

## ON DYNAMICS OF QUADRATIC STOCHASTIC OPERATORS: A SURVEY

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**Abstract.** We discuss the notion of Volterra,  $\ell$ -Volterra and separable quadratic stochastic operators defined on  $(m - 1)$ -dimensional simplex, where  $\ell \in \{0, 1, \dots, m\}$ . The  $\ell$ -Volterra operator is a Volterra operator if and only if  $\ell = m$ . We study the structure of the set of all Volterra and  $\ell$ -Volterra operators and describe their several fixed and periodic points. For  $m = 2$  and  $m = 3$  we describe behavior of trajectories of  $(m - 1)$ -Volterra operators. We also mention many remarks with comparisons of  $\ell$ -Volterra operators and Volterra ones. Also we discuss the dynamics of separable quadratic stochastic operators.

### 1 Introduction

There are many systems which are described by nonlinear operators. Quadratic is one of the simplest nonlinear cases. Quadratic dynamical systems have been proved to be a rich source of analysis for the investigation of dynamical properties and modeling in different domains, such as population dynamics in physics, economics and mathematics. On the other hand, the theory of Markov processes is a rapidly developing field with numerous applications to many branches of mathematics and physics. However, Markov processes fails to describe some physical and biological system. One of such system is given by Quadratic Stochastic Operators (QSOs) [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and they are related to population genetics. The main problem is, to study the behavior of trajectories of Quadratic Stochastic Operators (QSOs). The limit behavior and properties of trajectories of Quadratic Stochastic Operators and their applications to population genetics were studied.

A Quadratic Stochastic Operator (QSOs) has meaning of a population evolution Operator in biology which can be described as follows: Consider a population consisting of  $m$  species i.e.  $E = \{1, 2, \dots, m\}$ . Let  $(x^0) = (x_1^0, x_2^0, \dots, x_m^0)$  be the

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probability distribution of species in the initial generations, and  $p_{ij,k}$  the probability that individuals in the  $i$ th and  $j$ th species interbreed to produce an individual  $k$ . Then the probability distribution  $x' = (x'_1, x'_2, \dots, x'_m)$  of the species in the first generation can be found by the total probability,

$$\text{where } x'_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j. \quad (1.1)$$

This means that the association  $x^0 \rightarrow x'$  defines a map  $V$  called the evolution operator. The population evolves by starting from an arbitrary state  $x^0$ , then passing to the state  $x' = V(x^0)$ , similarly  $x'' = V(V(x^0)) = V^2(x^0)$  and so on. Hence states of the population are described by the following dynamical system

$$x^0, x' = V(x^0), x'' = V^2(x^0), x''' = V^3(x^0) \text{ and so on.}$$

Note that  $V$  defined in (1.1) is a nonlinear Quadratic Operator and if  $m \geq 3$ , then it is higher dimensional. The dynamics of Quadratic Operators were basically defined due to some recurrent rule which marks a possibility to study asymptotic behaviors of such operators. Under some conditions on coefficients of such operators we describe Lyapunov Functions on them. We will also describe a set of fixed points of the Volterra Operators [4, 9].

This article is arranged as follows: In section 2 we will discuss about the definition of Simplex, Quadratic Stochastic Operators (QSOs), the properties of Quadratic Stochastic Operators (QSOs). We will see that each Quadratic Stochastic Operators (QSOs) will be uniquely defined by a cubic matrix  $P = (p_{ij,k})_{i,j,k=1}^m$ .

In section 3 and 4 we study Volterra Quadratic Stochastic Operators, Canonical form of Volterra's discrete model, extremal points and the compactness of Volterra Quadratic Stochastic Operators, Lyapunov functions [2, 7], limits and critical points of Lyapunov functions of Volterra Quadratic Stochastic Operators.

In section 5 we study  $l$ -Volterra Quadratic Stochastic Operators, Canonical form of  $l$ -Volterra's discrete model, extremal points and the compactness of  $l$ -Volterra Quadratic Stochastic Operators, Lyapunov functions, for  $m = 2$  and  $3$  we describe behavior of trajectories of  $(m - 1)$ -Volterra operators.

Section 6 is devoted to the study of Separable Quadratic Stochastic Operators.

## 2 Preliminaries

In this chapter, we will study the definition of Simplex, Face of the Simplex, Relative Inside of the Face of Simplex and some examples. Also we will discuss some canonical form of Volterra Operators in discrete models.

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### 2.1 Simplex and Quadratic Stochastic Operators

In this Section we will give some basic definitions:

**Definition 1.** *The set*

$$S^{m-1} = \{x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m x_i = 1, 0 \leq x_i \leq 1\}$$

*is called Simplex with dimension  $m - 1$ .*

**Definition 2.** *The Quadratic Stochastic Operator is a mapping defined as*

$$V : S^{m-1} \rightarrow S^{m-1}$$

*such that  $V(x) = x'$  i.e.  $V(x_1, x_2, \dots, x_m) = (x'_1, x'_2, \dots, x'_m)$*

*where*

$$x'_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad \forall k = 1, 2, \dots, m, \tag{2.1}$$

*and  $p_{ij,k}$  is called coefficient of heredity, which satisfy the following conditions*

$$p_{ij,k} = p_{ji,k}, \quad 0 \leq p_{ij,k} \leq 1, \quad \sum_{k=1}^m p_{ij,k} = 1, \quad (i, j, k = 1, \dots, m). \tag{2.2}$$

*Note that each such operator can be uniquely defined by a cubic matrix  $P = (p_{ij,k})_{i,j,k=1}^m$ , where  $m \in \mathbb{R}$ .*

**Example 3.** *For the system  $E = \{1, 2\}$ , find the corresponding quadratic stochastic operator.*

**Solution:** Given that  $E = \{1, 2\}$  then the corresponding QSO is given by  $V(x_1, x_2) = (x'_1, x'_2)$ . Now by definition, we can find  $x'_1$  and  $x'_2$  as:

$$\begin{aligned} x'_1 &= \sum_{i,j=1}^2 P_{ij,1} x_i x_j \\ &= \sum_{i=1}^2 \sum_{j=1}^2 p_{ij,1} x_i x_j \\ &= \sum_{i=1}^2 [p_{i1,1} x_i x_1 + p_{i2,1} x_i x_2] \\ &= \sum_{i=1}^2 p_{i1,1} x_i x_1 + \sum_{i=1}^2 p_{i2,1} x_i x_2 \\ &= p_{11,1} x_1 x_1 + p_{21,1} x_2 x_1 + p_{12,1} x_1 x_2 + p_{22,1} x_2 x_2 \\ &= p_{11,1} x_1^2 + 2p_{12,1} x_1 x_2 + p_{22,1} x_2^2. \end{aligned}$$

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Similarly

$$\begin{aligned}
 x_2' &= \sum_{i,j=1}^2 P_{ij,2} x_i x_j \\
 &= \sum_{i=1}^2 \sum_{j=1}^2 p_{ij,2} x_i x_j \\
 &= \sum_{i=1}^2 [p_{i1,2} x_i x_1 + p_{i2,2} x_i x_2] \\
 &= \sum_{i=1}^2 p_{i1,2} x_i x_1 + \sum_{i=1}^2 p_{i2,2} x_i x_2 \\
 &= p_{11,2} x_1 x_1 + p_{21,2} x_2 x_1 + p_{12,2} x_1 x_2 + p_{22,2} x_2 x_2 \\
 &= p_{11,2} x_1^2 + 2p_{12,2} x_1 x_2 + p_{22,2} x_2^2.
 \end{aligned}$$

So

$$V(x_1, x_2) = (p_{11,1} x_1^2 + 2p_{12,1} x_1 x_2 + p_{22,1} x_2^2, p_{11,2} x_1^2 + 2p_{12,2} x_1 x_2 + p_{22,2} x_2^2).$$

**Example 4.** If  $m=2$  and  $i, j, k \in \{1, 2\}$ . Then

$$P = \left( \begin{array}{cc|cc} p_{11,1} & p_{12,1} & p_{21,1} & p_{22,1} \\ p_{11,2} & p_{12,2} & p_{21,2} & p_{22,2} \end{array} \right).$$

Thus the corresponding Quadratic Stochastic Operator (QSO) from above matrix

$$\text{for } k = 1 \text{ is } x_1' = p_{11,1} x_1^2 + 2p_{12,1} x_1 x_2 + p_{22,1} x_2^2,$$

and

$$\text{for } k = 2 \text{ is } x_2' = p_{11,2} x_1^2 + 2p_{12,2} x_1 x_2 + p_{22,2} x_2^2.$$

### 3 Volterra Operators

First to study the Canonical form of Volterra's discrete model.

#### 3.1 Canonical form of Volterra's discrete model

**Definition 5.** A Quadratic Stochastic Operator  $V : S^{m-1} \rightarrow S^{m-1}$  is called Volterra if

$$p_{ij,k} = 0, \forall k \notin \{i, j\}. \quad (3.1)$$

The biological treatment of condition (3.1) is clear i.e. the offspring repeats the genotype of one of its parents.

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**Example 6.** Let  $E = \{1, 2\}$ , then  $V(x_1, x_2) = (x'_1, x'_2)$ . Find  $x'_1$  and  $x'_2$ .

**Solution:** Since

$$x'_1 = p_{11,1}x_1^2 + 2p_{12,1}x_1x_2 + p_{22,1}x_2^2,$$

and

$$x'_2 = p_{11,2}x_1^2 + 2p_{12,2}x_1x_2 + p_{22,2}x_2^2,$$

but  $V$  is a Volterra QSO, so

$$p_{11,2} = p_{22,1} = 0.$$

Hence for Volterra QSO

$$x'_1 = p_{11,1}x_1^2 + 2p_{12,1}x_1x_2,$$

and

$$x'_2 = 2p_{12,2}x_1x_2 + p_{22,2}x_2^2.$$

The cubic matrix for the corresponding Volterra QSOs is

$$P = \left( \begin{array}{cc|cc} p_{11,1} & p_{12,1} & p_{21,1} & 0 \\ 0 & p_{12,2} & p_{21,2} & p_{22,2} \end{array} \right).$$

**Example 7.** Let  $E = \{1, 2, 3\}$ , then  $V(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$ . Find  $x'_1$ ,  $x'_2$  and  $x'_3$ .

**Solution:** For  $k=1$

$$\begin{aligned} x'_1 &= \sum_{i,j=1}^3 p_{ij,1}x_i x_j \\ &= \sum_{j=1}^3 \sum_{i=1}^3 p_{ij,1}x_i x_j \\ &= \sum_{j=1}^3 (p_{1j,1}x_1x_j + p_{2j,1}x_2x_j + p_{3j,1}x_3x_j) \\ &= \sum_{j=1}^3 p_{1j,1}x_1x_j + \sum_{j=1}^3 p_{2j,1}x_2x_j + \sum_{j=1}^3 p_{3j,1}x_3x_j \\ &= p_{11,1}x_1^2 + p_{12,1}x_1x_2 + p_{13,1}x_1x_3 + p_{21,1}x_1x_2 + p_{22,1}x_2^2 \\ &\quad + p_{23,1}x_2x_3 + p_{31,1}x_3x_1 + p_{32,1}x_3x_2 + p_{33,1}x_3^2 \\ &= p_{11,1}x_1^2 + 2(p_{12,1}x_1x_2 + p_{13,1}x_1x_3), \text{ since } p_{22,1} = p_{33,1} = p_{23,1} = p_{32,1} = 0 \\ &= x_1^2 + 2(p_{21,1}x_1x_2 + p_{31,1}x_1x_3), \text{ since } p_{11,1} = 1 \\ x'_1 &= x_1 \left( x_1 + 2 \sum_{i=2, i \neq 1}^3 p_{i1,1}x_i \right). \end{aligned}$$

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For  $k=2$

$$\begin{aligned}
 x_2' &= \sum_{i,j=1}^3 p_{ij,2} x_i x_j \\
 &= \sum_{j=1}^3 \sum_{i=1}^3 p_{ij,2} x_i x_j \\
 &= \sum_{j=1}^3 (p_{1j,2} x_1 x_j + p_{2j,2} x_2 x_j + p_{3j,2} x_3 x_j) \\
 &= \sum_{j=1}^3 p_{1j,2} x_1 x_j + \sum_{j=1}^3 p_{2j,2} x_2 x_j + \sum_{j=1}^3 p_{3j,2} x_3 x_j \\
 &= p_{11,2} x_1^2 + p_{12,2} x_1 x_2 + p_{13,2} x_1 x_3 + p_{21,2} x_1 x_2 + p_{22,2} x_2^2 \\
 &\quad + p_{23,2} x_2 x_3 + p_{31,2} x_3 x_1 + p_{32,2} x_3 x_2 + p_{33,2} x_3^2 \\
 &= p_{22,2} x_2^2 + 2(p_{12,2} x_1 x_2 + p_{32,2} x_3 x_2), \text{ since } p_{11,2} = p_{33,2} = p_{13,2} = p_{31,2} = 0 \\
 &= x_2^2 + 2(p_{12,2} x_1 x_2 + p_{32,2} x_3 x_2), \text{ since } p_{22,2} = 1 \\
 &= x_2(x_2 + 2(p_{12,2} x_1 + p_{32,2} x_3)) \\
 x_2' &= x_2 \left( x_2 + 2 \sum_{i=1, i \neq 2}^3 p_{i2,2} x_i \right).
 \end{aligned}$$

Similarly for  $k=3$

$$x_3' = x_3 \left( x_3 + 2 \sum_{i=1, i \neq 3}^2 p_{i3,3} x_i \right).$$

The cubic matrix for the corresponding operators is

$$P = \left( \begin{array}{ccc|ccc|ccc} p_{11,1} & p_{12,1} & p_{13,1} & p_{21,1} & 0 & 0 & p_{31,1} & 0 & 0 \\ 0 & p_{12,2} & 0 & p_{21,2} & p_{22,2} & p_{23,2} & 0 & p_{32,2} & 0 \\ 0 & 0 & p_{13,3} & 0 & 0 & p_{23,3} & p_{31,3} & p_{32,3} & p_{33,3} \end{array} \right).$$

The above examples are for  $m=2$  and  $m=3$ . So in general

$$x_k' = x_k \left( x_k + 2 \sum_{i=1, i \neq k}^m p_{ik,k} x_i \right).$$

It is called the canonical form of Volterra Quadratic Stochastic Operators and it will be discussed in the next Proposition 3.5.

**Proposition 8.** [6] *Let  $V$  be a Volterra QSO then it can be represented by*

$$x_k' = x_k \left( 1 + \sum_{i=1}^m a_{ki} x_i \right),$$

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where  $a_{ki} = 2p_{ik,k} - 1$  for  $i \neq k$  and  $a_{kk} = 0$ . Moreover  $a_{ki} = -a_{ik}$  and  $|a_{ki}| \leq 1$ .

*Proof.* As  $V$  is a Volterra QSO and by definition  $p_{kk,i} = 0$  for  $i \neq k$  but  $\sum_{i=1}^m p_{kk,i} = 1$ , so we have  $p_{kk,k} = 1$ . As we know that  $p_{ij,k} = p_{ji,k}$ , so by using these conditions we will get

$$x'_k = x_k \left( x_k + 2 \sum_{i=1, i \neq k}^m p_{ik,k} x_i \right). \tag{3.2}$$

Now as we know that

$$\begin{aligned} \sum_{i=1}^m x_i &= 1 \\ x_1 + x_2 + \dots + x_k + \dots + x_m &= 1 \\ \Rightarrow x_k &= 1 - \sum_{i=1, i \neq k}^m x_i. \end{aligned}$$

By putting the value of  $x_k$  in (3.2) we will get

$$\begin{aligned} x'_k &= x_k \left( 1 - \sum_{i=1}^m x_i + 2 \sum_{i=1}^m p_{ik,k} x_i \right) \\ x'_k &= x_k \left( 1 + \sum_{i=1, i \neq k}^m (2p_{ik,k} - 1) x_i \right). \end{aligned}$$

Let  $a_{ki} = 2p_{ik,k} - 1$  for  $i \neq k$  and  $a_{kk} = 0$ .

As the maximum value of  $a_{ki}$  is 1 and -1 i.e.  $|a_{ki}| \leq 1$ . Also  $0 \leq p_{ik,k} \leq 1$ . Finally by using the definition of Volterra QSO, we have  $p_{ik,k} + p_{ik,i} = 1$ . Hence,

$$a_{ki} + a_{ik} = 2p_{ik,k} - 1 + 2p_{ki,i} - 1 = 2(p_{ik,k} + p_{ki,i} - 1) = 2(1 - 1) = 0$$

i.e.

$$a_{ki} = -a_{ik}.$$

This completes the proof. □

**Remark 9.** It should be noted that  $\sum_{i=1}^m a_{ki} x_i$  is actually the multiplication of the matrices and  $a_{ki} = -a_{ik}$  shows the symmetry of the matrices. The Volterra Operator  $x'_k$  totally depends upon the matrix  $\sum_{i=1}^m a_{ki} x_i$ . Let us suppose  $A = (a_{ki})_{i,k=1}^m$  such that  $a_{kk} = 0 \forall k$  and  $a_{ki} = -a_{ik}$ . Keep in mind that  $A$  is a skew symmetric matrix with zeros on its main diagonal.

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**Example 10.** If  $E = \{1, 2\}$ . Then find  $x'_1$  and  $x'_2$  by using above proposition.

**Solution:** Since  $m=2$ , then  $A = (a_{ki})_{i,k=1}^2$  and  $a_{ki} = -aik$ . So this implies that

$$A = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{12}x_2 \\ -a_{12}x_1 \end{pmatrix}.$$

Then by using Proposition 3.5

$$x'_1 = x_1(1 + a_{12}x_2)$$

$$x'_2 = x_2(1 + a_{21}x_1).$$

### 3.2 Face of the Simplex

**Definition 11.** For any  $I \subseteq E = \{1, \dots, m\}$ , we define the Face  $\Gamma_I$  of the Simplex  $S^{m-1}$  by

$$\Gamma_I = \{x \in S^{m-1} : x_i = 0 \text{ for any } i \in I\}.$$

**Example 12.** If  $E = \{1, 2, 3\}$  and  $I = \{1, 2\}$ . Clearly  $I \subseteq E$ , Then find  $\Gamma_I$ .

**Solution:**

$$\begin{aligned} \Gamma_{\{1,2\}} &= \{x \in S^{3-1} : x_i = 0 \forall i \in I\} \\ &= \{x = (x_1, x_2, x_3) \in S^2 : x_1 = 0, x_2 = 0\} \\ &= \{x = (0, 0, x_3) : 0 + 0 + x_3 = 1\} \\ \Gamma_{\{1,2\}} &= (0, 0, 1), \end{aligned}$$

i.e.  $\Gamma_{\{1, 2\}} = (0, 0, 1)$  is the Face of the Simplex  $S^2$ .

**Definition 13.** The Relative Inside denoted by  $r_i\Gamma_I$  of  $\Gamma_I$  is defined as

$$r_i\Gamma_I = \{x \in \Gamma_I : 0 < x_i, \forall i \notin I\}.$$

**Example 14.** If  $E = \{1, 2, 3\}$  and  $I = \{1, 2\}$ . Clearly  $I \subseteq E$ , Then find the relative inside of  $\Gamma_I$ .

**Solution:** Since  $\Gamma_I = (0, 0, 1)$ , so the relative inside of  $\Gamma_I$  is

$$\begin{aligned} r_3\Gamma_{\{1,2\}} &= \{x = (x_1, x_2, x_3) \in \Gamma_{\{1,2\}} : x_3 > 0, \text{ since } 3 \notin I\} \\ &= \{x = (0, 0, x_3) \in \Gamma_I : x_3 > 0, \text{ since } 3 \notin I\} \\ r_3\Gamma_{\{1,2\}} &= \{(0, 0, 1) \in \Gamma_I : x_3 = 1 > 0, \text{ since } 3 \notin I\}. \end{aligned}$$

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**Definition 15.** A subset  $S$  of the domain  $U$  is called an invariant set under the mapping  $f$  if  $x \in S \Rightarrow f(x) \in S$ .

**Definition 16.** Let  $f : R \rightarrow R$  be any function. A point  $x \in R$  is called a fixed point if  $f(x) = x$ .

**Proposition 17.** [4], Let  $V$  be a Volterra QSO. Then the following assertions hold true

- (i) Any Face of  $S^{m-1}$  is invariant set with respect to  $V$ .
- (ii) The vertices of the Simplex  $S^{m-1}$  are fixed points of  $V$ .
- (iii) The Relative Inside of any Face of  $S^{m-1}$  is invariant with respect to  $V$ .

*Proof.* (i) Since  $V$  is a Volterra QSO. So if  $x_i = 0$ , then by Proposition 3.5

$$x'_i = x_i \left( 1 + \sum_{k=1}^m a_{ik}x_k \right), \tag{3.3}$$

by putting  $x_i = 0$  in (3.3), we get

$$\begin{aligned} x'_i &= (0) \left( 1 + \sum_{k=1}^m a_{ik}x_k \right) \\ x'_i &= 0. \end{aligned}$$

So

$$x_i = 0 \Rightarrow V(x_i) = 0.$$

And as  $x_i \in \Gamma_I$  and  $V : S^{m-1} \rightarrow S^{m-1}$ . So by definition of invariant set, any Face of  $S^{m-1}$  is invariant set with respect to  $V$  i.e.

$$V(\Gamma_I) \subseteq \Gamma_I.$$

(ii) Since  $V : S^{m-1} \rightarrow S^{m-1}$  and as in part(i)

$$x_i = 0 \Rightarrow V(x_i) = 0$$

i.e.  $V(x_i) = x_i$ . Also  $x_i \in S^{m-1}$  is an arbitrary point. So by definition of fixed points, the vertices of the Simplex  $S^{m-1}$  are fixed points of  $V$ .

(iii) Suppose  $x_k > 0$ , then by Proposition 3.5

$$x'_k = x_k(1 + a_{k1}x_1 + \dots + a_{kk-1}x_{k-1} + a_{kk}x_k + a_{kk+1}x_{k+1} + \dots + a_{km}x_m), \tag{3.4}$$

putting  $a_{kk} = 0$  in (3.4), we get

$$x'_k = x_k(1 + a_{k1}x_1 + \dots + a_{kk-1}x_{k-1} + a_{kk+1}x_{k+1} + \dots + a_{km}x_m). \tag{3.5}$$

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Now as we know that

$$\begin{aligned} |a_{ki}| &\leq 1 \\ -1 &\leq a_{ki} \leq 1 \\ a_{ki} &\geq -1. \end{aligned}$$

So (3.5) implies

$$x'_k \geq x_k(1 - x_1 - \cdots - x_{k-1} - x_{k+1} - \cdots - x_m). \quad (3.6)$$

Now since we know that

$$\begin{aligned} \sum_{i=1}^m x_i &= 1 \\ x_1 + \cdots + x_{k-1} + x_k + x_{k+1} + \cdots + x_m &= 1 \\ \Rightarrow x_k &= 1 - x_1 - \cdots - x_{k-1} - x_{k+1} - \cdots - x_m. \end{aligned}$$

So by putting the value of  $x_k$  in (3.6), we get

$$\begin{aligned} x'_k &\geq x_k(x_k) \\ x'_k &\geq x_k^2 \\ x'_k &> 0. \end{aligned}$$

So

$$x_k > 0 \Rightarrow V(x_k) > 0.$$

Hence by definition of invariant sets, the Relative Inside of any Face of  $S^{m-1}$  is invariant with respect to  $V$  i.e.

$$V(r_i\Gamma_I) \subseteq r_i\Gamma_I.$$

□

**Remark 18.** The set of all Volterra QSOs defined on the Simplex  $S^{m-1}$  is denoted by  $\mathfrak{R}$ .

**Definition 19.** Let  $S$  be a vector space over the real numbers. A set  $C$  in  $S$  is said to be convex for all  $x$  and  $y$  in  $C$  and all  $t$  in the interval  $[0,1]$ , the point  $(1-t)x + ty$  in  $C$ .

In other words, every point on the line segment connecting  $x$  and  $y$  is in  $C$ .

**Proposition 20.** The set  $\mathfrak{R}$  is a convex, centrally symmetric compact subset of  $\mathbb{R}^{\frac{n(n-1)}{2}}$ . The extremal points of  $\mathfrak{R}$  are Volterra Operators with  $a_{ki} \neq 1$  for  $k \neq i$  i.e.

$$Extr(\mathfrak{R}) = \{V \in \mathfrak{R} : a_{ki} = \pm 1, k \neq i\}.$$

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*Proof.* Let  $V_1, V_2$  be any two Volterra Operators. Then by proposition 3.5 we have

$$V_1(x_k) = x_k \left( 1 + \sum_{i=1}^m a'_{ki} x_i \right)$$

$$V_2(x_k) = x_k \left( 1 + \sum_{i=1}^m a''_{ki} x_i \right).$$

Let  $\alpha$  be any scalar in the interval  $[0, 1]$ . Then by definition of convex set,  $V = \alpha V_1 + (1 - \alpha)V_2$  can be written as

$$\begin{aligned} V(x_k) &= \alpha x_k \left( 1 + \sum_{i=1}^m a'_{ki} x_i \right) + (1 - \alpha) x_k \left( 1 + \sum_{i=1}^m a''_{ki} x_i \right) \\ &= x_k \left( \alpha + \sum_{i=1}^m \alpha a'_{ki} x_i \right) + x_k \left( (1 - \alpha) + \sum_{i=1}^m (1 - \alpha) a''_{ki} x_i \right) \\ &= x_k \left( \alpha + \sum_{i=1}^m \alpha a'_{ki} x_i + 1 - \alpha + \sum_{i=1}^m (1 - \alpha) a''_{ki} x_i \right) \\ &\Rightarrow V(x_k) = x_k \left( 1 + \sum_{i=1}^m (\alpha a'_{ki} + (1 - \alpha) a''_{ki}) x_i \right). \end{aligned} \tag{3.7}$$

Now again from Proposition 3.5

$$|a'_{ki}| \leq 1, \quad a'_{ki} = -a'_{ik}, \tag{3.8}$$

and

$$|a''_{ki}| \leq 1, \quad a''_{ki} = -a''_{ik}. \tag{3.9}$$

Now let

$$a_{ki} = \alpha a'_{ki} + (1 - \alpha) a''_{ki}.$$

So from (3.8) and (3.9), we get

$$|a_{ki}| \leq 1, \quad a_{ki} = -a_{ik}.$$

So (3.7) implies that

$$V(x_k) = x_k \left( 1 + \sum_{i=1}^m a_{ki} x_i \right).$$

So  $V = \alpha V_1 + (1 - \alpha)V_2$  is in the set  $\mathfrak{R}$ . Hence  $\mathfrak{R}$  is a convex set.

Now if  $V \in \mathfrak{R}$  is a Volterra Operator with coefficients  $a_{ki}$ , then  $V'$  is also a Volterra Operator with coefficients  $-a_{ki}$ . So  $\mathfrak{R}$  is centrally symmetric. Moreover, center of symmetry is identical operator with coefficients  $a_{ki} = 0$ .

\*\*\*\*\*

Let us assume  $a_{ki} \neq \pm 1$  for some  $k$  and  $i$  ( $k \neq i$ ). Suppose  $\alpha = a_{ki}$  and the matrix for the Volterra Operator  $V$  is

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

Since  $a_{ki} = -a_{ik}$  and we can assume that  $\alpha \geq 0$ . Here we will discuss two cases

**Case-1:**

If  $\alpha > 0$  then setting for  $k=1$  and  $i=3$ ,

$$a'_{ts} = \begin{cases} a_{ts}, & \text{if } (t, s) \neq (k, i) \\ 1, & \text{if } (t, s) = (k, i). \end{cases}$$

Then we have

$$A' = \begin{pmatrix} 0 & a_{12} & 1 \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

The above matrix is for Volterra Operator  $V_1$ . Now

$$a''_{ts} = \begin{cases} a_{ts}, & \text{if } (t, s) \neq (k, i) \\ 0, & \text{if } (t, s) = (k, i). \end{cases}$$

Then we have

$$A'' = \begin{pmatrix} 0 & a_{12} & 0 \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

The above matrix is for Volterra Operator  $V_2$ .

Now consider

$$\begin{aligned} \alpha A' + (1 - \alpha)A'' &= \begin{pmatrix} 0 & \alpha a_{12} & \alpha \\ -\alpha a_{12} & 0 & \alpha a_{23} \\ -\alpha & -\alpha a_{23} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & (1 - \alpha)a_{12} & 0 \\ -(1 - \alpha)a_{12} & 0 & (1 - \alpha)a_{23} \\ -(1 - \alpha)a_{13} & -(1 - \alpha)a_{23} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_{12} & \alpha \\ -a_{12} & 0 & a_{23} \\ -a_{13} & a_{23} & 0 \end{pmatrix} \\ \alpha A' + (1 - \alpha)A'' &= A. \end{aligned}$$

Hence we obtain

$$\alpha V_1 + (1 - \alpha)V_2 = V.$$

\*\*\*\*\*

**Case-2:**

If  $\alpha = 0$ , then setting for  $k=1$  and  $i=3$

$$a'_{ts} = \begin{cases} a_{ts}, & \text{if } (t, s) \neq (k, i) \\ 1, & \text{if } (t, s) = (k, i). \end{cases}$$

Then we have

$$A' = \begin{pmatrix} 0 & a_{12} & 1 \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

The above matrix is for Volterra Operator  $V_1$ . Now

$$a''_{ts} = \begin{cases} a_{ts}, & \text{if } (t, s) \neq (k, i) \\ -1, & \text{if } (t, s) = (k, i). \end{cases}$$

Then we have

$$A'' = \begin{pmatrix} 0 & a_{12} & -1 \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

The above matrix is for Volterra Operator  $V_2$ . Now consider

$$\begin{aligned} \frac{1}{2}A' + \frac{1}{2}A'' &= \begin{pmatrix} 0 & \frac{1}{2}a_{12} & \frac{1}{2} \\ -\frac{1}{2}a_{12} & 0 & \frac{1}{2}a_{23} \\ -\frac{1}{2}a_{13} & -\frac{1}{2}a_{23} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}a_{12} & -\frac{1}{2} \\ -\frac{1}{2}a_{12} & 0 & \frac{1}{2}a_{23} \\ -\frac{1}{2}a_{13} & -\frac{1}{2}a_{23} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_{12} & 0 \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix} \\ \frac{1}{2}A' + \frac{1}{2}A'' &= A. \end{aligned}$$

Hence we obtain

$$\frac{1}{2}V_1 + \frac{1}{2}V_2 = V.$$

So if  $a_{ki} \neq \pm 1$  for at least one  $(k, i)$ ,  $k \neq i$ , then  $V$  is an interior point of any line segment, not extremal point. So if  $a_{ki} = \pm 1 \forall k \neq i$ , then the equation  $\alpha V_1 + (1 - \alpha)V_2 = V$  is valid only if  $V_1 = V_2 = V$ . This completes the proof.  $\square$

### 4 Lyapunov Functions of Volterra Operators

In this Section we will study about the Lyapunov functions of Volterra Quadratic Stochastic Operators.

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Let  $V : S^{m-1} \rightarrow S^{m-1}$  and  $x^{(0)} \in S^{m-1}$  be the initial point. Then the trajectory of  $x^{(0)}$  is denoted by  $\{x^{(n)}\}$  and is defined as  $V(x^{(n)}) = x^{(n+1)}$ .

Let  $w(x^{(0)})$  denote the set of all limit points of the trajectory  $\{x^{(n)}\}$ . As  $S^{m-1}$  is a compact set, so  $w(x^{(0)}) \neq \phi$ .

**Definition 21.** A continuous function  $\varphi : S^{m-1} \rightarrow R$  is called a Lyapunov function for  $V$  if for any initial point  $x^{(0)} \in S^{m-1}$ , there exists  $\lim_{n \rightarrow \infty} \varphi(x^{(n)})$ .

Clearly if  $\lim_{n \rightarrow \infty} \varphi(x^{(n)}) = c$ , then  $\omega(x^{(0)}) \subseteq \varphi^{-1}(c)$ .

**Remark 22.** The set of fixed points of a Volterra Operator  $V$  is denoted by

$$Fix(V) = \{x \in S^{m-1} : V(x) = x\}.$$

**Proposition 23.** Let  $V$  be a Volterra Operator, then show that each point of a set  $P = \{x \in S^{m-1} : \sum_{i=1}^m a_{ki}x_i \geq 0, k = 1, \dots, m\}$  and set  $Q = \{x \in S^{m-1} : \sum_{i=1}^m a_{ki}x_i \leq 0, k = 1, \dots, m\}$  are fixed points i.e.  $P \subseteq Fix(V)$ ,  $Q \subseteq Fix(V)$ .

*Proof.* Let  $x \in P$ , then

$$\begin{aligned} \sum_{i=1}^m a_{ki}x_i &\geq 0 \\ \Rightarrow 1 + \sum_{i=1}^m a_{ki}x_i &\geq 1 \\ \Rightarrow x_k(1 + \sum_{i=1}^m a_{ki}x_i) &\geq x_k \\ \Rightarrow x'_k &\geq x_k \\ \Rightarrow x'_k = x_k, &\text{ since } x'_k > x_k \text{ is impossible.} \end{aligned}$$

This means that  $x_k$  is a fixed point but  $x_k$  is arbitrary. So each point of set  $P$  is a fixed point of  $V$ . Hence  $P \subseteq Fix(V)$ . Similarly we can show that each point of  $Q$  is a fixed point of  $V$  i.e.  $Q \subseteq Fix(V)$ .  $\square$

**Definition 24.** If  $b_i > 0$ ,  $p_i \geq 0$  and  $\sum_{i=1}^m p_i = 1$ , then

$$b_1^{p_1} b_2^{p_2} \dots b_m^{p_m} \leq p_1 b_1 + p_2 b_2 + \dots + p_m b_m.$$

The above inequality is called Young's inequality.

**Theorem 25.** [9], Let  $V$  is a Volterra Operator. If  $p = (p_1, \dots, p_m)$  such that  $\sum_{i=1}^m a_{ki}p_i \geq 0$ ,  $k = 1, 2, \dots, m$ , then  $\varphi(x) = x_1^{p_1} x_2^{p_2} \dots x_m^{p_m}$  is a Lyapunov function for  $V$ .

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*Proof.* Let  $x \in \text{int}(S^{m-1})$  i.e.  $x_i \neq 0; \forall i \in \{1, 2, \dots, m\}$ . Then

$$\begin{aligned} \varphi(V(x)) &= \varphi(x') \\ &= x_1^{p_1} x_2^{p_2} \dots x_m^{p_m} \\ &= [x_1(1 + \sum_{i=1}^m a_{1i}x_i)]^{p_1} [x_2(1 + \sum_{i=1}^m a_{2i}x_i)]^{p_2} \dots [x_m(1 + \sum_{i=1}^m a_{mi}x_i)]^{p_m} \\ &= x_1^{p_1} \dots x_m^{p_m} [(1 + \sum_{i=1}^m a_{1i}x_i)^{p_1} (1 + \sum_{i=1}^m a_{2i}x_i)^{p_2} \dots (1 + \sum_{i=1}^m a_{mi}x_i)^{p_m}] \\ \varphi(V(x)) &= \varphi(x) \prod_{k=1}^m \left[ 1 + \sum_{i=1}^m a_{ki}x_i \right]^{p_k}. \end{aligned}$$

Let  $b_k = 1 + \sum_{i=1}^m a_{ki}x_i \Rightarrow b_k > 0$  (because  $b_k = 0 \Leftrightarrow a_{ki} = -1 \forall k, i$ ). Thus by using Young's Inequality

$$\begin{aligned} \prod_{k=1}^m [1 + \sum_{i=1}^m a_{ki}x_i]^{p_k} &\leq \sum_{k=1}^m p_k [1 + \sum_{i=1}^m a_{ki}x_i] \\ &= \sum_{k=1}^m p_k + \sum_{k=1}^m p_k \sum_{i=1}^m a_{ki}x_i \\ &= 1 - \sum_{i=1}^m \left( \sum_{k=1}^m a_{ik}p_k \right) x_i, \text{ since } a_{ki} = -a_{ik} \text{ and } \sum_{k=1}^m p_k = 1 \end{aligned}$$

$$\Rightarrow \prod_{k=1}^m [1 + \sum_{i=1}^m a_{ki}x_i]^{p_k} \leq 1 - \sum_{i=1}^m \left( \sum_{k=1}^m a_{ik}p_k \right) x_i. \tag{4.1}$$

\*\*\*\*\*

Since given that

$$\begin{aligned}
 \sum_{k=1}^m a_{ik} p_k &\geq 0 \quad \forall i = 1, 2, \dots, m \\
 \Rightarrow \sum_{i=1}^m \left( \sum_{k=1}^m a_{ik} p_k \right) x_i &\geq 0 \\
 \Rightarrow 1 - \sum_{i=1}^m \left( \sum_{k=1}^m a_{ik} p_k \right) x_i &\leq 1 \\
 \Rightarrow \prod_{k=1}^m \left[ 1 + \sum_{i=1}^m a_{ki} x_i \right]^{p_k} &\leq 1, \text{ using (4.1)} \\
 \Rightarrow \varphi(x) \prod_{k=1}^m \left[ 1 + \sum_{i=1}^m a_{ki} x_i \right]^{p_k} &\leq \varphi(x) \\
 \Rightarrow \varphi(x') &\leq \varphi(x) \\
 \Rightarrow \varphi(x^{(n+1)}) &\leq \varphi(x^{(n)}).
 \end{aligned}$$

Thus  $\lim_{n \rightarrow \infty} \varphi(x^{(n)})$  exists (since the sequence is convergent). Hence  $\varphi(x) = x_1^{p_1} x_2^{p_2} \dots x_m^{p_m}$  is a Lyapunov function for V.  $\square$

**Proposition 26.** Let  $\varphi(x) = x_1^{p_1} x_2^{p_2} \dots x_m^{p_m}$ ,  $p \in P$ . Then prove that  $p$  is critical point i.e.

$$\max_{x \in S^{m-1}} \varphi(x) = \varphi(p).$$

*Proof.* For  $m=2$

$$\begin{aligned}
 \varphi(x) &= x_1^{p_1} x_2^{p_2} : x_1 + x_2 = 1, p_1 + p_2 = 1 \\
 \Rightarrow \varphi(x) &= x_1^{p_1} (1 - x_1)^{p_2},
 \end{aligned}$$

by differentiating both sides with respect to  $x$ , we get

$$\begin{aligned}
 \varphi'(x) &= p_1 x_1^{p_1-1} (1 - x_1)^{p_2} + x_1^{p_1} p_2 (1 - x_1)^{p_2-1} (-1) \\
 \Rightarrow \varphi'(x) &= p_1 x_1^{p_1-1} (1 - x_1)^{p_2} - x_1^{p_1} p_2 (1 - x_1)^{p_2-1},
 \end{aligned}$$

\*\*\*\*\*



for critical point put  $\varphi'(x) = 0$

$$\begin{aligned} \varphi'(x) &= 0 \\ \Rightarrow p_1 x_1^{p_1-1} (1-x_1)^{p_2} - x_1^{p_1} p_2 (1-x_1)^{p_2-1} &= 0 \\ \Rightarrow x_1^{p_1-1} (1-x_1)^{p_2-1} (p_1(1-x_1) - p_2 x_1) &= 0 \\ \Rightarrow (p_1(1-x_1) - p_2 x_1) &= 0, \text{ since } x_1^{p_1-1} \neq 0, (1-x_1)^{p_2-1} \neq 0 \\ \Rightarrow p_1(1-x_1) &= p_2 x_1 \\ \Rightarrow p_1 - p_1 x_1 &= p_2 x_1 \\ \Rightarrow p_1 &= p_1 x_1 + p_2 x_1 \\ \Rightarrow p_1 &= (1)x_1, \text{ where } p_1 + p_2 = 1 \\ \Rightarrow x_1 &= p_1 \\ \Rightarrow x_2 &= p_2, \end{aligned}$$

since  $p = (p_1, p_2)$  and  $x = (x_1, x_2)$ , so  $x = (x_1, x_2) = (p_1, p_2)$  is the critical point i.e.  $p$  is the critical point of  $\varphi(x)$  for  $m=2$ .

For  $m=3$

$$\begin{aligned} \varphi(x) &= x_1^{p_1} x_2^{p_2} x_3^{p_3} \\ \varphi(x) &= x_1^{p_1} x_2^{p_2} (1-x_1-x_2)^{p_3}, \end{aligned}$$

first by applying partial derivative with respect to  $x_1$  and putting  $\frac{\partial}{\partial x_1}(\varphi(x)) = 0$  i.e.

$$\begin{aligned} \frac{\partial}{\partial x_1} (x_1^{p_1} x_2^{p_2} (1-x_1-x_2)^{p_3}) &= 0 \\ \Rightarrow p_1 x_1^{p_1-1} x_2^{p_2} (1-x_1-x_2)^{p_3} - x_1^{p_1} x_2^{p_2} p_3 (1-x_1-x_2)^{p_3-1} &= 0 \\ \Rightarrow x_1^{p_1-1} x_2^{p_2} (1-x_1-x_2)^{p_3-1} (p_1(1-x_1-x_2) - x_1 p_3) &= 0 \\ \Rightarrow (p_1(1-x_1-x_2) - x_1 p_3) &= 0 \\ \Rightarrow p_1(1-x_1-x_2) &= x_1 p_3 \\ \Rightarrow p_1 x_3 &= x_1 p_3 \\ \Rightarrow x_1 &= x_3 \frac{p_1}{p_3}. \end{aligned}$$

Similarly by applying partial derivative with respect to  $x_2$  and putting  $\frac{\partial}{\partial x_2}(\varphi(x)) = 0$

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i.e.

$$\begin{aligned}
 \frac{\partial}{\partial x_2}(x_1^{p_1} x_2^{p_2} (1 - x_1 - x_2)^{p_3}) &= 0 \\
 \Rightarrow x_1^{p_1} p_2 x_2^{p_2-1} (1 - x_1 - x_2)^{p_3} - x_1^{p_1} x_2^{p_2} p_3 (1 - x_1 - x_2)^{p_3-1} &= 0 \\
 \Rightarrow x_1^{p_1} x_2^{p_2-1} (1 - x_1 - x_2)^{p_3-1} (p_2(1 - x_1 - x_2) - x_2 p_3) &= 0 \\
 \Rightarrow (p_2(1 - x_1 - x_2) - x_2 p_3) &= 0 \\
 \Rightarrow p_2(1 - x_1 - x_2) &= x_2 p_3 \\
 \Rightarrow p_2 x_3 &= x_2 p_3 \\
 \Rightarrow x_2 &= x_3 \frac{p_2}{p_3}.
 \end{aligned}$$

As we know that

$$\begin{aligned}
 x_1 + x_2 + x_3 &= 1 \\
 x_3 \frac{p_1}{p_3} + x_3 \frac{p_2}{p_3} + x_3 &= 1, \text{ using values of } x_1 \text{ and } x_2 \\
 \left( \frac{p_1 + p_2 + p_3}{p_3} \right) x_3 &= 1 \\
 \frac{x_3}{p_3} &= 1, \text{ as } p_1 + p_2 + p_3 = 1 \\
 x_3 &= p_3.
 \end{aligned}$$

Similarly

$$x_1 = p_1,$$

and

$$x_2 = p_2.$$

Thus  $x = (x_1, x_2, x_3) = (p_1, p_2, p_3) = p$  i.e.  $x = p$  is the critical point of  $\varphi(x)$  for  $m=3$ . Now we will prove it for general. Since given that

$$\begin{aligned}
 \varphi(x) &= x_1^{p_1} x_2^{p_2} \dots x_m^{p_m} \\
 \Rightarrow \varphi(x) &= x_1^{p_1} x_2^{p_2} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m}.
 \end{aligned}$$

Again by applying partial derivative with respect to  $x_1$  and

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putting  $\frac{\partial}{\partial x_1}(\varphi(x)) = 0$ ,

$$\begin{aligned}
 & \frac{\partial}{\partial x_1}(x_1^{p_1} x_2^{p_2} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m}) = 0 \\
 & \Rightarrow p_1 x_1^{p_1-1} x_2^{p_2} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m} - x_1^{p_1} x_2^{p_2} \dots p_m (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m-1} = 0 \\
 & \Rightarrow x_1^{p_1-1} x_2^{p_2} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m-1} (p_1(1 - x_1 - x_2 \dots x_{m-1}) - x_1 p_m) = 0 \\
 & \Rightarrow p_1(1 - x_1 - x_2 \dots x_{m-1}) - x_1 p_m = 0 \\
 & \Rightarrow p_1 x_m - x_1 p_m = 0 \\
 & \Rightarrow x_1 p_m = p_1 x_m \\
 & \Rightarrow x_1 = \frac{p_1}{p_m} x_m.
 \end{aligned}$$

Similarly by applying partial derivative with respect to  $x_2$  and then put  $\frac{\partial}{\partial x_2}(\varphi(x)) = 0$  i.e.

$$\begin{aligned}
 & \frac{\partial}{\partial x_2}(x_1^{p_1} x_2^{p_2} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m}) = 0 \\
 & x_1^{p_1} p_2 x_2^{p_2-1} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m} - x_1^{p_1} x_2^{p_2} \dots p_m (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m-1} = 0 \\
 & x_1^{p_1} x_2^{p_2-1} \dots (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m-1} (p_2(1 - x_1 - x_2 \dots x_{m-1}) - x_2 p_m) = 0 \\
 & p_2(1 - x_1 - x_2 \dots x_{m-1}) - x_2 p_m = 0 \\
 & p_2 x_m - x_2 p_m = 0 \\
 & x_2 p_m = p_2 x_m \\
 & x_2 = \frac{p_2}{p_m} x_m.
 \end{aligned}$$

Continuing in this way, by applying partial derivative with respect to  $x_{m-1}$  and then

\*\*\*\*\*

put  $\frac{\partial}{\partial x_{m-1}}(\varphi(x)) = 0$  i.e.

$$\begin{aligned}
 \frac{\partial}{\partial x_{m-1}}(x_1^{p_1} x_2^{p_2} \dots x_{m-1}^{p_{m-1}} (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m}) &= \\
 \Rightarrow x_1^{p_1} x_2^{p_2} \dots p_{m-1} x_{m-1}^{p_{m-1}-1} (1 - x_1 - x_2 - \dots - & \\
 x_{m-1})^{p_m} - x_1^{p_1} x_2^{p_2} \dots x_{m-1}^{p_{m-1}} p_m (1 - x_1 - x_2 - \dots - x_{m-1})^{p_m-1} &= 0 \\
 \Rightarrow x_1^{p_1} x_2^{p_2} \dots x_{m-1}^{p_{m-1}} (1 - x_1 - x_2 - \dots - & \\
 x_{m-1})^{p_m-1} (p_{m-1} (1 - x_1 - x_2 \dots - x_{m-1}) - x_{m-1} p_m) &= 0 \\
 \Rightarrow p_{m-1} (1 - x_1 - x_2 \dots x_{m-1}) - x_{m-1} p_m &= 0 \\
 \Rightarrow p_{m-1} x_m - x_{m-1} p_m &= 0 \\
 \Rightarrow x_{m-1} p_m &= p_{m-1} x_m \\
 \Rightarrow x_{m-1} &= \frac{p_{m-1}}{p_m} x_m.
 \end{aligned}$$

As we know that

$$\begin{aligned}
 \sum_{i=1}^m x_i &= 1 \\
 \Rightarrow x_1 + x_2 + \dots + x_{m-1} + x_m &= 1 \\
 \Rightarrow \frac{p_1}{p_m} x_m + \frac{p_2}{p_m} x_m + \dots + \frac{p_{m-1}}{p_m} x_m + x_m &= 1 \\
 \Rightarrow \left( \frac{p_1 + p_2 + \dots + p_{m-1} + p_m}{p_m} \right) x_m &= 1 \\
 \Rightarrow \frac{x_m}{p_m} &= 1 \\
 \Rightarrow x_m &= p_m.
 \end{aligned}$$

So

$$\begin{aligned}
 x_1 &= \frac{p_1}{p_m} x_m \\
 \Rightarrow x_1 &= \frac{p_1}{p_m} p_m \\
 \Rightarrow x_1 &= p_1.
 \end{aligned}$$

Similarly

$$\begin{aligned}
 x_2 &= \frac{p_2}{p_m} x_m \\
 \Rightarrow x_2 &= \frac{p_2}{p_m} p_m \\
 \Rightarrow x_2 &= p_2.
 \end{aligned}$$

\*\*\*\*\*

Continuing in this way, we will get

$$\begin{aligned} x_{m-1} &= \frac{p_{m-1}}{p_m} x_m \\ \Rightarrow &= \frac{p_{m-1}}{p_m} p_m \\ \Rightarrow x_{m-1} &= p_{m-1}. \end{aligned}$$

So  $x = (x_1, x_2, \dots, x_{m-1}, x_m) = (p_1, p_2, \dots, p_{m-1}, p_m) = p$  i.e.  $x = p$  is the critical point of  $\varphi(x)$  for each  $m$ . Hence

$$\max_{x \in S^{m-1}} \varphi(x) = \varphi(p).$$

□

## 5 l-volterra Quadratics Stochastic Operators

Now we shall give a new class of non-Volterra operators.

*l*-Volterra QSO. Fix  $\ell \in \{1, \dots, m\}$  and assume that elements  $P_{ij,k}$  of the matrix  $\mathbf{P}$  satisfy

$$P_{ij,k} = 0 \text{ if } k \notin \{i, j\} \text{ for any } k \in \{1, \dots, \ell\}, i, j \in E; \tag{5.1}$$

$$P_{ij,k} > 0 \text{ for at least one pair } (i, j), i \neq k, j \neq k \text{ for any } k \in \{\ell+1, \dots, m\}. \tag{5.2}$$

**Definition 27.** For any fixed  $\ell \in \{1, \dots, m\}$ , the QSO defined by (2.1), (2.2), (5.1) and (5.2) is called *l*-Volterra QSO.

Denote by  $\mathcal{V}_\ell$  the set of all *l*-Volterra QSOs.

**Remark 28.** 1. The condition (5.2) guarantees that  $\mathcal{V}_{\ell_1} \cap \mathcal{V}_{\ell_2} = \emptyset$  for any  $\ell_1 \neq \ell_2$ .

2. Note that *l*-Volterra QSO is Volterra if and only if  $\ell = m$ .

3. Quasi-Volterra operators (introduce above) are particular case of *l*-Volterra operators.

4. The class of *l*-Volterra QSO for a given  $\ell$  does not coincide with a class of non-Volterra QSOs mention above.

We shall use the following notations.

**Definition 29.** ([1], p. 215) A fixed point  $P$  for  $F : \mathbf{R}^m \rightarrow \mathbf{R}^m$  is called hyperbolic if the Jacobian matrix  $\mathbf{J}$  of the map  $F$  at the point  $P$  has no eigenvalues on the unit circle.

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There are three types of hyperbolic fixed points :

1.  $P$  is an *attracting* fixed point if all of the eigenvalues of  $\mathbf{J}(P)$  are less than one in absolute value.
2.  $P$  is a *repelling* fixed point if all of the eigenvalues of  $\mathbf{J}(P)$  are greater than one in absolute value.
3.  $P$  is a *saddle* point otherwise.

The following theorem is also very useful.

**Theorem 30.** ([1], p.217) Suppose  $F$  has a saddle fixed point  $P$ . There exist  $\varepsilon > 0$  and a smooth curve  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbf{R}^2$  such that  $\gamma(0) = P$ ;  $\gamma'(t) \neq 0$ ;  $\gamma'(0)$  is an unstable eigenvector for  $\mathbf{J}(P)$ ;  $\gamma$  is  $F^{-1}$ -invariant;  $F^{-n}(\gamma(t)) \rightarrow P$  as  $n \rightarrow \infty$ ; if  $|F^{-n}(q) - P| < \varepsilon$  for all  $n \geq 0$  then  $q = \gamma(t)$  for some  $t$ .

The curve  $\gamma$  is called the (local) unstable manifold at  $P$ . The theorem is true for stable sets as well as with the obvious modification. On the local manifold, all points tend to the fixed point under iteration of  $F$ .

### 5.1 Canonical form of l-Volterra QSO.

By definition for  $k = 1, \dots, \ell$  we have

$$x'_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j = P_{kk,k} x_k^2 + 2 \sum_{\substack{i=1 \\ i \neq k}}^m P_{ik,k} x_i x_k =$$

$$x_k \left( P_{kk,k} x_k + 2 \sum_{\substack{i=1 \\ i \neq k}}^m P_{ik,k} x_i \right).$$

Using  $x_k = 1 - \sum_{\substack{i=1 \\ i \neq k}}^m x_i$  we get

$$x'_k = x_k \left( P_{kk,k} + \sum_{\substack{i=1 \\ i \neq k}}^m (2P_{ik,k} - P_{kk,k}) x_i \right), \quad k = 1, \dots, \ell.$$

For  $k = \ell + 1, \dots, m$  we have

$$x'_k = x_k \left( P_{kk,k} + \sum_{\substack{i=1 \\ i \neq k}}^m (2P_{ik,k} - P_{kk,k}) x_i \right) + \sum_{\substack{i,j=1 \\ i \neq k \\ j \neq k}}^m P_{ij,k} x_i x_j.$$

\*\*\*\*\*

Denote  $a_{ki} = 2P_{ik,k} - P_{kk,k}$  then

$$\begin{cases} x'_k = x_k \left( a_{kk} + \sum_{\substack{i=1 \\ i \neq k}}^m a_{ki} x_i \right), & k = 1, \dots, \ell \\ x'_k = x_k \left( a_{kk} + \sum_{\substack{i=1 \\ i \neq k}}^m a_{ki} x_i \right) + \sum_{\substack{i,j=1 \\ i \neq k \\ j \neq k}}^m P_{ij,k} x_i x_j, & k = \ell + 1, \dots, m. \end{cases} \tag{5.3}$$

Note that  $0 \leq a_{kk} \leq 1$  and  $-a_{kk} \leq a_{ki} \leq 2 - a_{kk}$ ,  $i \neq k$ ,  $0 \leq P_{ij,k} \leq 1$ .

For any  $I \subset E = \{1, 2, \dots, m\}$  we define the face of the simplex  $S^{m-1}$ :

$$\Gamma_I = \{x \in S^{m-1} : x_i = 0 \text{ for any } i \in I\}.$$

**Proposition 31.** *Let  $V$  be a  $\ell$ -Volterra QSO. Then the following are true*

1. Any face  $\Gamma_I$  with  $I \subseteq \{1, \dots, \ell\}$  is invariant set with respect to  $V$ .
2. Let  $A_\ell = \{i \in \{1, \dots, \ell\} : a_{ii} > 0\}$ . For any  $I \subset A_\ell \cup \{\ell + 1, \dots, m\}$  the set  $T_I = \{x \in S^{m-1} : x_i > 0, \forall i \in I\}$  is invariant with respect to  $V$ .

*Proof.* 1. From (5.3) it follows that if  $x_i = 0$  then  $x'_i = 0$  for any  $i \in \{1, \dots, \ell\}$ . Hence  $V(\Gamma_I) \subset \Gamma_I$  if  $I \subset \{1, \dots, \ell\}$ .

2. Take  $I \subset A_\ell \cup \{\ell + 1, \dots, m\}$ . For  $k \in I \cap A_\ell$  by (5.3) and inequality  $-a_{kk} \leq a_{kj}$ ,  $j = 1, \dots, m$  we get

$$x'_k = x_k \left( a_{kk} + \sum_{\substack{j=1 \\ j \neq k}}^m a_{kj} x_j \right) \geq x_k \left( a_{kk} - a_{kk} \sum_{\substack{j=1 \\ j \neq k}}^m x_j \right) = a_{kk} x_k^2 > 0, \text{ since } x_k > 0 \text{ for } k \in I \cap A_\ell.$$

For  $k \in I \cap \{\ell + 1, \dots, m\}$  by (5.3) and condition (5.2) we have

$$x'_k = x_k \left( a_{kk} + \sum_{\substack{j=1 \\ j \neq k}}^m a_{kj} x_j \right) + \sum_{\substack{i,j=1 \\ i \neq k \\ j \neq k}}^m P_{ij,k} x_i x_j \geq x_k \left( a_{kk} - a_{kk} \sum_{\substack{j=1 \\ j \neq k}}^m x_j \right) + \sum_{\substack{i,j \in I \\ i \neq k \\ j \neq k}} P_{ij,k} x_i x_j > a_{kk} x_k^2 \geq 0,$$

\*\*\*\*\*

here we used  $\sum_{\substack{i,j \in I \\ i \neq k \\ j \neq k}} P_{ij,k} x_i x_j > 0$  which follows from condition (5.2) and  $x_i > 0, x_j > 0, \forall i, j \in I$ . Thus  $V(T_I) \subset T_I$  if  $I \subset A_\ell \cup \{\ell + 1, \dots, m\}$ . The proposition is proved. □

Denote  $e_i = (\delta_{1i}, \dots, \delta_{mi}) \in S^{m-1}, i = 1, \dots, m$  the vertices of the simplex  $S^{m-1}$ , where  $\delta_{ij}$  is the Kronecker's symbol.

**Proposition 32.** 1. The vertex  $e_i$  is a fixed point for a  $\ell$ -Volterra QSO iff  $P_{ii,i} = 1, (i = 1, \dots, m)$ .

2. For any collection  $I_s = \{e_{i_1}, \dots, e_{i_s}\} \subset \{e_{\ell+1}, \dots, e_m\}, (s \leq m - \ell)$  there exist a family  $\mathcal{V}_\ell(I_s) \subset \mathcal{V}_\ell$  such that  $\{e_{i_1}, \dots, e_{i_s}\}$  is a  $s$ -cycle for each  $V \in \mathcal{V}_\ell(I_s)$ .

*Proof.* 1. It is easy to see that if  $i \in \{1, \dots, \ell\}$  then

$$V(e_i) = (0, \dots, 0, P_{ii,i}, 0, \dots, 0, P_{ii,\ell+1}, \dots, P_{ii,m}) \text{ with } P_{ii,i} + \sum_{j=\ell+1}^m P_{ii,j} = 1$$

and if  $i \in \{\ell + 1, \dots, m\}$  then

$$V(e_i) = (0, \dots, 0, P_{ii,\ell+1}, \dots, P_{ii,m}) \text{ with } \sum_{j=\ell+1}^m P_{ii,j} = 1. \tag{5.4}$$

Thus  $V(e_i) = e_i$  iff  $P_{ii,i} = 1$ .

2. By (5.4) we have

$$V(e_{i_j}) = (0, \dots, 0, P_{i_j i_j, \ell+1}, \dots, P_{i_j i_j, m})$$

for any  $j = 1, \dots, s$ . In order to get  $V(e_{i_1}) = e_{i_2}$  we assume

$$P_{i_1 i_1, i_2} = 1, P_{i_1 i_1, j} = 0, j \neq i_2. \tag{5.5}$$

Then to get  $V(e_{i_2}) = e_{i_3}$  we assume

$$P_{i_2 i_2, i_3} = 1, P_{i_2 i_2, j} = 0, j \neq i_3. \tag{5.6}$$

Similarly to get  $V(e_{i_{s-1}}) = e_{i_s}$  we assume

$$P_{i_{s-1} i_{s-1}, i_s} = 1, P_{i_{s-1} i_{s-1}, j} = 0, j \neq i_s. \tag{5.7}$$

The last assumption follows from  $V(e_{i_s}) = e_{i_1}$  i.e

$$P_{i_s i_s, i_1} = 1, P_{i_s i_s, j} = 0, j \neq i_1. \tag{5.8}$$

Hence  $\mathcal{V}_\ell(I_s) = \{V \in \mathcal{V}_\ell : \text{the coefficients of } V \text{ satisfy (5.4) - (5.8)}\}$ . The proposition is proved. □

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For any set  $A$  denote by  $|A|$  its cardinality.

The next proposition gives a set of periodic orbits of  $\ell$ -Volterra QSOs.

**Proposition 33.** *For any  $I_1, \dots, I_q \subset \{\ell + 1, \dots, m\}$  such that  $I_i \cap I_j = \emptyset$  ( $i \neq j, i, j = 1, \dots, q$ ). There exists a family  $\mathcal{V}_\ell(I_1, \dots, I_q) \subset \mathcal{V}_\ell$  such that each collection  $\{e_i, i \in I_j\}, j = 1, \dots, q$  is a  $|I_j|$ - cycle for every  $V \in \mathcal{V}_\ell(I_1, \dots, I_q)$ .*

*Proof.* Since  $I_i \cap I_j = \emptyset, i \neq j$  the family can be constructed using Proposition 3.2 i.e.  $\mathcal{V}_\ell(I_1, \dots, I_q) = \bigcap_{i=1}^q \mathcal{V}_\ell(I_i)$ . □

**Remark 34.** 1. *There is not any  $\ell$ -Volterra operator with a periodic orbit  $\{e_{i_1}, \dots, e_{i_s}\} \subset \{e_1, \dots, e_\ell\}, 1 < s \leq \ell$ .*

2. *Propositions 5.8 and 5.10 show that  $\ell$ -Volterra operators have quite different behavior from the behavior of Volterra operators, since Volterra operators have no cyclic trajectories.*

Recall that  $\mathcal{V}_\ell$  is the set of all  $\ell$ -Volterra operators defined on  $S^{m-1}$ .

**Proposition 35.** 1. *The set  $\mathcal{V}_\ell$  is a convex, compact subset of  $\mathbf{R}^{\frac{m(m-1)(m-\ell+1)}{2}}$ .*

2. *The extremal points of  $\mathcal{V}_\ell$  are  $\ell$ -Volterra operators with  $P_{i,j,k} = 0$  or 1 for any  $i, j, k$  i.e.*

$$\text{Extr}(\mathcal{V}_\ell) = \{V \in \mathcal{V}_\ell : \text{the matrix } \mathbf{P} \text{ of } V \text{ contains only } 0 \text{ and } 1\}.$$

3. *If  $\ell = m$  then  $|\text{Extr}(\mathcal{V}_\ell)| = 2^{\frac{1}{2}m(m-1)}$ ; if  $\ell \leq m - 1$  then*

$$|\text{Extr}(\mathcal{V}_\ell)| = (m - \ell)^{\frac{1}{2}(m-\ell)(m-\ell+1)} (m - \ell + 1)^{(m-\ell+1)\ell} (m - \ell + 2)^{\frac{1}{2}\ell(\ell-1)}.$$

*Proof.* 1. Since we have one-to-one correspondence between the set of all QSOs and the set of all cubic matrices  $\mathbf{P}$ , we can consider a QSO  $V$  as a point of  $\mathbf{R}^{m(m^2-1)}$ . The number  $\frac{m(m-1)(m-\ell+1)}{2}$  is the number of independent elements of the matrix  $\mathbf{P}$  with the condition (5.1). Let  $V_1, V_2$  be two  $\ell$ -Volterra QSO i.e.  $V_1, V_2 \in \mathcal{V}_\ell$ . We shall prove that  $V = \lambda V_1 + (1 - \lambda)V_2 \in \mathcal{V}_\ell$  for any  $\lambda \in [0, 1]$ .

Let  $P_{ij,k}^{(1)}$  (resp.  $P_{ij,k}^{(2)}$ ) be coefficients of  $V_1$  (resp.  $V_2$ ). Then coefficients of  $V$  has the form

$$P_{ij,k} = \lambda P_{ij,k}^{(1)} + (1 - \lambda)P_{ij,k}^{(2)}. \tag{5.9}$$

By definition coefficients  $P_{ij,k}^{(1)}$  and  $P_{ij,k}^{(2)}$  satisfy conditions (5.1) and (5.2). Using (5.9) it is easy to check that  $P_{ij,k}$  also satisfy the condition (5.1) and (5.2).

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2. Assume  $V \in \mathcal{V}_\ell$  with  $P_{i_0 j_0 k_0} = \alpha \neq 0$  and 1 for some  $i_0, j_0, k_0$ . Construct two operators  $V_q$  with coefficients  $P_{ij,k}^{(q)}$ ,  $q = 1, 2$  as following

$$P_{ij,k}^{(1)} = \begin{cases} P_{ij,k} & \text{if } (i, j) \neq (i_0, j_0), \\ 1 & \text{if } (i, j, k) = (i_0, j_0, k_0), \\ 0 & \text{if } (i, j, k) = (i_0, j_0, k), \quad k \neq k_0, \end{cases}$$

$$P_{ij,k}^{(2)} = \begin{cases} P_{ij,k} & \text{if } (i, j) \neq (i_0, j_0), \\ 0 & \text{if } (i, j, k) = (i_0, j_0, k_0), \\ \frac{P_{ij,k}}{1-\alpha} & \text{if } (i, j, k) = (i_0, j_0, k), \quad k \neq k_0. \end{cases}$$

Then

$$\alpha P_{ij,k}^{(1)} + (1-\alpha) P_{ij,k}^{(2)} = \begin{cases} P_{ij,k} & \text{if } (i, j) \neq (i_0, j_0), \\ \alpha = P_{i_0 j_0 k_0} & \text{if } (i, j, k) = (i_0, j_0, k_0) \\ P_{ij,k} & \text{if } (i, j, k) = (i_0, j_0, k), \quad k \neq k_0. \end{cases} = P_{ij,k}, \quad (5.10)$$

Since  $\alpha > 0$ , from (5.10) we get  $P_{ij,k} = 0$  if and only if  $P_{ij,k}^{(1)} = 0$  and  $P_{ij,k}^{(2)} = 0$ . This means that  $V_1$  and  $V_2$  are  $\ell$ -Volterra operators. Hence  $V = \alpha V_1 + (1-\alpha)V_2$ . Thus if  $P_{ij,k} \in (0, 1)$  for some  $(i, j, k)$  then  $V$  is not an extremal point. Finally, if  $P_{ij,k} = 0$  or 1 for any  $(i, j, k)$  then the representation  $V = \lambda V_1 + (1-\lambda)V_2$ ,  $0 < \lambda < 1$  is possible only if  $V_1 = V_2 = V$ .

3. In order to compute cardinality of  $\text{Extr}(\mathcal{V}_\ell)$  we have to know which elements of the matrix  $\mathbf{P}$  can be 1.

Denote  $\mathcal{P}_{ij} = (P_{ij,1}, \dots, P_{ij,m})^t$  the  $(i, j)$ th column of  $\mathbf{P}$ , where  $(i, j) \in \mathcal{K} = \{(i, j) : 1 \leq i \leq j \leq m\}$ .

Let  $n_0(\mathcal{P}_{ij})$  be the number of elements of  $\mathcal{P}_{ij}$  which must be zero by conditions (2.2), (5.1), (5.2).

Put for  $\ell \in \{1, \dots, m\}$  :

$$\mathcal{A} \equiv \mathcal{A}_{em} = \{(i, j) \in \mathcal{K} : i \leq \ell, j \in \{i\} \cup \{\ell + 1, \dots, m\}\},$$

$$\mathcal{B} \equiv \mathcal{B}_{em} = \{(i, j) \in \mathcal{K} : i \leq \ell, j \leq \ell, i < j\},$$

$$\mathcal{C} \equiv \mathcal{C}_{em} = \{(i, j) \in \mathcal{K} : \ell < i \leq j\}.$$

Note that  $\mathcal{K} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ . If  $\ell = m$  then  $\mathcal{C} = \emptyset$ .

It is easy to see that

$$n_0(\mathcal{P}_{ij}) = \begin{cases} \ell - 1 & \text{if } (i, j) \in \mathcal{A} \\ \ell - 2 & \text{if } (i, j) \in \mathcal{B} \\ \ell & \text{if } (i, j) \in \mathcal{C} \end{cases}$$

\*\*\*\*\*

By condition (2.2) each column contains unique "1". We have  $m - n_0(\mathcal{P}_{ij})$  possibilities to write 1 in the column  $(i, j) \in \mathcal{K}$ . Thus

$$|\text{Extr}(\mathcal{V}_\ell)| = \prod_{(i,j) \in \mathcal{K}} (m - n_0(\mathcal{P}_{ij})) = (m - \ell + 1)^{|\mathcal{A}|} (m - \ell + 2)^{|\mathcal{B}|} (m - \ell)^{|\mathcal{C}|}.$$

This with

$$|\mathcal{A}| = (m - \ell + 1)\ell, \quad |\mathcal{B}| = \frac{1}{2}(\ell - 1)\ell, \quad |\mathcal{C}| = \frac{1}{2}(m - \ell + 1)(m - \ell),$$

would yield the formula. The proposition is proved. □

For the set  $\mathcal{V}$  of all QSOs we have  $\mathcal{V} \subset \mathbf{R}^{\frac{m(m^2-1)}{2}}$ . Note that  $\mathcal{V}$  also is a convex, compact set. Its extremal points also are operators with  $P_{ij,k} = 0$  or 1 only. It is easy to see that

$$|\text{Extr}(\mathcal{V}_m)| < |\text{Extr}(\mathcal{V}_{m-1})| < \dots < |\text{Extr}(\mathcal{V}_1)| < |\text{Extr}(\mathcal{V})| = m^{\frac{1}{2}m(m+1)}.$$

For example, if  $m = 3$  then

$$|\text{Extr}(\mathcal{V}_3)| = 8, \quad |\text{Extr}(\mathcal{V}_2)| = 48, \quad |\text{Extr}(\mathcal{V}_1)| = 216, \quad |\text{Extr}(\mathcal{V})| = 729.$$

The set  $\mathcal{V}$  can be written as  $\mathcal{V} = \bigcup_{\ell=0}^m \mathcal{V}_\ell$ . Here  $\mathcal{V}_0$  is the set of "0-Volterra QSO"s i.e for any  $k \in \{1, \dots, m\}$  there is at least one pair  $(i, j)$  with  $i \neq k$  and  $j \neq k$  such that  $P_{ij,k} > 0$ .

As it was mentioned above:  $\mathcal{V}_m$  is the set of all Volterra operators and  $\mathcal{V}_{\ell_1} \cap \mathcal{V}_{\ell_2} = \emptyset$  for any  $\ell_1 \neq \ell_2 \in \{0, \dots, m\}$ .

Thus to study dynamics of QSOs from  $\mathcal{V}$  it is enough to study the problem for each  $\mathcal{V}_\ell$ ,  $\ell = 0, \dots, m$ .

In general, the problem of study the behavior of  $V \in \mathcal{V}_\ell$  (for fixed  $\ell$ ) is also a difficult problem. So in the next sections we consider the problem for small dimensions (i.e  $m = 2, 3$ ) and  $\ell = 1, 2$ .

### 5.2 Dynamics of l-Volterra operators

In this section we discuss the dynamics of  $l$ -Volterra operators for  $m = 2$  and  $m = 3$ .

**Case  $m = 2$ :** In the case  $m = 2$  we have only

1-Volterra operator  $V : S^1 \rightarrow S^1$  such that

$$\begin{cases} x' = ax^2 + 2cxy \\ y' = bx^2 + 2dxy + y^2, \end{cases} \tag{5.11}$$

\*\*\*\*\*

where  $a, b, c, d \in [0, 1)$  (the case  $a = 1$  corresponds to Volterra operator),  $a + b = c + d = 1$ . Using  $x + y = 1$  from (5.11) we get a dynamical system generated by function  $f(x) = (a - 2c)x^2 + 2cx, x \in [0, 1], a \in [0, 1], c \in [0, 1]$ . By properties of  $f(x)$  one can prove the following

**Proposition 36.** 1. If  $c \leq \frac{1}{2}, \forall a \in [0, 1)$  the operator (5.11) has unique fixed point  $\lambda_0 = (0, 1)$  and for any initial point  $\lambda^0 = (x^0, y^0) \in S^1$  the trajectory  $\lambda^{(n)}$  goes to  $\lambda_0$  as  $n \rightarrow \infty$ .

2. If  $c > \frac{1}{2}, \forall a \in [0, 1)$  then (5.11) has two fixed points  $\lambda_0 = (0, 1)$  and  $\lambda^* = (\frac{2c-1}{2c-a}, \frac{1-a}{2c-a})$  the point  $\lambda_0$  is repeller. For any initial point  $\lambda^0 \in S^1 \setminus \{\lambda_0\}$  the trajectory  $\lambda^{(n)}$  tends to  $\lambda^*$  as  $n \rightarrow \infty$ .

**Case  $m = 3$ :** In case  $m = 3$  one has two  $\ell$ -Volterra operators (for  $\ell = 2$  and 1). Note that the case  $\ell = 1$  i.e the 1-Volterra QSO is complicated: for example, it is not easy to describe all fixed points of the operator. Here for simplicity we shall study the 2-Volterra operators.

Arbitrary 2-Volterra operator (for  $m = 3$ ) has the form :

$$\begin{cases} x' = x(a_1x + 2b_1y + 2c_1z) \\ y' = y(2b_2x + d_1y + 2e_1z) \\ z' = z(2c_2x + 2e_2y + z) + a_2x^2 + 2b_3xy + d_2y_2, \end{cases} \quad (5.12)$$

where

$$\begin{aligned} a_1 = P_{11,1}, \quad a_2 = P_{11,3}; \quad b_i = P_{12,i}, i = 1, 2, 3; \quad c_1 = P_{13,1}, \\ c_2 = P_{13,3}; \quad d_i = P_{22,i}, i = 2, 3; \quad e_i = P_{23,i}, i = 2, 3. \end{aligned} \quad (5.13)$$

To avoid many special cases and complicated formulas we consider the case

$$P_{11,1} = P_{22,2}, \quad P_{13,1} = P_{23,2}, \quad P_{12,1} = P_{12,2}. \quad (5.14)$$

This corresponds to a symmetric (with respect to permutations of 1 and 2) model.

Using  $x + y + z = 1$  and condition (5.14) the operator (5.12) can be written as

$$\begin{cases} x' = x(2c + (a - 2c)x + 2(b - c)y) \\ y' = y(2c + 2(b - c)x + (a - 2c)y), \end{cases} \quad (5.15)$$

where  $a = P_{11,1} \in [0, 1), b = P_{12,1} \in [0, \frac{1}{2}], c = P_{13,1} \in [0, 1]$ , and  $x, y \in [0, 1]$  such that  $x + y \leq 1$ .

**Remark 37.** The case  $a = P_{11,1} = P_{22,2} = 1$  corresponds to the Volterra case, so we consider only  $a \neq 1$ .

**Theorem 38.** 1. For  $c \leq \frac{1}{2}$  the operator (5.15) has unique fixed point  $\lambda_0 = (0, 0)$  which is global attractive point.

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2. Sets  $M_0 = \{\lambda = (x, y) : x = 0\}$ ,  $M_1 = \{\lambda = (x, y) : y = 0\}$ ,  $M_ = \{\lambda = (x, y) : x = y\}$ ,  $M_ > = \{\lambda = (x, y) : x > y\}$ ,  $M_ < = \{\lambda = (x, y) : x < y\}$  are invariant with respect to the operator (5.15).
3. For  $c > \frac{1}{2}$ ,  $a \neq 2b$  the operator (5.15) has four fixed points  $\lambda_0 = (0, 0)$ ,  $\lambda_1 = \left(0, \frac{2c-1}{2c-a}\right)$ ,  $\lambda_2 = \left(\frac{2c-1}{2c-a}, 0\right)$ ,  $\lambda_3 = \left(\frac{1-2c}{a+b-4c}, \frac{1-2c}{a+2b-4c}\right)$ . Moreover  $\lambda_0$  is repeller and

$$\lambda_1 \text{ and } \lambda_2 \text{ are } \begin{cases} \text{attractive, if } a > 2b \\ \text{saddle, if } a < 2b \end{cases}$$

$$\lambda_3 \text{ is } \begin{cases} \text{attractive, if } a < 2b \\ \text{saddle, if } a > 2b. \end{cases}$$

4. For  $c > \frac{1}{2}$ ,  $a = 2b$  the operator (5.15) has a repeller fixed point  $\lambda_0 = (0, 0)$  and continuum set of fixed points  $F = \{\lambda = (x, y) : x + y = \frac{2c-1}{2(c-b)}\}$ . The following line

$$I_\nu = \{\lambda = (x, y) : y = \nu x, x \in [0, 1]\}$$

is an invariant set for any  $\nu \in [0, \infty)$ . If  $\lambda^0 = (x^0, y^0)$  is an initial point with  $\frac{y^0}{x^0} = \nu$ , ( $x^0 \neq 0$ ) then its trajectory  $\lambda^{(n)}$  goes to  $\bar{\lambda}_\nu = \left(\frac{2c-1}{2(c-b)(1+\nu)}, \frac{(2c-1)\nu}{2(c-b)(1+\nu)}\right) \in I_\nu \cap F$  as  $n \rightarrow \infty$ ,  $\nu \in [0, \infty)$ , (if  $x^0 = 0$  then on invariant set  $M_0$  we have  $\lambda^{(n)} \rightarrow \lambda_1$ ).

5. If  $a < 2b$  then  $M_0$  (resp.  $M_1$ ) is the stable manifold of the saddle point  $\lambda_1$  (resp.  $\lambda_2$ ). If  $a > 2b$  then  $M_ =$  is the stable manifold of saddle point  $\lambda_3$ . There is an invariant curve  $\gamma$  passing through  $\lambda_1, \lambda_2, \lambda_3$  which is unstable manifold for the saddle points.

*Proof.* 1. Clearly  $\lambda_0 = (0, 0)$  is a fixed point for (5.15). Note that the Jacobian of (5.15) at  $(0, 0)$  has the form

$$\mathbf{J} = \begin{pmatrix} 2c & 0 \\ 0 & 2c \end{pmatrix},$$

so  $\lambda_0$  is an attractive if  $c < \frac{1}{2}$  and non-hyperbolic if  $c = \frac{1}{2}$ .

Now we shall prove (for  $c \leq \frac{1}{2}$ ) its global attractiveness. From the first equation of (5.15) we have

$$x' = x(ax + 2by + 2cz) \leq qx, \tag{5.16}$$

where  $q = \max\{a, 2b, 2c\}$ . By definition of the operator (5.15) and condition  $c \leq \frac{1}{2}$  we have  $q \leq 1$ . Consider two cases:

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**Case  $q < 1$ .** In this case from (5.16) we get  $x_{n+1} \leq qx_n \leq q^n x^0$ , where  $x_n$  is the first coordinate of the trajectory  $\lambda^{(n)} = V^n(l^0) = (x_n, y_n)$  with initial point  $\lambda^0 = (x^0, y^0)$ . Thus  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ . By symmetry of  $x$  and  $y$  we get  $y_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Case  $q = 1$ .** In this case we get  $x_{n+1} \leq x_n$ , hence

$$\lim_{n \rightarrow \infty} x_n = \alpha \geq 0 \text{ exists.}$$

Similarly,

$$\lim_{n \rightarrow \infty} y_n = \beta \text{ also exists.}$$

Thus the point  $(\alpha, \beta)$  must be a fixed point for the operator (5.15). Since  $\lambda_0 = (0, 0)$  is unique fixed point for  $c \leq \frac{1}{2}$  (we shall prove uniqueness in section (iii) of this proof), we get  $(\alpha, \beta) = (0, 0)$ .

**Remark 39.** *The argument used in the case  $q = 1$  also works for the case  $q < 1$ . But in the case  $q < 1$  we proved that the rate of convergence to  $\lambda_0$  is faster than  $q^n$ .*

2. Invariance of  $M_0, M_-, M_1$  are straightforward. Invariance of  $M_<, M_>$  follow from the following equality

$$x' - y' = (x - y)(2cz + a(x + y)), \text{ where } z = 1 - x - y \geq 0$$

which can be obtained from (5.15).

3. Clearly  $\lambda_0 = (0, 0)$  is a fixed point independently on parameters  $a, b, c$ . To get other fixed points consider several cases:

**Case  $x = 0, y \neq 0$ :** From the second equation one gets  $y = \frac{2c-1}{2c-a}$  which is between 0 and 1 iff  $c > \frac{1}{2}$ . Thus  $\lambda_1 = \left(0, \frac{2c-1}{2c-a}\right)$  is a fixed point.

**Case  $x \neq 0, y = 0$**  is similar to the previous case and gives  $\lambda_2 = \left(\frac{2c-1}{2c-a}, 0\right)$ .

**Case  $x \neq 0, y \neq 0$ :** From (5.15) one gets a system of linear equations, which has unique solution  $\lambda_4 = \left(\frac{2c-1}{4c-a-2b}, \frac{2c-1}{4c-a-2b}\right)$  (for  $c > \frac{1}{2}, a \neq 2b$ ). Note that if  $c \leq \frac{1}{2}$  then there is only  $\lambda_0$ .

To check the type of fixed points consider Jacobian at  $\lambda = (x, y)$

$$\mathbf{J}(\lambda) = \mathbf{J}(x, y) = \begin{pmatrix} 2c + 2(a - 2c)x + 2(b - c)y & 2(b - c)x \\ 2(b - c)y & 2c + 2(a - 2c)y + 2(b - c)x \end{pmatrix}. \quad (5.17)$$

\*\*\*\*\*

It is easy to see that the eigenvalues  $\mu_1(\lambda), \mu_2(\lambda)$  of (5.17) at fixed points are

$$\begin{aligned} \mu_1(\lambda_2) &= \mu_2(\lambda_1) = \mu_2(\lambda_3) = 2(1 - c) < 1, \\ |\mu_1(\lambda_1)| &= |\mu_2(\lambda_2)| = \left| \frac{2c(1 - a) + 2b(c - 1)}{2c - a} \right| = \begin{cases} < 1 & \text{if } a > 2b \\ > 1 & \text{if } a < 2b \end{cases} \\ |\mu_1(\lambda_3)| &= \left| \frac{4c(b - 1) + 2a(c - 1)}{a + 2b - 4c} \right| = \begin{cases} < 1 & \text{if } a < 2b \\ > 1 & \text{if } a > 2b \end{cases} \end{aligned}$$

This completes the proof of (iii).

4. For  $a = 2b$  the operator (5.15) has the following form

$$\begin{cases} x' = x(2c + 2(b - c)(x + y)) \\ y' = y(2c + 2(b - c)(x + y)). \end{cases} \tag{5.18}$$

It is easy to see that  $\lambda_0 = (0, 0)$  and any point of  $F = \left\{ \lambda = (x, y) : x + y = \frac{2c-1}{2(c-b)} \right\}$  is fixed point if  $c > \frac{1}{2}$ . Invariance of  $I_\nu$  follows easily from the following relation  $\frac{y'}{x'} = \frac{y}{x} = \nu$ . To check  $\lambda^{(n)} \rightarrow \bar{\lambda}_\nu$  for  $\lambda^0 \in I_\nu$ , consider restriction of operator (5.19) shown in Figures 1 – 4. on  $I_\nu$  which is  $x' = \varphi(x) = x(2c + 2(b - c)(1 + \nu)x)$ . The function  $\varphi$  has two fixed points  $x = 0$  and  $\bar{x} = \frac{1-2c}{2(b-c)(1+\nu)}$ . The point  $x = 0$  is repeller and  $\bar{x}$  is attractive independently on  $\nu$  since  $\varphi'(\bar{x}) = 2(1 - c) < 1$  for  $c > \frac{1}{2}$ . One can see that  $x^* \geq \bar{x}$  where  $x^*$  is the critical point i.e  $\varphi'(x^*) = 0$ . Now we shall take arbitrary  $x_0 \in (0, 1]$  and prove that  $x_n = \varphi(x_{n-1}), n \geq 1$  converges to  $\bar{x}$  as  $n \rightarrow \infty$ . Consider the following partition  $[0, 1] = \{0\} \cup (0, \bar{x}) \cup \{\bar{x}\} \cup (\bar{x}, x^*) \cup (x^*, 1]$ . For  $x \in (0, \bar{x})$  we have  $x < \varphi(x) < \bar{x}$ , consequently  $x_0 < x_n < x_{n+1} \leq \bar{x}$  i.e  $x_n$  converges and its limit is a fixed point of  $\varphi$ , since  $\varphi$  has unique fixed point  $\bar{x}$  in  $(0, 1]$  we conclude that the limit is  $\bar{x}$ . For  $x \in (\bar{x}, x^*)$  we have  $x > \varphi(x) > \bar{x}$ , consequently  $x_0 > x_n > x_{n+1} \geq \bar{x}$  i.e  $x_n$  converges and its limit is  $\bar{x}$ . If  $x_0 \in (x^*, 1]$  then it is easy to see that  $x_1 = \varphi(x_0) \in [0, x^*)$  so by above mentioned reasons we again have  $x_n \rightarrow \bar{x}$ . Hence  $\bar{x}$  is the global attractive point on  $I_\nu$ .

5. The existence of  $\gamma$  follows from Theorem 5.4. Other statements of 5 are straightforward. The theorem is proved. □

Note that 2-Volterra operator corresponding to (5.15) has the following form

$$\begin{cases} x' = x(ax + 2by + 2cz) \\ y' = y(2bx + ay + 2cz) \\ z' = 1 - 2c(x + y) - (a - 2c)(x^2 + y^2) - 4(b - c)xy \end{cases} \tag{5.19}$$

Using Theorem 5.2 we get phase portraits of the trajectories of (5.19) shown in Figures 1 – 4.

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**Remark 40.** One of the main goal by introducing the notion of  $\ell$ -Volterra operators was to give an example of QSO which has more rich dynamics than Volterra QSO. It is well known [2] that for Volterra operators if  $a_{ij} \neq 0$  ( $i \neq j$ ) then for any non-fixed initial point  $\lambda^0$  the set  $\omega(\lambda^0)$  of all limit points of the trajectory  $\{\lambda^{(n)}\}$  is subset of the boundary of simplex. But in our case Fig.3 and 4 show that the limit set is not subset of the boundary of  $S^2$ .

## 6 Separable Quadratic Stochastic Operators (SQSOs)

Separable Quadratic Stochastic Operators (SQSOs) were introduced in 2009 by U. A. Rozikov and S. Nazir. From the definition of Quadratic Stochastic Operator (QSO), we know that

$$x'_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad p_{ij,k} = p_{ji,k}, \quad 0 \leq p_{ij,k} \leq 1, \quad \sum_{k=1}^m p_{ij,k} = 1.$$

U. A. Rozikov and S. Nazir supposed that if  $p_{ij,k} = a_{ik} b_{jk}$  such that  $0 \leq a_{ik} b_{jk} \leq 1$ ,  $\sum_{k=1}^m a_{ik} b_{jk} = 1$ ,  $\forall i, j \in \{1, \dots, m\}$ . So

$$\begin{aligned} x'_k &= \sum_{i,j=1}^m a_{ik} b_{jk} x_i x_j \quad \forall k \in \{1, \dots, m\} \\ \Rightarrow x'_k &= \sum_{i=1}^m a_{ik} x_i \sum_{j=1}^m b_{jk} x_j, \end{aligned}$$

where  $0 \leq a_{ik} b_{jk} \leq 1$ ,  $\sum_{k=1}^m a_{ik} b_{jk} = 1$ ,  $\forall i, j \in \{1, \dots, m\}$ .

**Definition 41.** The Separable Quadratic Stochastic Operator (SQSO) is defined as

$$x'_k = \sum_{i=1}^m a_{ik} x_i \sum_{j=1}^m b_{jk} x_j, \quad (6.1)$$

where  $0 \leq a_{ik} b_{jk} \leq 1$ ,  $\sum_{k=1}^m a_{ik} b_{jk} = 1$ ,  $\forall i, j \in \{1, \dots, m\}$ .

The Quadratic Stochastic Operator defined in (6.1) is called Separable Quadratic Stochastic Operator (SQSO).

From the conditions  $p_{ij,k} \geq 0$  and  $\sum_{k=1}^m p_{ij,k} = 1 \quad \forall i, j$  it follows the condition on matrices A and B that  $a_{ik} b_{jk} \geq 0$ ,

$$AB^T = \mathbf{1}, \quad (6.2)$$

\*\*\*\*\*



where  $B^T$  is the transpose of matrix B and  $\mathbf{1}$  is a matrix with all entries 1. Let  $a^{(i)}$  denote the  $i$ th row of matrix A and  $b^{(j)}$  denote the  $j$ th row of matrix B or the  $j$ th column of  $B^T$ . Then from (6.2), we get

$$a^{(i)}b^{(j)} = 1, \forall i, j \in \{1, 2, \dots, m\}.$$

For a fixed  $j$ , (6.2) implies that

$$Ab^{(j)T} = (1, 1, \dots, 1)^T, \tag{6.3}$$

where  $(1, 1, \dots, 1)^T$  is a column vector. Now if determinant of matrix A is non-zero i.e.  $\det(A) \neq 0$ , then (6.3) implies that

$$b^{(j)T} = A^{-1}(1, 1, \dots, 1)^T, \forall j \in \{1, 2, \dots, m\}. \tag{6.4}$$

From (6.4) we will get the identical rows of matrix B. It means that if  $\det(A) \neq 0$ , then the rows of matrix B will be the same. Similarly if  $\det(B) \neq 0$ , then the rows of matrix A will be the same. So here, we will discuss three cases.

*Case-1:* If  $\det(A) = \det(B) = 0$ , then both A and B have identical rows.

*Case-2:* If  $\det(A) \neq 0$ , then the matrix B has identical rows. Similarly if  $\det(B) \neq 0$ , then the matrix A has identical rows.

*Case-3:* If  $\det(A) = \det(B) = 0$  but both A and B have non-identical rows.

Now if we want to find a Separable Quadratic Stochastic Operator (SQSO) for case-1, then consider

$$\begin{aligned} x'_k &= \sum_{i=1}^m a_{ik}x_i \sum_{j=1}^m b_{jk}x_j \\ &= (a_{1k}x_1 + a_{2k}x_2 + \dots + a_{mk}x_m)(b_{1k}x_1 + b_{2k}x_2 + \dots + b_{mk}x_m) \\ &= (a_{1k}x_1 + a_{1k}x_2 + \dots + a_{1k}x_m)(b_{1k}x_1 + b_{1k}x_2 + \dots + b_{1k}x_m) \\ &= a_{1k}b_{1k}(x_1 + x_2 + \dots + x_m)(x_1 + x_2 + \dots + x_m) \\ &= a_{1k}b_{1k}(1)(1) \\ x'_k &= a_{1k}b_{1k}. \end{aligned}$$

So  $V(x) = (x') = (x'_1, x'_2, \dots, x'_m) = (a_{11}b_{11}, a_{12}b_{12}, \dots, a_{1m}b_{1m})$  i.e. in case-1, we have a constant Separable Quadratic Stochastic Operator (SQSO). Similarly in case-2 if B has identical rows, then we have a linear Separable Quadratic Stochastic Operator (SQSO) i.e.

$$x'_k = b_{1k}(a_{1k}x_1 + a_{2k}x_2 + \dots + a_{mk}x_m).$$

**Example 42.** For the system  $E = \{1, 2\}$ , find the corresponding Separable Quadratic Stochastic Operator (SQSO).

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**Solution:** Given that  $E = \{1, 2\}$  then the corresponding SQSO is given by  $V(x_1, x_2) = (x'_1, x'_2)$ . Now by definition, we can find  $x'_1$  and  $x'_2$  as:

$$\begin{aligned} x'_1 &= \sum_{i,j=1}^2 (a_{i1}b_{j1}) x_i x_j \\ &= \sum_{i=1}^2 a_{i1} x_i \sum_{j=1}^2 b_{j1} x_j \\ &= (a_{11}x_1 + a_{21}x_2)(b_{11}x_1 + b_{21}x_2) \\ &= a_{11}b_{11}x_1^2 + a_{11}b_{21}x_1x_2 + a_{21}b_{11}x_1x_2 + a_{21}b_{21}x_2^2. \end{aligned}$$

Similarly

$$\begin{aligned} x'_2 &= \sum_{i,j=1}^2 (a_{i2}b_{j2}x_i x_j) \\ &= \sum_{i=1}^2 a_{i2} x_i \sum_{j=1}^2 b_{j2} x_j \\ &= (a_{12}x_1 + a_{22}x_2)(b_{12}x_1 + b_{22}x_2) \\ &= a_{12}b_{12}x_1^2 + a_{12}b_{22}x_1x_2 + a_{22}b_{12}x_1x_2 + a_{22}b_{22}x_2^2. \end{aligned}$$

So

$$V(x_1) = a_{11}b_{11}x_1^2 + a_{11}b_{21}x_1x_2 + a_{21}b_{11}x_1x_2 + a_{21}b_{21}x_2^2,$$

and

$$V(x_2) = a_{12}b_{12}x_1^2 + a_{12}b_{22}x_1x_2 + a_{22}b_{12}x_1x_2 + a_{22}b_{22}x_2^2.$$

**Example 43.** For the system  $E = \{1, 2, 3\}$ , find the corresponding Separable Quadratic Stochastic Operator (SQSO).

**Solution:** Given that  $E = \{1, 2, 3\}$  then the corresponding SQSO is given by  $V(x_1, x_2) = (x'_1, x'_2)$ . Now by definition, we can find  $x'_1$ ,  $x'_2$  and  $x'_3$  as:

$$\begin{aligned} x'_1 &= \sum_{i,j=1}^3 (a_{i1}b_{j1}) x_i x_j \\ &= \sum_{i=1}^3 a_{i1} x_i \sum_{j=1}^3 b_{j1} x_j \\ &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3)(b_{11}x_1 + b_{21}x_2 + b_{31}x_3) \\ x'_1 &= a_{11}b_{11}x_1^2 + a_{11}b_{21}x_1x_2 + a_{11}b_{31}x_1x_3 + a_{21}b_{11}x_1x_2 \\ &+ a_{21}b_{21}x_2^2 + a_{21}b_{31}x_2x_3 + a_{31}b_{11}x_1x_3 + a_{31}b_{21}x_2x_3 + a_{31}b_{31}x_3^2. \end{aligned}$$

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Similarly

$$\begin{aligned}
 x_2' &= \sum_{i,j=1}^3 (a_{i2}b_{j2}) x_i x_j \\
 &= \sum_{i=1}^3 a_{i2} x_i \sum_{j=1}^3 b_{j2} x_j \\
 &= (a_{12}x_1 + a_{22}x_2 + a_{32}x_3)(b_{12}x_1 + b_{22}x_2 + b_{32}x_3) \\
 x_2' &= a_{12}b_{12}x_1^2 + a_{12}b_{22}x_1x_2 + a_{12}b_{32}x_1x_3 + a_{22}b_{12}x_1x_2 \\
 &+ a_{22}b_{22}x_2^2 + a_{22}b_{32}x_2x_3 + a_{32}b_{12}x_1x_3 + a_{32}b_{22}x_2x_3 + a_{32}b_{32}x_3^2,
 \end{aligned}$$

and

$$\begin{aligned}
 x_3' &= \sum_{i,j=1}^3 (a_{i3}b_{j3}) x_i x_j \\
 &= \sum_{i=1}^3 a_{i3} x_i \sum_{j=1}^3 b_{j3} x_j \\
 &= (a_{13}x_1 + a_{23}x_2 + a_{33}x_3)(b_{13}x_1 + b_{23}x_2 + b_{33}x_3) \\
 x_3' &= a_{13}b_{13}x_1^2 + a_{13}b_{23}x_1x_2 + a_{13}b_{33}x_1x_3 + a_{23}b_{13}x_1x_2 \\
 &+ a_{23}b_{23}x_2^2 + a_{23}b_{33}x_2x_3 + a_{33}b_{13}x_1x_3 + a_{33}b_{23}x_2x_3 + a_{33}b_{33}x_3^2.
 \end{aligned}$$

So

$$\begin{aligned}
 V(x_1) = x_1' &= a_{11}b_{11}x_1^2 + a_{11}b_{21}x_1x_2 + a_{11}b_{31}x_1x_3 + a_{21}b_{11}x_1x_2 \\
 &+ a_{21}b_{21}x_2^2 + a_{21}b_{31}x_2x_3 + a_{31}b_{11}x_1x_3 + a_{31}b_{21}x_2x_3 + a_{31}b_{31}x_3^2,
 \end{aligned}$$

$$\begin{aligned}
 V(x_2) = x_2' &= a_{12}b_{12}x_1^2 + a_{12}b_{22}x_1x_2 + a_{12}b_{32}x_1x_3 + a_{22}b_{12}x_1x_2 \\
 &+ a_{22}b_{22}x_2^2 + a_{22}b_{32}x_2x_3 + a_{32}b_{12}x_1x_3 + a_{32}b_{22}x_2x_3 + a_{32}b_{32}x_3^2,
 \end{aligned}$$

and

$$\begin{aligned}
 V(x_3) = x_3' &= a_{13}b_{13}x_1^2 + a_{13}b_{23}x_1x_2 + a_{13}b_{33}x_1x_3 + a_{23}b_{13}x_1x_2 \\
 &+ a_{23}b_{23}x_2^2 + a_{23}b_{33}x_2x_3 + a_{33}b_{13}x_1x_3 + a_{33}b_{23}x_2x_3 + a_{33}b_{33}x_3^2.
 \end{aligned}$$

### 6.1 Lyapunov Function of SQSOs

In this Section we will discuss about the Lyapunov function of Separable Quadratic Stochastic Operators (SQSOs). By using this Lyapunov function we will be able to describe the upper estimates for the set of limit points of Separable Quadratic Stochastic Operators (SQSOs).

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**Theorem 44.** [11], Let  $V$  is Separable Quadratic Stochastic Operator (SQSO), then the function  $\varphi_c : S^{m-1} \rightarrow R$  defined by  $\varphi_c(x) = \sum_{k=1}^m c_k x_k$  is a Lyapunov function if  $c = (c_1, c_2, \dots, c_m)^T$  satisfies  $c_i \geq 0 \forall i \in \{1, \dots, m\}$  and either  $Ac \leq Ic$  or  $Bc \leq Ic$ , where  $A = (a_{ij})_{i,j=1}^m$ ,  $B = (b_{ij})_{i,j=1}^m$ ,  $0 \leq a_{ij}, b_{ij} \leq 1$ .

*Proof.* Since we know that

$$\begin{aligned}
 \varphi_c(x') &= \sum_{k=1}^m c_k x'_k \\
 &= \sum_{k=1}^m c_k \left( \sum_{i=1}^m a_{ik} x_i \right) \left( \sum_{j=1}^m b_{jk} x_j \right) \\
 &= \sum_{k=1}^m c_k \sum_{i,j=1}^m (a_{ik} b_{jk}) x_i x_j \\
 &\leq \sum_{k=1}^m c_k \left( \sum_{i=1}^m a_{ik} x_i \sum_{j=1}^m x_j \right) \\
 &= \sum_{i=1}^m \left( x_i \sum_{k=1}^m c_k a_{ik} \right) \\
 &= \sum_{i=1}^m \left( \sum_{k=1}^m c_k a_{ik} \right) x_i \\
 &\leq \sum_{i=1}^m c_i x_i \\
 &= \varphi_c(x) \\
 \Rightarrow \varphi_c(x') &\leq \varphi_c(x).
 \end{aligned}$$

Continuing in this way, we will get

$$\varphi_c(x^{n+1}) \leq \varphi_c(x^n),$$

which is a non-increasing monotonic and bounded sequence. So it means that  $\varphi_c(x)$  is a Lyapunov function.  $\square$

## 6.2 Properties of $F(A)$

In this Section we will discuss some properties of the particular set  $F(A) = \{B = (b_{jk})_{j,k=1}^m : 0 \leq a_{ik} b_{jk} \leq 1, AB^T = \mathbf{1}\}$ .

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**Theorem 45.** [14], Let  $A = (a_{ik})_{i,k=1}^m$  is a fixed matrix, consider the set  $F(A) = \{B = (b_{jk})_{j,k=1}^m : 0 \leq a_{ik}b_{jk} \leq 1, AB^T = \mathbf{1}\}$ . The set  $F(A)$  has the following properties

1.  $F(A)$  is convex.
2. The extremal points of  $F(A)$  are those matrices  $B$  whose entries are either zeroes or ones.
3. If  $F(A)$  is non-empty then at least one of the entry of each row of matrix  $A$  must be non-zero.
4.  $F(A)$  is not closed with respect to addition. If  $B = (b_{jk})_{j,k=1}^m$  and  $C = (c_{lk})_{l,k=1}^m$  are contained in  $F(A)$  then  $BC$  is also contained in  $F(A)$  if

$$\sum_{k=1}^m b_{jk} = 1 \text{ and } \sum_{r=1}^m a_{ik}b_{jr}c_{rk} \geq 0, \forall i, j, k \in \{1, \dots, m\}.$$

5.  $B \in F(A) \Leftrightarrow A \in F(B)$ .

*Proof.* 1. Let  $X, Y \in F(A)$  such that  $X = (b'_{jk})_{j,k=1}^m, Y = (b''_{jk})_{j,k=1}^m$ . Let  $\alpha \in [0, 1]$  and consider  $B = (b_{jk})_{j,k=1}^m$  such that

$$\alpha(b'_{jk}) + (1 - \alpha)(b''_{jk}) = b_{jk}, \tag{6.5}$$

multiplying both sides of (6.5) by  $a_{ik}$ , we get

$$\alpha a_{ik}b'_{jk} + (1 - \alpha)a_{ik}b''_{jk} = a_{ik}b_{jk}.$$

By definition we know that  $0 \leq a_{ik}b'_{jk} \leq 1, 0 \leq a_{ik}b''_{jk} \leq 1$ , thus  $0 \leq a_{ik}b_{jk} \leq 1$  holds. Let  $a^{(i)} = (a_{i1}, a_{i2}, \dots, a_{im})$  be the  $i$ th row of  $A, b^{(j)'} = (b'_{j1}, b'_{j2}, \dots, b'_{jm})$  be the  $j$ th row of  $X$  and  $b^{(j)''} = (b''_{j1}, b''_{j2}, \dots, b''_{jm})$  be the  $j$ th row of  $Y$ . Then  $a^{(i)}b^{(j)'} = 1, \forall i, j$  and  $a^{(i)}b^{(j)''} = 1, \forall i, j$ . Let  $b^{(j)}$  be the  $j$ th row of  $B$ , then (6.5) implies that

$$\alpha b^{(j)'} + (1 - \alpha)b^{(j)''} = b^{(j)}, \forall j. \tag{6.6}$$

Again by multiplying both sides of (6.6) by  $a^{(i)}$ , we get

$$\begin{aligned} a^{(i)}b^{(j)} &= \alpha a^{(i)}b^{(j)'} + (1 - \alpha)a^{(i)}b^{(j)''} \\ &= \alpha(1) + (1 - \alpha)(1) \\ &= \alpha + 1 - \alpha \\ &= 1 \\ \Rightarrow a^{(i)}b^{(j)} &= 1. \end{aligned}$$

\*\*\*\*\*

Thus  $B = (b_{jk})_{j,k=1}^m \in F(A)$ . Hence  $F(A)$  is a convex set.

2. Let  $0 < \alpha < 1$  and  $B \in F(A)$  such that  $B = (b_{jk})_{j,k=1}^m$  with  $b_{jk} = \alpha$  for some  $j = j_0, k = k_0$ . Consider two matrices  $B_1 = (b_{jk}^{(1)})_{j,k=1}^m$  and  $B_2 = (b_{jk}^{(2)})_{j,k=1}^m$  with

$$b_{jk}^{(1)} = \begin{cases} b_{jk}, & \text{if } (j, k) \neq (j_0, k_0) \\ 1, & \text{if } (j, k) = (j_0, k_0). \end{cases}$$

$$b_{jk}^{(2)} = \begin{cases} b_{jk}, & \text{if } (j, k) \neq (j_0, k_0) \\ 0, & \text{if } (j, k) = (j_0, k_0). \end{cases}$$

If  $(j, k) \neq (j_0, k_0)$ , then

$$\begin{aligned} \alpha b_{jk}^{(1)} + (1 - \alpha)b_{jk}^{(2)} &= \alpha b_{jk} + (1 - \alpha)b_{jk} \\ \Rightarrow \alpha b_{jk}^{(1)} + (1 - \alpha)b_{jk}^{(2)} &= b_{jk}. \end{aligned}$$

If  $(j, k) = (j_0, k_0)$ , then

$$\begin{aligned} \alpha b_{jk}^{(1)} + (1 - \alpha)b_{jk}^{(2)} &= \alpha + (1 - \alpha)(0) \\ \Rightarrow \alpha b_{jk}^{(1)} + (1 - \alpha)b_{jk}^{(2)} &= \alpha \\ \Rightarrow \alpha b_{jk}^{(1)} + (1 - \alpha)b_{jk}^{(2)} &= b_{jk}. \end{aligned}$$

As  $\alpha \in (0, 1)$ , hence  $B = \alpha B_1 + (1 - \alpha)B_2$  i.e.  $B$  is not an extremal point which is a contradiction. Hence the extremal points of  $F(A)$  are those matrices whose entries are either zeros or ones.

3. If all the entries of matrix  $A$  are zero, then the condition  $AB^T = \mathbf{1}$  does not hold. So if the set  $F(A)$  is non empty then at least one of the entry of each row of matrix  $A$  must be non-zero such that the condition  $AB^T = \mathbf{1}$  holds.

4. Let  $X = (b_{jk}^{(1)})_{j,k=1}^m, Y = (b_{jk}^{(2)})_{j,k=1}^m \in F(A)$  such that  $AX^T = \mathbf{1}$  and  $BY^T = \mathbf{1}$ . Then

$$\begin{aligned} A(X + Y)^T &= A(X^T + Y^T) \\ &= AX^T + AY^T \\ &= \mathbf{1} + \mathbf{1} \\ \Rightarrow A(X + Y)^T &\neq \mathbf{1}. \end{aligned}$$

So  $X + Y \notin F(A)$  i.e.  $F(A)$  is not closed with respect to addition.

Now

$$\begin{aligned} A(XY)^T &= A(Y^T X^T) \\ &= (AY^T)X^T \\ &= \mathbf{1}X^T. \end{aligned}$$

\*\*\*\*\*

Now  $\mathbf{1}X^T = \mathbf{1}$  implies that  $\sum_{k=1}^m (b_{jk}^{(1)}) = 1, \forall j \in \{1, 2, \dots, m\}$ . Also we know that each entry  $(XY)_{j,k}$  of product matrix  $XY$  is given by  $(XY)_{j,k} = \sum_{r=1}^m b_{jr}^{(1)}b_{rk}^{(2)}$ . Thus  $XY \in F(A)$  if the following properties are satisfied:

$$\sum_{k=1}^m (b_{jk}^{(1)}) = 1 \text{ and } \sum_{r=1}^m a_{ik}b_{jr}^{(1)}b_{rk}^{(2)} \geq 0, \forall i, j, k \in \{1, 2, \dots, m\}.$$

5. Let  $B \in F(A)$ , then

$$\begin{aligned} AB^T &= \mathbf{1} \\ \Leftrightarrow (AB^T)^T &= \mathbf{1}^T \\ \Leftrightarrow (B^T)^T A^T &= \mathbf{1}^T \\ \Leftrightarrow BA^T &= \mathbf{1} \\ \Leftrightarrow A &\in F(B). \end{aligned}$$

So  $B \in F(A)$  if and only if  $A \in F(B)$ . □

### 6.3 Skew-Symmetric Matrix and SQSO

In this Section we will discuss about the relation between Skew-Symmetric Matrix and Separable Quadratic Stochastic Operator (SQSO).

**Proposition 46.** [14], If  $A = (a_{ik})_{i,k=1}^3$  is a skew symmetric matrix with  $a_{ii} = 0$ . Then the solution of the system

$$A(b^{(j)})^T = (1, 1, 1); \quad j = 1, 2, 3$$

exists if and only if  $a_{23} = a_{13} - a_{12}$ . Moreover, the solution is

$$(b^{(j)})^T = \left( b_{1j}, \frac{1 + a_{13}b_{1j}}{a_{12} - a_{13}}, \frac{1 + a_{12}b_{1j}}{a_{13} - a_{12}} \right) \quad \forall j = 1, 2, 3,$$

where  $b^{(j)}$  is a row of matrix  $B = (b_{jk})_{j,k=1}^3$ . In each case show that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{13} - a_{12} \\ -a_{13} & -(a_{13} - a_{12}) & 0 \end{pmatrix}.$$

*Proof.* Since

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

\*\*\*\*\*

Let  $a_{12} = a$ ,  $a_{13} = b$ ,  $a_{23} = c$ ,  $b_{11} = \alpha$ ,  $b_{12} = \beta$ ,  $b_{13} = \gamma$ ,  $b_{21} = \xi$ ,  $b_{22} = \zeta$ ,  $b_{23} = \eta$ ,  $b_{31} = x$ ,  $b_{32} = y$  and  $b_{33} = z$ . So the above matrices A, B will become

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, B = \begin{pmatrix} \alpha & \beta & \gamma \\ \xi & \zeta & \eta \\ x & y & z \end{pmatrix}.$$

And we know that

$$\Rightarrow \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \alpha & \xi & x \\ \beta & \zeta & y \\ \gamma & \eta & z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

by multiplying and comparing, we get

$$a\beta + b\gamma = 1, -a\alpha + c\gamma = 1, -b\alpha - c\beta = 1 \quad (6.7)$$

$$a\xi + b\eta = 1, -a\xi + c\eta = 1, -b\xi - c\zeta = 1 \quad (6.8)$$

$$ay + bz = 1, -ax + cz = 1, -bx - cy = 1. \quad (6.9)$$

From (6.7), we get

$$\beta = \frac{1 - b\gamma}{a}, \alpha = \frac{c\gamma - 1}{a}, \beta = \frac{1 + b\alpha}{-c}. \quad (6.10)$$

From (6.10), it is clear that

$$\begin{aligned} \frac{1 - b\gamma}{a} &= \frac{1 + b\alpha}{-c} \\ \Rightarrow \frac{1 - b\gamma}{a} &= \frac{1 + \frac{b(c\gamma - 1)}{a}}{-c} \\ \Rightarrow -c(1 - b\gamma) &= a \left( 1 + \frac{b(c\gamma - 1)}{a} \right) \\ \Rightarrow -c + cb\gamma &= a + bc\gamma - b \\ \Rightarrow -c &= a - b \\ \Rightarrow c &= b - a \\ \Rightarrow a_{23} &= a_{13} - a_{12}. \end{aligned}$$

Now from (6.8), we get

$$\zeta = \frac{1 - b\eta}{a}, \xi = \frac{c\eta - 1}{a}, \zeta = \frac{1 + b\xi}{-c}. \quad (6.11)$$

\*\*\*\*\*



And it is clear from (6.11) that

$$\begin{aligned} \frac{1 - b\eta}{a} &= \frac{1 + b\xi}{-c} \\ \Rightarrow \frac{1 - b\eta}{a} &= \frac{1 + \frac{b(c\eta - 1)}{a}}{-c} \\ \Rightarrow -c(1 - b\eta) &= a + b(c\eta - 1) \\ \Rightarrow -c + cb\eta &= a + bc\eta - b \\ \Rightarrow -c &= a - b \\ \Rightarrow c &= b - a \\ \Rightarrow a_{23} &= a_{13} - a_{12}. \end{aligned}$$

Now from (6.9), we get

$$y = \frac{1 - bz}{a}, \quad x = \frac{cz - 1}{a}, \quad y = \frac{1 + bx}{-c}. \tag{6.12}$$

And again it is clear from (6.12) that

$$\begin{aligned} \frac{1 - bz}{a} &= \frac{1 + bx}{-c} \\ \Rightarrow \frac{1 - bz}{a} &= \frac{1 + \frac{b(cz - 1)}{a}}{-c} \\ \Rightarrow -c(1 - bz) &= a + b(cz - 1) \\ \Rightarrow -c + cbz &= a + bcz - b \\ \Rightarrow -c &= a - b \\ \Rightarrow c &= b - a \\ \Rightarrow a_{23} &= a_{13} - a_{12}. \end{aligned}$$

So in each and every case

$$a_{23} = a_{13} - a_{12}$$

i.e.

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{13} - a_{12} \\ -a_{13} & -(a_{13} - a_{12}) & 0 \end{pmatrix}.$$

Now as given that

$$A(b^{(j)})^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad j = 1, 2, 3, \tag{6.13}$$

\*\*\*\*\*

where  $b^{(j)}$  is a row of matrix  $B = (b_{jk})_{j,k=1}^3$ . Now for  $j = 1$ , (6.13) implies that

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (6.14)$$

So (6.14) implies that

$$a_{12}b_{12} + a_{13}b_{13} = 1 \quad (6.15)$$

$$-a_{12}b_{11} + (a_{13} - a_{12})b_{13} = 1 \quad (6.16)$$

$$-a_{13}b_{11} + (a_{12} - a_{13})b_{12} = 1. \quad (6.17)$$

Now from (6.16) and (6.17), we get

$$b_{12} = \frac{1 + a_{13}b_{11}}{a_{12} - a_{13}}, \quad b_{13} = \frac{1 + a_{12}b_{11}}{a_{13} - a_{12}},$$

by putting the values of  $b_{12}$  and  $b_{13}$  in (6.15), we get

$$b_{11} = b_{11}.$$

Hence for  $j = 1$ , the solution is

$$(b^{(1)})^T = \left( b_{11}, \frac{1 + a_{13}b_{11}}{a_{12} - a_{13}}, \frac{1 + a_{12}b_{11}}{a_{13} - a_{12}} \right).$$

In the similar way, for  $j = 2$  and  $j = 3$ , the solution is

$$(b^{(2)})^T = \left( b_{12}, \frac{1 + a_{13}b_{12}}{a_{12} - a_{13}}, \frac{1 + a_{12}b_{12}}{a_{13} - a_{12}} \right), \quad \text{for } j = 2,$$

$$(b^{(3)})^T = \left( b_{13}, \frac{1 + a_{13}b_{13}}{a_{12} - a_{13}}, \frac{1 + a_{12}b_{13}}{a_{13} - a_{12}} \right), \quad \text{for } j = 3.$$

So in general, the solution is

$$(b^{(j)})^T = \left( b_{1j}, \frac{1 + a_{13}b_{1j}}{a_{12} - a_{13}}, \frac{1 + a_{12}b_{1j}}{a_{13} - a_{12}} \right) \quad \forall j = 1, 2, 3.$$

This completes the proof.  $\square$

**Proposition 47.** [14], *If the matrix  $A$  is same as in Proposition 6.11 then show that Separable Quadratic Stochastic Operator (SQSO) is a linear operator.*

*Proof.* From Proposition 6.11 we have a result

$$(b^{(j)})^T = \left( b_{1j}, \frac{1 + a_{13}b_{1j}}{a_{12} - a_{13}}, \frac{1 + a_{12}b_{1j}}{a_{13} - a_{12}} \right) \quad \forall j = 1, 2, 3.$$

By the definition of Separable Quadratic Stochastic Operator (SQSO), we know that  $0 \leq a_{ik}b_{jk} \leq 1$ ,  $\forall i, j, k \in \{1, \dots, m\}$ . So we will consider the following cases.

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Case-1:-

If  $a_{12} > 0$ ,  $a_{13} > 0$  such that  $a_{13} < a_{12}$ , then

$$\begin{aligned} 0 &\leq a_{13}b_{13} \\ \Rightarrow 0 &\leq a_{13} \left( \frac{1 + a_{12}b_{11}}{a_{13} - a_{12}} \right) \\ \Rightarrow b_{11} &\leq \frac{-1}{a_{12}}. \end{aligned}$$

Similarly from  $a_{13}b_{23} \geq 0$  and  $a_{13}b_{33} \geq 0$ , we will get  $b_{21} \leq \frac{-1}{a_{12}}$  and  $b_{31} \leq \frac{-1}{a_{12}}$  respectively. So we have a result

$$b_{11} \leq \frac{-1}{a_{12}}, b_{21} \leq \frac{-1}{a_{12}}, b_{31} \leq \frac{-1}{a_{12}}. \tag{6.18}$$

In a similar way from  $a_{23}b_{13} \geq 0$ ,  $a_{23}b_{23} \geq 0$  and  $a_{23}b_{33} \geq 0$ , we have a result

$$b_{11} \geq \frac{-1}{a_{12}}, b_{21} \geq \frac{-1}{a_{12}}, b_{31} \geq \frac{-1}{a_{12}}. \tag{6.19}$$

So from (6.18) and (6.19), we have a result

$$b_{11} = b_{21} = b_{31} = \frac{-1}{a_{12}}.$$

Hence

$$(b^{(1)})^T = \left( \frac{-1}{a_{12}}, \frac{1}{a_{12}}, 0 \right).$$

Also  $(b^{(1)})^T = (b^{(2)})^T = (b^{(3)})^T$ . Hence the matrix B has identical rows. So in this case, Separable Quadratic Stochastic Operator (SQSO) is a linear operator.

Case-2:-

We can easily check that for  $a_{12} > 0$  and  $a_{13} > 0$  with  $a_{13} > a_{12}$ . We get

$$(b^{(1)})^T = (b^{(2)})^T = (b^{(3)})^T = \left( \frac{-1}{a_{13}}, 0, \frac{1}{a_{13}} \right).$$

So again in this case Separable Quadratic Stochastic Operator (SQSO) is a linear operator. Similarly all the remaining cases will give the same result i.e. matrix B has identical rows. So in each and every case Separable Quadratic Stochastic Operator (SQSO) is a linear operator.  $\square$

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**Remark 48.** The general form of  $m \times m$  skew symmetric matrix  $A$  with the same conditions as in Proposition 6.11 is:

$$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & \cdots & \cdots & a_m \\ -a_2 & 0 & a_3 - a_2 & a_4 - a_2 & \cdots & \cdots & a_m - a_2 \\ -a_3 & a_2 - a_3 & 0 & a_4 - a_3 & \cdots & \cdots & a_m - a_3 \\ -a_4 & a_2 - a_4 & a_3 - a_4 & 0 & \cdots & \cdots & a_m - a_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ -a_m & a_2 - a_m & a_3 - a_m & a_4 - a_m & \cdots & \cdots & 0 \end{pmatrix} \tag{6.20}$$

**Theorem 49.** Let  $A, B$  be two matrices satisfying  $P_{i,j,k} = a_{ik}b_{jk}$ . If  $A$  is a skew symmetric matrix which has the form (6.20). Then  $B$  is a matrix with identical rows. Moreover, each row of  $B$  contains at most two non zero elements, one of which is positive and second one is negative.

*Proof.* Since we know that

$$a_{ik}b_{jk} \geq 0, \forall i, j, k \in E. \tag{6.21}$$

This implies that

$$a_{ki}b_{ji} \geq 0, k \neq i \forall i, j, k \in E. \tag{6.22}$$

(6.21) and (6.22) implies that

$$a_{ik}a_{ki}b_{jk}b_{ji} \geq 0, \forall i, j, k \in E, \tag{6.23}$$

but  $a_{ik}a_{ki} < 0, i \neq k$ , so from (6.23) we get

$$b_{jk}b_{ji} \leq 0, \forall i, j, k \in E. \tag{6.24}$$

It follows from (6.24) that for a fixed  $j$ , there exist  $k_0, i_0$  such that  $b_{jk_0} \geq 0, b_{ji_0} < 0$  and  $b_{jp} = 0$  for  $p \neq i_0, k_0$ . Also if  $a_{ik} < 0 \forall i, i \neq k$  then  $b_{jk} \leq 0 \forall j \neq k$ . Moreover if  $a_{ik} > 0 \forall i, i \neq k$  then  $b_{jk} \geq 0 \forall j \neq k$ . If there exist  $i_0, i'$  such that  $a_{i_0k} > 0$  and  $a_{i'k} < 0$  then  $b_{jk} = 0 \forall j \in E$ .

Let us suppose that

$$a_{i_0k} = \min_{2 \leq i \leq m} \{a_i\}, a_{i'k} = \max_{2 \leq i \leq m} \{a_i\}.$$

Then column  $i_0$  of  $A$  contains all non-positive numbers and column  $i'$  of  $A$  contains all non-negative elements. By above mentioned property  $B$  has the following form:

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 & \beta_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \alpha_2 & 0 & \cdots & 0 & \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \alpha_3 & 0 & \cdots & 0 & \beta_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_m & 0 & \cdots & 0 & \beta_m & 0 & \cdots & 0 \end{pmatrix}$$

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where for column  $(\alpha_1, \alpha_2, \dots, \alpha_m)$  all  $\alpha_i \leq 0$  for any  $i \in E$  and for column  $(\beta_1, \beta_2, \dots, \beta_m)$  all  $\beta_i \geq 0$  for any  $i \in E$ . Thus  $B$  is a matrix with identical rows. From condition  $AB^T = 1$  (where  $A$  is given in (6.20)) we must have the following system:

$$\begin{aligned} a_{i_0}\alpha_i + a_{i'}\beta_i &= 1, \text{ for any } i \in E, \\ (a_{i_0} - a_{i'})\alpha_i + (a_{i'} - a_j)\beta_i &= 1, \text{ for any } i, j \in E. \end{aligned}$$

From above system it is easy to obtained that  $a_j(\alpha_i + \beta_i) = 0$  i.e.  $\alpha_i = -\beta_i$ , from  $a_{i_0}\alpha_i + a_{i'}\beta_i = 1$ , we have  $\alpha_i(a_{i_0} - a_{i'}) = 1$ . Hence  $\alpha_i = \frac{1}{a_{i_0} - a_{i'}}$  and  $\beta_i = \frac{1}{a_{i'} - a_{i_0}}$ .  $\square$

**Lemma 50.** *Any skew symmetric matrix  $A$  with  $a_{ij} \neq 0$  for  $i \neq j$  has at most one positive and at most one negative column. Moreover one of the following holds*

- (i) *The matrix  $A$  has no positive and no negative column.*
- (ii) *The matrix  $A$  has a positive column but has no negative one.*
- (iii) *The matrix  $A$  has a negative column but has no positive column.*
- (iv) *The matrix  $A$  has one negative column and one positive column.*

**Theorem 51.** *If for a skew symmetric matrix  $A$  one of the conditions (i) to (iii) of Lemma 6.18. holds. Then there is no matrix  $B$  satisfying  $Ab^{(j)T} = (1, 1, \dots, 1)$ . In case (iv) of Lemma 6.18. there is a solution  $B$  satisfying  $Ab^{(j)T} = (1, 1, \dots, 1)$  if and only if the positive column  $a^{(k_0)}$  and the negative column  $a^{(k_1)}$  of the matrix  $A$  satisfy the condition*

$$a_{ik_1} - a_{ik_0} = a_{k_0k_1}, \quad \forall i \in E.$$

Moreover,  $B$  has all identical rows.

*Proof.* As we know that the condition  $a_{ik}b_{jk} \geq 0$  implies that if column  $a^{(k)}$  of  $A$  has a positive element as well as a negative element then  $b^{(k)}$  column of  $B$  contains only zeros. Thus condition (i) of Lemma 6.18. gives  $B = 0$ , which does not satisfy the properties of Separable Quadratic Stochastic Operator (SQSO). In condition (ii) of Lemma 6.18.  $B$  has unique non-zero column which is positive too. If  $a^{(k)}$  is a positive column for  $A$  then  $b^{(k)}$  is positive for  $B$  and  $AB^T = 1$  implies that  $\sum_p a_{ip}b_{jp} = 1, \forall i, j \in E$  i.e.  $a_{ik}b_{jk} = 1 \forall i, j \in E$ . For  $i = k$  we have  $0b_{jk} = 1$  which is not possible, thus in this case there exists no  $B$ . Condition (iii) of Lemma 6.18. is also similar to condition (ii) of Lemma 6.18. i.e. also here, there exists no  $B$ . Now let us consider Condition (iv) of Lemma 6.18. Assume that  $a^{(k_0)}$  is positive and  $a^{(k_1)}$  is negative column of  $A$ . Then  $b^{(k_0)}$  and  $b^{(k_1)}$  are positive and negative columns in  $B$  respectively, all other columns of  $B$  are zeros. Also we have

$$a_{ik_0}b_{jk_0} + a_{ik_1}b_{jk_1} = 1, \quad \forall i, j \in E, \tag{6.25}$$

which for  $i = k_0$  gives that  $a_{k_0k_1}b_{jk_1} = 1 \Rightarrow b_{jk_1} = \frac{1}{a_{k_0k_1}} \forall j$  and for  $i = k_1$  gives that  $a_{k_1k_0}b_{jk_0} = 1 \Rightarrow b_{jk_0} = -\frac{1}{a_{k_0k_1}} \forall j$ . So from (6.25) we get

$$a_{k_0i} + a_{ik_1} = a_{k_0k_1}. \tag{6.26}$$

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From (6.26), A has the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1k_0-1} & a_{1k_0} & a_{1k_0+1} & \cdots & (a_{k_0k_1} + a_{1k_0}) & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2k_0-1} & a_{2k_0} & a_{1k_0+1} & \cdots & (a_{k_0k_1} + a_{2k_0}) & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mK_0-1} & a_{mk_0} & a_{1k_0+1} & \cdots & (a_{k_0k_1} + a_{mk_0}) & \cdots & a_{mm} \end{pmatrix}$$

where all the elements of column  $(a_{1k_0}, a_{2k_0}, \dots, a_{mk_0})$  are positive and all the elements of column  $(a_{k_0k_1} + a_{1k_0}, a_{k_0k_1} + a_{2k_0}, \dots, a_{k_0k_1} + a_{mk_0})$  are negative. Consequently, rows of matrix B are identical and are equal to

$$\left( 0, \dots, 0, \frac{-1}{a_{k_0k_1}}, 0, \dots, 0, \frac{1}{a_{k_0k_1}}, 0, \dots, 0 \right).$$

□

### 6.4 Limit Points of SQSO

In this Section we will discuss about the limit points of Separable Quadratic Stochastic Operators (SQSOs).

If the Separable Quadratic Stochastic Operator (SQSO) is constant, then the set of limit points for all  $x^0 \in S^{m-1}$  will be

$$\omega(x^0) = \{(b_{11}a_{11}, \dots, b_{1m}a_{1m})\}.$$

Now if Separable Quadratic Stochastic Operator (SQSO) is a linear operator, then the set of limit points  $\omega(x^0)$  becomes dependent on  $x^0$  and on the properties of matrix A. For ergodic case, the set  $\omega(x^0)$  is a singleton set but for periodic case, the set  $\omega(x^0)$  can be a finite set. Let us denote

$$\Lambda = \{d \in \mathbb{R}^m : 0 \leq d_i, \sum_{i=1}^m d_i > 0, Ac \leq Ic \text{ or } Bc \leq Ic\}.$$

We know by Theorem 6.5 that  $\psi_d$  is a Lyapunov function for any  $d \in \Lambda$  i.e. by definition of Lyapunov functions, for any initial point  $x^0 \in S^{m-1}$ , we get

$$\lim_{n \rightarrow \infty} \psi_d(x^n) = \lambda_d(x^0), \quad d \in \Lambda.$$

Thus we can say that  $\omega(x^0) \subseteq \{x \in S^{m-1} : \psi_d(x^n) = \lambda_d(x^0)\}$  for any  $d \in \Lambda$  and this implies that

$$\omega(x^0) \subseteq \bigcap_{d \in \Lambda} \{x \in S^{m-1} : \psi_d(x) = \lambda_d(x^0)\}. \tag{6.27}$$

Let us suppose that there are m distinct vectors  $d^{(1)}, \dots, d^{(m)}$  in  $\Lambda$  such that the determinant of  $\Lambda$  is non-zero and  $\Lambda$  is a  $m \times m$  matrix with rows  $d^{(j)}$ ,  $j = 1, \dots, m$ .

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Then the system of equations  $\psi_{d^{(j)}}(x) = \lambda_{d^{(j)}}(x^0)$ ,  $j = 1, \dots, m$  has a unique solution  $x = \bar{x}$  and from (6.27), we have  $\omega(x^0) = \{\bar{x}\}$ . Let us suppose that

$$G = \bigcap_{d \in \Lambda} \{x \in S^{m-1} : \psi_d(x) = \lambda_d(x^0)\}.$$

If we have no collection of  $m$  distinct vectors  $d^{(1)}, \dots, d^{(m)}$  in  $\Lambda$  with determinant of  $\Lambda$  non-zero, then  $G$  is an uncountable set. Keep in mind that  $G$  is always a non empty set, since  $\omega(x^0)$  is a non-empty set because  $\{x^{(n)}\}_{n=0}^{\infty} \subseteq S^{(m-1)}$  and as we know that  $S^{(m-1)}$  is a compact set.

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