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APPROXIMATIONS FOR UNIFORMLY CONTINUOUS FUNCTIONS ON GROUPOIDS

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Abstract. The purpose of this paper is to prove an approximation/extension theorem for a family of partial functions on a groupoid satisfying a uniform compatibility condition. In the particular case of a trivial groupoid $G = X \times X$ and a singleton family we recover the well-known result of Katětov: every bounded uniformly continuous real-valued function f defined on a subspace of a uniform space X has a bounded uniformly continuous extension to X.

1 Introduction

The notion of groupoid generalizes the notion of group by replacing the binary operation with a partial function. More precise, a groupoid is a set G endowed with partial product operation $(x, y) \mapsto xy \ \left[: G^{(2)} \to G\right]$ (where $G^{(2)} \subset G \times G$) and an inversion operation $x \mapsto x^{-1}$ $[: G \to G]$ satisfying appropriate versions of the group axioms:

G1 If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy)\, z = x\,(yz)$.

G2 $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.

G3 For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$, then $(zx)\, x^{-1} = z$.

G4 For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

We use the same definition, notation and terminology concerning groupoids as in [2]: $r(x) = xx^{-1}$, $d(x) = x^{-1}x$, $G^{(0)} = r(G) = d(G)$, $G^u = r^{-1}(\{u\})$, $G_u = d^{-1}(\{u\})$, $G^u_v = G^u \cap G_v$.

Definition 1 ([2, Definition 2.1]). *Let G be a groupoid. By a G-uniformity we mean a collection $\{W\}_{W \in \mathcal{W}}$ of subsets of G satisfying the following conditions:*

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1. *$G^{(0)} \subset W \subset G$ for all $W \in \mathcal{W}$.*

2. *If W_1, $W_2 \in \mathcal{W}$, then there is $W_3 \subset W_1 \cap W_2$ such that $W_3 \in \mathcal{W}$.*

3. *For every $W_1 \in \mathcal{W}$ there is $W_2 \in \mathcal{W}$ such that $W_2 W_2 \subset W_1$.*

4. *$W = W^{-1}$for all $W \in \mathcal{W}$.*

Let us remark that for $G = X \times X$ (viewed as a trivial groupoid under the operations: $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$) a G-uniformity is a fundamental system of symmetric entourages of a uniform structure on X.

Definition 2 ([2, Definition 3.2])**.** *Let G be a groupoid endowed with a G-uniformity $\mathcal{W}$, $A \subset G$ and E be a Banach space. A function $h : A \to E$ is said to be* uniformly continuous on fibres *if and only if for each $\varepsilon > 0$ there is $W_\varepsilon \in \mathcal{W}$ such that:*

$$\|h(x) - h(sxt)\| < \varepsilon \text{ for all } s, t \in W_\varepsilon \text{ and } x \in A \cap G_{r(t)}^{d(s)} \text{ such that } sxt \in A.$$

Obviously, if $f, g : G \to \mathbb{R}$ are uniformly continuous on fibres, then $|f|$, $\overline{f}$, $f+g$ are uniformly continuous on fibres. If $f, g : G \to \mathbb{R}$ are bounded uniformly continuous on fibres functions, then fg is a bounded uniformly continuous on fibres function.

The purpose of this paper is to prove an approximation/extension theorem for a family of partial functions $\{f_x\}_{x\in H}$ satisfying a uniform compatibility condition ($f_x : S_x \to \mathbb{R}$, where $S_x \subset G$ for all $x \in H$ and G is a groupoid). As a particular case, we obtain that if S is a subspace of a groupoid G endowed with a G-uniformity, then every bounded uniformly continuous on fibres real-valued function $f : S \to \mathbb{R}$ has a bounded uniformly continuous on fibres extension to G. Furthermore if $G = X \times X$ (viewed as the trivial groupoid on X), we recover the well-known result of Katětov [3, Theorem 3].

2 Approximations for uniformly continuous on fibres functions

We shall use a consequence of the following theorem proved in [2]:

Theorem 3 ([2, Theorem 2.5])**.** *Let G be a groupoid, $\mathcal{W}$ be a G-uniformity (in the sense of Definition 1) and let*

$$I = \left\{ \frac{1}{2^n},\ n \in \mathbb{N} \right\}.$$

Let us consider an I-indexed family $\{W_i\}_{i\in I}$ satisfying the following properties:

1. *$W_i \in \mathcal{W}$ for all $i \in I$.*

2. $W_i W_i \subset W_{2i}$ *for all* $i \in I$, $i \leq \frac{1}{2}$.

Then for every subset A *of* G *there is a function* $f = f_{A,\mathcal{W}_I} : G \to [0,\ 1]$ *satisfying the following conditions:*

1. *If* $n \in \mathbb{N}$, $n \geq 2$, $x \in G$ *and* $y \in W_{1/2^n} x W_{1/2^n}$, *then* $|f(x) - f(y)| < \frac{1}{2^{n-2}}$. *Consequently,* f *is uniformly continuous on fibres (in the sense of Definition 2).*
2. $f(x) = 0$ *for all* $x \in A$.
3. $f(x) = 1$ *for all* $x \notin WAW$.
4. *If* $A = A^{-1}$, *then* $f(x) = f\left(x^{-1}\right)$ *for all* $x \in G$.
5. *If* G *is endowed with a topology such that* $W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$ *is open for all* $i_1,\ i_2,\ ...,\ i_k \in I$, $i_k < i_{k-1} < ... < i_1 < 1$, *then* f *is upper semi-continuous.*
6. *For all* $n \in \mathbb{N}$, , $n \geq 2$, *we have* $W_{1/2^{n+1}} A W_{1/2^{n+1}} \subset \left\{x : f(x) < \frac{1}{2^n}\right\} \subset W_{1/2^{n-1}} A W_{1/2^{n-1}}$.
7. *If* $A = G^{(0)}$, *then* $f(xy) \leq 3f(x) + f(y)$ *for all* $(x, y) \in G^{(2)}$.
8. *If* $A = G^{(0)}$, *then* $f(xy) \leq 2(f(x) + f(y))$ *for all* $(x, y) \in G^{(2)}$.
9. *If* $A = G^{(0)}$, *then* $f(x_1 x_2 ... x_n) \leq 3(f(x_1) + f(x_2) + ... + f(x_n))$ *for all* $n \in \mathbb{N}$ *and* $x_1, x_2, ..., x_n \in G$ *such that* $d(x_i) = r(x_{i+1})$ *for all* $i \in \{1, 2, ..., n-1\}$.
10. *If* $A = G^{(0)}$ *and for every* $x \in G \setminus G^{(0)}$ *there is* $i_x \in I$ *such that* $x \notin W_{i_x}$ *(or equivalently,* $\bigcap\limits_n W_{1/2^n} = G^{(0)}$*), then* $f^{-1}(\{0\}) = G^{(0)}$.

Corollary 4. *Let* G *be a groupoid endowed with a* G*-uniformity* $\mathcal{W}$ *(in the sense of Definition 1). If* A *and* B *are two subsets of* G *with the property that there is* $W \in \mathcal{W}$ *such that* $WAW \subset B$, *then there is a uniformly continuous on fibres function* $f : G \to [0,\ 1]$ *such that* $f(x) = 1$ *for all* $x \in A$ *and* $f(x) = 0$ *for all* $x \notin B$.

Proof. Let $C = G \setminus B$ and notice that $C \subset G \setminus WAW$. By Theorem 3 there is a uniformly continuous on fibres function $f : G \to [0,1]$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \notin WAW$ and thus for all $x \in C$. □

Lemma 5. *Let* G *be a groupoid endowed with a* G*-uniformity* $\mathcal{W}$ *(in the sense of Definition 1). Let* $S \subset G$ *and* $f : S \to \mathbb{R}$ *be a function that is uniformly continuous on fibres. Let* $a < b$ *be two real constants and let*

$$\begin{aligned} A &= \{x \in S : f(x) \leq a\} \\ B &= \{x \in S : \ f(x) \geq b\}. \end{aligned}$$

Then there is $W \in \mathcal{W}$ such that $WAW \cap B = \emptyset$.

Proof. Since f is uniformly continuous on fibres, there is $W \in \mathcal{W}$ such that □

$$|f(x) - f(sxt)| < (b-a) \quad \text{for all } s,t \in W \text{ and } x \in A \cap G_{r(t)}^{d(s)}.$$

Thus if $s,t \in W$ and $x \in A \cap G_{r(t)}^{d(s)}$, then

$$f(sxt) = f(sxt) - f(x) + f(x) < b - a + a = b.$$

Consequently, $sxt \notin B$.

Theorem 6. *Let G be a groupoid endowed with a G-uniformity $\mathcal{W}$ (in the sense of Definition 1). Let $\{S_x\}_{x\in H}$ be a family of subsets of G and $\{f_x\}_{x\in H}$ be a family of functions $f_x : S_x \to \mathbb{R}$ satisfying the following conditions:*

c1. $\sup_{x\in H} \sup_{z\in S_x} |f_x(z)| < \infty$.

c2. *There is a family $\{H_\varepsilon\}_{\varepsilon>0}$ of subsets of H and there is a family $\{W_\varepsilon^H\}_{\varepsilon>0} \subset \mathcal{W}$ such that $\bigcap_{\varepsilon>0} H_\varepsilon \neq \emptyset$ and*

$$|f_y(szt) - f_x(z)| < \varepsilon$$

for all $x,y \in H_\varepsilon$, $s,t \in W_\varepsilon^H$ and $z \in G_{r(t)}^{d(s)} \cap S_x$ with the property that $szt \in S_y$.

If $c > 0$ is such that $c \geq \sup_{x\in H} \sup_{z\in S_c} |f(z)|$, then there is a bounded uniformly continuous on fibres function $h : G \to \mathbb{R}$ such that

1. *$|h| \leq c$ on G.*
2. *For all positive integers n and all $x \in \bigcap_{i=1}^{n+1} H_{2^i c/3^i}$, $|f_x - h| \leq \frac{2^{n+2}c}{3^{n+1}}$ on S_x.*
3. *For all $x_0 \in \bigcap_{\varepsilon>0} H_\varepsilon$, $h = f_{x_0}$ on S_{x_0}.*

Proof. We use a similar reasoning as in the proof of Tietze Extension Theorem (see https://proofwiki.org/wiki/Tietze_Extension_Theorem for instance). Let $c > 0$ be such that $c \geq \sup_{x\in H} \sup_{z\in S_c} |f(z)|$. Let us denote $J_0 = H_{2c/3}$ and let

$$\begin{aligned} A_0 &= \bigcup_{x\in J_0} \left\{ z \in S_x : f_x(z) \leq -\frac{c}{3} \right\} \\ B_0 &= \bigcup_{x\in J_0} \left\{ z \in S_x : \ f_x(z) \geq \frac{c}{3} \right\}. \end{aligned}$$

There is $W_0 = W_{2c/3}^H \in \mathcal{W}$ such that

$$|f_y(szt) - f_x(z)| < \frac{2c}{3}$$

for all $s,t \in W_0$, $x,y \in J_0$ and $z \in S_x \cap G_{r(t)}^{d(s)}$ such that $szt \in S_y$. Thus if $x,y \in J_0$, $s,t \in W_0$ and $z \in A_0 \cap G_{r(t)}^{d(s)} \cap S_x$ is such that $szt \in S_y$, then

$$f_y(szt) = f_y(szt) - f_x(z) + f_x(z) < \frac{2c}{3} - \frac{c}{3} = \frac{c}{3}.$$

Hence $sxt \notin B_0$. Consequently, $W_0 A_0 W_0 \cap B_0 = \emptyset$. By Corollary 4 there is a uniformly continuous on fibres function $f_0 : G \to [0,1]$ such that $f_0(x) = 0$ for all $x \in A_0$ and $f_0(x) = 1$ for all $x \in B_0$. Let $g_0 : G \to \mathbb{R}$ be defined by $g_0(x) = \frac{2c}{3} f_0(x) - \frac{c}{3}$ for all $x \in G$. Then $-\frac{c}{3} \le g_0 \le \frac{c}{3}$, $g(x) = -\frac{c}{3}$ for $x \in A_0$ and $g_0(x) = \frac{c}{3}$ for $x \in B_0$. Hence

$$\begin{aligned} |g_0| &\le \frac{c}{3} \text{ on } G \\ |f_x - g_0| &\le \frac{2c}{3} \text{ on } S_x \text{ for all } x \in J_0. \end{aligned}$$

Since g_0 is uniformly continuous on fibres, there is $W_{g,\varepsilon} \in \mathcal{W}$ such that

$$|g_0(szt) - g_0(z)| < \frac{\varepsilon}{3}$$

for all $s,t \in W_{g,\varepsilon}$ and $z \in G_{r(t)}^{d(s)}$. Thus if $x,y \in H_{2\varepsilon/3} \cap J_0$, $s,t \in W_{2\varepsilon/3}^{H} \cap W_{g,\varepsilon}$ and $z \in A_0 \cap G_{r(t)}^{d(s)} \cap S_x$ is such that $szt \in S_y$, then we have

$$\begin{aligned} |f_y(szt) - g_0(szt) - (f_x(z) - g_0(z))| &\le |f_y(szt) - f_x(z)| + |g_0(szt) - g_0(z)| \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence the family $\{f_x - g_0\}_{x \in J_0}$ satisfies the hypotheses of the theorem. Let us repeat the procedure with the family $\{f_x - g_0\}_{x \in J_0}$ instead of $\{f_x\}_{x \in H}$, $\left\{H_{2\varepsilon/3} \cap J_0\right\}_{\varepsilon}$ instead of $\{H_\varepsilon\}_\varepsilon$ and $\frac{2c}{3}$ instead of c. We obtain a function $g_1 : G \to \mathbb{R}$ such that

$$\begin{aligned} |g_1| &\le \frac{2c}{9} \text{ on } G \\ |f_x - g_0 - g_1| &\le \frac{4c}{9} \text{ on } S_x \text{ for all } x \in J_1 = H_{4c/9} \cap J_0. \end{aligned}$$

Thus we can inductively generate functions g_0, g_1, ..., g_n, ... such that

$$\begin{aligned} |g_n| &\le \frac{2^n c}{3^{n+1}} \text{ on } G \\ |f_x - g_0 - g_1 - \ldots - g_n| &\le \frac{2^{n+1} c}{3^{n+1}} \text{ on } S_x \text{ for all } x \in H_{2^{n+1}c/3^{n+1}} \cap J_n. \end{aligned}$$

Since $|g_n| \leq \frac{2^n}{3^n}\frac{c}{3}$ for all n, it follows that the series $\sum\limits_{n\geq 0} g_n$ converges absolutely and uniformly on G to a real-valued function h satisfying $|h| \leq c$ on G. Since all g_n are uniformly continuous on fibres, h is uniformly continuous on fibres. For all $x \in H_{2^{n+1}c/3^{n+1}} \cap J_n$ we have

$$|f_x - h| \leq |f_x - g_0 - g_1 - ... - g_n| + \sum\nolimits_{k=n+1}^{\infty} |g_k| \leq \frac{2^{n+1}c}{3^{n+1}} + \frac{2^{n+1}c}{3^{n+1}} = \frac{2^{n+2}c}{3^{n+1}}$$

and consequently, $f_{x_0} = h$ on S_{x_0} for all $x_0 \in \bigcap_{\varepsilon>0} H_\varepsilon$. Moreover

$$\sup_{x\in G} |h(x)| \leq c.$$

□

Corollary 7. *Let G be a groupoid endowed with a G-uniformity $\mathcal{W}$ (in the sense of Definition 1). Let $S \subset G$ and $f : S \to \mathbb{R}$ be a bounded function that is uniformly continuous on fibres. Then there is a bounded uniformly continuous on fibres function $h : G \to \mathbb{R}$ such that $h(x) = f(x)$ for all $x \in S$. Moreover h can be chosen such that $\sup_{x\in S} f(x) = \sup_{x\in G} |h(x)|$.*

Proof. The family for which the only one element is $\{f : S \to \mathbb{R}\}$ satisfies the hypotheses of Theorem 6. □

Corollary 8. *Let X be a uniform space and let $\mathcal{U}$ be a fundamental system of symmetric entourages of the uniformity on X. Let $\{g_j\}_{j\in J}$ be a family of functions $g_j : S_j \to \mathbb{R}$, where $S_j \subset X$ for all $j \in J$. Let us assume that the family $\{g_j\}_{j\in J}$ satisfies the following conditions:*

c1. $\sup_{j\in J} \sup_{j\in S_j} |g_j(z)| < \infty$.

c2. *There is a family $\{J_\varepsilon\}_{\varepsilon>0}$ of subsets of J and there is a family $\left\{U_\varepsilon^J\right\}_{\varepsilon>0} \subset \mathcal{U}$ such that $\bigcap_{\varepsilon>0} J_\varepsilon \neq \emptyset$ and*

$$|g_j(x) - g_k(y)| < \varepsilon$$

for all $j, k \in J_\varepsilon$ and $(x, y) \in U_\varepsilon^J$ with the property that $x \in S_j$ and $y \in S_k$.

If $c > 0$ is such that $c \geq \sup_{x\in H} \sup_{z\in S_c} |f(z)|$, then there is a bounded uniformly continuous on fibres function $h : X \to \mathbb{R}$ such that

1. *$|h| \leq c$ on X.*

2. *For all positive integers n and all $j \in \bigcap_{i=1}^{n+1} J_{2^i c/3^i}$, $|g_j - h| \leq \frac{2^{n+2}c}{3^{n+1}}$ on S_j.*

3. For all $j_0 \in \bigcap_{\varepsilon>0} J_\varepsilon$*,* $h = g_{j_0}$ *on* S_{j_0}*.*

Proof. Let us consider the trivial groupoid $G = X \times X$. Then $\mathcal{U}$ satisfies conditions $1-4$ from Definition 1. Thus $\mathcal{U}$ is G-uniformity on the groupoid $G = X \times X$. Let $x \in X$.

For each $j \in J$, let $f_j : \{x\} \times S_j \to \mathbb{R}$ be defined by $f_j(x,y) = g_j(y)$ for all $(x,y) \in \{x\} \times S_j$. Then

$$|f_j((x,x)(x,z)(z,y)) - f_k(x,z)| = |g_j(y) - g_k(z)| < \varepsilon$$

for all $j,k \in J_\varepsilon$ and $(z,y) \in U_\varepsilon^J$ with the property that $y \in S_j$ and $z \in S_k$. Thus the family $\{f_j\}_{j\in J}$ satisfies the hypotheses of Theorem 6. □

Remark 9. *If X is a uniform space and $g : S \to \mathbb{R}$ is a bounded uniformly continuous function, where $S \subset X$ is endowed with the uniform structure coming from X, then, applying the preceding corollary to the singleton family $\{g\}$, there is a bounded uniformly continuous function $h : X \to \mathbb{R}$ such that $h(x) = f(x)$ for all $x \in S$. Moreover h can be chosen such that $\sup_{x\in S}|f(x)| = \sup_{x\in G}|h(x)|$. Thus we obtain [3, Theorem 3].*

A topological groupoid is a groupoid G together with a topology on G such that the product operation $(x,y) \mapsto xy$ $\left[: G^{(2)} \to G\right]$ (where $G^{(2)} \subset G \times G$ is endowed with the topology induced by the product topology on $G \times G$) and the inversion operation $x \mapsto x^{-1}$ $[: G \to G]$ are continuous functions.

Lemma 10. *Let G be a topological groupoid and $\mathcal{W}$ be a family of neighborhoods of $G^{(0)}$. Let us assume that topology on G has the property that for each $x \in G$ and each neighborhood V of x there is $W \in \mathcal{W}$ and there is a neighborhood U of x such that $WUW \subset V$. Then for each $W_1 \in \mathcal{W}$ and each $x \in G$, there is $W_2 \in \mathcal{W}$ and there is a neighborhood V of x such that $V^{-1}W_2V \subset W_1$.*

Proof. For each $W_1 \in \mathcal{W}$ and each $x \in G$, there is a neighborhood V_1 of x such that $V_1^{-1}V_1 \subset W_1$. Furthermore there is $W_2 \in \mathcal{W}$ and there is a neighborhood V of x such that $W_2VW_2 \subset V_1$. Hence $(W_2VW_2)^{-1}W_2VW_2 \subset V_1^{-1}V_1 \subset W_1$. Consequently, $V^{-1}W_2V \subset W_2^{-1}V^{-1}W_2^{-1}W_2VW_2 \subset V_1^{-1}V_1 \subset W_1$. □

Remark 11. *Every locally Hausdorff, locally compact groupoid G (in the sense of [4, p. 6]) satisfies the hypothesis of the preceding lemma ([4, Lemma 2.10],[4, Lemma 2.14]) with $\mathcal{W}$ a fundamental system of diagonally compact ([4, p. 10]) neighborhoods of $G^{(0)}$.*

Proposition 12. *Let G be a topological groupoid and $\mathcal{W}$ be a family of neighborhoods of $G^{(0)}$ satisfying conditions $1-4$ in Definition 1. Let us assume that the topology of G has the property that for each $x \in G$ and each neighborhood V of x there is $W \in \mathcal{W}$ and there is a neighborhood U of x such that $WUW \subset V$. Let $x_0 \in G$,* *let V_0 be a neighborhood of x_0 and let $f : G^{d(V_0)} \to \mathbb{R}$ be a bounded function that is uniformly continuous on fibres with respect to the G-uniformity $\mathcal{W}$. For each $x \in V_0$, let us define $f_x : G^{r(x)} \to \mathbb{R}$, $f_x(y) = f\left(x^{-1}y\right)$ for all $y \in G^{r(x)}$. Then there is a bounded uniformly continuous on fibres function $h : G \to \mathbb{R}$ (with respect to the G-uniformity $\mathcal{W}$) such that*

1. $\sup_{z\in G} |h(z)| \le \sup_{z\in G^{d(V_0)}} |f(z)|$.
2. $h = f_{x_0}$ *on* $G^{r(x_0)}$.
3. *For each* $\varepsilon > 0$ *there is a neighborhood* $U_\varepsilon \subset V_0$ *of* x_0 *such that for all* $x \in U_\varepsilon$, $|f_x - h| < \varepsilon$ *on* $G^{r(x)}$.

Proof. Since f is uniformly continuous on fibres, it follows that for each $\varepsilon > 0$, there is $W_{f,\varepsilon} \in \mathcal{W}$ such that $|f(szt) - f(z)| < \varepsilon$ for all $s, t \in W_{f,\varepsilon}$ and $z \in G_{r(t)}^{d(s)} \cap G^{d(V_0)}$. Furthermore there is $W_{f,\varepsilon,x_0} \in \mathcal{W}$ $(W_{f,\varepsilon,x_0} \subset W_{f,\varepsilon})$ and there is a neighborhood $V_\varepsilon \subset V_0$ of x_0 such that $V_\varepsilon^{-1} W_{f,\varepsilon,x_0} V_\varepsilon \subset W_{f,\varepsilon}$. For all $x, y \in V_\varepsilon$, $s, t \in W_{f,\varepsilon,x_0}$ and $z \in G_{r(t)}^{d(s)} \cap G^{r(x)}$ with the property that $szt \in G^{r(y)}$, we have

$$\begin{aligned} |f_y(szt) - f_x(z)| &= \left|f\left(y^{-1}szt\right) - f\left(x^{-1}z\right)\right| = \left|f\left(y^{-1}sxx^{-1}zt\right) - f\left(x^{-1}z\right)\right| = \\ &= \left|f\left(s'x^{-1}zt\right) - f\left(x^{-1}z\right)\right| < \varepsilon \end{aligned}$$

because $s' = y^{-1}sx \in V_\varepsilon^{-1} W_{f,\varepsilon,x_0} V_\varepsilon \subset W_{f,\varepsilon}$ and $t \in W_{f,\varepsilon,x_0} \subset W_{f,\varepsilon}$. Thus $\{f_x\}_{x\in V_0}$ satisfies the hypotheses of Theorem 6 with $H = V_0$, $H_\varepsilon = V_\varepsilon$, $W_\varepsilon^H = W_{f,\varepsilon,x_0}$ and $c = \sup_{z\in G^{d(V_0)}} |f(z)|$. Consequently, there is a bounded uniformly continuous on fibres function $h : G \to \mathbb{R}$ such that

i) $|h| \le c$ on G.

ii) For all positive integers n and all $x \in \bigcap_{i=1}^{n+1} V_{2^i c/3^i}$, $|f_x - h| \le \frac{2^{n+2}c}{3^{n+1}}$ on $G^{r(x)}$.

iii) For all $x \in \bigcap_{\varepsilon>0} V_\varepsilon$, $h = f_x$ on $G^{r(x)}$.

Since $x_0 \in \bigcap_{\varepsilon>0} V_\varepsilon$, it follows that $h = f_{x_0}$ on $G^{r(x_0)}$. Let $\varepsilon > 0$ and let n_ε be a positive integer such that $\frac{2^{n_\varepsilon+2}c}{3^{n_\varepsilon+1}} < \varepsilon$. If $x \in U_\varepsilon = \bigcap_{i=1}^{n_\varepsilon+1} V_{2^i c/3^i}$ then $|f_x - h| < \varepsilon$ on $G^{r(x)}$. □

Remark 13. *Any topological groupoid that is paracompact admits a fundamental system $\mathcal{W}$ of neighborhoods that is a G-uniformity compatible with the topology of fibres [5]. The same is true for a topological groupoid with paracompact unit space [1].*

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