into $\mathbb{R}^2$).

We may introduce an exterior derivative:

Definition 86. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in C^\infty_{\mathbb{C}}(U)$ a function. Define the smooth $(0,1)$-form $\overline{\partial} f$ by*

$$\overline{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

For a smooth $(0,m)$-form ω given by (5.1), define the smooth $(0,m+1)$-form $\overline{\partial}\omega$ by

$$\overline{\partial}\omega := \sum_{j_1=1,\dots,j_m=1}^{n} (\overline{\partial}\omega_{j_1,\dots,j_m}) \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_m}.$$

We note the connection with the Cauchy-Riemann conditions – a real-differentiable function $f\colon U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic on U if and only if $\overline{\partial} f = 0$.

In some instances we will see that it is important to find a function $\phi \in C^\infty_{\mathbb{C}}(U)$ such that $\overline{\partial}\phi = \omega$, where ω is a prescribed smooth $(0,1)$-form. Roughly speaking, this will allow us to adjust smooth functions in order to make them holomorphic. If $\overline{\partial}\phi = \omega$ then applying $\overline{\partial}$ to both sides we have $\overline{\partial}^2\phi = \overline{\partial}\omega$, and it is readily verified that $\overline{\partial}^2 = 0$, so a necessary condition for the equation $\overline{\partial}\phi = \omega$ to be solvable (for ϕ) is that $\overline{\partial}\omega = 0$. In 1965 Hörmander proved, using Hilbert space theory, that when U is a pseudoconvex domain this condition is also sufficient [10]. For our purposes a weaker result suffices, where we assume the domain is smoothly bounded and strictly pseudoconvex:

Theorem 87 (Hörmander's theorem)**.** *Let $U \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain and let ω be a smooth $(0,1)$-form on U with $\overline{\partial}\omega = 0$. Then there exists $\phi \in C^\infty_{\mathbb{C}}(U)$ with $\overline{\partial}\phi = \omega$.*

We refer the reader to [12, chapter 4] for a proof.

5.2 The Oka-Weil approximation theorem

We will require the Oka-Weil approximation theorem, which is an important theorem of complex analysis in its own right:

Theorem 88 (Oka-Weil theorem)**.** *Let $U \subset \mathbb{C}^n$ be a domain of holomorphy and $K \subset U$ a compact subset with $K = \hat{K}$ (where $\hat{K} := \hat{K}_{H(U)}$ is the holomorphically convex hull of K). Then for all $f \in H(K)$ and $\epsilon > 0$ there exists $g \in H(U)$ such that $\|f - g\|_K < \epsilon$.*

Recall that for a function to be holomorphic on an arbitrary set C (not necessarily open) it must be holomorphic on a neighbourhood of C. Before we discuss the proof of this theorem we will introduce some useful terminology:

Definition 89. *Let $A \subset B \subset \mathbb{C}^n$ be sets such that for any $f \in H(A)$, any compact $K \subset A$ and any $\epsilon > 0$ there exists $g \in H(B)$ such that $\|f - g\|_K < \epsilon$. Then we say A* ***is Runge in*** *B or (A, B) is a* ***Runge pair****.*

Note that if A is compact, for A to be Runge in B it is necessary and sufficient that for all $f \in H(A)$ and $\epsilon > 0$ there exists $g \in H(B)$ such that $\|f - g\|_A < \epsilon$. Thus the Oka-Weil approximation theorem states that if K is a compact subset of a domain of holomorphy U which satisfies $K = \hat{K}_{H(U)}$ then (K, U) is a Runge pair. We have the following result which encapsulates the typical application of Runge pairs:

Proposition 90. *Let $\{K_j\}_{j\geq 1}$ be a sequence of compact sets in $\mathbb{C}^n$ such that $K_j \subset \mathring{K}_{j+1}$ and K_j is Runge in K_{j+1} for all $j \geq 1$. Then K_1 is Runge in $U := \bigcup_{j\geq 1} \mathring{K}_j$.*

Proof. Let $f \in H(K_1)$ be a function and $\epsilon > 0$. Let $g_1 := f \in H(K_1)$. Since K_1 is Runge in K_2 there is $g_2 \in H(K_2)$ such that $\|g_2 - g_1\|_{K_1} < \epsilon/2$. Similarly, K_2 is Runge in K_3 so there is $g_3 \in H(K_3)$ such that $\|g_3 - g_2\|_{K_2} < \epsilon/4$. In this way we obtain, for all $j \geq 1$, functions $g_j \in H(K_j)$ such that $\|g_{j+1} - g_j\|_{K_j} < \epsilon/2^j$.

Now consider the series $g(z) := g_1(z) + \sum_{j=1}^{\infty}(g_{j+1}(z) - g_j(z))$. For any $J \geq 1$ we may write $g(z) = g_J(z) + \sum_{j=J}^{\infty}(g_{j+1}(z) - g_j(z))$, so the value of $g(z)$ depends only on the tail of the sequence $\{g_j(z)\}_{j\geq 1}$. In particular, since for all $z \in U$ there exists $J \geq 1$ such that $j \geq J$ implies $z \in K_j$, g is well-defined on all of U (provided the series converges). Let $J \geq 1$. For any $j \geq J$ we have $\|g_{j+1} - g_j\|_{K_J} \leq \|g_{j+1} - g_j\|_{K_j} < \epsilon/2^j$, so the series converges uniformly on K_J by the Weierstrass M-test. It follows that g is holomorphic on $\mathring{K}_J$. This is true for each $J \geq 1$, so $g \in H(U)$. Moreover, for any $z \in K_1$ we have

$$|f(z) - g(z)| = |g(z) - g_1(z)| = \left|\sum_{j=1}^{\infty}(g_{j+1}(z) - g_j(z))\right| < \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon,$$

which implies $\|f - g\|_{K_1} < \epsilon$ as required. □

Before we prove the Oka-Weil theorem we need some intermediate facts. For the following three results and their proofs let $D := \overline{B(0,1)} \subset \mathbb{C}$.

Lemma 91. *Let $K \subset \mathbb{C}^n$ be a compact analytic polyhedron and U a neighbourhood of $K \times D$. Then there is an open set X consisting of a union of disjoint smoothly bounded strictly pseudoconvex domains such that $K \times D \subset X \subset U$.*

Proof. Let $\alpha > 1$ be a real number and $W \subset \mathbb{C}^n$ a neighbourhood of K such that $W \times B(0, \alpha) \subset U$ (this is possible by compactness of K and D). By Proposition 43 there is an open set of holomorphy V such that $K \subset V \subset W$. Each component of V is a domain of holomorphy, so by Proposition 38 each component of $V \times B(0, \alpha)$ is a domain of holomorphy and hence a pseudoconvex domain. Consider such a component Y, so $Y \cap (K \times D)$ is compact (as $K \times D$ does not intersect the boundary of Y). By Proposition 77, Y is given by an increasing union of smoothly bounded strictly pseudoconvex domains, so in particular there is a domain X_Y of this union such that $Y \cap (K \times D) \subset X_Y \subset Y$. Repeating this procedure for each component Y of $V \times B(0, \alpha)$ and taking the union X of the strictly pseudoconvex domains X_Y yields the desired open set $X \subset V \times B(0, \alpha) \subset U$ (note the domains X_Y are disjoint because the components Y of $V \times B(0, \alpha)$ are, by definition, disjoint). □

Lemma 92. *Let $K \subset \mathbb{C}^n$ be compact and $G \in H(K \times D)$. Then for (z, w) in a neighbourhood of $K \times D$, $G(z, w) = \sum_{j=1}^{\infty} a_j(z) w^j$ where for $j \geq 1$ we have $a_j \in H(K)$. Moreover, the series converges uniformly on $K \times D$.*

Proof. We have $G \in H(K_o \times D_o)$ where K_o and $D_o := B(0, \alpha) \subset \mathbb{C}$ (where $\alpha > 1$) are neighbourhoods of K and D respectively. For each $j \geq 1$ let $a_j(z) := \frac{1}{j!} \frac{\partial^j G}{\partial w^j}\big|_{(z,0)}$, so because the partial derivatives of a holomorphic function are holomorphic it follows that $a_j \in H(K_o) \subset H(K)$. For each fixed $z \in K_o$ the function $w \mapsto G(z, w)$ is holomorphic on D_o (and by reducing α if necessary we may assume $w \mapsto G(z, w)$ is continuous on $\overline{D_o}$), so $G(z, w) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j G}{\partial w^j}\big|_{(z,0)} w^j = \sum_{j=0}^{\infty} a_j(z) w^j$ for all $z \in K_o$ and $w \in D_o$.

Next we show uniform convergence on $K \times D$. For fixed $z \in K$ the function $w \mapsto G(z, w)$ is continuous on $\overline{D_o}$ and holomorphic on D_o, so by the Cauchy estimate

$$|a_j(z)| = \left| \frac{1}{j!} \frac{\partial^j G}{\partial w^j}\bigg|_{(z,0)} \right| \leq \frac{\|G\|_{\{z\} \times \overline{D_o}}}{\alpha^j} \leq \frac{\|G\|_{K \times \overline{D_o}}}{\alpha^j}.$$

Thus on $K \times D$ the terms of the series for G are bounded in norm by $\|G\|_{K \times \overline{D_o}} (w/\alpha)^j$, which are terms of a convergent geometric series (because $|w| \leq 1 < \alpha$). By the Weierstrass M-test it follows that the series for G converges uniformly on $K \times D$. □

Lemma 93. *Let $U \subset \mathbb{C}^n$ be a domain, $K \subset U$ a compact analytic polyhedron, $f \in H(U)$ and $L := \{z \in K : |f(z)| \leq 1\}$. Then L is Runge in K.*

Proof. Let $g \in H(L)$ and $\epsilon > 0$. We are to find $h \in H(K)$ such that $\|g - h\|_L < \epsilon$.

Since $g \in H(L)$ we have $g \in H(L_o)$ for some neighbourhood $L_o \subset U$ of L. Choose a bounded open V such that $L \subset V \subset \overline{V} \subset L_o$. If $K \setminus V = \emptyset$ then $K \subset V \subset L_o$, meaning $g \in H(K)$, so if we set $h := g$ then $h \in H(K)$ and $\|g - h\|_L = 0 < \epsilon$, as required. For the rest of the proof assume that $K \setminus V \neq \emptyset$. Let $\chi : \mathbb{C}^n \to \mathbb{R}$ be C^∞ and compactly supported in L_o with $\chi(z) = 1$ for $z \in V$. We have $\min_{z \in K \setminus V}\{|f(z)|\} > 1$, so there is a neighbourhood $W \subset U$ of $K \setminus V$ and a number $\alpha > 1$ such that $|f(z)| > \alpha$ for all $z \in W$. Let $\omega(z, w) := \frac{g(z) \overline{\partial} \chi(z)}{f(z) - w}$ for $z \in W \cup V$ and $w \in B(0, \alpha) \subset \mathbb{C}$. If $f(z) = w$ then $z \notin W$ and thus $z \in V$, so $\overline{\partial} \chi = 0$ in a neighbourhood of z. Therefore ω is a smooth $(0, 1)$-form, and because f and g are holomorphic we have $\overline{\partial} \omega = 0$ on $(W \cup V) \times B(0, \alpha)$.

Clearly $(W \cup V) \times B(0, \alpha)$ is a neighbourhood of $K \times D$, so by Lemma 91 there is an open set X given by a union of disjoint smoothly bounded strictly pseudoconvex domains such that $K \times D \subset X \subset (W \cup V) \times B(0, \alpha)$. Thus we may apply Hörmander's theorem to the restriction of ω to each component of X, and this will yield a function $\phi \in C^\infty_{\mathbb{C}}(X)$ such that $\overline{\partial}\phi = \omega$, so $g(z)\overline{\partial}\chi(z) = (f(z) - w)\overline{\partial}\phi(z, w)$.

Let $G\colon X \to \mathbb{C}$ be given by $G(z, w) := g(z)\chi(z) - (f(z) - w)\phi(z, w)$, so G is $C^\infty_{\mathbb{C}}$ and we see that $\overline{\partial}G(z, w) = g(z)\overline{\partial}\chi(z) - (f(z) - w)\overline{\partial}\phi(z, w) = 0$, so $G \in H(X)$. By Lemma 92 there are functions $\{a_j\}_{j\geq 1} \subset H(K)$ such that $G(z, w) = \sum_{j=1}^{\infty} a_j(z)w^j$ in a neighbourhood of $K \times D$, with uniform convergence on $K \times D$. For $z \in L$ we have $f(z) \in D$, and thus $g(z) = G(z, f(z)) = \sum_{j=1}^{\infty} a_j(z)f(z)^j$ with uniform convergence on L. Therefore for some large $m \geq 1$ the function $h(z) := \sum_{j=1}^{m} a_j(z)f(z)^j \in H(K)$ satisfies $\|g - h\|_L < \epsilon$, as required.

It follows that L is Runge in K. □

This lemma admits the following easy generalisation:

Corollary 94. *Let $U \subset \mathbb{C}^n$ be a domain, and let $K \subset U$ and $L \subset K$ be compact analytic polyhedra such that the frame of L is in $H(U)$. Then L is Runge in K.*

Proof. There exist functions $f_1, \ldots, f_m \in H(U)$ such that $L = \{z \in K\colon |f_j(z)| \leq 1, 1 \leq j \leq m\}$. Let $L_0 := K$, $L_1 := \{z \in L_0\colon |f_1(z)| \leq 1\}, \ldots, L_m := \{z \in L_{m-1}\colon |f_m(z)| \leq 1\}$. Note that each L_j (for $0 \leq j \leq m$) is a compact analytic polyhedron, and $L_m = L$. Now let $g \in H(L) = H(L_m)$ and $\epsilon > 0$. By Lemma 93 there exists $h_{m-1} \in H(L_{m-1})$ such that $\|g - h_{m-1}\|_L < \epsilon/m$. By Lemma 93 again there exists $h_{m-2} \in H(L_{m-2})$ such that $\|h_{m-1} - h_{m-2}\|_{L_{m-1}} < \epsilon/m$ and thus $\|g - h_{m-2}\|_L < 2\epsilon/m$. Repeating this argument we obtain functions $h_j \in H(L_j)$ such that $\|g - h_j\|_L < (m - j)\epsilon/m$ for $0 \leq j < m$. That is, there exists $h := h_0 \in H(L_0) = H(K)$ such that $\|g - h\|_L < \epsilon$, so L is Runge in K. □

This is the fundamental approximation result on which the proof of the Oka-Weil theorem is based, as it yields the following:

Proposition 95. *Let $U \subset \mathbb{C}^n$ be a domain of holomorphy and $K \subset U$ a compact analytic polyhedron with frame in $H(U)$. Then K is Runge in U.*

Proof. Let $\{K_j\}_{j\geq 1}$ be a sequence of compact sets with $K \subset \mathring{K}_j \subset K_j \subset \mathring{K}_{j+1}$ for all $j \geq 1$ and $\bigcup_{j\geq 1} \mathring{K}_j = U$ (take, for instance, $K_j := \{z \in U\colon d(z, \partial U) \geq r/j$ and $|z| \leq R + j\}$ for each $j \geq 1$, where $r := d(K, \partial U)/2 > 0$ and $R := \max_{z\in K}\{|z|\}$). For each $j \geq 1$ the hull $\hat{K}_j$ is compact (because U is holomorphically convex) and satisfies $\hat{\hat{K}}_j = \hat{K}_j$, so by Proposition 44 there exist compact analytic polyhedra L_j with frames in $H(U)$ such that $\hat{K}_j \subset L_j \subset U$ for each $j \geq 1$. Clearly $U = \bigcup_{j\geq 1} \mathring{L}_j$ (since $\mathring{K}_j \subset \mathring{L}_j \subset U$ for all $j \geq 1$), so by passing to a subsequence if necessary we may assume $L_j \subset \mathring{L}_{j+1}$ for each $j \geq 1$. Let $L_0 := K$, and consider the sequence

$\{L_j\}_{j\geq 0}$. For each $j \geq 0$ we have $L_j \subset \mathring{L}_{j+1}$, where L_j and L_{j+1} are compact analytic polyhedra with frames in $H(U)$, so L_j is Runge in L_{j+1} by Corollary 94. But we also have $U = \bigcup_{j\geq 0} \mathring{L}_j$, so by Proposition 90 it follows that $L_0 = K$ is Runge in U. □

The Oka-Weil theorem is now easily proved:

Proof of Theorem 88. We have a domain of holomorphy $U \subset \mathbb{C}^n$ and a compact subset $K \subset U$ with $K = \hat{K}$, and we are to show K is Runge in U. Let $f \in H(K)$ and $\epsilon > 0$. It is enough to find $g \in H(U)$ so that $\|f - g\|_K < \epsilon$. Since $f \in H(K)$ there is a neighbourhood K_o of K such that $f \in H(K_o)$. By Proposition 44 there is a compact analytic polyhedron L with frame in $H(U)$ such that $K \subset L \subset K_o$. By Proposition 95 we know that L is Runge in U, so since $f \in H(L)$ there is a function $g \in H(U)$ such that $\|f - g\|_K < \epsilon$, as required. □

We have a useful corollary:

Corollary 96. *Let $U \subset \mathbb{C}^n$ be a domain of holomorphy, $K \subset U$ a compact subset and $V \subset U$ a neighbourhood of K such that $\hat{K} \cap \partial V = \emptyset$. Then $\hat{K} \subset V$.*

Proof. Let $W := \overline{V}^c$ be the exterior of V, so $\hat{K} \subset V \cup W$. Let $f \in H(V \cup W)$ be identically equal to 0 on V and to 1 on W, so by the Oka-Weil theorem (and the fact that $\hat{\hat{K}} = \hat{K}$) there exists $g \in H(U)$ with $\|f - g\|_{\hat{K}} < 1/2$. We have $f \equiv 0$ on K, so $\|g\|_K < 1/2$. If $z \in W \cap \hat{K}$ then $f(z) = 1$, so $|g(z)| > 1/2 > \|g\|_K$ and hence $z \notin \hat{K}$, which is a contradiction. Thus $W \cap \hat{K} = \emptyset$, so $\hat{K} \subset V$. □

5.3 The Behnke-Stein theorem

Recall from Section 3 the Behnke-Stein theorem:

Theorem 39 (Behnke-Stein theorem)**.** *Let $\{U_j\}_{j\geq 1}$ be a sequence of domains of holomorphy such that $U_j \subset U_{j+1}$ for all $j \geq 1$. Then $U := \bigcup_{j\geq 1} U_j$ is a domain of holomorphy.*

To prove the Behnke-Stein theorem we will invoke the following lemmas (note that if $U = \emptyset$ or $U = \mathbb{C}^n$ the assertions of the results in this subsection are trivial, so in the proofs we will assume this is not the case):

Lemma 97. *Let $\{K_j\}_{j\geq 1}$ be a sequence of compact sets in $\mathbb{C}^n$ such that $K_j \subset K_{j+1}$ and K_j is Runge in K_{j+1} for all $j \geq 1$. Let $U := \bigcup_{j\geq 1} \mathring{K}_j$, and suppose there is a sequence $\{U_j\}_{j\geq 1}$ of domains of holomorphy such that $K_j \subset U_j \subset U_{j+1} \subset U$ for each $j \geq 1$. Then U is a domain of holomorphy.*

Proof. Using the fact that $U = \bigcup_{j\geq 1} \mathring{K}_j$ we may pass to subsequences if necessary and assume, in addition to the hypotheses, that $K_j \subset \mathring{K}_{j+1}$ for all $j \geq 1$. By Proposition 90 it follows that K_j is Runge in U for all $j \geq 1$.

Note that $U = \bigcup_{j\geq 1} U_j$, so U is connected and hence a domain. We will show that U is holomorphically convex. Let $K \subset U$ be compact, and let $a \in U$ with $\rho(a, \partial U) < \rho(K, \partial U)$. We can show $a \notin \hat{K}$. Let $j \geq 1$ be sufficiently large that $\{a\} \cup K \subset K_j$, and let $k > j$ be sufficiently large that $\rho(a, \partial U_k) < \rho(K, \partial U_k)$ (this is possible because as $k \to \infty$ we have $\rho(a, \partial U_k) \to \rho(a, \partial U)$ and $\rho(K, \partial U_k) \to \rho(K, \partial U)$). By Lemma 37 we have $a \notin \hat{K}_{H(U_k)}$, so there is $f \in H(U_k) \subset H(K_j)$ such that $\epsilon := |f(a)| - \|f\|_K > 0$. Since K_j is Runge in U there is $g \in H(U)$ such that $\|f - g\|_{K_j} < \epsilon/2$. Therefore $|g(a)| - \|g\|_K > |f(a)| - \|f\|_K - 2\epsilon/2 = 0$, so $a \notin \hat{K}$. Therefore $\rho(\hat{K}, \partial U) \geq \rho(K, \partial U) > 0$, so $\hat{K}$ is compact. This is true for each compact $K \subset U$, so U is holomorphically convex and hence a domain of holomorphy. □

We have another lemma, which will allow us to consider only bounded domains in the proof of the Behnke-Stein theorem:

Lemma 98. *Let $U \subset \mathbb{C}^n$ be a domain, let $z_0 \in U$, and suppose the component of $U \cap \Delta(z_0, r)$ containing z_0 is a domain of holomorphy for all $r > 0$. Then U is a domain of holomorphy.*

Proof. For each $j \geq 1$ let U_j be the component of $U \cap \Delta(z_0, j)$ containing z_0. We first make an observation. Suppose $K \subset U_j$ is compact, so by holomorphic convexity of U_{j+1} the hull $\hat{K}_{H(U_{j+1})}$ is compact. We also know $\hat{K}_{H(U_{j+1})} \subset \hat{K}_{H(\Delta(z_0,j+1))}$ since $U_{j+1} \subset \Delta(z_0, j+1)$. Furthermore, $\rho(\hat{K}_{H(\Delta(z_0,j+1))}, \partial\Delta(z_0, j+1)) = \rho(K, \partial\Delta(z_0, j+1)) > 1$ (we have used Lemma 37 for the equality) which implies $\hat{K}_{H(\Delta(z_0,j+1))} \subset \Delta(z_0, j)$. Therefore $\hat{K}_{H(U_{j+1})} \subset \Delta(z_0, j)$, and obviously $\hat{K}_{H(U_{j+1})} \subset U_{j+1} \subset U$, meaning $\hat{K}_{H(U_{j+1})} \subset U \cap \Delta(z_0, j)$. It follows that $\hat{K}_{H(U_{j+1})} \cap \partial U_j = \emptyset$ (since U_j is a component of $U \cap \Delta(z_0, j)$), so by Corollary 96 we have $\hat{K}_{H(U_{j+1})} \subset U_j$. That is, if $K \subset U_j$ is compact then $\hat{K}_{H(U_{j+1})}$ is a compact subset of U_j.

Let $\alpha := \rho(z_0, \partial U)/2$. For each $j \geq 1$ let $L_j \subset U_j$ be the component of $\{z \in U \colon \rho(z, \partial U) \geq \alpha/j \text{ and } \rho(z, z_0) \leq j - 1\}$ containing z_0, so L_j is compact, and let $K_j := (\hat{L}_j)_{H(U_{j+1})} \subset U_j$ (where we have used the observation at the start of the proof). Clearly $U = \bigcup_{j\geq 1} \mathring{L}_j$ and $\mathring{L}_j \subset \mathring{K}_j \subset U$ for all $j \geq 1$, so $U = \bigcup_{j\geq 1} \mathring{K}_j$. We also have $K_j \subset K_{j+1}$ and $K_j \subset U_j \subset U_{j+1} \subset U$ for all $j \geq 1$, and that each U_j is a domain of holomorphy, so by Lemma 97 to show that U is a domain of holomorphy it is enough to show that K_j is Runge in K_{j+1} for each j. But for each $j \geq 1$, K_j coincides with its $H(U_{j+1})$-convex hull, so K_j is Runge in U_{j+1} by the Oka-Weil theorem, and so obviously K_j is Runge in $K_{j+1} \subset U_{j+1}$. Therefore U is a domain of holomorphy. □

We need one more lemma:

Lemma 99. *Let $U \subset \mathbb{C}^n$ be a domain of holomorphy and let $z_0 \in U$. Then for all $\epsilon > 0$ with $\epsilon < \rho(z_0, \partial U)$, the component of $\{z \in U : \rho(z, \partial U) > \epsilon\}$ containing z_0 is a domain of holomorphy.*

Proof. Suppose $0 < \epsilon < \rho(z_0, \partial U)$. Let the component of $\{z \in U : \rho(z, \partial U) > \epsilon\}$ containing z_0 be V. Let $K \subset V$ be compact, so by holomorphic convexity of U and Lemma 37 the hull $\hat{K}_{H(U)}$ satisfies $\rho(\hat{K}_{H(U)}, \partial U) = \rho(K, \partial U) > \epsilon$, meaning $\hat{K}_{H(U)} \subset \{z \in U : \rho(z, \partial U) > \epsilon\}$. It follows that $\hat{K}_{H(U)} \cap \partial V \neq \emptyset$, so by Corollary 96 we see that $\hat{K}_{H(U)}$ is a compact subset of V, and obviously $\hat{K}_{H(V)} \subset \hat{K}_{H(U)}$, so $\hat{K}_{H(V)}$ is compact. This is true for each $K \subset V$, so V is holomorphically convex and hence a domain of holomorphy. □

Finally we can prove the Behnke-Stein theorem:

Proof of Theorem 39. We have a sequence $\{U_j\}_{j \geq 1}$ of domains of holomorphy in $\mathbb{C}^n$ such that $U_j \subset U_{j+1}$ for all $j \geq 1$. We are to show $U := \bigcup_{j \geq 1} U_j$ is a domain of holomorphy. First suppose U is bounded. Choose $z_0 \in U_1$, set $\epsilon := \rho(z_0, \partial U_1)/2$ and replace each U_j with the component of $\{z \in U_j : \rho(z, \partial U_j) > \epsilon/j\}$ containing z_0, so we still have that each U_j is a domain of holomorphy (by Lemma 99) and that $U = \bigcup_{j \geq 1} U_j$, but now $\overline{U_j} \subset U_{j+1}$ for all $j \geq 1$ and in particular $\rho(U_j, \partial U) > 0$.

Now we construct a subsequence $\{V_k\}_{k \geq 1}$ of $\{U_j\}_{j \geq 1}$ satisfying, for all $k \geq 2$, the inequality

$$\max_{z \in \partial V_k} \rho(z, \partial V_{k+1}) < \rho(V_{k-1}, \partial V_{k+1}). \tag{5.2}$$

Let $V_1 := U_1$, choose $j_2 > 1$ sufficiently large that $\{z \in U : \rho(z, \partial U) \geq \rho(V_1, \partial U)\} \subset U_{j_2}$ and let $V_2 := U_{j_2}$, so $\max_{z \in \partial V_2} \rho(z, \partial U) < \rho(V_1, \partial U)$. As $j \to \infty$ we have $\rho(z, \partial U_j) \to \rho(z, \partial U)$ (for every $z \in \partial V_2$) and $\rho(V_1, \partial U_j) \to \rho(V_1, \partial U)$, so since ∂V_2 is compact it follows from basic metric space theory that there exists $J > j_2$ such that $j \geq J$ implies $\max_{z \in \partial V_2} \rho(z, \partial U_j) < \rho(V_1, \partial U_j)$. Now, as with our choice of j_2, choose $j_3 \geq J$ sufficiently large that $\{z \in U : \rho(z, \partial U) \geq \rho(V_2, \partial U)\} \subset U_{j_3}$ and let $V_3 := U_{j_3}$, so (5.2) is satisfied for $k = 2$ and $\max_{z \in \partial V_3} \rho(z, \partial U) < \rho(V_2, \partial U)$.

Now we proceed inductively. Let $m \geq 3$, and suppose we have sets $\{V_k\}_{1 \leq k \leq m}$ (with $V_k = U_{j_k}$ for each k) such that (5.2) is satisfied for $2 \leq k \leq m-1$ and $\max_{z \in \partial V_m} \rho(z, \partial U) < \rho(V_{m-1}, \partial U)$ (the above argument yields such a construction in the base case $m = 3$). As above, there exists $J > j_m$ such that $j \geq J$ implies $\max_{z \in \partial V_m} \rho(z, \partial U_j) < \rho(V_{m-1}, \partial U_j)$. Again, as above we may choose $j_{m+1} \geq J$ so that $\{z \in U : \rho(z, \partial U) \geq \rho(V_m, \partial U)\} \subset U_{j_{m+1}}$ and set $V_{m+1} := U_{j_{m+1}}$, which yields the desired properties (5.2) and $\max_{z \in \partial V_{m+1}} \rho(z, \partial U) < \rho(V_m, \partial U)$. Thus we have constructed the appropriate sets $\{V_k\}_{1 \leq k \leq m+1}$.

Repeating this process we obtain the subsequence $\{V_k\}_{k \geq 1}$ satisfying (5.2) for all $k \geq 2$. For $k \geq 1$ let $L_k := \overline{V_k}$ and for $k \geq 2$ let $K_k := (\hat{L}_{k-1})_{H(V_{k+1})}$, so K_k is a compact subset of V_{k+1} by holomorphic convexity of V_{k+1}. Let $k \geq 2$ and suppose $z \in \partial V_k$. Then by (5.2) $\rho(z, \partial V_{k+1}) < \rho(V_{k-1}, \partial V_{k+1}) = \rho(L_{k-1}, \partial V_{k+1}) = \rho(K_k, \partial V_{k+1})$ (for the last equality we have used Lemma 37), so $z \notin K_k$. That is, $K_k \cap \partial V_k = \emptyset$. Since $L_{k-1} \subset V_k$, Corollary 96 implies that $K_k \subset V_k$.

Thus $K_k \subset K_{k+1}$ and $K_k \subset V_k \subset V_{k+1} \subset U$ for all $k \geq 2$, and clearly $U = \bigcup_{k\geq 2} \mathring{K}_k$, so by Lemma 97 to show U is a domain of holomorphy it is enough to show K_k is Runge in K_{k+1} for all $k \geq 2$. But K_k coincides with its $H(V_{k+1})$-convex hull, so by the Oka-Weil approximation theorem K_k is Runge in V_{k+1}, and because $K_{k+1} \subset V_{k+1}$ it follows that K_k is Runge in K_{k+1}. Therefore U is a domain of holomorphy.

Now suppose U is unbounded, and let $z_0 \in U_1$ and $r > 0$. For each $j \geq 1$ let W_j be the component of $U_j \cap \Delta(z_0, r)$ containing z_0, so $W_j \subset W_{j+1}$ for all $j \geq 1$, each W_j is a domain of holomorphy by Proposition 36, and $\bigcup_{j\geq 1} W_j$ is the component of $U \cap \Delta(z_0, r)$ containing z_0. The above argument shows this (bounded) component is a domain of holomorphy, and this is true for each $r > 0$, so by Lemma 98 we conclude that U is a domain of holomorphy. □

Observe that the proof of this relatively simple fact about domains of holomorphy is long and complicated, while the corresponding result for pseudoconvex domains is almost trivial (see Lemma 67). This indicates the power of a more easily verified characterisation of domains of holomorphy – if we had already solved the Levi problem, the Behnke-Stein theorem would follow immediately from Lemma 67.

5.4 Construction of unbounded functions

Finally we are in a position to complete the solution to the Levi problem. First we show that smoothly bounded strictly pseudoconvex domains are domains of holomorphy, and for this it is enough (by Theorem 25) to show that at every boundary point of such a domain there is a function holomorphic on the domain and tending to ∞ at that point. The following result ensures that such functions always exist:

Proposition 100. *Let $U \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain and let $a \in \partial U$. Then there exists $f \in H(U)$ tending to ∞ at a.*

To prove this we require an intermediate result, which in turn requires Taylor's theorem expressed in a particular form. Let $V \subset \mathbb{C}^n$ be open, $a \in V$ and $f\colon V \to \mathbb{R}$ a C^2 function. Regarding f as a function of real variables, applying Taylor's theorem, and then writing the derivatives with respect to real variables in terms of those with respect to complex variables, we find that

$$f(z) = f(a) + 2\Re \left\langle z - a, \overline{\nabla f(a)} \right\rangle + \Re\left(\Lambda_{z-a} f(a)\right) + \Delta_{z-a} f(a) + r(z), \qquad (5.3)$$

where $\Lambda_{z-a} f(a) := \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial z_k}\Big|_a (z_j - a_j)(z_k - a_k)$ and $r(z)/|z-a|^2 \to 0$ as $z \to a$.

Lemma 101. *Let $U \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain and let $a \in \partial U$. Then there is a function $g \in H(\mathbb{C}^n)$ and a neighbourhood V of a such that $g(a) = 0$ and $g(z) \neq 0$ for $z \in \overline{U} \cap V \setminus \{a\}$.*

Proof. Let $f\colon W \to \mathbb{R}$ be a C^∞ strictly plurisubharmonic global defining function for U, and for $z \in \mathbb{C}^n$ set $g(z) := 2\left\langle z-a, \overline{\nabla f(a)}\right\rangle + \Lambda_{z-a} f(a)$, so clearly $g \in H(\mathbb{C}^n)$ and $g(a) = 0$. By Taylor's theorem we have $f(z) = \Re(g(z)) + \Delta_{z-a} f(a) + r(z)$, where $r(z)/|z-a|^2 \to 0$ as $z \to a$. Since f is strictly plurisubharmonic we know that $\Delta_\delta f(a) > 0$ for all $\delta \in \mathbb{C}^n \setminus \{0\}$, and since $\delta \mapsto \Delta_\delta f(a)$ is continuous and $\{\delta \in \mathbb{C}^n \colon |\delta| = 1\}$ is compact there is some $M > 0$ such that $\Delta_\delta f(a) \geq M$ for all $\delta \in \mathbb{C}^n$ with $|\delta| = 1$. Since $r(z)/|z-a|^2 \to 0$ as $z \to 0$ there is a neighbourhood V of a such that $|r(z)|/|z-a|^2 < M/2$ for $z \in V$.

Suppose $z \in \overline{U} \cap V \setminus \{a\}$. Then

$$\begin{aligned} 0 \geq f(z) = \Re(g(z)) + \Delta_{z-a} f(a) + r(z) &\geq \Re(g(z)) + |z-a|^2 M - |z-a|^2 M/2 \\ &= \Re(g(z)) + |z-a|^2 M/2, \end{aligned}$$

and since $|z-a|^2 M/2 > 0$ it follows that $\Re(g(z)) < 0$ and thus $g(z) \neq 0$, as required. $\square$

Now we can prove Proposition 100.

Proof of Proposition 100. By Lemma 101 there exists a function $g \in H(\mathbb{C}^n)$ and a neighbourhood V of a such that $g(a) = 0$ and $g(z) \neq 0$ for $z \in \overline{U} \cap V \setminus \{a\}$. Passing to a subset of V we may assume that $g(z) \neq 0$ for $z \in \overline{U} \cap \overline{V} \setminus \{a\}$, so in particular there is a neighbourhood Y of $\overline{U} \setminus \{a\}$ such that g is non-vanishing on $V \cap Y$. Let W be a bounded neighbourhood of a such that $\overline{W} \subset V$, and replace Y with $Y \cup W$, so now Y is a neighbourhood of $\overline{U}$ and g is non-vanishing on $V \cap (Y \setminus W)$.

Let $\chi\colon \mathbb{C}^n \to \mathbb{R}$ be a C^∞ function compactly supported in V and satisfying $\chi(z) = 1$ for $z \in W$. Let $\omega := \frac{\overline{\partial}\chi}{g}$, so because $\overline{\partial}\chi$ is zero on a neighbourhood of $W \cup V^c$ and g is non-vanishing on $V \cap (Y \setminus W)$ we see that ω is a smooth $(0,1)$-form on Y. Furthermore, since g is holomorphic we have $\overline{\partial}\omega = 0$.

Now we will replace Y with a smoothly bounded strictly pseudoconvex domain. Let $g\colon Z \to \mathbb{R}$ (where Z is a neighbourhood of $\overline{U}$) be a C^∞ strictly plurisubharmonic global defining function for U, so $g^{-1}((-\infty, \epsilon)) \subset Y$ for sufficiently small $\epsilon > 0$. In view of Lemmas 79 and 80 by decreasing ϵ if necessary we may assume $z \mapsto g(z) - \epsilon$ is a global defining function for the component of $g^{-1}((-\infty, \epsilon))$ containing U. Replace Y with this component, so Y is now a smoothly bounded strictly pseudoconvex domain such that ω is smooth on Y and $\overline{U} \subset Y$.

By Hörmander's theorem there exists $\phi \in C^\infty_{\mathbb{C}}(Y)$ with $\overline{\partial}\phi = \frac{\overline{\partial}\chi}{g}$, so the function $f\colon U \to \mathbb{C}$ given by $f(z) := \frac{\chi(z)}{g(z)} - \phi(z)$ is $C^\infty_{\mathbb{C}}$ (as g is non-vanishing on U), and since $\overline{\partial} f = \frac{\overline{\partial}\chi}{g} - \overline{\partial}\phi = 0$ it follows that $f \in H(U)$. As $z \to a$ (with $z \in U$) we have

$g(z) \to 0$, $\chi(z) \to 1$ and $\phi(z) \to \phi(a) \in \mathbb{C}$, so $f(z) \to \infty$. Thus f tends to ∞ at a, as required. □

In conjunction with Theorem 25, this immediately implies:

Corollary 102. *If $U \subset \mathbb{C}^n$ is a smoothly bounded strictly pseudoconvex domain then U is a domain of holomorphy.*

The required result now follows easily:

Theorem 103. *If $U \subset \mathbb{C}^n$ is a pseudoconvex domain then U is a domain of holomorphy.*

Proof. By Proposition 77 there exists a sequence $\{U_j\}_{j\geq 1}$ of smoothly bounded strictly pseudoconvex domains such that $U_j \subset U_{j+1}$ for all $j \geq 1$ and $U = \bigcup_{j\geq 1} U_j$, and by Corollary 102 each U_j is a domain of holomorphy, so by the Behnke-Stein theorem U is a domain of holomorphy. □

With the results of previous sections in mind we see that the domains of holomorphy, holomorphically convex domains, pseudoconvex domains, locally pseudoconvex domains and domains satisfying the continuity principle are identical. In the case of domains with twice-differentiable boundaries, these domains are precisely the Levi pseudoconvex domains. This completes the solution to the Levi problem.

5.5 Generalisations

Here we have considered the Levi problem and its related concepts in their simplest setting, namely for domains in $\mathbb{C}^n$. Such domains are the most basic type of *complex manifold* – these are topological manifolds which admit atlases with biholomorphic transition maps, and on these spaces one may naturally define the notion of a holomorphic function (see, for example, [23, subsection 12]). The natural generalisation of a domain of holomorphy to general complex manifolds is a *Stein manifold*, which is a complex manifold satisfying holomorphic convexity, defined as in Section 3, and holomorphic separability, which means that for any two points in the manifold there is a holomorphic function attaining different values at the points (see [23, page 223] for an introduction). The Levi problem in this setting asks for necessary and sufficient conditions for a complex manifold to be Stein. In 1953 Oka solved the Levi problem for Riemann domains spread over $\mathbb{C}^n$ (which are special types of complex manifold) [21], and in 1958 Grauert generalised this result to general complex manifolds [7]. This was further generalised by Narasimhan in 1961 who proved that a complex space is a Stein space if and only if it admits a strictly plurisubharmonic exhaustion function [18]. A *complex space* is a generalisation of a complex manifold which allows the presence of singularities, and a *Stein space* is the corresponding generalisation of a Stein manifold (see chapters 5 and 7 of the textbook *Analytic Functions of Several Complex Variables* by Gunning and Rossi [8] for an introduction). In the intervening years solutions to the Levi problem for even more general spaces have been presented – see, for example, the article [24] for a more detailed survey of these results and their implications. To conclude, we refer the reader to the website [25] for a comprehensive list of historical references.

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Harry J. Slatyer
Department of Quantum Science, The Australian National University,
Canberra, ACT 0200, Australia.
e-mail: harry.slatyer@anu.edu.au

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