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SOLUTIONS TO A FIRST ORDER HYPERBOLIC SYSTEM

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Abstract The study of small perturbations in the shock initiation of an inviscid compressible fluid with chemical reaction leads to a first order hyperbolic system of two equations. The order zero approximation of the system involves only constant coefficients. Here, we study a variation of this hyperbolic system and generalize it so that not all coefficients are constants. The boundary conditions in the first quadrant $(t, x > 0)$, where x is the spatial variable and t is time, include data along $x = 0$ and a proportionality relation between the dependent variables along $t = 0$. Using the characteristics of the system, we obtain explicit solutions.

1 Introduction

In the work of David Logan [1], the forced, linear, one dimensional hyperbolic system,

$$\begin{pmatrix} -1 & a^2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \; x > 0, t > 0, \tag{1.1}$$

where $0 < a < 1$ is a constant, is studied under the assumption that the boundary condition,

$$u(0, t) = 0, \quad t \geq 0, \tag{1.2}$$

and the initial condition,

$$u(x, 0) = \alpha v(x, 0), \quad x > 0, \tag{1.3}$$

are satisfied. Here $\alpha > 0$ is also a constant. The problem was originally studied by Fickett [2] in relation to the work on a condensed explosive to determine the rate of chemical energy release from detonating the explosive. The linearization of the underlying mathematical model led to a hyperbolic system which very closely

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resembles the system in (1.1). In our work presented here, we use the method of David Logan [1] to solve a variation of the system in (1.1) namely,

$$\begin{pmatrix} -1 & -a \\ -a & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \ x > 0, t > 0, \tag{1.4}$$

with the assumption that $0 < a < 1$ depends on the variable t. The boundary and initial conditions are kept the same as in (1.2) and (1.3), namely,

$$u(0,t) = 0, \quad t \geq 0, \quad u(x,0) = \gamma v(x,0), \quad x > 0, \tag{1.5}$$

where we have replaced constant α, $0 < \alpha < 1$, in (1.3) with the constant γ, $-1 < \gamma < 0$. Our goal is to first use the method of characteristics to find v along $x = 0$. This is then used to find the solution $u(x,t)$ to (1.4), (1.5) at an arbitrary point (x,t) in the quadrant $x > 0, t > 0$.

2 The Characteristics

The eigenvalues of the coefficient matrix in (1.4) are $a - 1$ and $-(a + 1)$ with the corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. Let $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} a-1 & 0 \\ 0 & -(a+1) \end{pmatrix}$ and note that $PDP^{-1} = \begin{pmatrix} -1 & -a \\ -a & -1 \end{pmatrix}$. Now rewrite the equation (1.4) in the following manner,

$$PDP^{-1} \begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \ x > 0, t > 0, \tag{2.1}$$

Multiply equation (2.1) by P^{-1} to obtain,

$$DP^{-1} \begin{pmatrix} u \\ v \end{pmatrix}_t + P^{-1} \begin{pmatrix} u \\ v \end{pmatrix}_x = P^{-1} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \ x > 0, t > 0, \tag{2.2}$$

The system (2.2) in expanded form will be a pair of equations as follows,

$$\begin{array}{c} (u_t - v_t) + \frac{1}{a-1}(u_x - v_x) = \frac{f-g}{a-1}, \\ (u_t + v_t) - \frac{1}{a+1}(u_x - v_x) = \frac{f+g}{-(a+1)}. \end{array} \tag{2.3}$$

Consider the functions u and v along the curves C^+: $\frac{dx}{dt} = \frac{1}{a-1}$ and C^-: $\frac{dx}{dt} = \frac{-1}{a+1}$ respectively. Then equations in (2.3) along these curves can be written as,

$$\begin{array}{c} \frac{du}{dt} - \frac{dv}{dt} = \frac{f-g}{a-1}, \\ \frac{du}{dt} + \frac{dv}{dt} = \frac{f+g}{-(a+1)}. \end{array} \tag{2.4}$$

We assume that the negative characteristics C^+ and C^- both meet at the same point $(x_0, 0)$ on the x-axis and cross the t-axis at points $(0, \alpha)$ and $(0, \beta)$, $\alpha < \beta$. They therefore must satisfy,

$$\begin{aligned} C^+ : x_1(t) = \textstyle\int_t^\alpha \frac{1}{a-1} d\tau, \\ C^- : x_2(t) = \textstyle\int_t^\beta \frac{1}{-(1+a)} d\tau. \end{aligned} \tag{2.5}$$

To ensure that $x_1(t)$ and $x_2(t)$ meet at $t = 0$ we impose the condition,

$$\int_0^\alpha \frac{1}{a-1} d\tau = \int_0^\beta \frac{1}{-(1+a)} d\tau. \tag{2.6}$$

An example of a function a that allows the condition (2.6) to be satisfied is,

$$a(t) = \frac{1}{\sqrt{t+2}}, \quad t \geq 0. \tag{2.7}$$

A pair of characteristics corresponding to this particular choice of a are,

$$x_1(t) = 2\log\left(\frac{\sqrt{2}-1}{\sqrt{t+2}-1}\right) - 2\sqrt{t+2} - t + 2\sqrt{2} + 1, \quad t \geq 0, \tag{2.8}$$

and

$$x_2(t) = 2\log\left(\frac{\sqrt{2}+1}{\sqrt{t+2}+1}\right) + 2\sqrt{t+2} - t - 2\sqrt{2} + 1, \quad t \geq 0. \tag{2.9}$$

In the quadrant $x \geq 0$ and $t \geq 0$ the curves meet at $x_0 = 1$ on the x-axis, and at $\alpha \approx 0.318$ and $\beta \approx 1.60$ on the t-axis. See figure 1.

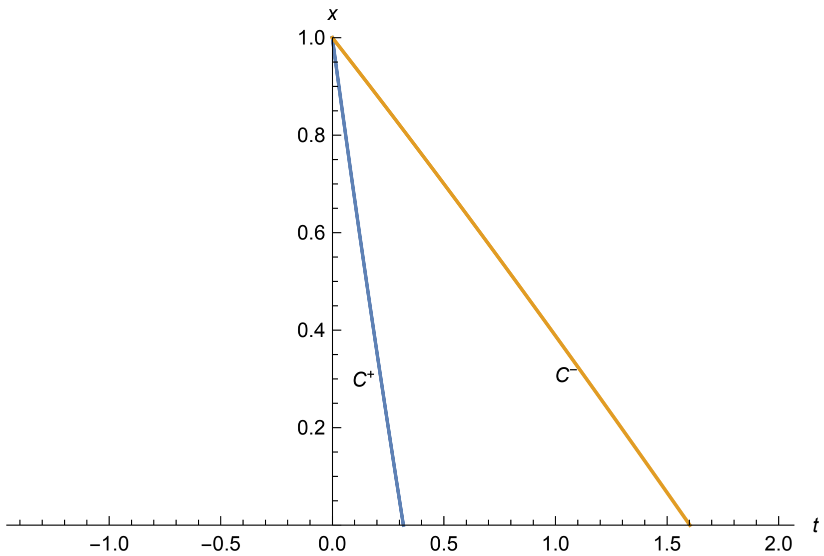

Figure 1: In the quadrant $x \geq 0$ and $t \geq 0$ the characteristic curves C^+ and C^- through (x,t) and $(0,\alpha)$, $(0,\beta)$ on the t-axis.

Now let's integrate equations in (2.3) along the curves C^+ and C^- respectively. We have,

$$u(x,t) - v(x,t) - \int_0^t \frac{f(\tau) - g(\tau)}{a(\tau) - 1} d\tau = k_1, \quad on \quad C^+, \tag{2.10}$$

and

$$u(x,t) + v(x,t) - \int_0^t \frac{f(\tau) + g(\tau)}{-(a(\tau) + 1)} d\tau = k_2, \quad on \quad C^-, \tag{2.11}$$

where k_1 and k_2 are arbitrary constants. From (2.10), (2.11) and the boundary condition in (1.5), at the point $(x_0, 0)$ on the x-axis we must have,

$$\begin{aligned} u(x_0,0) - v(x_0,0) &= k_1, \\ u(x_0,0) + v(x_0,0) &= k_2, \\ u(x_0,0)) &= \gamma v(x_0,0). \end{aligned} \tag{2.12}$$

From the equations in (2.12) we find k_1 and k_2 to be,

$$k_1 = (\gamma - 1)v(x_0, 0), \tag{2.13}$$

$$k_2 = (\gamma + 1)v(x_0, 0). \tag{2.14}$$

Also, the expression on the left-hand-side of (2.10) has the same values at the two points $(x_0, 0)$ and $(0, \alpha)$. The same is true about the expression in (2.11) at the points $(x_0, 0)$ and $(0, \beta)$. These mean,

$$u(x_0,0) - v(x_0,0) = u(0,\alpha) - v(0,\alpha) - \int_0^{\alpha} \frac{f-g}{a-1} d\tau, \tag{2.15}$$

and

$$u(x_0,0) + v(x_0,0) = u(0,\beta) + v(0,\beta) - \int_0^{\beta} \frac{f+g}{-(a+1)} d\tau. \tag{2.16}$$

Apply the condition in (1.5), where we assume that $u = 0$ along the t-axis, to (2.15)-(2.16) and then add the resulting equations to find,

$$2u(x_0,0) = -v(0,\alpha) + v(0,\beta)) - \int_0^{\alpha} \frac{f-g}{a-1} d\tau - \int_0^{\beta} \frac{f+g}{-(a+1)} d\tau. \tag{2.17}$$

Do the same as above but this time subtract the equation (2.15) from (2.16) to find

$$2v(x_0,0) = v(0,\alpha) + v(0,\beta)) + \int_0^{\alpha} \frac{f-g}{a-1} d\tau - \int_0^{\beta} \frac{f+g}{-(a+1)} d\tau. \tag{2.18}$$

Add (2.17) to (2.18) to get,

$$u(x_0,0) + v(x_0,0) = v(0,\beta) - \int_0^{\beta} \frac{f+g}{-(a+1)} d\tau. \tag{2.19}$$

Use the initial condition in (1.5), $u(x_0, 0) = \gamma v(x_0, 0)$, in (2.19) to find,

$$(\gamma + 1)v(x_0,0) = v(0,\beta) - \int_0^{\beta} \frac{f+g}{-(a+1)} d\tau. \tag{2.20}$$

Subtract (2.17) from (2.18) and use the initial condition in (1.5) to get,

$$(1 - \gamma)v(x_0,0) = v(0,\alpha) + \int_0^{\alpha} \frac{f-g}{a-1} d\tau. \tag{2.21}$$

Equating $v(x_0, 0)$ in (2.20) and (2.21) we have,

$$v(0,\alpha) = \frac{1-\gamma}{1+\gamma}\left(v(0,\beta) - \int_0^\beta \frac{f+g}{-(a+1)}d\tau\right) - \int_0^\alpha \frac{f-g}{a-1}d\tau. \tag{2.22}$$

Rewriting (2.22),

$$v(0,\beta) = \frac{1+\gamma}{1-\gamma}v(0,\alpha) + \frac{1+\gamma}{1-\gamma}\int_0^\alpha \frac{f-g}{a-1}d\tau + \int_0^\beta \frac{f+g}{-(a+1)}d\tau. \tag{2.23}$$

Let $\frac{\alpha}{\beta} = r$, then $0 < r < 1$. Also let $\frac{1+\gamma}{1-\gamma} = \delta$, then for $-1 < \gamma < 0$ we will have $0 < \delta < 1$ and (2.23) can be written in the form,

$$v(0,\beta) = \delta v(0, r\beta) + \delta \int_0^{r\beta} \frac{f-g}{a-1}d\tau + \int_0^\beta \frac{f+g}{-(a+1)}d\tau. \tag{2.24}$$

Substitute $r\beta$ for β in (2.24),

$$v(0,r\beta) = \delta v(0, r^2\beta) + \delta \int_0^{r^2\beta} \frac{f-g}{a-1}d\tau + \int_0^{r\beta} \frac{f+g}{-(a+1)}d\tau. \tag{2.25}$$

Replace $v(0, r\beta)$ in (2.24) with the right hand side of (2.25),

$$\begin{aligned} v(0,\beta) = \delta(\delta v(0, r^2\beta) + \delta \textstyle\int_0^{r^2\beta} \frac{f-g}{a-1}d\tau + \int_0^{r\beta} \frac{f+g}{-(a+1)}d\tau) \\ +\delta \textstyle\int_0^{r\beta} \frac{f-g}{a-1}d\tau + \int_0^{\beta} \frac{f+g}{-(a+1)}d\tau. \end{aligned} \tag{2.26}$$

Repeated replacements of β with $r\beta$ as above leads to a general statement for $v(0, \beta)$ as follows,

$$v(0,\beta) = \delta^n v(0, r^n\beta) + \sum_{i=1}^{n} \delta^i \int_0^{r^i\beta} \frac{f-g}{a-1}d\tau + \sum_{i=0}^{n-1} \delta^i \int_0^{r^i\beta} \frac{f+g}{-(a+1)}d\tau \tag{2.27}$$

Letting $n \to \infty$ we obtain,

$$v(0,\beta) = \sum_{i=1}^{\infty} \delta^i \int_0^{r^i\beta} \frac{f-g}{a-1}d\tau + \sum_{i=0}^{\infty} \delta^i \int_0^{r^i\beta} \frac{f+g}{-(a+1)}d\tau. \tag{2.28}$$

If we assume that the functions $\frac{f-g}{a-1}$ and $\frac{f+g}{-(a+1)}$ are bounded then the series in (2.28) converge and the value of v along $x = 0$ is computed. To sum up, we state the following lemma.

Lemma 1. *Let $\beta > \alpha > 0$ be the points of intersections of the characteristics C^- and C^+ respectively, as defined in (2.5), with the t-axis and $\delta = \frac{1+\gamma}{1-\gamma}$. Suppose f and g are piecewise continuous on $0 \leq t < \infty$ and the condition (2.6) is satisfied. Then under the conditions of boundedness of the functions $\frac{f-g}{a-1}$ and $\frac{f+g}{-(a+1)}$ for $t \geq 0$ and $x \geq 0$ the value of v at the point $(0, \beta)$ can be computed and is given by the identity (2.28).*

3 The Solution of the System

Suppose (x, t) is a point in the quadrant $x \geq 0, t \geq 0$ and the characteristics C^+, C^- pass through it and the points $(0, \alpha)$ and $(0, \beta)$ respectively, as shown in figure 2. Integrating the equations in (2.4) from α to t on C^+ and β to t on C^- we have,

$$u(x,t) - v(x,t) - \int_{\alpha}^{t} \frac{f(\tau) - g(\tau)}{a(\tau) - 1} d\tau = c_1, \quad on \quad C^+, \tag{3.1}$$

$$u(x,t) + v(x,t) - \int_{\beta}^{t} \frac{f(\tau) + g(\tau)}{-(a(\tau) + 1)} d\tau = c_2, \quad on \quad C^-. \tag{3.2}$$

Since the expressions on the left of (3.1), (3.2) are constants on the characteristics, their values at the points $(0, \alpha)$ and $(0, \beta)$ are the same. This means,

$$u(x,t) - v(x,t) - \int_{\alpha}^{t} \frac{f(\tau) - g(\tau)}{a(\tau) - 1} d\tau = u(0,\alpha) - v(0,\alpha), \quad on \quad C^+, \tag{3.3}$$

$$u(x,t) + v(x,t) - \int_{\beta}^{t} \frac{f(\tau) + g(\tau)}{-(a(\tau) + 1)} d\tau = u(0,\beta) + v(0,\beta), \quad on \quad C^-. \tag{3.4}$$

Using the boundary condition in (1.5) we have $u(0, \alpha) = 0 = u(0, \beta)$ in the identities (3.3), (3.4). Adding these equations and solving for $u(x, t)$ we have,

$$u(x,t) = \frac{1}{2}[\int_{\alpha}^{t} \frac{f(\tau) - g(\tau)}{a(\tau) - 1} d\tau + \int_{\beta}^{t} \frac{f(\tau) + g(\tau)}{-(a(\tau) + 1)} d\tau + v(0,\beta) - v(0,\alpha)]. \tag{3.5}$$

In order to write $u(x, t)$ in terms of the integrals of the forcing data f, g only we use the identity (2.22), where we let $\frac{1+\gamma}{1-\gamma} = \delta$ in (2.22), to express $v(0, \alpha)$ in the following form,

$$v(0,\alpha) = \frac{1}{\delta}(v(0,\beta) - \int_{0}^{\beta} \frac{f + g}{-(a + 1)} d\tau) - \int_{0}^{\alpha} \frac{f - g}{a - 1} d\tau. \tag{3.6}$$

Also from (2.28) we have the value of $v(0, \beta)$ in terms of the forcing data. If we substitute this in the above equation we have $v(0, \alpha)$ completely determined. After a little simplification we have,

$$v(0,\alpha) = -\int_{0}^{\alpha} \frac{f - g}{a - 1} d\tau + \sum_{i=1}^{\infty} \delta^{i-1} (\int_{0}^{r^i\beta} \frac{f - g}{a - 1} d\tau + \int_{0}^{r^i\beta} \frac{f + g}{-(a + 1)} d\tau). \tag{3.7}$$

Now we can compute $u(x, t)$ by substituting $v(0, \alpha)$ and $v(0, \beta)$ from (3.7) and (2.28), respectively, into (3.5). The formula, after some simplification is,

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}[\textstyle\int_0^t (\frac{f(\tau)-g(\tau)}{a(\tau)-1} + \frac{f(\tau)+g(\tau)}{-(a(\tau)+1)}) d\tau \\ &+ \textstyle\sum_{i=1}^{\infty} ((\delta^i - \delta^{i-1}) \int_0^{r^i\beta} (\frac{f-g}{a-1} + \frac{f+g}{-(a+1)}) d\tau)], \quad x, t > 0. \end{aligned} \tag{3.8}$$

To find $v(x,t)$, we eliminate $u(x,t)$ from the equations (3.3), (3.4) and repeat a process similar to the one above for the computation of $u(x,t)$ to find,

$$\begin{aligned} v(x,t) &= \tfrac{1}{2}[-\int_0^t (\tfrac{f(\tau)-g(\tau)}{a(\tau)-1} + \tfrac{f(\tau)+g(\tau)}{(a(\tau)+1)})d\tau \\ &+ \sum_{i=1}^{\infty}((\delta^i + \delta^{i-1})\int_0^{r^i\beta}(\tfrac{f-g}{a-1} + \tfrac{f+g}{-(a+1)})d\tau)], \quad x,t > 0. \end{aligned} \tag{3.9}$$

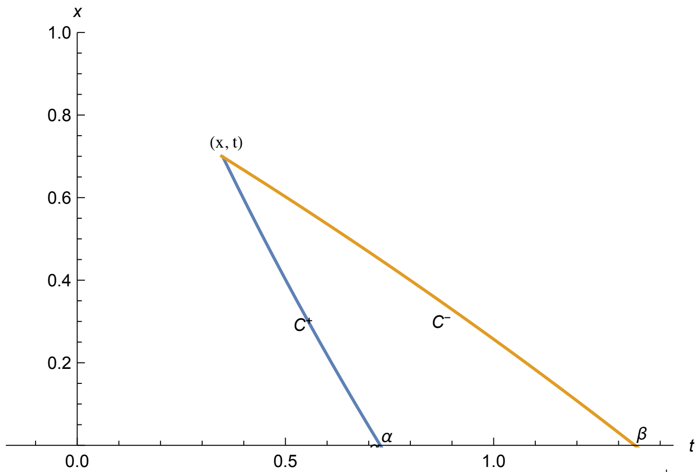

Figure 2: In the quadrant $x \geq 0$ and $t \geq 0$ the characteristic curves C^+ and C^- through (x,t) and $(0,\alpha)$, $(0,\beta)$ on the t-axis.

We express these results in the following theorem,

Theorem 2. *Under the assumptions of Lemma 1 the system (1.4) along with the initial and boundary conditions in (1.5) has the solution given by (3.8), (3.9). The parameter* $0 < r < 1$, *is* $r = \frac{\alpha}{\beta}$.

4 Asymptotic Response

As in [1], we consider the physical response of the system to the case where the forcing functions f, g become zero after certain times. We should expect that under appropriate conditions the response of the system will be finite as well. For this purpose we make the following assumptions, $f(t) = 0$ for $t > T_1$ and $g(t) = 0$ for $t > T_2$. Let,

$$M_1 = \sup_{[0,T_1]} |f(t)|, \quad M_2 = \sup_{[0,T_2]} |g(t)|. \tag{4.1}$$

Also for $0 < a(t) < 1$ we assume,

$$M = \sup_{[0,\infty)} a(t), \quad m = \inf_{[0,\infty)} a(t). \tag{4.2}$$

Then $0 < M, m < 1$. Let,

$$T = \max\{T_1, T_2\}. \tag{4.3}$$

Then,

$$|\int_0^T \frac{f-g}{a-1}| \leq \frac{T}{1-M}(M_1 + M_2), \tag{4.4}$$

and

$$|\int_0^T \frac{f+g}{-(a-1)}| \leq \frac{T}{1+m}(M_1 + M_2). \tag{4.5}$$

Upon using the inequalities (4.4), (4.5) in the solutions for u and v in (3.8), (3.9) we will have,

$$|u(x,t)| \leq T(M_1 + M_2)(\frac{1}{1-\delta})(\frac{1}{1-M} + \frac{1}{1+m}), \tag{4.6}$$

$$|v(x,t)| \leq T(M_1 + M_2)(\frac{1}{1-\delta})(\frac{1}{1-M} + \frac{1}{1+m}). \tag{4.7}$$

We state the above result as a theorem.

Theorem 3. *Assume that f, g and a in equation (1.4) satisfy the conditions (4.1), (4.2). Then the solutions of the system (1.4) under the conditions in (1.5) are bounded with bounds give by (4.6), (4.7).*

As can be seen from the inequalities (4.6) and (4.7), when M_1 and M_2 are small, i.e., the forcing functions are small short lived pulses, the bounds on the solutions u, v are small. To see the asymptotic behavior of the solutions for large t, we consider a square impulse wave for f, and allow g to be zero for all time,

$$f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases} \tag{4.8}$$

$$g(t) = 0, \quad t \geq 0. \tag{4.9}$$

Also define,

$$n_1 = \min\{n \in Z^+ | r^{n+1}\beta < 1 < r^n\beta\} \tag{4.10}$$

Writing the solutions (3.8), (3.9), for u and v with data given by, (4.8), (4.9), and considering (4.10) we have,

$$u(x,t) = \begin{cases} \frac{1}{2}[\int_0^t(\frac{1}{a-1} + \frac{1}{-(a+1)})d\tau + \sum_{i=1}^{n_1}[(\delta^i - \delta^{i-1}) \\ \int_0^1(\frac{1}{a-1} + \frac{1}{-(a+1)})dt] \\ + \sum_{i=n_1+1}^{\infty}[(\delta^i - \delta^{i-1}) \\ \int_0^{r^i\beta}(\frac{1}{a-1)} + \frac{1}{-(a+1)})dt]], & x \geq 0, \quad 0 \leq t \leq 1, \\ \frac{1}{2}[\int_0^1(\frac{1}{a-1} + \frac{1}{-(a+1)})d\tau + \sum_{i=1}^{n_1}[(\delta^i - \delta^{i-1}) \\ \int_0^1(\frac{1}{a-1} + \frac{1}{-(a+1)})dt] \\ + \sum_{i=n_1+1}^{\infty}[(\delta^i - \delta^{i-1}) \\ \int_0^{r^i\beta}(\frac{1}{a-1)} + \frac{1}{-(a+1)})dt]], & x \geq 0, \quad t > 1. \end{cases} \tag{4.11}$$

Similarly,

$$v(x,t)=\begin{cases}\frac{1}{2}[\int_0^t(-\frac{1}{a-1}+\frac{1}{-(a+1)})d\tau+\sum_{i=1}^{n_1}[(\delta^i+\delta^{i-1})\\ \int_0^1(\frac{1}{a-1}+\frac{1}{-(a+1)})dt]\\ +\sum_{i=n_1+1}^{\infty}[(\delta^i+\delta^{i-1})\\ \int_0^{r^i\beta}(\frac{1}{a-1)}+\frac{1}{-(a+1)})dt]], \quad x\geq 0, \quad 0\leq t\leq 1,\\ \frac{1}{2}[\int_0^1(-\frac{1}{a-1}+\frac{1}{-(a+1)})d\tau+\sum_{i=1}^{n_1}[(\delta^i+\delta^{i-1})\\ \int_0^1(\frac{1}{a-1}+\frac{1}{-(a+1)})dt]\\ +\sum_{i=n_1+1}^{\infty}[(\delta^i+\delta^{i-1})\\ \int_0^{r^i\beta}(\frac{1}{a-1)}+\frac{1}{-(a+1)})dt]], \quad x\geq 0, \quad t>1.\end{cases} \tag{4.12}$$

The components of the solutions (4.11), (4.12) above, show that for large $t>1$ both u and v are of order $O(\delta^{n_1})$. To illustrate this further we consider the specific function $a(t)=\frac{1}{\sqrt{t+2}}$. Then the solutions u,v are computed to be,

$$u(x,t)=\begin{cases}-\ln(t+1)-t-(\ln 2+1)(\delta^{n_1}-1)\\ -\sum_{i=n_1+1}^{\infty}[(\delta^i-\delta^{i-1})\\ (\ln(r^i\beta+1)+r^i\beta)], \quad x\geq 0, \quad 0\leq t\leq 1,\\ -\delta^{n_1}(\ln 2+1)-\sum_{i=n_1+1}^{\infty}[(\delta^i-\delta^{i-1})\\ (\ln(r^i\beta+1)+r^i\beta)], \quad x\geq 0, \quad t>1.\end{cases} \tag{4.13}$$

and,

$$v(x,t)=\begin{cases}\ln(\frac{\sqrt{t+2}-1}{\sqrt{t+2}+1})+2\sqrt{t+2}+\ln(\frac{\sqrt{2}+1}{\sqrt{2}-1})-\sqrt{2}\\ -\frac{(1+\delta)(1-\delta^{n_1})}{1-\delta}(\ln 2+1)\\ -\sum_{i=n_1+1}^{\infty}[(\delta^i+\delta^{i-1})\\ (\ln(r^i\beta+1)+r^i\beta)], \quad x\geq 0, \quad 0\leq t\leq 1,\\ -\ln((\sqrt{3}+2)(3-2\sqrt{2}))+2\sqrt{3}-2\sqrt{2}\\ -(\frac{(1+\delta)(1-\delta^{n_1})}{1-\delta})(\ln 2+1)-\sum_{i=n_1+1}^{\infty}[(\delta^i+\delta^{i-1})\\ (\ln(r^i\beta+1)+r^i\beta)], \quad x\geq 0, \quad t>1.\end{cases} \tag{4.14}$$

From the formulas in (4.13), (4.14) we have the following estimates on u,v for large $t>1$,

$$-(\ln(2)+1)\delta^{n_1}<u(x,t)<(\ln(\frac{r^{n_1+1}}{2})+r^{n_1+1}\beta-1)\delta^{n_1} \tag{4.15}$$

and,

$$-(\ln(r^{n_1+1}\beta+1)+1)(\frac{1+\delta}{1-\delta})\delta^{n_1}<v(x,t)-v_{\infty}<0, \tag{4.16}$$

where v_{∞} is

$$v_{\infty}=-\ln((\sqrt{3}+2)(3-2\sqrt{2}))+2\sqrt{3}-2\sqrt{2}-(\frac{(1+\delta)(1-\delta^{n_1})}{1-\delta})(\ln 2+1). \tag{4.17}$$

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