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q-DERIVATIVE ON p-VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH CONNECTED SETS

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Abstract. In this article, two subfamilies of p–valent meromorphic functions by means of q–derivative are defined. With that, we study coefficient inequality, distortion bounds and convex family of these subclasses. Also connected sets structure is investigated.

1 Introduction

Let $\sum_p$ denotes the class of p–valently meromorphic functions of the type:

$$f(z) = \frac{A}{z^p} + \sum_{k=p}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Gasper and Rahman [1] defined the q–derivative $(0 < q < 1)$ of a meromorphic function f of the form (1.1), by:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \qquad (z \in \mathbb{U}^*), \tag{1.2}$$

see also [2].

From (1.1) and (1.2), we get:

$$D_q f(z) = \frac{-A[p]_q}{q^p z^{p+1}} + \sum_{k=p}^{\infty} a_k z^{k-1} [k]_q, \tag{1.3}$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \ldots + q^{k-1}. \tag{1.4}$$

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As $q \to 1^-$, then $[k]_q \to k$ and $[p]_q \to p$, so we conclude:

$$\lim_{q\to 1^-} D_q f(z) = f'(z), \qquad (z \in \mathbb{U}^*),$$

see [2].

For $0 < q < 1$, $0 < \alpha < 1$, $0 \leqslant \lambda \leqslant 1$ and $0 < \beta < 1$, let $\sum_p^q(\alpha,\beta,\lambda)$ be the subclass of $\sum_p$ consisting of functions f of the form (1.1) and satisfying the condition:

$$\left| \frac{z^{p+2}\big(D_q f(z)\big)' - A(p+1)q^{-p}[p]_q}{\lambda z^{p+1}\big(D_q f(z)\big) - A[p]_q q^{-p} + (1+\lambda)\alpha A q^{-p}[p]_q} \right| < \beta, \tag{1.5}$$

where $D_q f(z)$ is given in (1.3).

2 Main result

First, we state coefficient estimates for functions belongs to $\sum_p^q(\alpha,\beta,\lambda)$.

Theorem 1. *Let $f(z) \in \sum_p$, then $f(z) \in \sum_p^q(\alpha,\beta,\lambda)$ if and only if:*

$$\sum_{k=p}^{\infty} \frac{[k]_q(k-1+\lambda\beta)q^p}{\beta(\lambda+1)A[p]_q(1-\alpha)} a_k \leqslant 1. \tag{2.1}$$

The result is sharp for $F(z)$ given by:

$$F(z) = \frac{A}{z^p} + \frac{\beta(\lambda+1)Aq^{-p}[p]_q(1-\alpha)}{[k]_q(k-1+\lambda\beta)} z^{k+p}. \tag{2.2}$$

Proof. Let $f(z) \in \sum_p^q(\alpha,\beta,\lambda)$, then (1.5) holds true. So by replacing (1.3) in (1.5), we have:

$$\left| \frac{\sum\limits_{k=p}^{\infty}(k-1)[k]_q a_k z^{k+p}}{A(\lambda+1)q^{-p}[p]_q(\alpha-1) + \sum\limits_{k=p}^{\infty}\lambda[k]_q a_k z^{k+p}} \right| < \beta.$$

Since $\mathrm{Re}\{z\} \leqslant |z|$, for all z, therefore:

$$\mathrm{Re}\left\{ \frac{\sum\limits_{k=p}^{\infty}(k-1)[k]_q a_k z^{k+p}}{A(\lambda+1)q^{-p}[p]_q(1-\alpha) - \sum\limits_{k=p}^{\infty}\lambda[k]_q a_k z^{k+p}} \right\} < \beta.$$

By letting $z \to 1^-$ through real values, we get:

$$\sum_{k=p}^{\infty}[k]_q(k-1+\lambda\beta)a_k \leqslant A\beta(\lambda+1)q^{-p}[p]_q(1-\alpha).$$

Conversely, let (2.1) holds true. It is enough to show that:

$$X = \left|z^{p+2}\big(D_q f(z)\big)' - A(p+1)q^{-p}[p]_q\right| \\ - \beta\left|\lambda z^{p+1}\big(D_q f(z)\big) - A[p]_q q^{-p} + (1+\lambda)\alpha A q^{-p}[p]_q\right| < 0.$$

But for $0 < |z| = r < 1$, we have:

$$\begin{aligned} X &= \left|\sum_{k=p}^{\infty}(k-1)[k]_q a_k z^{k+p}\right| - \beta\left|A(\lambda+1)q^{-p}[p]_q(1-\alpha) - \sum_{k=p}^{\infty}\lambda[k]_q a_k z^{k+p}\right| \\ &\leqslant \sum_{k=p}^{\infty}(k-1)[k]_q|a_k|r^{k+p} - \beta A(\lambda+1)[p]_q(1-\alpha) + \sum_{k=p}^{\infty}\lambda\beta[k]_q|a_k|r^{k+p} \\ &\leqslant \sum_{k=p}^{\infty}[k]_q(k-1+\lambda\beta)|a_k|r^{k+p} - \beta A(\lambda+1)q^{-p}[p]_q(1-\alpha). \end{aligned}$$

Since, the above inequality holds for all r $(0 < r < 1)$, by letting $r \to 1^-$ and applying (2.1), we obtain $X \leqslant 0$, and this gives the required result. Hence the proof is complete. □

3 Properties of subfamilies of $\sum_p$

In this section, we define two subclasses of $\sum_p$ and obtain some important properties of these subclasses.

For a given $x \in \mathbb{R}$ such that $0 < x < 1$, let $\sum_p(1)$ be a subclass of $\sum_p$ satisfying the condition:

$$x^p f(x) = 1, \tag{3.1}$$

and $\sum_p 2$ be a subclass of $\sum_p$ satisfying the condition:

$$-x^{p+1}f'(x) = p, \tag{3.2}$$

and

$$\textstyle\sum_{p,j}^{q}(\alpha,\beta,\lambda,x) = \sum_p^q(\alpha,\beta,\lambda) \cap \sum_p(j), \qquad (j = 1, 2). \tag{3.3}$$

Such as these subclasses are studied in [3].

Theorem 2. *Let $f(z)$ be introduced by (1.1). Then $f(z) \in \sum_{p,1}^{q}(\alpha, \beta, \lambda, x)$ if and only if:*

$$\sum_{k=p}^{\infty}\Big(\frac{[k]_q(k-1+\lambda\beta)q^p}{\beta(\lambda+1)[p]_q(1-\alpha)} + x^{k+p}\Big)a_k \leqslant 1. \tag{3.4}$$

Proof. Since $f(z) \in \sum_{p,1}^{q}(\alpha, \beta, \lambda, x)$, so we have:

$$x^p f(x) = x^p\Big(\frac{A}{x} + \sum_{k=p}^{\infty} a_k x^k\Big) = 1,$$

thus

$$A = 1 - \sum_{k=p}^{\infty} a_k x^{k+p}. \tag{3.5}$$

By putting this value of A in Theorem 1, we get:

$$\sum_{k=p}^{\infty}[k]_q(k-1+\lambda\beta)q^p a_k \leqslant \beta(\lambda+1)[p]_q(1-\alpha)\Big(1 - \sum_{k=p}^{\infty} a_k x^{k+p}\Big),$$

or

$$\sum_{k=p}^{\infty}[k]_q(k-1+\lambda\beta)q^p a_k + \sum_{k=p}^{\infty}\beta(\lambda+1)[p]_q(1-\alpha)a_k x^{k+p}$$
$$\leqslant \beta(\lambda+1)[p]_q(1-\alpha),$$

or

$$\sum_{k=p}^{\infty}\Big(\frac{[k]_q(k-1+\lambda\beta)q^p}{\beta(\lambda+1)[p]_q(1-\alpha)} + x^{k+p}\Big)a_k \leqslant 1.$$

Hence, we get the desired assertion. □

Theorem 3. *Let $f(z)$ be defined by (1.1). Then $f(z) \in \sum_{p,2}^{q}(\alpha, \beta, \lambda, x)$ if and only if:*

$$\sum_{k=p}^{\infty}\Big(\frac{[k]_q(k-1+\lambda\beta)q^p}{\beta(\lambda+1)[p]_q(1-\alpha)} - \frac{k}{p}x^{k+p}\Big)a_k \leqslant 1. \tag{3.6}$$

Proof. Since $-x^{p+1}f'(x) = p$, we have:

$$A = 1 + \sum_{k=p}^{\infty}\frac{k}{p}a_k x^{k+p}. \tag{3.7}$$

Replacing A in (2.1), gives the required result. □

Corollary 4. *Let* $f(z)$ *of the type* (1.1) *be in the class* $\sum_{p,1}^{q}(\alpha,\beta,\lambda,x)$*, then:*

$$a_k \leqslant \frac{\beta(\lambda+1)[p]_q(1-\alpha)}{[k]_q(k-1+\lambda\beta)q^p+\beta(\lambda+1)[p]_q(1-\alpha)x^{k+p}}. \tag{3.8}$$

Corollary 5. *Let* $f(z)$ *of the form* (1.1) *be in the class* $\sum_{p,2}^{q}(\alpha,\beta,\lambda,x)$*, then:*

$$a_k \leqslant \frac{\beta(\lambda+1)[p]_q(1-\alpha)}{\frac{p}{k}[k]_q(k-1+\lambda\beta)q^p-\beta(\lambda+1)[p]_q(1-\alpha)x^{k+p}}. \tag{3.9}$$

Now, for $j=1,2$*, we obtain distortion bounds of the classes* $\sum_{p,j}^{q}(\alpha,\beta,\lambda,x)$*.*

Theorem 6. *Let* $f(z)\in\sum_{p,1}^{q}(\alpha,\beta,\lambda,x)$*, then for* $0<|z|=r<1$*:*

$$|f(z)| \geqslant \frac{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)r^{2p}}{r^p\big((p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p}\big)}.$$

The result is sharp for:

$$G_1(z) = \frac{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)z^{2p}}{z^p\big((p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p}\big)}.$$

Proof. Since $f(z)\in\sum_{p,1}^{q}(\alpha,\beta,\lambda,x)$, so by (3.4), we have:

$$\sum_{k=p}^{\infty} a_k \leq \frac{\beta(\lambda+1)(1-\alpha)}{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p}}.$$

From (3.5), we have:

$$A = 1-\sum_{k=p}^{\infty} a_k x^{k+p} \geqslant \frac{(p-1+\lambda\beta)q^p}{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p}}.$$

Hence

$$\begin{aligned}|f(z)| &= \left|Az^{-p}+\sum_{k=p}^{\infty}a_k z^k\right| \\ &\geqslant Ar^{-p}-r^p\sum_{k=p}^{\infty}a_k \\ &\geqslant \frac{(p-1+\lambda\beta)q^p}{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p+1}}r^{-p} \\ &- r^p\Big(\frac{\beta(\lambda+1)(1-\alpha)}{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p}}\Big) \\ &= \frac{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)r^{2p}}{r^p\big((p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)x^{2p}\big)}.\end{aligned}$$

So, the proof is complete. □

Theorem 7. *If $f(z) \in \sum_{p,2}^{q}(\alpha,\beta,\lambda,x)$, then for $0<|z|=r<1$:*

$$|f(z)| \leqslant \frac{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)r^{2p}}{r^p\big((p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}\big)}.$$

The result is sharp for:

$$G_2(z) = \frac{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)z^{2p}}{z^p\big((p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}\big)}.$$

Proof. From (3.6), we have:

$$\sum_{k=p}^{\infty} a_k \leqslant \frac{\beta(\lambda+1)(1-\alpha)}{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}}.$$

By using (3.7), we get:

$$\begin{aligned} A &= 1+\sum_{k=p}^{\infty}\frac{k}{p}a_k x^{k+p} \\ &\leqslant \frac{(p-1+\lambda\beta)q^p}{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}}. \end{aligned}$$

Therefore

$$\begin{aligned} |f(z)| &= \left|Az^{-p}+\sum_{k=p}^{\infty}a_k z^k\right| \leqslant Ar^{-p}+r^p\sum_{k=p}^{\infty}a_k \\ &\leqslant \frac{(p-1+\lambda\beta)q^p}{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}}r^{-p} \\ &+ r^p\Big(\frac{\beta(\lambda+1)(1-\alpha)}{(p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}}\Big) \\ &= \frac{(p-1+\lambda\beta)q^p+\beta(\lambda+1)(1-\alpha)r^{2p}}{r^p\big((p-1+\lambda\beta)q^p-\beta(\lambda+1)(1-\alpha)x^{2p}\big)}. \end{aligned}$$

□

4 Convexity and connected sets

In this section, we show that $\sum_{p,j}^{q}(\alpha,\beta,\lambda,x)$ for $j=1,2$ are convex. Also convex family and connected set structures are investigated.

Theorem 8. *The classes $\sum_{p,j}^{q}(\alpha,\beta,\lambda,x)$ for $j=1,2$ are convex sets.*

Proof. Let

$$f_t(z) = A_t z^{-p} + \sum_{k=p}^{\infty} a_{k,t} z^k, \tag{4.1}$$

be in the class $\sum_{p,1}^{q}(\alpha, \beta, \lambda, x)$. It is enough to prove that:

$$F(z) = \sum_{t=0}^{m} d_t f_t(z), \qquad (d_t \geqslant 0), \tag{4.2}$$

is also in the same class, where $\sum\limits_{t=0}^{m} d_t = 1$. Replacing (4.1) in (4.2), we get:

$$\begin{aligned} F(z) &= \sum_{t=0}^{m} d_t \Big(A_t z^{-p} + \sum_{k=p}^{\infty} a_{k,t} z^k \Big) \\ &= \sum_{t=0}^{m} d_t A_t z^{-p} + \sum_{k=p}^{\infty} \Big(\sum_{t=0}^{m} d_t a_{k,t} \Big) z^k \\ &= B z^{-p} + \sum_{k=p}^{\infty} S_k z^k, \end{aligned}$$

where $B = \sum\limits_{t=0}^{m} d_t A_t$ and $S_k = \sum\limits_{t=0}^{m} d_t a_{k,t}$.

Since $f_t(z) \in \sum_{p,1}^{q}(\alpha, \beta, \lambda, x)$ for $t = 0, 1, \dots, m$, by using (3.4), we have:

$$\sum_{k=p}^{\infty} \Big(\frac{[k]_q (k - 1 + \lambda\beta) q^p}{\beta(\lambda + 1)[p]_q (1 - \alpha)} + z^{k+p} \Big) a_{k,t} \leqslant 1.$$

But

$$\begin{aligned} &\sum_{k=p}^{\infty} \Big(\frac{[k]_q (k - 1 + \lambda\beta) q^p}{\beta(\lambda + 1)[p]_q (1 - \alpha)} + x^{k+p} \Big) S_k \\ &= \sum_{k=p}^{\infty} \Big(\frac{[k]_q (k - 1 + \lambda\beta) q^p}{\beta(\lambda + 1)[p]_q (1 - \alpha)} + x^{k+p} \Big) \Big(\sum_{t=0}^{m} d_t a_{k,t} \Big) \\ &= \sum_{t=0}^{m} d_t \Big(\sum_{k=p}^{\infty} \Big(\frac{[k]_q (k - 1 + \lambda\beta) q^p}{\beta(\lambda + 1)[p]_q (1 - \alpha)} + x^{k+p} \Big) a_{k,t} \Big) \\ &\leqslant \sum_{t=0}^{m} d_t = 1. \end{aligned}$$

So the proof by Theorem 2, is complete. □

By using the same technique, we can prove the same property for the class $\sum_{p,2}^{q}(\alpha,\beta,\lambda,x)$, and so we get the required result.

Now, we investigate about connected set structure. Let J be a non-empty subset of $[0,1]$. We define:

$$\textstyle\sum_{p,1}^{q}(\alpha,\beta,\lambda,J)=\bigcup_{x_2\in J}\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_2).$$

If J has only one element, then $\sum_{p,1}^{q}(\alpha,\beta,\lambda,J)$ is known to be a convex family by Theorem 8.

For proving the main theorem of this article, we need the following lemma.

Lemma 9. *If $f(z)\in\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_1)\cap\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_2)$, where x_1 and x_2 are positive numbers and $x_1\neq x_2$, then $f(z)=z^{-p}$.*

Proof. If $f(z)\in\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_1)\cap\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_2)$, and $f(z)=Az^{-p}+\sum\limits_{k=p}^{\infty}a_kz^k$. Then:

$$A=1-\sum_{k=p}^{\infty}a_kx_1^{k+p}=1-\sum_{k=p}^{\infty}a_kx_2^{k+p},$$

or

$$\sum_{k=p}^{\infty}a_k\left(x_1^{k+p}-x_2^{k+p}\right)=0.$$

Since $a_k\geqslant 0$, $x_1>0$ and $x_2>0$, we conclude that $a_k=0$ for each $k\geqslant 0$ and so $f(z)=z^{-p}$. □

Theorem 10. *If J is contained in $[0,1]$, then $\sum_{p,1}^{q}(\alpha,\beta,\lambda,J)$ is a convex family if and only if J is connected.*

Proof. Let J be connected and $x_1,x_2\in J$ with $x_1\leqslant x_2$. It is sufficient to show that for $f(z)$ and $g(z)$ given by:

$$f(z)=Az^{-p}+\sum_{k=p}^{\infty}a_kz^k\in\textstyle\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_1), \tag{4.3}$$

$$g(z)=Bz^{-p}+\sum_{k=p}^{\infty}b_kz^k\in\textstyle\sum_{p,1}^{q}(\alpha,\beta,\lambda,x_2), \tag{4.4}$$

and $0\leqslant\mu\leqslant 1$, there exists a $x_1\leqslant y\leqslant x_2$ such that:

$$h(z)=\mu f(z)+(1-\mu)g(z)\in\textstyle\sum_{p,1}^{q}(\alpha,\beta,\lambda,y).$$

By (3.5), we have:

$$A=1-\sum_{k=p}^{\infty}a_kx_1^{k+p},\qquad B=1-\sum_{k=p}^{\infty}b_kx_2^{k+p}.$$

Therefore, we get:

$$\begin{aligned} H(z) &= z^p h(z) = z^p\big(\mu f(z) + (1-\mu) g(z)\big) \\ &= \mu A + \sum_{k=p}^{\infty} \mu a_k z^{k+p} + (1-\mu)B + \sum_{k=p}^{\infty}(1-\mu) b_k z^{k+p} \\ &= \mu - \sum_{k=p}^{\infty} \mu a_k x_1^{k+p} + \sum_{k=p}^{\infty} \mu a_k z^{k+p} \\ &+ (1-\mu) - \sum_{k=p}^{\infty}(1-\mu) b_k x_2^{k+p} + \sum_{k=p}^{\infty}(1-\mu) b_k z^{k+p} \\ &= 1 + \mu \sum_{k=p}^{\infty} \big(z^{k+p} - x_1^{k+p}\big) a_k + (1-\mu) \sum_{k=p}^{\infty} \big(z^{k+p} - x_2^{k+p}\big) b_k. \end{aligned} \tag{4.5}$$

But it is trivial that $H(x_1) \leqslant 1$ and $H(x_2) \geqslant 1$.

Then there exists $y \in [x_1, x_2]$ such that $H(y) = 1$. So:

$$y^p h(y) = 1. \tag{4.6}$$

Thus $h(z) \in \sum_p(1)$. On the other hand by (4.5), (4.6) and (3.4), we have:

$$\begin{aligned} &\sum_{k=p}^{\infty} \Big(\frac{[k]_q (k-1+\lambda\beta) q^p}{\beta(\lambda+1)[p]_q(1-\alpha)} + y^{k+p}\Big)\big(\mu a_k + (1-\mu) b_k\big) \\ &= \mu \sum_{k=p}^{\infty} \Big(\frac{[k]_q (k-1+\lambda\beta) q^p}{\beta(\lambda+1)[p]_q(1-\alpha)} + x_1^{k+p}\Big) a_k \\ &+ (1-\mu) \sum_{k=p}^{\infty} \Big(\frac{[k]_q (k-1+\lambda\beta) q^p}{\beta(\lambda+1)[p]_q(1-\alpha)} + x_2^{k+p}\Big) b_k \\ &\leqslant \mu + (1-\mu) = 1. \end{aligned}$$

Hence $h(z) \in \sum_{p,1}^{q}(\alpha, \beta, \lambda, y)$. Since x_1, x_2 and y are arbitrary, the family $\sum_{p,1}^{q}(\alpha, \beta, \lambda, J)$ is convex.

Conversely, if J is not connected, then there exists x_1, x_2 and y such that $x_1 < y < x_2$ and $x_1, x_2 \in J$ but $y \notin J$. If $f(z) \in \sum_{p,1}^{q}(\alpha, \beta, \lambda, x_1)$ and $g(z) \in \sum_{p,1}^{q}(\alpha, \beta, \lambda, x_2)$, then by Lemma 9, $f(z)$ and $g(z)$ are not both equal to z^{-p}, then for fixed y and $0 \leqslant \mu \leqslant 1$ by (4.5), we obtain:

$$H(y, \mu) = 1 + \mu \sum_{k=p}^{\infty} \big(z^{k+p} - x_1^{k+p}\big) a_k + (1-\mu) \sum_{k=p}^{\infty} \big(z^{k+p} - x_2^{k+p}\big) b_k.$$

Since $H(y, 0) < 1$ and $H(y, 1) > 1$, there exists μ^*, $0 < \mu^* < 1$, such that $H(y, \mu^*) = 1$ or $yh(y) = 1$, where $h(z) = \mu^* f(z) + (1-\mu^*) g(z)$.

Thus $h(z) \in \sum_{p,1}^{q}(\alpha,\beta,\lambda,y)$. From Lemma 9, we have $h(z) \notin \sum_{p,1}^{q}(\alpha,\beta,\lambda,J)$. Since $y \in J$ and $h(z) \neq z^{-p}$, this implies that the family $\sum_{p,1}^{q}(\alpha,\beta,\lambda,J)$ is not convex, which is a contradiction and so the proof is complete. □

Remark 11. *With the same definition and using the same techniques, we obtain the same property for the class* $\sum_{p,1}^{q}(\alpha,\beta,\lambda,J)$.

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