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ON FINSLER SPACES WITH UNIFIED MAIN SCALAR $L^2C = \beta$

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Abstract. The purpose of present paper is to study the T-tensor of such a Finsler space with the condition $L^2(\alpha,\beta)C = \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_iy^i$ and get some important theorems. We shall also obtain the condition for such a Finsler space to be a Landsberg space or Berwald space. The notations and terminologies are referred to the monograph [7].

1 Introduction

M. Matsumoto, Shibata, Asanov and Kiransov [1, 9] have treated non-Riemannian Finsler spces with vanishing T-tensor are said to satisfy T-condition. If a Finsler space M^n satisfies the T-condition, then the function L^2C^2 of M^n is reduced to a function of the position only (i.e. $L^2C^2 = f(x)$) where L is the metric function and C^2 is the square of the length of the torsion vector C_i. For example, if the metric tensor g_{ij} has such a special form as $g_{ij} = Q^{ls}_{ij}l_ls_s$ as in [1] then the function L^2C^2 becomes zero i.e. $L^2C^2 = f(x) = 0$, because in this case the T-condition is satisfied automatically and $C_i = 0$. Ikeda [4] investigated the interplay between the condition $L^2C^2 = f(x)$ and the vanishing of the Tensor T and has been considering on the properties of those Finsler space. Pandey, Chaubey and Mishra [10] studied Finsler spaces with unified main scalar LC of the form $L^2C^2 = f(y) + g(x)$ i.e. some known function of x and y .

In the present paper we shall study the T-tensor of such a Finsler space with the condition $L^2C = \beta$, where $\alpha^2 = a_{ij}y^iy^j$ and $\beta = b_i(x)y^i$ and get some important theorems.We shall also obtain the condition for such a Finsler space to be a Landsberg space or Berwald space and then show that a Landsberg space (respectively, Berwald space) satisfying the condition $L^2C = \beta$ reduces to a Berwald space.

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2 The condition $L^2C = \beta$

Let $F^n = (M^n; L)$ be an n-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L^2(x,y) = g_{ij}(x,y)y^iy^j$ is the fundamental function defined on the manifold $T(M)\backslash 0$ of nonzero tangent vectors. The notations l_i, h_{ij} and C_{ijk} denote the unit vector (i.e.$l^i = \frac{y^i}{L}$), the angular metric tensor and the (h) hv - torsion tensor (the Cartan torsion tensor), respectively. The T-tensor T_{ijkl} is defined by $T_{ijkl} = LC_{ijk}|_l + C_{ijk}l_l + C_{ijl}l_k + C_{ilk}l_j + C_{ljk}l_i$ and the torsion vector C_i is given by $C_i = g^{jk}C_{ijk}$, where one symbol $|_l$ denote the v-covariant differentiation and g^{jk} is one reciprocal tensor of g_{jk}.
Assume that the function L^2C is a non-zero function of position and direction s.t. $L^2C = \beta$. The differentiation of this equation by y^i yields

$$L^2C|_i + 2Cy_i = \phi_i \tag{2.1}$$

where the symbol $|_i$ denote the differentiation by y^i and $\phi_i = \frac{\partial\beta}{\partial y^i} = b_i$

Since $C^2 = g^{ij}C_iC_j$ then $T_{ij}(= g^{kl}T_{ijkl}) = LC_i|_j + C_il_j + C_jl_i$

Since $C^2 = g^{ij}C^iC^j$, then $C^2|_h = 2g^{ij}C^iC_iC_j = 2C^iC_{i|h}$

$\Rightarrow 2CC|_h = 2C^iC_i|_h$

$$C|_h = \frac{C^i}{C}C_{i|h} \tag{2.2}$$

from (2.1) and (2.2) we get

$$2CLl_i + L^2\tfrac{C^h}{C}C_h|_i = \phi_i$$

$$2C^2Ll_i + L^2C^hC_h|_i = C\phi_i \tag{2.3}$$

$$\because T_{ij} = LC_i|_j + l_iC_j + l_jC_i$$
$$\therefore C^iT_{ih} = LC^iC_i|_h + C^il_iC_h + C^il_hC_i$$

$$C^iT_{ih} = LC^iC_i|_h + C^2l_h(\because C^iC_i = C^2) \tag{2.4}$$

from (2.3) and (2.4), we get

$$2LC^iT_{ih} - L^2C^iC_{i|_h} = C\phi_h \tag{2.5}$$

Conversely, let $\quad 2LC^iT_{ih} - L^2C^iC_{i|_h} = C\phi_h$

$$\Rightarrow \qquad 2LC^i[LC_i|_h + l_iC_h + l_hC_i] - L^2C^iC_{i|_h} = C\phi_h$$

$$\Rightarrow \qquad 2L^2C^iC_i|_h + 2LC^2l_h - L^2C^iC_{i|_h} = C\phi_h$$

$$\Rightarrow \qquad L^2C^iC_i|_h + 2LC^2l_h = C\phi_h$$

$$\Rightarrow \qquad \tfrac{L^2}{C}C^iC_i|_h + 2LCl_h = \phi_h$$

$$\Rightarrow \qquad (L^2C)|_h = \phi_h$$

Integrating, we get

$$L^2C = \beta \tag{2.6}$$

Thus, we have

Theorem 1. *For a n-dimensional Finsler space the unified scalar $L^2C = \beta$, if and only if the T - tensor satisfies the condition*

$$T_{ij}C^j = \tfrac{\phi_i}{2L}$$

Again for a two dimensional Finsler space the T - tensor [7] can be written as,

$$T_{hijk} = I_2 m_h m_i m_j m_k \text{ and } LC_{ijk} = I m_i m_j m_k$$

this implies that $\qquad LC = I$

Since $\qquad (L^2C)|_i = \phi_i$ this implies that

$$2Ll_iC + L^2C\ |_i = \phi_i$$

Thus $\qquad T_{hijk} = \frac{\phi^r m_r}{2L} m_h m_i m_j m_k$

Corollary 2. *In two dimensional Finsler space with the unified scalar $L^2C(L^2C = \beta)$ satisfies T-condition iff ϕ_i is parallel to l_i i.e.*

$$\phi_i = \lambda l_i, \textit{ for some scalar function } \lambda.$$

Differentiating this equation w.r.t.y^j, we get

$$\frac{\partial \phi_i}{\partial y^j} = \frac{\partial \lambda}{\partial y^j} l_i + \lambda L^{-1} h_{ij}$$

Now contracting this equation w.r.t. y^i, we get

$$\frac{\partial \phi_i}{\partial y^j} y^i = L\frac{\partial \lambda}{\partial y^j}$$

this implies that $\qquad L\frac{\partial \lambda}{\partial y^j} = -\phi_j$

$$\because \qquad \phi_i y^i = \frac{\partial \phi_i}{\partial y^i} y^i = 0 \qquad \Rightarrow \quad \frac{\partial}{\partial y^j}(\phi_i y^i) = \frac{\partial \phi_i}{\partial y^j} y^i + \phi_i \delta^i_j$$

this implies that $L\frac{\partial\lambda}{\partial y^j} = -\phi_j = -\lambda l_j$
Integrating, we get

$$\lambda = \frac{\psi(\beta)}{L}$$

where $\psi(\beta)$ is any arbitrary function of β
Again,

$$\phi_i = \frac{\psi(\beta)}{L} l_i = \frac{\psi(\beta)}{L} \frac{\partial L}{\partial y^i}$$

$$\frac{\partial\beta}{\partial y^i} = \frac{\psi(\beta)}{\psi(\beta)L} \frac{\partial L(\psi(\beta))}{\partial y^i}$$

On integration above equation, we get

$$\phi(\beta) = L^2C = \beta = \psi(\beta)log(L\psi(\beta)) + p(\beta) \tag{2.7}$$

where $p(\beta)$ is also any arbitrary function of β
therefore from (2.7), we get

$$L = \frac{1}{\psi(\beta)} e^{\frac{\phi(\beta)-p(\beta)}{\psi(\beta)}}$$

thus, we have

Theorem 3. *If a two dimensional Finsler space with $L^2C = \beta$ satisfies T-condition then the metric function L is given by*

$$L = \frac{1}{\psi(\beta)} e^{\frac{\phi(\beta)-p(\beta)}{\psi(\beta)}}$$

where ψ and p both are arbitrary function of β.

In C-reducible Finsler space the T-tensor [7] can be written as

$$T_{hijk} = \frac{LC^\star}{n^2-1}\pi_{hijk}(h_{hi}h_{jk}), \tag{2.8}$$

where $C^\star = g^{ij}C_i|_j$ and π_{hijk} represents cyclic permutation of the indices h,i,j,k. Contracting (2.8) by g^{jk}, we get

$$T_{hi} = \frac{LC^\star}{n-1} h_{hi}$$

$$\Rightarrow \quad \phi_h = \frac{2L^2C^\star C_h}{n-1}$$

thus

$$\phi_h C^h = \frac{2L^2C^\star C^2}{n-1} \tag{2.9}$$

Corollary 4. *For an n-dimensional C-reducible Finsler space with unified scalar $L^2C = \beta$ satisfies T-condition if ϕ_i is perpendicular to C^i.*

3 Landsberg and Berwald spaces satisfying the condition $L^2C = \beta$

We assume that a Finsler space M^n satisfies the condition $L^2C = \beta$. Now from equation (2.1) the important tensor which will be used later are given by

$$g_{ij} = -\frac{L^2C|_i|_j}{2C} - \frac{L}{C}(l_jC|_i + l_iC|_j) \tag{3.1}$$

$$\begin{aligned} C_{ijk} = -\frac{L^2C|_i|_j|_k}{4C} - \frac{1}{C}[h_{jk}C|_i + h_{ik}C|_j \\ +l_jl_kC|_i + l_il_kC|_j] - \frac{L}{C}[l_iC|_j|_k \\ +l_jC|_i|_k + \frac{l_kC|_i|_j}{2}] + \frac{L}{C^2}[l_iC|_jC|_k \\ +l_jC|_iC|_k - LC|_kC|_i|_j]. \end{aligned} \tag{3.2}$$

Now taking h-covariant derivative w.r.t h both side, we get

$$\begin{aligned} C_{ijk|h} = -\frac{L^2}{4C}C|_i|_j|_{k||h} - \frac{C_{|h}}{C}[C_{ijk} + \frac{1}{C}[y^iC|_jC|_k + y^jC|_iC|_k \\ -L^2C|_kC|_i|_j] - \frac{1}{C}[y^iC|_j|_{k|h} + y^jC|_i|_{k|h} + \frac{y^kC|_i|_{j|h}}{2} \\ +h_{jk}C|_{i|h} + h_{ik}C|_{j|h} + l_jl_kC|_{i|h} + l_il_kC|_{j|h}] \\ +\frac{1}{C^2}[y^iC|_{j|h}C|_k + y^iC|_jC|_{k|h} + y^jC|_{i|h}C|_k \\ +y^jC|_iC|_{k|h} - L^2C|_{k|h}C|_i|_j - L^2C|_kC|_iC|_{j|h}][\because l_iL = y^i] \end{aligned} \tag{3.3}$$

contracting above equation by y^h, we get

$$\begin{aligned} P_{ijk} = -\frac{L^2}{4C}C|_i|_j|_{k|o} - \frac{C_{|0}}{C}[C_{ijk} + \frac{1}{C}(y^iC|_jC|_k + y^jC|_iC|_k \\ -L^2C|_kC|_i|_j)] - \frac{1}{C}[y^iC|_j|_{k|0} + y^jC|_i|_{k|0} + \frac{y^kC|_i|_{j|0}}{2} \\ +h_{jk}C|_{i|0} + h_{ik}C|_{j|0} + l_jl_kC|_{i|0} + l_il_kC|_{j|0}] \\ +\frac{1}{C^2}[y^iC|_{j|0}C|_k + y^iC|_jC|_{k|0} + y^jC|_{i|0}C|_k + y^jC|_iC|_{k|0} \\ -L^2C|_{k|0}C|_i|_j - L^2C|_kC|_iC|_{j|0}] \end{aligned} \tag{3.4}$$

where P_{ijk} is the $(v)hv-$ torsion tensor, the symbol $|_i$ denotes h-covariant differentiation and the index $'0'$ means the contraction by y^i. The above equation (3.3)(respectively, (3.4)) gives the result that the condition
$C_{ijk|l} = 0$ (res. $P_{ijk} = 0$) is equivalent to $C|_i|_j|_{k|h} = 0$ (res. $C|_i|_j|_{k|0} = 0$). Then, we have

Theorem 5. *If an n-dimensional Finsler space M^n satisfies the condition $L^2C = \beta$, then the necessary and sufficient condition for M^n to be a Berwald space is that $C|_i|_j|_{k|l} = 0$ holds good. In this case the function L^2C is constant.*

Theorem 6. *If an n-dimensional Finsler space M^n satisfies the condition $L^2C = \beta$, then the necessary and sufficient condition for M^n to be a Landsberg space is that $C|_i|_j|_{k|0} = 0$ holds good.*

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