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EXISTENCE THEORY AND HYERS-ULAM STABILITY FOR A COUPLE SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We discuss the existence and uniqueness of solutions for a coupled system of fractional differential equations by the help of some fixed point theorems. Further, we investigate the Hyers-Ulam stability results for the proposed problem. An example is also included to illustrate the established results.

1 Introduction

Fractional differential equations have recently proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering, physics and economics. We can find numerous applications in viscoelasticity, electrochemistry, electrical networks, control theory, biosciences, electromagnetic, signal processes, mechanics and diffusion processes see [22, 23, 25, 27]. Significant developments in fractional differential equations can be find in the monographs of Kilbas et al.[22], Miller and Ross [25], Lakshmikantham et al. [23], Podlubny [27]. Ordinary differential equations and fractional differential equations have been studied by many authors, for detail see [1, 2, 3, 4, 5, 6, 7, 8, 32, 34]. In all these articles the concerned results were obtained via classical fixed point theorems like Banach contraction principal, Leray-Schauder fixed point theorems....

An other aspect of fractional differential equations that has got attentions from researchers is committed to the stability analysis of differential equations for classical and fractional order. Stability analysis plays a significant role in the optimization and numerical analysis of fractional differential equations. Historically, stability was importantly given by Ulam (1940) [29], which was formally introduced by Hyers in 1941 [19] using Banach spaces. Obloza [26] was the first to investigate the Ulam-Hyers stability for linear differential equations. Later, this result was generalized

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and extended by Rassias, Jung and others, for instance see [20, 21, 24, 28, 33].

Urs [30] studied the Hyers-Ulam stability to a system of periodic boundary value problem of classical differential equations

$$\begin{cases} \frac{d\alpha(t)}{dt} = f(t, \alpha(t)) + g(t, \beta(t)), \ t \in [0, T], \\ \frac{d\beta(t)}{dt} = f(t, \beta(t)) + g(t, \alpha(t)), \ t \in [0, T], \\ \quad \alpha(0) = \alpha(T), \quad \beta(0) = \beta(T), \end{cases}$$

where the nonlinear function $f, g \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

Motivated by the above works, in this work, we investigate a nonlinear coupled system of fractional order differential equations

$$\begin{cases} {}^cD^q_{0+}u(t) = f(t, u(t), v(t)), \\ {}^cD^p_{0+}v(t) = h(t, u(t), v(t)), \\ u(0) = u'(0) = 0, \quad u''(0) = \alpha u(1), \\ v(0) = v'(0) = 0, \quad v''(0) = \alpha v(1), \end{cases} \quad t \in [0, 1], \tag{1.1}$$

where f, $h : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $2 < q < 3$, $2 < p < 3$, ${}^cD^q_{0+}$ denotes the Caputo's fractional derivative. We establish some adequate conditions for the existence and uniqueness of solution to system (1.1) by using Leray-Schauder fixed point theorem and Banach contraction type. Further, we investigate the Hyers-Ulam stability results for the proposed problem. An example is given as an applicable of the obtained results.

2 Preliminaries

In this section, we present some definitions and lemmas from fractional calculus theory, which will be needed later.

Definition 1. *If $g \in C([a, b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by*

$$I^\alpha_{a+}g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2. *Let $\alpha \geq 0, n = [\alpha] + 1$. If $f \in C^n[a, b]$ then the Caputo fractional derivative of order α of f defined by ${}^cD^\alpha_{a+}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^n(s)}{(t-s)^{\alpha-n+1}} ds$ exists almost everywhere on $[a, b]$ ($[\alpha]$ is the entire part of α).*

Lemma 3. *For $\alpha > 0$, $g \in C([0, 1], \mathbb{R})$, the homogenous fractional differential equation ${}^cD^\alpha_{a+}g(t) = 0$ has a solution $g(t) = c_1 + c_2t + c_3t^2 + ... + c_nt^{n-1}$, where, $c_i \in \mathbb{R}$, $i = 0, ..., n$, and $n = [\alpha] + 1$.*

Lemma 4. *Let* $p, q \geq 0$, $f \in L_1[a,b]$. *Then* $I_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{p+q} f(t) = I_{0+}^{q} I_{0+}^{p} f(t)$ *and* ${}^{c}D_{a+}^{q} I_{0+}^{q} f(t) = f(t)$, *for all* $t \in [a,b]$.

Now, we present the necessary definition from the theory of cone in Banach spaces.

Definition 5. *A nonempty subset* P *of a Banach space* E *is called a cone if* P *is convex, closed and satisfies the conditions*
(i) $\alpha x \in P$ *for all* $x \in P$ *and* $\alpha \in \mathbb{R}_+$,
(ii) $x, -x \in P$ *implies* $x = 0$.

Definition 6. *A mapping is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Theorem 7. *[15] Consider a Banach space* E *together with a cone* $K \subset E$ *and if* $\Omega \subset K$ *is relatively open set with* $0 \in \Omega$. *Let* $T : \overline{\Omega} \to E$ *be a completely continuous operator. Then either there exists* (1) *the operator* T *has a fixed point in* Ω, *or* (2) *there exist* $u \in \partial\Omega$ *and* $\lambda \in (0,1)$ *such that* $u = \lambda T(u)$

Definition 8. *[30, 31] Let* E *be a Banach space such that* $T : E \to E$ *is a continues operator. Then the fixed point equation given by*

$$u = T(u) \tag{2.1}$$

is called Hyers-Ulam stable if for the inequality provided as

$$|u - T(u)| \leq \varepsilon, \ \ t \in [0,1], \tag{2.2}$$

there exists a constant $\beta > 0$ *such that for each solution* $u \in C([0,1], \mathbb{R})$ *there exists a unique solution* $\eta \in C([0,1], \mathbb{R})$ *of the operator* (2.1) *with*

$$|u(t) - \eta(t)| \leq \beta\varepsilon \tag{2.3}$$

Similarly, the operator equation (2.1) *is generalized Hyers-Ulam stable if there exist a non decreasing mapping* $\Psi_T \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\Psi_T(0) = 0$, *such that for every solution* $\eta \in E$ *of the inequality* (2.2), *there exist a unique solution* $u \in E$ *of* (2.1) *which satisfies*

$$|u(t) - \eta(t)| \leq \Psi_T(\varepsilon), \quad t \in [0,1]. \tag{2.4}$$

3 Existence and Uniqueness Theorems

We start by solving an auxiliary problem which allows us to get the expression of the solution.

Lemma 9. *Assuming that* $\alpha \neq 2$ *and* $y \in C([0,1],\mathbb{R})$. *Then the problem*

$$\begin{cases} {}^{c}D_{0+}^{q}u(t) = y(t), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u''(0) = \alpha u(1), \end{cases} \tag{3.1}$$

has a unique solution given by:

$$u(t) = \frac{1}{\Gamma(q)} \int_0^1 G_1(t,s)\, y(s) ds, \tag{3.2}$$

where

$$G_1(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} + \frac{\alpha}{2-\alpha} t^2 (1-s)^{q-1}, & 0 \le s \le t, \\ \frac{\alpha}{2-\alpha} t^2 (1-s)^{q-1}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Proof. Using Lemmas 3 and 4, we get

$$u(t) = I_{0+}^{q} y(t) + a + bt + ct^2. \tag{3.4}$$

The boundary condition $u(0) = 0$ implies that $a = 0$. Differentiating both sides of (1.1) and using the initial condition $u'(0) = 0$, it yields $b = 0$. The condition $u''(0) = \alpha u(1)$, $u''(0) = 2c = \alpha u(1)$, $2c = \alpha[I_{0+}^{q} y(1) + c]$, $2c - \alpha c = \alpha I_{0+}^{q} y(1)$, and $c = \dfrac{\alpha}{2-\alpha} I_{0+}^{q} y(1)$. Substituting a, b and c by their values in (2), we obtain

$$u(t) = I_{0+}^{q} y(t) + \frac{\alpha}{2-\alpha} t^2 I_{0+}^{q} y(1) \tag{3.5}$$

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds + \frac{\alpha}{2-\alpha} \frac{1}{\Gamma(q)} t^2 \int_t^1 (t-s)^{q-1} y(s) ds \tag{3.6}$$

then $u(t) = \int_0^1 G_1(t,s) y(s) ds$, Where $G_1(t,s)$ is the Green's function given in (1) .

Thank to lemma 9, an equivalent system of Fredholm integral equations to the proposed system (1.1) is given by

$$\begin{cases} u(t) = \int_0^1 G_1(t,s) f(s, u(s), v(s))\, ds, \\ v(t) = \int_0^1 G_2(t,s) h(s, u(s), v(s))\, ds \end{cases} \tag{3.7}$$

where

$$G_2(t,s) = \frac{1}{\Gamma(p)} \begin{cases} (t-s)^{p-1} + \frac{\alpha}{2-\alpha} t^2 (1-s)^{p-1}, & 0 \le s \le t, \\ \frac{\alpha}{2-\alpha} t^2 (1-s)^{p-1}, & 0 \le t \le s \le 1. \end{cases} \tag{3.8}$$

□

Lemma 10. *The functions* $G_1(t,s), G_2(t,s)$ *satisfy the following properties:*

(*i*) $G_i(t,s)$, $i = 1, 2$ *is continues over* $[0,1] \times [0,1]$ *for all* $t,\ s \in [0,1]$,

(*ii*) $\max_{t\in[0,1]} |G_1(t,s)| = \frac{2}{2-\alpha} \frac{(1-s)^{q-1}}{\Gamma(q)} = G_1(1,s)$, $\max_{t\in[0,1]} |G_2(t,s)| = \frac{2}{2-\alpha} \frac{(1-s)^{p-1}}{\Gamma(p)} = G_2(1,s)$, $s \in [0,1]$,

(*iii*) $\max_{t\in[0,1]} \int_0^1 |G_1(t,s)|\, ds \le \frac{2}{(2-\alpha)\Gamma(q+1)}$, $\max_{t\in[0,1]} \int_0^1 |G_2(t,s)|\, ds \le \frac{2}{(2-\alpha)\Gamma(p+1)}$.

Proof. The proof of (i), (ii) and (iii) is easy, then we omit it. □

Define the Banach space $E = \{u | u \in C[0,1]\}$ equipped with the norm $\|u\|_E = \max_{t\in[0,1]} |u(t)|$. Similary, the norm on the product space is define by $\|(u,v)\|_{E\times E} = \|u\|_E + \|v\|_E$

We define the cone $K \subset E \times E$ by

$$K = \{(u,v) \in E \times E : u(t) \geq 0, v(t) \geq 0, t \in [0,1]\}.$$

Definition 11. *Suppose that $f, h \in ([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are continuous. Then $(u,v) \in E \times E$ is a solution of system (1.1) if and only if $(u,v) \in E \times E$ satisfies the system (3.7).*

Define the integral operator $T : E \times E \to E \times E$ by

$$T(u,v)(t) = \left(\int_0^1 G_1(t,s) f(s, u(s), v(s))\, ds, \int_0^1 G_2(t,s) h(s, u(s), v(s))\, ds \right)$$

$$= (T_1 u, T_2 v)(t). \tag{3.9}$$

Then the fixed point of the operator T coincide with the solution of the coupled system (1.1).

Theorem 12. *Let $f, h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then the operator $T : K \to K$ defined in (3.9) is completely continuous.*

Proof. It is obvious that T is continuous since $f, h, G_1(t,s), G_2(t,s)$ are continuous. Let us prove that $T : K \to K$ is completely continuous.

Claim 1. $T(B_r)$ is uniformly bounded, where $B_r = \{(u,v) \in K, \|(u,v)\| \leq r\}$.

Since the functions f and h are continuous, then there exist constants c, m such that $\max_{t\in[0,1]} |f(t, u(t), v(t)| = c$ and $\max_{t\in[0,1]} |h(t, u(t), v(t)| = m$ for any $(u,v) \in B_r$. By virtue of Lemma 10 we obtain

$$|T_1(t, u(t), v(t))| = \left| \int_0^1 G_1(t,s) f(s, u(s), v(s))\, ds \right|$$

$$\leq \int_0^1 |G_1(t,s)| |f(s, u(s), v(s))|\, ds \leq \frac{2c}{(2-\alpha)\Gamma(q+1)}. \tag{3.10}$$

and

$$|T_2(t, u(t), v(t))| = \left| \int_0^1 G_2(t,s) h(s, u(s), v(s))\, ds \right|$$

$$\leq \int_0^1 |G_2(t,s)| h(s, u(s), v(s))\, ds \leq \frac{2m}{(2-\alpha)\Gamma(p+1)}. \tag{3.11}$$

which implies that

$$\|T_1\,(u,v)\|_E \leq \frac{2c}{(2-\alpha)\,\Gamma(q+1)} \tag{3.12}$$

and

$$\|T_2\,(u,v)\|_E \leq \frac{2m}{(2-\alpha)\,\Gamma(p+1)}. \tag{3.13}$$

Thus from (3.12) and (3.13), one has

$$\begin{aligned} \|T_1\,(u,v)\|_E + \|T_2\,(u,v)\|_E &\leq \\ \frac{2c}{(2-\alpha)\,\Gamma(q+1)} + \frac{2m}{(2-\alpha)\,\Gamma(p+1)} &= \omega \end{aligned}$$

Then

$$\|T(u,v)\|_{E\times E} \leq \omega$$

Hence T is uniformly bounded.

Claim 2. T is equicontinuous. We have for any $(u,v) \in B_r$, and let $t_1 < t_2 \in [0,1]$,

$$|T_1\,(u,v)\,(t_2) - T_1\,(u,v)\,(t_1)| \leq \int_0^1 |G_1(t_2,s) - G_1(t_1,s)|\,|f\,(s,u(s),v\,(s))|\,ds \tag{3.14}$$

$$\begin{aligned} &\leq \frac{2c}{(2-\alpha)\,\Gamma\,(q)} \left(t_2^2 - t_1^2\right) \int_0^1 (1-s)^{q-1} ds \\ &\quad + \frac{2c}{(2-\alpha)\,\Gamma\,(q)} \left[\int_0^{t_2} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] ds + \int_{t_1}^{t_2} (t_2-s)^{q-1} ds\right] \\ &\leq \frac{c}{\Gamma\,(q+1)} \left[\left(t_2^2 - t_1^2\right) + t_2^q - t_1^q + (t_1-t_2)^q + (t_2-t_1)^q\right]. \end{aligned}$$

Similarly, one can show that

$$\begin{aligned} |T_2\,(u,v)\,(t_2) - T_2\,(u,v)\,(t_1)| &\leq \int_0^1 |G_2(t_2,s) - G_2(t_1,s)|\,f\,(s,u(s),v\,(s))\,ds \\ &\leq \frac{2m}{(2-\alpha)\,\Gamma\,(p)} \left(t_2^2 - t_1^2\right) \int_0^1 (1-s)^{p-1} ds \\ &\quad + \frac{2m}{(2-\alpha)\,\Gamma\,(p)} \left[\int_0^{t_2} \left[(t_2-s)^{p-1} - (t_1-s)^{p-1}\right] ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-s)^{p-1} ds\right] \end{aligned}$$

$$\leq \frac{m}{\Gamma\,(p+1)} \left[\left(t_2^2 - t_1^2\right) + t_2^p - t_1^p + (t_1-t_2)^p + (t_2-t_1)^p\right]. \tag{3.15}$$

The right hand sides of (3.14) and (3.15) tend to zero, when $t_1 \to t_2$, therefore, we conclude by Arzela Ascoli theorem, that T is equicontinuous. We deduce then that T is completely continuous operator. □

Theorem 13. *Suppose that the following hypotheses hold*

(H_1) $\frac{2}{(2-\alpha)}\left(\frac{\Theta_f}{\Gamma(q+1)}+\frac{\Theta_h}{\Gamma(p+1)}\right)<1$

(H_2) *There exist constants* Θ_f, $\Theta_h > 0$ *for all* $u, v, \mu, \sigma \in \mathbb{R}$ *such that*

$$|f(t,u,v) - f(t,\mu,\sigma)| \leq \Theta_f \left[|u-\mu| + |v-\sigma|\right]$$

$$|h(t,u,v) - h(t,\mu,\sigma)| \leq \Theta_h \left[|u-\mu| + |v-\sigma|\right]$$

then the coupled system (1.1) *has a unique solution.*

Proof. Let (u,v), $(\mu,\sigma) \in K$ and consider

$$\begin{aligned} |T_1(u,v)(t) - T_1(\mu,\sigma)(t)| &= \left|\int_0^1 G_1(t,s)\left[(f(s,u(s),v(s))) - (f(s,\mu(s),\sigma(s)))\right] ds\right| \\ &\leq \int_0^1 |G_1(t,s)| \left[|(f(s,u(s),v(s))) - (f(s,\mu(s),\sigma(s)))|\right] ds \end{aligned}$$

$$\|T_1(u,v) - T_1(\mu,\sigma)\|_E \leq \frac{2}{(2-\alpha)} \frac{\Theta_f}{\Gamma(q+1)}. \tag{3.16}$$

Similarly, we can obtain

$$\|T_2(u,v) - T_2(\mu,\sigma)\|_E \leq \frac{2}{(2-\alpha)} \frac{\Theta_h}{\Gamma(p+1)} \tag{3.17}$$

From (3.16) and (3.17), we get

$$\|T(u,v) - T(\mu,\sigma)\|_{E\times E} \leq \frac{2}{(2-\alpha)}\left(\frac{\Theta_f}{\Gamma(q+1)} + \frac{\Theta_h}{\Gamma(p+1)}\right) < 1 \tag{3.18}$$

Therefore, T is a contraction operator and has a unique fixed point which is the corresponding unique solution of (1.1). □

Theorem 14. *Assume that* $f, h : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *are continuous and the functions* φ_i, ρ_i, $(i = 0,1,2) : (0,1) \to [0,\infty)$ *satisfy the following inequalities*

(H_3)

$$\begin{aligned} |f(t,u(t),v(t))| &\leq \varphi_0(t) + \varphi_1(t)|u(t)| + \varphi_2(t)|v(t)| \\ |h(t,u(t),v(t))| &\leq \rho_0(t) + \rho_1(t)|u(t)| + \rho_2(t)|v(t)|; \end{aligned}$$

(H_4)

$$0 < \int_0^1 G_1(1,s)\varphi_0(s)\, ds < \infty, \qquad \int_0^1 G_1(1,s)\left[\varphi_1(s) + \varphi_2(s)\right] ds < 1;$$

$$0 < \int_0^1 G_2(1,s)\rho_0(s)\,ds < \infty, \qquad \int_0^1 G_2(1,s)\left[\rho_1(s)+\rho_2(s)\right]ds < 1.$$

Then the coupled system (1.1) has at least one solution.

Proof. Define the set B_r as $B_r = \left\{(u,v) \in E\times E : \|(u,v)\|_{E\times E} < r\right\}$,
where $\max\left\{\left(\frac{2\int_0^1 G_1(1,s)\varphi_0(s)ds}{1-2\int_0^1 G_1(1,s)[\varphi_1(s)+\varphi_2(s)]ds}\right), \left(\frac{2\int_0^1 G_2(1,s)\rho_0(s)ds}{1-2\int_0^1 G_2(1,s)[\rho_1(s)+\rho_2(s)]ds}\right)\right\} < r.$

Define $T : \overline{B_r} \to K$ as in (3.9) which is completely continues. Let $(u,v) \in B_r$, so $\|(u,v)\|_{E\times E} < r$. Then

$$\begin{aligned}\|T_1(u,v)\|_E &\le \max_{t\in[0,1]}\int_0^1 |G_1(t,s)|\,|f(s,u(s),v(s))|\,ds\\ &\le \max_{t\in[0,1]}\int_0^1 |G_1(t,s)|\,\varphi_0(s)\,ds + \max_{t\in[0,1]}\int_0^1 |G_1(t,s)|\left[\varphi_1(s)\,|u(s)|+\varphi_2(s)\,|v(s)|\right]ds\end{aligned}$$

$$\le \int_0^1 G_1(1,s)\varphi_0(s)\,ds + r\int_0^1 G_1(1,s)\left[\varphi_1(s)+\varphi_2(s)\right]ds \le \frac{r}{2}. \tag{3.19}$$

$$(u,v) = \lambda T(u,v)\,, \lambda \in (0,1)\,. \tag{3.20}$$

Then in view of (3.20) for $\lambda \in (0,1)$, we get

$$\begin{aligned}\|u\|_E &= \|\lambda T_1(u,v)\|_E = \lambda \max_{t\in[0,1]}\left|\int_0^1 G_1(t,s)f(s,u(s),v(s))\,ds\right|\\ &< \max_{t\in[0,1]}\int_0^1 G_1(t,s)\left[\varphi_0(s)+\varphi_1(s)\,|u(s)|+\varphi_2(s)\,|v(s)|\right]ds\end{aligned}$$

$$\le \int_0^1 G_1(1,s)\varphi_0(s)\,ds + r\int_0^1 G_1(1,s)\left[\varphi_1(s)+\varphi_2(s)\right]ds \le \frac{r}{2}. \tag{3.21}$$

Similarly, we can obtain

$$\|v\|_E = \lambda \max_{t\in[0,1]}\left|\int_0^1 G_2(t,s)h(s,u(s),v(s))\,ds\right| \le \frac{r}{2}. \tag{3.22}$$

Then the coupled system (1.1) has at least one solution. □

4 Hyers-Ulam stability analysis of BVP(1)

Here we study Hyers-Ulam stability for the coupled system of FDEs (1.1).

For some positive $\phi_i > 0$, $i = 1, 2$, consider the system of inequalities given by

$$\begin{cases} \left|{}^c D^q_{0^+} u(t) - f\left(t, u(t), v\left(t\right)\right)\right| \leq \phi_1, \quad t \in [0,1], \\ \left|{}^c D^p_{0^+} v(t) - h\left(t, u(t), v\left(t\right)\right)\right| \leq \phi_2, \quad t \in [0,1]. \end{cases} \tag{4.1}$$

The coupled system (1) is Hyers-Ulam stable if there exists a non zero positive real number C such that for each solution $(u, v) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ of the system of inequalities (4.1), there exists a unique solution $(\mu, \sigma) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ with the assumption (H_5)

$$\left|(u, v)(t) - (\mu, \sigma)(t)\right| \leq C\phi\ , \ t \in [0,1], \tag{4.2}$$

with $C = \frac{C_1 + C_2}{1 - (\Theta_f C_1 + \Theta_h C_2)}$ and $\phi = \max\{\phi_1, \phi_2\}$.

Remark 15. *We say that $(u, v) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ is a solution of the system of inequalities (4.1) if there exist functions $\alpha, \beta \in C([0,1], \mathbb{R})$ that depend upon u, v respectively such that*

(i) $|\alpha(t)| \leq \phi_1$, $|\beta(t)| \leq \phi_2$, $t \in [0,1]$,

(ii) and

$$\begin{cases} \left|{}^c D^q_{0^+} u(t) = f\left(t, u(t), v\left(t\right)\right)\right| + \alpha\left(t\right), \quad t \in [0,1], \\ \left|{}^c D^p_{0^+} v(t) - h\left(t, u(t), v\left(t\right)\right)\right| + \beta\left(t\right), \quad t \in [0,1]. \end{cases}$$

Lemma 16. *Let $(u, v) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ be the solution of the system of inequalities (4.1). If there exist constants $C_1 > 0, C_2 > 0$, then the following estimates hold*

$$\begin{cases} \left|u(t) - \int_0^1 G_1(t,s) f\left(s, u(s), v\left(s\right)\right) ds\right| \leq C_1\phi_1, \quad t \in [0,1], \\ \left|v(t) - \int_0^1 G_2(t,s) h\left(s, u(s), v\left(s\right)\right) ds\right| \leq C_2\phi_2, \ t \in [0,1], \end{cases}$$

Proof. From (ii) of Remark 15, we have that

$$\begin{cases} {}^c D^q_{0^+} u(t) = f\left(t, u(t), v\left(t\right)\right) + \alpha\left(t\right), \quad t \in [0,1], \\ {}^c D^p_{0^+} v(t) = h\left(t, u(t), v\left(t\right)\right) + \beta\left(t\right), \quad t \in [0,1], \\ u\left(0\right) = u'(0) = 0, \quad u''\left(0\right) = \alpha u\left(1\right), \\ v\left(0\right) = v'(0) = 0, \quad v''\left(0\right) = \alpha v\left(1\right), \end{cases} \tag{4.3}$$

Then, in view of Lemma 3, the solution of (4.3) is given by

$$\begin{cases} u(t) = \int_0^1 G_1(t,s) f\left(s, u(s), v\left(s\right)\right) ds + \int_0^1 G_1(t,s)\alpha\left(s\right) ds, \ t \in [0,1], \\ v(t) = \int_0^1 G_2(t,s) h\left(s, u(s), v\left(s\right)\right) ds + \int_0^1 G_2(t,s)\beta\left(s\right) ds, \ t \in [0,1]. \end{cases} \tag{4.4}$$

From the first equation of system (4.4), we have

$$\begin{aligned}\left|u(t)-\int_0^1 G_1(t,s)f\left(s,u(s),v\left(s\right)\right)ds\right| &= \left|\int_0^1 G_1(t,s)\alpha\left(s\right)ds\right| \\ &\leq \max_{t\in[0,1]}\left|\int_0^1 G_1(t,s)\right|\left|\alpha\left(s\right)\right|ds \\ &\leq C_1\phi_1, \end{aligned} \tag{4.5}$$

where $C_1=\frac{2}{(2-\alpha)\Gamma(q+1)}$
Similarly, repeating the above procedure for the second equation of (4.4), we get

$$\left|v(t)-\int_0^1 G_2(t,s)h\left(s,u(s),v\left(s\right)\right)ds\right|\leq C_2\phi_2, \tag{4.6}$$

where $C_2=\frac{2}{(2-\alpha)\Gamma(p+1)}$. □

Theorem 17. *Under the assumption* (H_5), *the coupled system* (1) *is Hyers-Ulam stable if*

$$C\phi<1,\ \ \textit{with}\ C=\frac{C_1+C_2}{1-\left(\Theta_f C_1+\Theta_h C_2\right)}\ ,\ \phi=\max\left\{\phi_1,\phi_2\right\}\ \ \textit{and}\ \ \left(\Theta_f C_1+\Theta_h C_2\right)\neq 1.$$

Consequently, the coupled system (1.1) *is generalized Hyers-Ulam stable.*

Proof. Let $(u,v)\in C\left([0,1],\mathbb{R}\right)\times C\left([0,1],\mathbb{R}\right)$ be the solution of the system of inequalities given by

$$\left|{}^cD_{0^+}^p u(t)-f\left(t,u(t),v\left(t\right)\right)\right|\leq\phi_1,\quad t\in[0,1], \tag{4.7}$$

$$\left|{}^cD_{0^+}^p v(t)-h\left(t,u(t),v\left(t\right)\right)\right|\ \leq\ \phi_2,\quad t\in[0,1].$$

and $(\mu,\sigma)\in C\left([0,1],\mathbb{R}\right)\times C\left([0,1],\mathbb{R}\right)$ be the unique solution for the system of FDE:

$$\begin{cases} {}^cD_{0^+}^p\mu(t)-f\left(t,\mu(t),\sigma\left(t\right)\right)=0,\ t\in[0,1], \\ {}^cD_{0^+}^p\sigma(t)-h\left(t,\mu(t),\sigma\left(t\right)\right)=0,\ t\in[0,1], \\ \mu\left(0\right)=\mu'(0)=0,\quad \mu''\left(0\right)=\alpha\mu\left(1\right), \\ \sigma\left(0\right)=\sigma'(0)=0,\quad \sigma''\left(0\right)=\alpha\sigma\left(1\right), \end{cases} \tag{4.8}$$

Then, by Lemma 3, we may write the solution of (4.8) as

$$\begin{cases} \mu(t)=\int_0^1 G_1(t,s)f\left(s,\mu(s),\sigma\left(s\right)\right)ds,\ t\in[0,1], \\ \sigma(t)=\int_0^1 G_2(t,s)h\left(s,\mu(s),\sigma\left(s\right)\right)ds,\ t\in[0,1]. \end{cases} \tag{4.9}$$

Using Lemma 16 and considering

$$\begin{aligned}|u(t)-\mu(t)| &= \left|u(t)-\int_0^1 G_1(t,s)f\left(s,\mu(s),\sigma\left(s\right)\right)ds\right| \\ &\leq \left|u(t)-\int_0^1 G_1(t,s)f\left(s,u(s),v\left(s\right)\right)ds\right| \\ &\quad+\left|\int_0^1 G_1(t,s)f\left(s,u(s),v\left(s\right)\right)ds-\int_0^1 G_1(t,s)f\left(s,\mu(s),\sigma\left(s\right)\right)ds\right| \\ &\leq C_1\phi_1+\Theta_f\int_0^1 |G_1(t,s)|\left[|u(s)-\mu(s)|+|v\left(s\right)-\sigma\left(s\right)|\right]ds.\end{aligned}$$

From which, we have

$$\|u-\mu\|_E \leq C_1\phi_1+\Theta_f C_1\left[\|u-\mu\|_E+\|v-\sigma\|_E\right]. \tag{4.10}$$

Similarly from the second equation of (4.8) and (4.9), we have

$$\|v-\sigma\|_E \leq C_2\phi_2+\Theta_h C_2\left[\|u-\mu\|_E+\|v-\sigma\|_E\right]. \tag{4.11}$$

Now from (4.10) and (4.11), and taking $\max\{\phi_1,\phi_2\}=\phi$, we have

$$\|u-\mu\|_E+\|v-\sigma\|_E \leq C_1\phi+C_2\phi+\left(\Theta_f C_1+\Theta_h C_2\right)\left[\|u-\mu\|_E+\|v-\sigma\|_E\right]$$

$$\|(u,v)-(\mu,\sigma)\|_{E\times E} \leq \frac{C_1+C_2}{1-\left(\Theta_f C_1+\Theta_h C_2\right)}\phi = C\phi, \text{ where } \left(\Theta_f C_1+\Theta_h C_2\right)\neq 1. \tag{4.12}$$

Thus the coupled system (1) has an Hyers-Ulam stable solution. □

Example 18. *Let us consider the following system of fractional boundary value problem*

$$\begin{cases} {}^cD_{0+}^{\frac{5}{2}}u(t)=\frac{t}{2}+e^{-t}\left[\frac{1+\sin|u(t)|}{t+10}+\frac{1+\cos|v(t)|}{t+10}\right], \quad 0<t<1, \\ {}^cD_{0+}^{\frac{5}{2}}v(t)=\frac{1+t}{2}+e^{-t^2}\left[\frac{1+\cos|u(t)|}{t+30}+\frac{1+\sin|v(t)|}{t+30}\right], \quad 0<t<1, \\ u\left(0\right)=u'(0)=0, \quad u''\left(0\right)=\frac{1}{2}u\left(1\right), \\ v\left(0\right)=v'(0)=0, \quad v''\left(0\right)=\frac{1}{2}v\left(1\right), \end{cases}$$

where $q=p=\frac{5}{2}$, $\alpha=\frac{1}{2}$, *by calculus we obtain*

$$\max_{t\in[0,1]}\int_0^1 |G_1(t,s)|\,ds \leq \frac{32}{45\sqrt{\pi}}=C_1, \quad \max_{t\in[0,1]}\int_0^1 |G_2(t,s)|\,ds \leq \frac{32}{45\sqrt{\pi}}=C_2.$$

Also $\Theta_f=\frac{1}{10}$, $\Theta_h=\frac{1}{30}$. *Therefore, we see that*

$$\frac{2}{2-\alpha}\left(\frac{\Theta_f}{\Gamma\left(q+1\right)}+\frac{\Theta_h}{\Gamma\left(p+1\right)}\right)=\frac{4}{3}\frac{16}{225\sqrt{\pi}}=\frac{64}{675\sqrt{\pi}}<1.$$

Hence, in view of Theorem 13, the coupled system has a unique solution. Similarly the conditions of Theorem 17 are easy to verify. Further as

$$\frac{2}{2-\alpha}\left(\frac{\Theta_f}{\Gamma(q+1)}+\frac{\Theta_h}{\Gamma(p+1)}\right)=C_1\Theta_f+C_2\Theta_h=\frac{64}{675\sqrt{\pi}}\neq 1$$

so in view oh Theorem 17, the condition of Hyers-Ulam stability are also satisfied. So the solution of the coupled system (1.1) *is Hyers-Ulam stable.*

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