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GENERALIZED ALGEBRAIC COMPLETELY INTEGRABLE SYSTEMS

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Abstract. We tackle in this paper the study of generalized algebraic completely integrable systems. Some interesting cases of integrable systems appear as coverings of algebraic completely integrable systems. The manifolds invariant by the complex flows are coverings of Abelian varieties and these systems are called algebraic completely integrable in the generalized sense. The later are completely integrable in the sense of Arnold-Liouville. We shall see how some algebraic completely integrable systems can be constructed from known algebraic completely integrable in the generalized sense. A large class of algebraic completely integrable systems in the generalized sense, are part of new algebraic completely integrable systems. We discuss some interesting and well known examples : a 4-dimensional algebraically integrable system in the generalized sense as part of a 5-dimensional algebraically integrable system, the Hénon-Heiles and a 5-dimensional system, the RDG potential and a 5-dimensional system, the Goryachev-Chaplygin top and a 7-dimensional system, the Lagrange top, the (generalized) Yang-Mills system and cyclic covering of Abelian varieties.

1 Introduction and generalities

Consider Hamiltonian vector field of the form

$$X_H : \dot{z} = J\frac{\partial H}{\partial z} \equiv f(z),\ z \in \mathbb{R}^m, \quad \left(\cdot \equiv \frac{d}{dt}\right) \tag{1.1}$$

where H is the Hamiltonian and $J = J(z)$ is a skew-symmetric matrix with polynomial entries in z, for which the corresponding Poisson bracket

$$\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial z}, J\frac{\partial H_j}{\partial z} \right\rangle,$$

satisfies the Jacobi identities.

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Definition 1. *The system* (1.1) *with polynomial right hand side will be called algebraic complete integrable (in abbreviated form : a.c.i.) in the sense of Adler-van Moerbeke [3, 4, 5, 31] when the following conditions hold.*

i) The system admits $n+k$ independent polynomial invariants $H_1, ..., H_{n+k}$ of which k invariants (Casimir functions) lead to zero vector fields

$$J\frac{\partial H_i}{\partial z}(z) = 0, \quad 1 \leq i \leq k,$$

the $n = (m-k)/2$ remaining ones $H_{k+1} = H, ..., H_{k+n}$ are in involution (i.e., $\{H_i, H_j\} = 0$),which give rise to n commuting vector fields. For generic c_i, the invariant manifolds (level surfaces) $\bigcap_{i=1}^{n+k} \{z \in \mathbb{R}^m : H_i = c_i\}$ are assumed compact and connected. According to the Arnold-Liouville theorem [5], there exists a diffeomorphism

$$\bigcap_{i=1}^{n+k} \{z \in \mathbb{R}^m : H_i = c_i\} \longrightarrow \mathbb{R}^n/Lattice,$$

and the solutions of the system (1.1) *are straight lines motions on these real tori.*

ii) The (affine) invariant manifolds (level surfaces) thought of as lying in $\mathbb{C}^m$,

$$\mathcal{A} = \bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i = c_i\},$$

are related, for generic c_i, to Abelian varieties $T^n = \mathbb{C}^n/Lattice$ (complex algebraic tori) as follows : $\mathcal{A} = T^n \backslash \mathcal{D}$, where $\mathcal{D}$ is a (Painlevé) divisor (codimension one subvarieties) in T^n. Algebraic means that the torus can be defined as an intersection $\bigcap_i \{Z \in \mathbb{P}^N : P_i(Z) = 0\}$, involving a large number of homogeneous polynomials P_i. In the natural coordinates $(t_1, ..., t_n)$ of T^n coming from $\mathbb{C}^n$, the coordinates $z_i = z_i(t_1, ..., t_n)$ are meromorphic and $\mathcal{D}$ is the minimal divisor on T^n where the variables z_i blow up. Moreover, the Hamiltonian flows (1.1) *(run with complex time) are straight-line motions on T^n.*

If the Hamiltonian flow (1.1) is a.c.i., it means that the variables z_i are meromorphic on the torus T^n and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset T^n$. By the a.c.i. definition, the flow (1.1) is a straight line motion in T^n and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants c_i defining the torus T^n, the total count is therefore

Mumford gave one in his Tata lectures [35], which includes the noncompact case as well.

$m-1 = dim$ $(phase\ space)-1$ parameters. The fait that algebraic complete integrable systems possess $(m-1)$-dimensional families of Laurent solutions, was implicitly used, as known, by Kowalewski [24] in her classification of integrable rigid body motions. Such a necessary condition for algebraic complete integrability can be formulated as follows [4] : If the Hamiltonian system (1.1) (with invariant tori not containing elliptic curves) is algebraic complete integrable, then each z_i blows up after a finite (complex) time, and for every z_i, there is a family of solutions

$$z_i = \sum_{j=0}^{\infty} z_i^{(j)} t^{j-k_i}, \quad k_i \in \mathbb{Z}, \quad \text{some } k_i > 0, \tag{1.2}$$

depending on dim $(phase\ space)-1 = m-1$ free parameters. Moreover, the system (1.1) possesses families of Laurent solutions depending on $m-2$, $m-3$,...,$m-n$ free parameters. The coefficients of each one of these Laurent solutions are rational functions on affine algebraic varieties of dimensions $m-1$, $m-2$, $m-3$,...,$m-n$.

How to complete the affine variety $\mathcal{A} = \bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m, H_i = c_i\}$, into an Abelian variety? A naive guess would be to take the natural compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$ by projectivizing the equations. Indeed, this can never work for a general reason: an Abelian variety $\widetilde{\mathcal{A}}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^n$ by n-dim $\widetilde{\mathcal{A}}$ global polynomial homogeneous equations. In other words, if $\mathcal{A}$ is to be the affine part of an Abelian variety, $\overline{\mathcal{A}}$ must have a singularity somewhere along the locus at infinity. The trajectories of the vector fields (1.1) hit every point of the singular locus at infinity and ignore the smooth locus at infinity. In fact, the existence of meromorphic solutions to the differential equations (1.1) depending on some free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor. More precisely, around the points of hitting, the system of differential equations (1.1) admit a Laurent expansion solution depending on $m-1$ free parameters and in order to regularize the flow at infinity, we use these parameters to blowing up the variety $\overline{\mathcal{A}}$ along the singular locus at infinity. The new complex variety obtained in this fashion is compact, smooth and has commuting vector fields on it; it is therefore an Abelian variety.

The system (1.1) with $k+n$ polynomial invariants has a coherent tree of Laurent solutions, when it has families of Laurent solutions in t, depending on $n-1$, $n-2$,...,$m-n$ free parameters. Adler and van Moerbeke [4] have shown that if the system possesses several families of $(n-1)$-dimensional Laurent solutions (principal Painlevé solutions) they must fit together in a coherent way and as we mentioned above, the system must possess $(n-2)$-, $(n-3)$-,... dimensional Laurent solutions (lower Painlevé solutions), which are the gluing agents of the $(n-1)$-dimensional family. The gluing occurs via a rational change of coordinates in which the lower parameter solutions are seen to be genuine limits of the higher parameter solutions an which in turn appears due to a remarkable propriety of algebraic complete integrable systems; they can be put into quadratic form both in the original variables and in their ratios. As a whole, the full set of Painlevé solutions glue together to form a fiber bundle with singular base. A partial converse to the above condition can be formulated as follows [4] : If the Hamiltonian system (1.1) satisfies the condition $i)$ in the definition 1 of algebraic complete integrability and if it possesses a coherent tree of Laurent solutions, then the system is algebraic complete integrable and there are no other $m-1$-dimensional Laurent solutions but those provided by the coherent set.

We assume that the divisor is very ample and in addition projectively normal (see [4] for definitions when needed). Consider a point $p \in \mathcal{D}$, a chart U_j around p on the torus and a function y_j in $\mathcal{L}(\mathcal{D})$ having a pole of maximal order at p. Then the vector $(1/y_j, y_1/y_j, \ldots, y_N/y_j)$ provides a good system of coordinates in U_j. Then taking the derivative with regard to one of the flows

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \frac{\dot{y}_i y_j - y_i \dot{y}_j}{y_j^2}, \quad 1 \leq j \leq N,$$

are finite on U_j as well. Therefore, since y_j^2 has a double pole along $\mathcal{D}$, the numerator must also have a double pole (at worst), i.e., $\dot{y}_i y_j - y_i \dot{y}_j \in \mathcal{L}(2\mathcal{D})$. Hence, when $\mathcal{D}$ is projectively normal, we have that

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \sum_{k,l} a_{k,l} \left(\frac{y_k}{y_j}\right)\left(\frac{y_l}{y_j}\right),$$

i.e., the ratios y_i/y_j form a closed system of coordinates under differentiation. At the bad points, the concept of projective normality play an important role : this enables one to show that y_i/y_j is a bona fide Taylor series starting from every point in a neighborhood of the point in question.

Moreover, the Laurent solutions provide an effective tool for find the constants of the motion. For that, just search polynomials H_i of z, having the property that evaluated along all the Laurent solutions $z(t)$ they have no polar part. Indeed, since an invariant function of the flow does not blow up along a Laurent solution, the series obtained by substituting the formal solutions (1.2) into the invariants should, in particular, have no polar part. The polynomial functions $H_i(z(t))$ being holomorphic and bounded in every direction of a compact space, (i.e., bounded along all principle solutions), are thus constant by a Liouville type of argument. It thus an important ingredient in this argument to use all the generic solutions. To make these informal arguments rigorous is an outstanding question of the subject.

Assume Hamiltonian flows to be weight-homogeneous with a weight $s_i \in \mathbb{N}$, going with each variable z_i, i.e.,

$$f_i\left(\alpha^{s_1} z_1, ..., \alpha^{s_m} z_m\right) = \alpha^{s_i+1} f_i\left(z_1, ..., z_m\right), \ \forall \alpha \in \mathbb{C}.$$

Observe that then the constants of the motion H can be chosen to be weight-homogeneous :

$$H\left(\alpha^{s_1} z_1, \ldots, \alpha^{s_m} z_m\right)=\alpha^k H\left(z_1, \ldots, z_m\right), \ k \in \mathbb{Z}.$$

The study of the algebraic complete integrability of Hamiltonian systems, includes several passages to prove rigorously. Here we mention the main passages, leaving the detail when studying the different problems in the following sections. We saw that if the flow is algebraically completely integrable, the differential equations (1.1) must admits Laurent series solutions (1.2) depending on $m-1$ free parameters. We must have $k_i=s_i$ and coefficients in the series must satisfy at the 0^{th}step non-linear equations,

$$f_i\left(z_1^{(0)}, \ldots, z_m^{(0)}\right)+g_i z_i^{(0)}=0, \ 1 \leq i \leq m, \tag{1.3}$$

and at the kthstep, linear systems of equations :

$$(L-kI)\, z^{(k)}=\left\{\begin{array}{cc} 0 & \text{for } k=1 \\ \text{some polynomial in} & z^{(1)}, \ldots, z^{(k-1)} \text{ for } k>1, \end{array}\right. \tag{1.4}$$

where

$$L= \text{ Jacobian map of (1.3)}=\frac{\partial f}{\partial z}+gI \left.\right|_{z=z^{(0)}} .$$

If $m-1$ free parameters are to appear in the Laurent series, they must either come from the non-linear equations (1.3) or from the eigenvalue problem (1.4), i.e., L must have at least $m-1$ integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity k of the constant H must be an eigenvalue of L. Moreover the formal series solutions are convergent as a consequence of the majorant method. Thus, the first step is to show the existence of the Laurent solutions, which requires an argument precisely every time k is an integer eigenvalue of L and therefore $L-kI$ is not invertible. One shows the existence of the remaining constants of the motion in involution so as to reach the number $n+k$. Then you have to prove that for given $c_1, \ldots, c_m$, the set

$$\mathcal{D} \equiv\left\{\begin{array}{l} x_i(t)=t^{-\nu_i}\left(x_i^{(0)}+x_i^{(1)} t+x_i^{(2)} t^2+\cdots\right), 1 \leq i \leq m \\ \text{Laurent solutions such that} : H_j\left(x_i(t)\right)=c_j+\text{Taylor part} \end{array}\right\}$$

defines one or several $n-1$ dimensional algebraic varieties (Painlevé divisor) having the property that $\bigcap_{i=1}^{n+k}\left\{z \in \mathbb{C}^m : H_i=c_i\right\} \cup \mathcal{D}$, is a smooth compact, connected variety with n commuting vector fields independent at every point, i.e., a complex algebraic torus $\mathbb{C}^n / \textit{Lattice}$. The flows $J \frac{\partial H_{k+i}}{\partial z}, \ldots, J \frac{\partial H_{k+n}}{\partial z}$ are straight line motions on this torus. Let's point out and we'll see all this in more detail later, that having computed the space of functions $\mathcal{L}(\mathcal{D})$ with simple poles at worst along the expansions, it is often important to compute the space of functions $\mathcal{L}(k\mathcal{D})$ of functions having k-fold poles at worst along the expansions. These functions play a crucial role in the study of the procedure for embedding the invariant tori into projective space.

There are many examples of differential equations which have the weak Painlevé property that all movable singularities of the general solution have only a finite number of branches and some integrable systems appear as coverings of algebraic completely integrable systems. The manifolds invariant by the complex flows are coverings of Abelian varieties and these systems are called *algebraic completely integrable in the generalized sense*. These systems are Liouville integrable and by the Arnold-Liouville theorem, the compact connected manifolds invariant by the real flows are tori; the real parts of complex affine coverings of Abelian varieties. Most of these systems of differential equations possess solutions which are Laurent series of $t^{1/n}$ (t being complex time) and whose coefficients depend rationally on certain algebraic parameters. In other words, for these systems just replace in the definition 1 the condition (ii) by the following :

(iii) *the invariant manifolds* $\mathcal{A}$ *are related to an* l*-fold cover* $\widetilde{T}^n$ *of* T^n *ramified along a divisor* $\mathcal{D}$ *in* T^n *as follows :* $\mathcal{A} = \widetilde{T}^n \backslash \mathcal{D}$.

Also we shall see how some algebraic completely integrable systems can be constructed from known algebraic completely integrable in the generalized sense. We will see that a large class of algebraic completely integrable systems in the generalized sense, are part of new algebraic completely integrable systems.

Example 2. *Consider the following differential equations*

$$\dot{x} = y^3, \qquad \dot{y} = -x^3. \tag{1.5}$$

These equations can be written as a Hamiltonian vector field

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad z = (x, y)^\intercal, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with the Hamiltonian

$$H = \frac{1}{4}(x^4 + y^4) = a.$$

This system is obviously completely integrable and can be solved in terms of Abelian integrals. Indeed, we deduce from the equations $\dot{x} = y^3$, $\frac{1}{4}(x^4 + y^4) = a$, *the integral form*

$$t = \int \frac{dx}{(a - x^4)^{3/4}} + t_0.$$

The system (1.5) *admits four* 1*-dimensional families of Laurent solutions in* $\sqrt{t}$, *depending on one free parameter and they are explicitly given as follows*

$$x = \frac{1}{\sqrt{t}}\left(x_0 + x_1 t + x_2 t^2 + \cdots\right), \qquad y = \frac{1}{\sqrt{t}}\left(y_0 + y_1 t + y_2 t^2 + \cdots\right),$$

where

$$x_0 + 2y_0^3 = 0, \qquad -y_0 + 2x_0^3 = 0, \qquad x_1 = y_1 = 0,$$
$$-x_2 + 2y_0^2 y_2 = 0, \qquad y_2 + 2x_0^2 x_2 = 0.$$

Hence,

$$(2x_0 y_0)^2 = -1, \quad \left(\frac{y_0}{x_0}\right)^4 = -1, \quad x_1 = y_1 = 0,$$

and x_2, y_2 *depend on one free parameter. We have just seen that it possible for the variables* x, y *to contain square root terms of the type* $\sqrt{t}$, *which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing some new variables* z_1, z_2, z_3, *which restores the Painlevé property to the system. A simple inspection of Laurent series above, suggests choosing* $z_1 = x^2$, $z_2 = y^2$, $z_3 = xy$. *And using the first integrals* $H = a$, *and differential equations* (1.5), *we obtain a new system of differential equations in three unknowns* z_1, z_2, z_3, *having two quadrics invariants* F_1, F_2 *:*

$$\dot{z}_1 = 2z_2 z_3, \qquad \dot{z}_2 = -2z_1 z_3, \qquad \dot{z}_3 = z_2^2 - z_1^2$$

and

$$F_1 = z_1^2 + z_2^2 = 4a, \qquad F_2 = z_1^2 - z_2^2 + z_3^2 = b.$$

The intersection

$$\mathcal{A} = \left\{ z \equiv (z_1, z_2, z_3) \in \mathbb{C}^3 : F_1(z) = 4a, F_2(z) = b \right\},$$

is an elliptic curve :

$$\mathcal{E} : z_2^2 = -z_1^2 + 4a, \quad z_3^2 = -2z_1^2 + 4a + b.$$

Note that the equation : $x^4 + y^4 = 4a$ *defines a Riemann surface of genus* 3 *but is not a torus. An equivalent description of* $x^4 + y^4 = 4a$ *is given by*

$$\left\{ z_2^2 = -z_1^2 + 4a, z_3^2 = -2z_1^2 + 4a + b \right\}, \quad \left\{ x^2 = z_1, y^2 = z_2, xy = z_3 \right\},$$

as a double cover of $\mathcal{E}$ *ramified at the four points where* $z_i = \infty$. *Consequently, the invariant surface completes into a double cover of an elliptic curve ramified at the points where the variables blow up. This example corresponds to definition* (i), (iii) *and we shall see later more complicated examples but very interesting problems. Consider finally the change of variable :*

$$z_1 = \frac{1}{2}(m_2 - m_1), \quad z_2 = \frac{1}{2}(m_1 + m_2), \quad z_3 = m_3.$$

Taking the derivative and using the differential equations above for z_1, z_2, z_3, *leads to the following system of differential equations :*

$$\dot{m}_1 = -2m_2 m_3, \qquad \dot{m}_2 = 2m_1 m_3, \qquad \dot{m}_3 = m_1 m_2.$$

We see the resemblance with the equations of the Euler rigid body motion.

It was shown in series of publications [1, 42, 43], that θ-divisor can serve as a carrier of integrability. Let $\mathcal{H}$ be a hyperelliptic curve of genus g and $\mathrm{Jac}(\mathcal{H})=\mathbb{C}^g/\Lambda$ its Jacobian variety where Λ is a lattice of maximal rank in $\mathbb{C}^g$. Let

$$\mathcal{A}_k:\mathrm{Sym}^k(\mathcal{H})\longrightarrow \mathrm{Jac}(\mathcal{H}),(P_1,...,P_k)\longmapsto\sum_{j=1}^k\int_\infty^{P_j}(\omega_1,...,\omega_g)\mathrm{mod}.\Lambda, 0\leq k\leq g,$$

be the Abel map where $(\omega_1,...,\omega_g)$ is a canonical basis of the space of differentials of the first kind on $\mathcal{H}$. The theta divisor Θ is a subvariety of $\mathrm{Jac}(\mathcal{H})$ defined as $\Theta\equiv\mathcal{A}\left[\mathrm{Sym}^{g-1}(\mathcal{H})\right]/\Lambda$. By Θ_k we will denote the subvariety (strata) of $\mathrm{Jac}(\mathcal{H})$ defined by $\Theta_k\equiv\mathcal{A}_k\left[\mathrm{Sym}^k(\mathcal{H})\right]/\Lambda$ and we have the stratification

$$\{O\}\subset\Theta_0\subset\Theta_1\subset\Theta_2\subset ...\subset\Theta_{g-1}\subset\Theta_g=\mathrm{Jac}(\mathcal{H}),$$

where O is the origin of $\mathrm{Jac}(\mathcal{H})$. It was shown in [42], that these stratifications of the Jacobian are connected with stratifications of the Sato Grassmannian, via an extension of Krichever's map and some remarks on the relation between Laurent solutions for the Master systems and stratifications of the Jacobian of a hyperelliptic curve. One find in [43] a study about Lie-Poisson structure in the Jacobian which indicates that invariant manifolds associated with Poisson brackets can be identified with these strata. Some problems were considered in [43] and [1], where a connection was established with the flows on these strata. Such varieties or their open subsets often appear as coverings of complex invariants manifolds of finite dimensional integrable systems (Hénon-Heiles and Neumann systems).

Let us consider the Ramani-Dorizzi-Grammaticos (RDG) series of integrable potentials [37, 20] :

$$V(x,y)=\sum_{k=0}^{[m/2]}2^{m-2i}\left(\begin{array}{c}m-i\\ i\end{array}\right)x^{2i}y^{m-2i},\quad m=1,2,...$$

It can be straightforwardly proven that a Hamiltonian H :

$$H=\frac{1}{2}(p_x^2+p_y^2)+\alpha_m V_m,,\quad m=1,2,...$$

containing V is Liouville integrable, with an additional first integral :

$$F=p_x(xp_y-yp_x)+\alpha_m x^2V_{m-1},\quad m=1,2,...$$

The study of cases $m=1$ and $m=2$ is easy. The study of other cases is not obvious. For the case $m=3$, one obtains the Hénon-Heiles system we will see in section 3. The case $m=4$, corresponds the system that will be studied in section 4. However, the case $m=5$, corresponds to a system with an Hamiltonian of the form

$$H=\frac{1}{2}(p_x^2+p_y^2)+y^5+x^2y^3+\frac{3}{16}x^4y.$$

The corresponding Hamiltonian system admits a second first integral :

$$F = -p_x^2 y + p_x p_y x - \frac{1}{2}x^2y^4 + \frac{3}{8}x^4y^2 + \frac{1}{32}x^6,$$

and admits three 3-dimensional families solutions x, y, which are Laurent series of $t^{1/3}$: $x = at^{-\frac{1}{3}}$, $x = bt^{-\frac{2}{3}}$, $b^3 = -\frac{2}{9}$, but for which there are no polynomial P such that $P(x(t), y(t), \dot{x}(t), \dot{y}(t))$ is Laurent series in t.

We introduce a practical method for generating some new integrable systems from known ones. For the algebraic integrable systems in the generalized sense, the Laurent series solutions contain square root terms of the type $t^{-1/n}$ which are strictly not allowed by the Painlevé test (i.e. the general solutions should have no movable singularities other than poles in the complex plane). However, for some problems these terms are trivially removed by introducing some new variables, which restores the Painlevé property to the system. By inspection of the Laurent solutions of the algebraic integrable systems in the generalized sense, we look for polynomials in the variables defining these systems, without fractional exponents. In fact, for many problems, obtaining these new variables is not a problem, just use (by simple inspection) the first terms of the Laurent solutions. These new variables belong to the space $\mathcal{L}(\mathcal{D})$ where $\mathcal{D}$ is a divisor on a Abelian variety T^n which completes the affine defined by the intersection of the invariants of the new algebraically completely integrable system. In all the problems we have studied, we find that the known algebraically integrable systems in the generalized sense are part of new algebraically integrable systems.

Let

$$\dot{x} = J\frac{\partial H}{\partial x}, \quad x \in \mathbb{C}^m,$$

be an algebraically integrable system in the generalized sense. The Laurent series solutions of this system contain fractional exponents and the manifolds invariant by the complex flows are coverings of Abelian varieties. We might conjecture (with some additional conditions to be determined) from the problems discussed further that this system is part of a new algebraically integrable system in $m+1$ variables. In other words, there is a new algebraically integrable system

$$\dot{z} = J\frac{\partial \mathbf{H}}{\partial z}, \quad z \in \mathbb{C}^{m+1},$$

i.e., whose solutions expressible in terms of theta functions are associated with an Abelian variety with divisor on it and the Hamiltonian flows are linear on this Abelian variety.

2 A 4-dimensional integrable system in the generalized sense

We consider the following Hamiltonian [21],

$$H_1 \equiv H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{a}{2}(q_1^2 + 4q_2^2) + \frac{1}{4}q_1^4 + 4q_2^4 + 3q_1^2 q_2^2, \tag{2.1}$$

($a =$ constant), the corresponding system, i.e.,

$$\ddot{q}_1 = -(a + q_1^2 + 6q_2^2)q_1, \qquad \ddot{q}_2 = -2(2a + 3q_1^2 + 8q_2^2)q_2, \tag{2.2}$$

is integrable, the second integral is

$$H_2 = aq_1^2 q_2 + q_1^4 q_2 + 2q_1^2 q_2^3 - q_2 p_1^2 + q_1 p_1 p_2. \tag{2.3}$$

Recall that a system $\dot{z} = f(z)$ is weight-homogeneous with a weight ν_k going with each variable z_k if $f_k(\lambda^{\nu_i} z_1, \ldots, \lambda^{\nu_m} z_m) = \lambda^{\nu_k+1} f_k' z_1, \ldots, z_m)$, for all $\lambda \in \mathbb{C}$. The system (2.2) is weight-homogeneous with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that H_1 and H_2 have weight 4 and 5 respectively. When one examines all possible singularities of this system, one finds that it possible for the variable q_1 to contain square root terms of the type $\sqrt{t}$.

Theorem 3. *a) The system (2.2) possesses 3-dimensional family of Laurent solutions*

$$(q_1, q_2, p_1, p_2) = \left(t^{-1/2}, t^{-1}, t^{-3/2}, t^{-2}\right) \times \text{ a Taylor series,}$$

depending on three free parameters u, v and w.

b) Let $\mathcal{A}$ be the invariant surface defined by the two constants of motion

$$\mathcal{A} = \bigcap_{k=1}^{2} \{z = (q_1, q_2, p_1, p_2) \in \mathbb{C}^4 : H_k(z) = b_k\}, \tag{2.4}$$

for generic $(b_1, b_2) \in \mathbb{C}^2$. These Laurent solutions restricted to the surface $\mathcal{A}$ (2.4) are parameterized by two smooth curves $\mathcal{C}_{\varepsilon=\pm i}$ (2.6) of genus 4.

c) The system of differential equations (2.2) can be written as follows

$$\frac{ds_1}{\sqrt{P_6(s_1)}} - \frac{ds_2}{\sqrt{P_6(s_2)}} = 0, \qquad \frac{s_1 ds_1}{\sqrt{P_6(s_1)}} - \frac{s_2 ds_2}{\sqrt{P_6(s_2)}} = dt,$$

where $P_6(s)$ is a polynomial of degree 6 of the form

$$P_6(s) = s\left(-8s^5 - 4as^3 + 2b_1 s + b_2\right),$$

and the flow can be linearized in terms of genus 2 hyperelliptic functions.

Proof. *a*) The first fact to observe is that if the system is to have Laurent solutions depending on four free parameters, the Laurent decomposition of such asymptotic solutions must have the following form

$$
\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots\right),\\
q_2 &= \frac{1}{t}\left(b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + \cdots\right),
\end{aligned}
$$

and $p_1 = \dot{q}_1$, $p_2 = \dot{q}_2$. Putting these expansions into (2.2), solving inductively for the $q_k^{(j)} (k = 1, 2)$, one finds at the 0^{th} step a free parameter u, at the 2^{th} step a second free parameter v and the remaining one w at the 4^{th} step. There are precisely two such families, labelled by $\varepsilon = \pm i$, and they are explicitly given as follows

$$
\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}(u - \frac{1}{2}u^3 t + v t^2 + u^2(-\frac{11}{16}u^5 + \frac{1}{3}au + v)t^3\\
&\quad + \frac{u}{4}(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3 v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w)t^4 + \cdots),\\
q_2 &= \frac{\varepsilon\sqrt{2}}{4t}(1 + u^2 t + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{8}u(24v - u^5)t^3 - 2\varepsilon\sqrt{2}wt^4 + \cdots),\\
p_1 &= \frac{1}{t\sqrt{t}}(-\frac{1}{2}u - \frac{1}{4}u^3 t + \frac{3}{2}vt^2 + \frac{5}{2}u^2(-\frac{11}{16}u^5 + \frac{1}{3}au + v)t^3\\
&\quad + \frac{7u}{8}(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3 v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w)t^4 + \cdots),\\
p_2 &= \frac{\varepsilon\sqrt{2}}{4t^2}(-1 + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{4}u(24v - u^5)t^3 - 6\varepsilon\sqrt{2}wt^4 + \cdots).
\end{aligned}
\tag{2.5}
$$

The formal series solutions (2.5) are convergent as a consequence of the majorant method.

b) By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter w linearly, leading to an equation connecting the two remaining parameters u and v :

$$
2v^2 + \frac{1}{6}\left(15u^4 - 8a\right)uv - \frac{39}{32}u^{10} + \frac{7}{6}au^6 + \frac{2}{9}\left(a^2 + 9b_1\right)u^2 - \varepsilon\sqrt{2}b_2 = 0, \tag{2.6}
$$

this defines two smooth curves $\mathcal{C}_\varepsilon$ $(\varepsilon = \pm i)$. The curve $\mathcal{C}_\varepsilon$ has 10 branch points given by the solution of the equation :

$$
\frac{39}{64}u^{10} - \frac{7}{12}au^6 - \frac{1}{9}a^2u^2 - b_1u^2 + \frac{1}{2}\varepsilon\sqrt{2}b_2 = 0.
$$

According to Hurwitz' formula, the genus g of $\mathcal{C}_\varepsilon$ is $g = -2 + 1 + \frac{10}{2} = 4$.

c) We set

$$
q_2 = s_1 + s_2, \quad q_1^2 = -4s_1s_2, \quad p_2 = \dot{s}_1 + \dot{s}_2, \quad q_1p_1 = -2(\dot{s}_1s_2 + s_1\dot{s}_2).
$$

The latter equation together with the second implies that

$$p_1^2 = -\frac{(\dot{s}_1 s_2 + s_1 \dot{s}_2)^2}{s_1 s_2}.$$

In term of these new variables, the above differential equations take the following form

$$\begin{aligned}
&(s_1 - s_2)\left(s_2(\dot{s}_1)^2 - s_1(\dot{s}_2)^2\right) - 2b_1 s_1 s_2 \\
&\qquad +4s_1 s_2\left(2s_1^4 + 2s_1^3 s_2 + 2s_1^2 s_2^2 + 2s_1 s_2^3 + 2s_2^4 + as_1^2 + as_1 s_2 + as_2^2\right) = 0, \\
&(s_1 - s_2)\left(s_2^2(\dot{s}_1)^2 - s_1^2(\dot{s}_2)^2\right) + 4s_1^2 s_2^2\,(s_1 + s_2)\left(a + 2s_1^2 + 2s_2^2\right) + b_2 s_1 s_2 = 0.
\end{aligned}$$

These equations are solved linearly for $(\dot{s}_1)^2$ and $(\dot{s}_2)^2$ as

$$(\dot{s}_1)^2 = \frac{s_1\left(-8s_1^5 - 4as_1^3 + 2b_1 s_1 + b_2\right)}{(s_1 - s_2)^2}, \quad (\dot{s}_2)^2 = \frac{s_2\left(-8s_2^5 - 4as_2^3 + 2b_1 s_2 + b_2\right)}{(s_1 - s_2)^2},$$

which leads immediately to the following equations for s_1 and s_2 :

$$\dot{s}_1 = \frac{ds_1}{dt} = \frac{\sqrt{P_6(s_1)}}{s_1 - s_2}, \qquad \dot{s}_2 = \frac{ds_2}{dt} = \frac{\sqrt{P_6(s_2)}}{s_2 - s_1},$$

where $P_6(s) \equiv s(-8s^5 - 4as^3 + 2b_1 s + b_2)$. These equations can be integrated by the Abelian mapping

$$\mathcal{H} \longrightarrow Jac(\mathcal{H}) = \mathbb{C}^2/\Lambda, \quad (p_1, p_2) \longmapsto (\xi_1, \xi_2),$$

where the hyperelliptic curve $\mathcal{H}$ of genus 2 is given by $\zeta^2 = P_6(s)$, Λ is the lattice generated by the vectors $n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2, \Omega$ is the matrix of period of the curve $\mathcal{H}$, $p_1 = (s_1, \sqrt{P_6(s_1)})$, $p_2 = (s_2, \sqrt{P_6(s_2)})$,

$$\xi_1 = \int_{p_0}^{p_1} \omega_1 + \int_{p_0}^{p_2} \omega_1, \qquad \xi_2 = \int_{p_0}^{p_1} \omega_2 + \int_{p_0}^{p_2} \omega_2,$$

where p_0 is a fixed point and (ω_1, ω_2) is a canonical basis of holomorphic differentials on $\mathcal{H}$, i.e.,

$$\omega_1 = \frac{ds}{\sqrt{P_6(s)}}, \qquad \omega_2 = \frac{sds}{\sqrt{P_6(s)}}.$$

We have

$$\frac{ds_1}{\sqrt{P_6(s_1)}} - \frac{ds_2}{\sqrt{P_6(s_2)}} = 0, \qquad \frac{s_1 ds_1}{\sqrt{P_6(s_1)}} - \frac{s_2 ds_2}{\sqrt{P_6(s_2)}} = dt,$$

and hence the problem can be integrated in terms of genus 2 hyperelliptic functions of time. This ends the proof of the theorem. □

We will now embed the system (2.2) in a system of five equations in five unknowns having three quartic invariants. We will show how to explicitly construct this integrable system in five unknowns $(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5$ from the above integrable system in $(q_1, q_2, p_1, p_2) \in \mathbb{C}^4$. This system admits three quartic invariants and it is described how the invariant variety corresponding to fixed generic values of these invariants is compactified in an Abelian surface. On the zero level of some of these invariants the system reduces to the natural mechanical system (2.2). We have seen that the asymptotic solutions of the system (2.2) contain square root terms of the type $t^{1/n}$, which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing some new variables $z_1, \ldots, z_5$, which restores the Painlevé property to the system. Indeed, let

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C}^5, (q_1, q_2, p_1, p_2) \longmapsto (z_1, z_2, z_3, z_4, z_5), \tag{2.7}$$

be a morphism on the affine variety $\mathcal{A}$ (2.4) where $z_1, \ldots, z_5$ are defined as

$$z_1 = q_1^2, \quad z_2 = q_2, \quad z_3 = p_2, \quad z_4 = q_1 p_1, \quad z_5 = 2q_1^2 q_2^2 + p_1^2.$$

Obtaining these new variables is not a problem, just use the first terms of the Laurent solutions (2.5). The morphism (2.7) maps the vector field (2.2) into the system

$$\begin{array}{rclr}
\dot{z}_1 & = & 2z_4, & \dot{z}_3 = -4az_2 - 6z_1 z_2 - 16z_2^3, \\
\dot{z}_2 & = & z_3, & \dot{z}_4 = -az_1 - z_1^2 - 8z_1 z_2^2 + z_5, \\
\dot{z}_5 & = & -8z_2^2 z_4 - 2az_4 - 2z_1 z_4 + 4z_1 z_2 z_3, &
\end{array} \tag{2.8}$$

in five unknowns having three quartic invariants

$$\begin{array}{rcl}
F_1 & = & \frac{1}{2}z_5 + 2z_1 z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}az_1 + 2az_2^2 + \frac{1}{4}z_1^2 + 4z_2^4, \\
F_2 & = & az_1 z_2 + z_1^2 z_2 + 4z_1 z_2^3 - z_2 z_5 + z_3 z_4, \\
F_3 & = & z_1 z_5 - 2z_1^2 z_2^2 - z_4^2.
\end{array} \tag{2.9}$$

To obtain rapidly these three first integrals, just use the two first integrals H_1 (2.1), H_2 (2.3) and differential equations (2.2). This system is completely integrable and the Hamiltonian structure is defined by the Poisson bracket :

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^{5} J_{kl} \frac{\partial F}{\partial z_k} \frac{\partial H}{\partial z_l},$$

where $\frac{\partial H}{\partial z} = \left(\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3}, \frac{\partial H}{\partial z_4}, \frac{\partial H}{\partial z_5} \right)^\top$, and

$$J = \begin{pmatrix}
0 & 0 & 0 & 2z_1 & 4z_4 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & -4z_1 z_2 \\
-2z_1 & 0 & 0 & 0 & 2z_5 - 8z_1 z_2^2 \\
-4z_4 & 0 & 4z_1 z_2 & -2z_5 + 8z_1 z_2^2 & 0
\end{pmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The system (2.8) can be written as

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

where $H = F_1$. The second flow commuting with the first is regulated by the equations $\dot{z} = J\frac{\partial F_2}{\partial z}$, $z = (z_1, z_2, z_3, z_4, z_5)^\top$. These vector fields are in involution, i.e., $\{F_1, F_2\} = 0$, and the remaining one is Casimir, i.e., $J\frac{\partial F_3}{\partial z} = 0$. Therefore, we have the following result :

Theorem 4. *The system* (2.8) *possesses three quartic invariants* (2.9) *and is completely integrable in the sense of Liouville.*

Let $\mathcal{B}$ be the complex affine variety defined by

$$\mathcal{B} = \bigcap_{k=1}^{2} \{z : F_k(z) = c_k\} \subset \mathbb{C}^5, \tag{2.10}$$

for generic $(c_1, c_2, c_3) \in \mathbb{C}^3$.

Theorem 5. *The affine variety $\mathcal{B}$* (2.10) *defined by putting these invariants equal to generic constants, is a double cover of a Kummer surface* (2.11). *The system* (2.8) *can be integrated in genus* 2 *hyperelliptic functions.*

Proof. Note that $\sigma : (z_1, z_2, z_3, z_4, z_5) \longmapsto (z_1, z_2, -z_3, -z_4, z_5)$, is an involution on $\mathcal{B}$. The quotient $\mathcal{B}/\sigma$ is a Kummer surface defined by

$$p(z_1, z_2)\, z_5^2 + q(z_1, z_2)\, z_5 + r(z_1, z_2) = 0, \tag{2.11}$$

where

$$\begin{aligned}
p(z_1, z_2) &= z_2^2 + z_1, \\
q(z_1, z_2) &= \frac{1}{2}z_1^3 + 2az_1z_2^2 + az_1^2 - 2c_1z_1 + 2c_2z_2 - c_3, \\
r(z_1, z_2) &= -8c_3z_2^4 + \left(a^2 + 4c_1\right) z_1^2z_2^2 - 8c_2z_1z_2^3 - 2c_2z_1^2z_2 - 4c_3z_1z_2^2 \\
&\quad -\frac{1}{2}c_3z_1^2 - 4ac_3z_2^2 - 2ac_2z_1z_2 - ac_3z_1 + c_2^2 + 2c_1c_3.
\end{aligned}$$

Using $F_1 = c_1$, we have

$$z_5 = 2c_1 - 4z_1z_2^2 - z_3^2 - az_1 - 4az_2^2 - \frac{1}{2}z_1^2 - 8z_2^4,$$

and substituting this into $F_2 = c_2$, $F_3 = c_3$, (2.9) yields the system

$$\begin{aligned}
2az_1z_2 + \frac{3}{2}z_1^2z_2 + 8z_1z_2^3 - 2c_1z_2 + z_2z_3^2 + 4az_2^3 + 8z_2^5 + z_3z_4 &= c_2, \\
2c_1z_1 - 6z_1^2z_2^2 - z_1z_3^2 - az_1^2 - 4az_1z_2^2 - \frac{1}{2}z_1^3 - 8z_1z_2^4 - z_4^2 &= c_3.
\end{aligned}$$

We introduce two coordinates s_1, s_2 as follows

$$z_1 = -4s_1s_2, \quad z_2 = s_1 + s_2, \quad z_3 = \dot{s}_1 + \dot{s}_2, \quad z_4 = -2\left(\dot{s}_1 s_2 + s_1\dot{s}_2\right).$$

Upon substituting this parametrization, the above system turns into

$$\begin{aligned}
&(s_1 - s_2)\left((\dot{s}_1)^2 - (\dot{s}_2)^2\right) + 8(s_1 + s_2)\left(s_1^4 + s_2^4 + s_1^2 s_2^2\right) \\
&+4a(s_1 + s_2)\left(s_1^2 + s_2^2\right) - 2c_1(s_1 + s_2) - c_2 = 0, \\
&(s_1 - s_2)\left(s_2(\dot{s}_1)^2 - s_1(\dot{s}_2)^2\right) + 32s_1s_2\left(s_1^4 + s_2^4 + s_1^2 s_2^2\right) \\
&+32s_1^2s_2^2\left(s_1^2 + s_2^2\right) + 16as_1s_2\left(s_1^2 + s_2^2\right) + 16as_1^2s_2^2 - 8c_1s_1s_2 - c_3 = 0.
\end{aligned}$$

These equations are solved linearly for $\dot{s}_1^2$ and $\dot{s}_2^2$ as

$$\begin{aligned}
(\dot{s}_1)^2 &= \frac{-32s_1^6 - 16as_1^4 + 8c_1s_1^2 + 4c_2s_1 - c_3}{4(s_2 - s_1)^2}, \\
(\dot{s}_2)^2 &= \frac{-32s_2^6 - 16as_2^4 + 8c_1s_2^2 + 4c_2s_2 - c_3}{4(s_2 - s_1)^2},
\end{aligned}$$

and can be integrated by means of the Abel map $\mathcal{H} \longrightarrow Jac(\mathcal{H})$, where the hyperelliptic curve $\mathcal{H}$ of genus 2 is given by an equation

$$w^2 = -32s^6 - 16as^4 + 8c_1s^2 + 4c_2s - c_3.$$

This completes the proof. □

Theorem 6. *The system (2.8) possesses Laurent series solutions which depend on 4 free parameters : α, β, γ and θ,*

$$\begin{aligned}
z_1 &= \frac{1}{t}(\alpha - \alpha^2 t + \beta t^2 + \frac{1}{6}\alpha(3\beta - 9\alpha^3 + 4a\alpha)t^3 + \gamma t^4 + \cdots), \\
z_2 &= \frac{\varepsilon\sqrt{2}}{4t}(1 + \alpha t + \frac{1}{3}(-3\alpha^2 + 2a)t^2 + \frac{1}{2}(3\beta - \alpha^3)t^3 - 2\varepsilon\sqrt{2}\theta t^4 + \cdots), \\
z_3 &= \frac{\varepsilon\sqrt{2}}{4t^2}(-1 + \frac{1}{3}(-3\alpha^2 + 2a)t^2 + (3\beta - \alpha^3)t^3 - 6\varepsilon\sqrt{2}\theta t^4 + \cdots), \qquad (2.12) \\
z_4 &= \frac{1}{2t^2}(-\alpha + \beta t^2 + \frac{1}{3}\alpha(3\beta - 9\alpha^3 + 4a\alpha)t^3 + 3\gamma t^4 + \cdots), \\
z_5 &= \frac{1}{t}(-\frac{1}{3}a\alpha + \alpha^3 - \beta + (3\alpha^4 - a\alpha^2 - 3\alpha\beta)t \\
&\quad +(4\varepsilon\sqrt{2}\alpha\theta + 2\gamma + \frac{8}{3}a\alpha^3 - \frac{1}{3}a\beta - \alpha^2\beta - 3\alpha^5 - \frac{4}{9}a^2\alpha)t^2 + \cdots),
\end{aligned}$$

with $\varepsilon = \pm i$. These meromorphic solutions restricted to the invariant surface $\mathcal{B}$ (2.10) are parameterized by two isomorphic hyperelliptic curves $\mathcal{H}_{\varepsilon=\pm i}$ of genus 2 :

$$\beta^2 + \frac{2}{3}\left(3\alpha^2 - 2a\right)\alpha\beta - 3\alpha^6 + \frac{8}{3}a\alpha^4 + \frac{4}{9}\left(a^2 + 9c_1\right)\alpha^2 - 2\varepsilon\sqrt{2}c_2\alpha + c_3 = 0, \quad (2.13)$$

Proof. The first fact to observe is that if the system is to have Laurent solutions depending on four free parameters, the Laurent decomposition of such asymptotic solutions must have the following form

$$z_1 = \sum_{j=0}^{\infty} z_1^{(j)} t^{j-1}, \qquad z_2 = \sum_{j=0}^{\infty} z_2^{(j)} t^{j-2}, \qquad z_5 = \sum_{j=0}^{\infty} z_5^{(j)} t^{j-3,}$$

and $z_3 = \dot{z}_2$, $z_4 = \dfrac{\dot{z}_1}{2}$. Putting these expansions into

$$\begin{aligned} \ddot{z}_1 &= -2az_1 - 2z_1^2 - 16z_1z_2^2 + 2z_5, \\ \ddot{z}_2 &= -4az_2 - 6z_1z_2 - 16z_2^3, \\ \dot{z}_5 &= -8z_2^2z_4 - 2az_4 - 2z_1z_4 + 4z_1z_2z_3, \end{aligned}$$

deduced from (2.2), solving inductively for the $z_k^{(j)} (k = 1, 2, 5)$, one finds at the 0^{th} step (resp. 2^{th} step) a free parameter α (resp. β) and the two remaining ones γ, θ at the 4^{th} step. More precisely, we have the solutions (2.12) with $\varepsilon = \pm i$. Using the majorant method, we can show that the formal Laurent series solutions are convergent. Substituting the solutions (2.12) into $F_1 = c_1$, $F_2 = c_2$ and $F_3 = c_3$, and equating the t^0-terms yields

$$\begin{aligned} F_1 &= \frac{15}{8}\alpha^4 - \frac{5}{6}a\alpha^2 - \frac{5}{4}\alpha\beta - \frac{7}{36}a^2 - \frac{5}{4}\varepsilon\sqrt{2}\theta = c_1, \\ F_2 &= \varepsilon\sqrt{2}(\frac{1}{4}\alpha^5 - \gamma + \frac{\varepsilon\sqrt{2}}{2}\alpha\theta - \frac{2}{3}a\alpha^3 + \frac{1}{3}a\beta + \frac{1}{6}a^2 + \frac{1}{2}\alpha^2\beta) = c_2, \\ F_3 &= -\frac{11}{2}\alpha^6 - \beta^2 + 4\alpha\gamma + 3\alpha^2\varepsilon\sqrt{2}\theta + \alpha^3\beta - \frac{1}{3}a^2\alpha^2 + \frac{10}{3}a\alpha^4 = c_3. \end{aligned}$$

Eliminating γ and θ from these equations, leads to the equation (2.13) connecting the two remaining parameters α and β. According to Hurwitz's formula, this defines two isomorphic smooth hyperelliptic curves $\mathcal{H}_\varepsilon$ ($\varepsilon = \pm i$) of genus 2, which finishes the proof of the theorem. □

In order to embed $\mathcal{H}_\varepsilon$ into some projective space, one of the key underlying principles used is the Kodaira embedding theorem [15, 34], which states that a smooth complex manifold can be smoothly embedded into projective space $\mathbb{P}^N(\mathbb{C})$ with the set of functions having a pole of order k along positive divisor on the manifold, provided k is large enough; fortunately, for Abelian varieties, k need not be larger than three according to Lefshetz [15, 34]. These functions are easily constructed from the Laurent solutions (2.12) by looking for polynomials in the phase variables which in the expansions have at most a k-fold pole. The nature of the expansions and some algebraic proprieties of Abelian varieties provide a recipe for when to terminate our search for such functions, thus making the procedure implementable. Precisely, we wish to find a set of polynomial functions $\{f_0, \ldots, f_N\}$, of increasing degree in the original variables $z_1, \ldots, z_5$, having the property that the embedding $\mathcal{D}$ of $\mathcal{H}_i + \mathcal{H}_{-i}$ into $\mathbb{P}^N$ via those functions satisfies the relation: geometric genus $(2\mathcal{D}) \equiv g(2\mathcal{D}) = N + 2$. A this point, it may be not so clear why the curve $\mathcal{D}$ must really live on an Abelian surface. Let us say, for the moment, that the equations of the divisor $\mathcal{D}$ (i.e., the place where the solutions blow up), as a curve traced on the Abelian surface $\widetilde{\mathcal{B}}$ (to be constructed below), must be understood as relations connecting the free parameters as they appear firstly in the expansions (2.12). In the present situation, this means that (2.13) must be understood as relations connecting α and β. Let

$$L^{(r)} = \left\{ \begin{array}{l} \text{polynomials} \quad f = f(z, \ldots, z_5) \\ \text{of degre} \leq r, \quad \text{with at worst a} \\ \text{double pole along} \quad \mathcal{H}_i + \mathcal{H}_{-i} \\ \text{and with} \quad z_1, \ldots, z_5 \quad \text{as in (2.13)} \end{array} \right\} / [F_k = c_k, k = 1, 2, 3],$$

and let $(f_0, f_1, \ldots, f_{N_r})$ be a basis of $L^{(r)}$. We look for r such that : $g(2\mathcal{D}^{(r)}) = N_r + 2$, $2\mathcal{D}^{(r)} \subset \mathbb{P}^{N_r}(\mathbb{C})$. We shall show (theorem 7, b)) that it is unnecessary to go beyond $r = 4$.

Theorem 7. *a) The spaces $L^{(r)}$, nested according to weighted degree, are generated as*

$$L^{(1)} = \{f_0, f_1, f_2, f_3, f_4, f_5\}, \qquad L^{(2)} = L^{(1)} \oplus \{f_6, f_8, f_9, f_{10}, f_{11}, f_{12}\},$$
$$L^{(3)} = L^{(2)}, \qquad L^{(4)} = L^{(3)} \oplus \{f_{13}, f_{14}, f_{15}\},$$

where

$$f_0 = 1, \quad f_1 = z_1 = \frac{\alpha}{t} + \cdots, \quad f_2 = z_2 = \frac{\varepsilon\sqrt{2}}{t} + \cdots, \quad f_3 = z_3 = -\frac{\varepsilon\sqrt{2}}{4t^2} + \cdots,$$

$$f_4 = z_4 = -\frac{\alpha}{2t^2} + \cdots, \quad f_5 = z_5 = -\frac{\eta}{3t} + \cdots, \quad f_6 = z_1^2 = \frac{\alpha^2}{t^2} + \cdots,$$

$$f_7 = z_2^2 = -\frac{1}{8t^2} + \cdots, \quad f_8 = z_5^2 = \frac{\eta^2}{9t^2} + \cdots, \quad f_9 = z_1 z_2 = \frac{\varepsilon\sqrt{2}\alpha}{4t^2} + \cdots,$$

$$f_{10} = z_1 z_5 = -\frac{\alpha\eta}{3t^2} + \cdots, \quad f_{11} = z_2 z_5 = -\frac{\varepsilon\sqrt{2}\eta}{12t^2} + \cdots,$$

$$f_{12} = W(z_1, z_2) = -\frac{\varepsilon\sqrt{2}\alpha^2}{2t^2} + \cdots, \qquad f_{13} = W(z_1, z_5) = \frac{4\alpha^2\eta}{3t^2} + \cdots,$$

$$f_{14} = W(z_2, z_5) = \frac{\varepsilon\sqrt{2}\alpha\eta}{6t^2} + \cdots, \qquad f_{15} = (z_3 - 2\varepsilon\sqrt{2}z_2^2)^2 = -\frac{\alpha^2}{2t^2} + \cdots,$$

with $W(z_j, z_k) = \dot{z}_j z_k - z_j \dot{z}_k$ (Wronskian of z_k and z_j) and $\eta \equiv 3\beta - 3\alpha^3 + a\alpha$.

b) The space $L^{(4)}$ provides an embedding of $\mathcal{D}^{(4)}$ into projective space $\mathbb{P}^{15}$ and $\mathcal{D}^{(4)}$ (resp. $2\mathcal{D}^{(4)}$) has genus 5 (resp. 17).

Proof. The proof of $a)$ is straightforward and can be done by inspection of the expansions (2.12). Note also that the functions z_1, z_2, z_5 behave as $1/t$ and if we consider the derivatives of the ratios $\frac{z_1}{z_2}$, $\frac{z_1}{z_5}$, $\frac{z_2}{z_5}$, the Wronskians $W(z_1, z_2)$, $W(z_1, z_5)$, $W(z_2, z_5)$, must behave as $\frac{1}{t^2}$ since z_2^2, z_5^2 behave as $\frac{1}{t^2}$.

$b)$ Note that $\dim L^{(1)} = 6$, $\dim L^{(2)} = dim L^{(3)} = 13$, $\dim L^{(4)} = 16$. It turns out that neither $L^{(1)}$, nor $L^{(2)}$, nor $L^{(3)}$, yields a curve of the right genus; in fact $g(2\mathcal{D}^{(r)}) \neq \dim L^{(r)} + 1$, $r = 1, 2, 3$. For instance, the embedding into $\mathbb{P}^5(\mathbb{C})$ via $L^{(1)}$ does not separate the sheets, so we proceed to $L^{(2)}$ and we consider the corresponding embedding into $\mathbb{P}^{12}(\mathbb{C})$. For finite values of α and β, the curves $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are disjoint; dividing the vector $(f_0, \ldots, f_{12})$ by f_7 and taking the limit $t \to 0$, to yield

$$\left[0:0:0:2\varepsilon\sqrt{2}:4\alpha:0:-8\alpha^2:1:-\frac{8}{9}\eta^2:-2\varepsilon\sqrt{2}\alpha:\frac{8}{3}\alpha\eta:\frac{2\varepsilon\sqrt{2}}{3}\eta:4\varepsilon\sqrt{2}\alpha^2\right].$$

The curve (2.13) has two points covering $\alpha = \infty$, at which $\eta \equiv 3\beta - 3\alpha^3 + a\alpha$ behaves as follows :

$$\begin{aligned}\eta &= -6\alpha^3 + 3a\alpha \pm 3\sqrt{4\alpha^6 - 4a\alpha^4 - 4c_1\alpha^2 + 2\varepsilon\sqrt{2}c_2\alpha - c_3},\\ &= \begin{cases} -\frac{3(a^2+4c_1)}{4\alpha} + \text{lower order terms, picking the + sign,} \\ -12\alpha^3 + o(\alpha), \text{ picking the - sign.}\end{cases}\end{aligned}$$

By picking the - sign and by dividing the vector $(f_0, \ldots, f_{12})$ by f_8, the corresponding point is mapped into the point

$$[0:0:0:0:0:0:0:0:1:0:0:0:0],$$

in $\mathbb{P}^{12}(\mathbb{C})$ which is independent of ε, whereas picking the + sign leads to two different points, according to the sign of ε. Thus, adding at least 2 to the genus of each curve, so that $g(2\mathcal{D}^{(2)}) - 2 > 12$, $2\mathcal{D}^{(2)} \subset \mathbb{P}^{12}(\mathbb{C}) \neq \mathbb{P}^{g-2}(\mathbb{C})$, which contradicts the fact that $N_r = g(2\mathcal{D}^{(2)}) - 2$. The embedding via $L^{(2)}$ (or $L^{(3)}$) is unacceptable as well. Consider now the embedding $2\mathcal{D}^{(4)}$ into $\mathbb{P}^{15}(\mathbb{C})$ using the 16 functions $f_0, \ldots, f_{15}$ of $L^{(4)}$. It is easily seen that these functions separate all points of the curve (except perhaps for the points at ∞) : The curves $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are disjoint for finite values of α and β; dividing the vector $(f_0, \ldots, f_{15})$ by f_7 and taking the limit $t \to 0$, to yield

$$\left[0:0:0:2\varepsilon\sqrt{2}:4\alpha:0:-8\alpha^2:1:-\frac{8}{9}\eta^2:-2\varepsilon\sqrt{2}\alpha:\frac{8}{3}\alpha\eta:\frac{2\varepsilon\sqrt{2}}{3}\eta:\right.$$
$$\left.4\varepsilon\sqrt{2}\alpha^2:-\frac{32}{3}\alpha^2\eta:-\frac{4\varepsilon\sqrt{2}}{3}\alpha\eta:4\alpha^2\right].$$

About the point $\alpha = \infty$, it is appropriate to divide by g_8; then by picking the sign - in η above, the corresponding point is mapped into the point

$$[0:0:0:0:0:0:0:0:1:0:0:0:0:0:0:0],$$

in $\mathbb{P}^{15}(\mathbb{C})$ which is independent of ε, whereas picking the + sign leads to two different points, according to the sign of ε. Hence, the divisor $\mathcal{D}^{(4)}$ obtained in this way has genus 5 and thus $g(2\mathcal{D}^{(4)})$ has genus 17 and $2\mathcal{D}^{(4)} \subset \mathbb{P}^{15}(\mathbb{C}) = \mathbb{P}^{g-2}(\mathbb{C})$, as desired. This ends the proof of the theorem. □

Let $L = L^{(4)}$, $\mathcal{D} = \mathcal{D}^{(4)}$ and $\mathcal{S} = 2\mathcal{D}^{(4)} \subset \mathbb{P}^{15}(\mathbb{C})$. Next we wish to construct a surface strip around $\mathcal{S}$ which will support the commuting vector fields. In fact, $\mathcal{S}$ has a good chance to be very ample divisor on an Abelian surface, still to be constructed.

Theorem 8. *The variety $\mathcal{B}$ (2.10) generically is the affine part of an Abelian surface $\widetilde{\mathcal{B}}$, more precisely the Jacobian of a genus 2 curve. The reduced divisor at infinity $\widetilde{\mathcal{B}}\backslash\mathcal{B} = \mathcal{H}_i + \mathcal{H}_{-i}$, consists of two smooth isomorphic genus 2 curves $\mathcal{H}_\varepsilon$ (2.13). The system of differential equations (2.8) is algebraic complete integrable and the corresponding flows evolve on $\widetilde{\mathcal{B}}$.*

Proof. We need to attaches the affine part of the intersection of the three invariants (2.9) so as to obtain a smooth compact connected surface in $\mathbb{P}^{15}(\mathbb{C})$. We will repeat the same reasoning used previously. To be precise, the orbits of the vector field (2.8) running through $\mathcal{S}$ form a smooth surface Σ near $\mathcal{S}$ such that $\Sigma\backslash\mathcal{B} \subseteq \widetilde{\mathcal{B}}$ and the variety $\widetilde{\mathcal{B}} = \mathcal{B} \cup \Sigma$ is smooth, compact and connected. Indeed, let

$$\psi(t,p) = \{z(t) = (z_1(t), \ldots, z_5(t)) : t \in \mathbb{C}, 0 < |t| < \varepsilon\},$$

be the orbit of the vector field (2.8) going through the point $p \in \mathcal{S}$. Let $\Sigma_p \subset \mathbb{P}^{15}(\mathbb{C})$ be the surface element formed by the divisor $\mathcal{S}$ and the orbits going through p, and set $\Sigma \equiv \cup_{p\in\mathcal{S}}\Sigma_p$. Consider the curve $\mathcal{S}' = \mathcal{H} \cap \Sigma$ where $\mathcal{H} \subset \mathbb{P}^{15}(\mathbb{C})$ is a hyperplane transversal to the direction of the flow. If $\mathcal{S}'$ is smooth, then using the implicit function theorem the surface Σ is smooth. But if $\mathcal{S}'$ is singular at 0, then Σ would be singular along the trajectory ($t-$axis) which go immediately into the affine part $\mathcal{B}$. Hence, $\mathcal{B}$ would be singular which is a contradiction because $\mathcal{B}$ is the fibre of a morphism from $\mathbb{C}^5$ to $\mathbb{C}^2$ and so smooth for almost all the three constants of the motion c_k. Next, let $\overline{\mathcal{B}}$ be the projective closure of B into $\mathbb{P}^5(\mathbb{C})$, let $Z = [Z_0 : Z_1 : \ldots : Z_5] \in \mathbb{P}^5(\mathbb{C})$ and let $I = \overline{\mathcal{B}} \cap \{Z_0 = 0\}$ be the locus at infinity. Consider the following map

$$\overline{\mathcal{B}} \subseteq \mathbb{P}^5(\mathbb{C}) \longrightarrow \mathbb{P}^{15}(\mathbb{C}), \quad Z \longmapsto f(Z),$$

where $f = (f_0, f_1, ..., f_{15}) \in L(\mathcal{S})$ and let $\widetilde{B} = f(\overline{\mathcal{B}})$. In a neighborhood $V(p) \subseteq \mathbb{P}^{15}(\mathbb{C})$ of p, we have $\Sigma_p = \widetilde{\mathcal{B}}$ and $\Sigma_p\backslash\mathcal{S} \subseteq \mathcal{B}$. Otherwise there would exist an element of surface $\Sigma_p' \subseteq \widetilde{\mathcal{B}}$ such that $\Sigma_p \cap \Sigma_p' = (t - \text{axis})$, orbit $\psi(t,p) = (t - axis) \backslash\, p \subseteq \mathcal{B}$, and hence $\mathcal{B}$ would be singular along the $t-$axis which is impossible. Since the variety $\overline{\mathcal{B}} \cap \{Z_0 \neq 0\}$ is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\overline{\mathcal{B}}$ is also irreducible, all hyperplane sections are connected and hence I is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{P}^5(\mathbb{C}) \times \mathbb{P}^{15}(\mathbb{C})$, of the map f, which is irreducible together with $\overline{\mathcal{B}}$. It follows from the irreducibility of I that a generic hyperplane section $\Gamma_f \cap \{\mathcal{H}_{gen.} \times \mathbb{P}^{15}(\mathbb{C})\}$ is irreducible, hence the special hyperplane section $\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}(\mathbb{C})\}$ is connected and the projection map

$$proj_{\mathbb{P}^{15}(\mathbb{C})}\{\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}(\mathbb{C})\}\} = f(I) \equiv \mathcal{S},$$

is connected. Hence, the variety $\mathcal{B} \cup \Sigma = \widetilde{\mathcal{B}}$ is compact, connected and embeds smoothly into $\mathbb{P}^{15}(\mathbb{C})$ via f. We wish to show that $\widetilde{\mathcal{B}}$ is an Abelian surface equipped with two everywhere independent commuting vector fields. Let ϕ^{τ_1} and ϕ^{τ_2} be the flows corresponding to vector fields X_{F_1} and X_{F_2}. The latter are generated respectively by F_1, F_2. For $p \in \mathcal{S}$ and for small $\varepsilon > 0$, $\phi^{\tau_1}(p), \forall \tau_1, 0 < |\tau_1| < \varepsilon$, is well defined and $\phi^{\tau_1}(p) \in \mathcal{B}$. Then we may define ϕ^{τ_2} on $\mathcal{B}$ by

$$\phi^{\tau_2}(q) = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q), \quad q \in U(p) = \phi^{-\tau_1}(U(\phi^{\tau_1}(p))),$$

where $U(p)$ is a neighborhood of p. By commutativity one can see that ϕ^{τ_2} is independent of τ_1;

$$\phi^{-\tau_1-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1+\varepsilon_1}(q) = \phi^{-\tau_1}\phi^{-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1}\phi^{\varepsilon_1}(q) = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q).$$

$\phi^{\tau_2}(q)$ is holomorphic away from $\mathcal{S}$. This because $\phi^{\tau_2}\phi^{\tau_1}(q)$ is holomorphic away from $\mathcal{S}$ and that ϕ^{τ_1} is holomorphic in $U(p)$ and maps bi-holomorphically $U(p)$ onto $U(\phi^{\tau_1}(p))$. Now, since the flows ϕ^{τ_1} and ϕ^{τ_2} are holomorphic and independent on $\mathcal{S}$, we can show along the same lines as in the Arnold-Liouville theorem [5] that $\widetilde{\mathcal{B}}$ is a complex torus $\mathbb{C}^2/lattice$ and so in particular $\widetilde{\mathcal{B}}$ is a Kähler variety. And that will done, by considering the local diffeomorphism $\mathbb{C}^2 \longrightarrow \widetilde{\mathcal{B}}$, $(\tau_1, \tau_2) \longmapsto \phi^{\tau_1}\phi^{\tau_2}(p)$, for a fixed origin $p \in \mathcal{B}$. The additive subgroup $\{(\tau_1, \tau_2) \in \mathbb{C}^2 : \phi^{\tau_1}\phi^{\tau_2}(p) = p\}$ is a lattice of $\mathbb{C}^2$, hence $\mathbb{C}^2/lattice \longrightarrow \widetilde{\mathcal{B}}$ is a biholomorphic diffeomorphism and $\widetilde{\mathcal{B}}$ is a Kähler variety with Kähler metric given by $d\tau_1 \otimes d\overline{\tau}_1 + d\tau_2 \otimes d\overline{\tau}_2$. A compact complex Kähler variety having the required number as (its dimension) of independent meromorphic functions is a projective variety [34]. In fact, here we have $\widetilde{\mathcal{B}} \subseteq \mathbb{P}^{15}(\mathbb{C})$. Thus $\widetilde{\mathcal{B}}$ is both a projective variety and a complex torus $\mathbb{C}^2/lattice$ and hence an Abelian surface as a consequence of Chow theorem. By the classification theory of ample line bundles on Abelian varieties, $\widetilde{\mathcal{B}} \simeq \mathbb{C}^2/L_\Omega$ with period lattice given by the columns of the matrix

$$\begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}, \quad Im \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0,$$

and $\delta_1\delta_2 = g(\mathcal{H}_\varepsilon) - 1 = 1$, implying $\delta_1 = \delta_2 = 1$. Thus $\widetilde{\mathcal{B}}$ is principally polarized and it is the Jacobian of the hyperelliptic curve $\mathcal{H}_\varepsilon$. This completes the proof of the theorem. $\square$

Observe that the reflection σ on the affine variety $\mathcal{B}$ amounts to the flip

$$\sigma : (z_1, z_2, z_3, z_4, z_5) \longmapsto (z_1, z_2, -z_3, -z_4, z_5),$$

changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1, t_2) \longmapsto (-t_1, -t_2)$ on $\widetilde{\mathcal{B}}$, where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2}. In addition, the involution σ acts on the parameters of the Laurent solution (2.12) as follows

$$\sigma : (t, \alpha, \beta, \gamma, \theta, \varepsilon) \longmapsto (-t, -\alpha, -\beta, -\gamma, -\theta, -\varepsilon),$$

interchanges the curves $\mathcal{H}_{\varepsilon=\pm i}$ (2.13). Geometrically, this involution interchanges $\mathcal{H}_i$ and $\mathcal{H}_{-i}$, i.e., $\mathcal{H}_{-i} = \sigma\mathcal{H}_i$. We have shown that this system is part of a system of differential equation in five unknowns having three constants of motion. The asymptotic solution (2.5) can be read off from (2.12) and the change of variable

$$q_1 = \sqrt{z_1}, \quad q_2 = z_2, \quad p_1 = \frac{z_4}{q_1}, \quad p_2 = z_3.$$

The function z_1 has a simple pole along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ and a double zero along a hyperelliptic curve of genus 2 defining a double cover of $\widetilde{B}$ ramified along $\mathcal{H}_i + \mathcal{H}_{-i}$. Applying the method explained in [36], we have the

Theorem 9. *The invariant surface $\mathcal{A}$ (2.4) can be completed as a cyclic double cover $\overline{\mathcal{A}}$ of the Abelian surface $\widetilde{\mathcal{B}}$ (Jacobian of a genus 2 curve), ramified along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$. The system (2.2) is algebraic complete integrable in the generalized sense. Moreover, $\overline{\mathcal{A}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$ and the resolution $\widetilde{\mathcal{A}}$ of $\overline{\mathcal{A}}$ is a surface of general type with invariants : $\mathcal{X}(\widetilde{\mathcal{A}}) = 1$ and $p_g(\widetilde{\mathcal{A}}) = 2$.*

Proof. We have shown that the morphism φ (2.7) maps the vector field (2.2) into an algebraic completely integrable system (2.8) in five unknowns and the affine variety $\mathcal{A}$ (2.4) onto the affine part $\mathcal{B}$ (2.10) of an Abelian variety $\widetilde{\mathcal{B}}$. More precisely the Jacobian of a genus 2 curve with $\widetilde{\mathcal{B}}\backslash\mathcal{B} = \mathcal{H}_i + \mathcal{H}_{-i}$. Observe that $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an unramified cover. The curves $\mathcal{C}_\varepsilon$ (2.6) play an important role in the construction of a compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$. Let us denote by G a cyclic group of two elements $\{-1, 1\}$ on $V_\varepsilon^j = U_\varepsilon^j \times \{\tau \in \mathbb{C} : 0 < |\tau| < \delta\}$, where $\tau = \sqrt{t}$ and U_ε^j is an affine chart of $\mathcal{C}_\varepsilon$ for which the Laurent solutions (2.5) are defined. The action of G is defined by $(-1) \circ (u, v, \tau) = (-u, -v, -\tau)$, and is without fixed points in V_ε^j. So we can identify the quotient V_ε^j/G with the image of the smooth map $h_\varepsilon^j : V_\varepsilon^j \longrightarrow \mathcal{A}$ defined by the expansions (2.12). We have

$$(-1, 1).(u, v, \tau) = (-u, -v, \tau), \quad (1, -1).(u, v, \tau) = (u, v, -\tau),$$

i.e., $G \times G$ acts separately on each coordinate. Thus, identifying V_ε^j/G^2 with the image of $\varphi \circ h_\varepsilon^j$ in $\mathcal{B}$. Note that $\mathcal{A}_\varepsilon^j = V_\varepsilon^j/G$ is smooth (except for a finite number of points) and the coherence of the A^j_ε follows from the coherence of V^j_ε and the action of G. Now by taking $\mathcal{A}$ and by gluing on various varieties $A^j_\varepsilon\backslash\{\text{some points}\}$, we obtain a smooth complex manifold $\widehat{\mathcal{A}}$ which is a double cover of the Abelian variety $\widetilde{\mathcal{B}}$ ramified along $\mathcal{H}_i+\mathcal{H}_{-i}$, and therefore can be completed to an algebraic cyclic cover of $\widetilde{\mathcal{B}}$. To see what happens to the missing points, we must investigate the image of $\mathcal{C}_\varepsilon\times\{0\}$ in $\cup\mathcal{A}^j_\varepsilon$. The quotient $\mathcal{C}_\varepsilon\times\{0\}/G$ is birationally equivalent to the smooth hyperelliptic curve Γ_ε of genus 2 :

$$2w^2+\frac{1}{6}\left(15z^2-8a\right)zw+z\left(-\frac{39}{32}z^5+\frac{7}{6}az^3+\frac{2}{9}(a^2+9b_1)z-\varepsilon\sqrt{2}b_2\right)=0,$$

where $w=uv$, $z=u^2$. The curve Γ_ε is birationally equivalent to $\mathcal{H}_\varepsilon$. The only points of $\mathcal{C}_\varepsilon$ fixed under $(u,v)\longmapsto(-u,-v)$ are the two points to infinity which correspond to the ramification points of the following map

$$\mathcal{C}_\varepsilon\times\{0\}\overset{2-1}{\longrightarrow}\Gamma_\varepsilon:(u,v)\longmapsto(z,w),$$

and coincides with the points at ∞ of the curve $\mathcal{H}_\varepsilon$. Then the variety $\widehat{\mathcal{A}}$ constructed above is birationally equivalent to the compactification $\overline{\mathcal{A}}$ of the generic invariant surface $\mathcal{A}$. So $\overline{\mathcal{A}}$ is a cyclic double cover of the Abelian surface $\widetilde{\mathcal{B}}$ (the Jacobian of a genus 2 curve) ramified along the divisor $\mathcal{H}_i+\mathcal{H}_{-i}$, where $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ intersect each other in a tacnode. It follows that the system (2.2) is algebraic complete integrable in the generalized sense. Moreover, $\overline{\mathcal{A}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i+\mathcal{H}_{-i}$. In term of an appropriate local holomorphic coordinate system (x,y,z), the local analytic equation about this singularity is $x^4+y^2+z^2=0$. Now, let $\widetilde{\mathcal{A}}$ be the resolution of singularities of $\overline{\mathcal{A}}$, $\mathcal{X}(\widetilde{\mathcal{A}})$ be the Euler characteristic of $\widetilde{\mathcal{A}}$ and $p_g(\widetilde{\mathcal{A}})$ the geometric genus of $\widetilde{\mathcal{A}}$. Then $\widetilde{\mathcal{A}}$ is a surface of general type with invariants : $\mathcal{X}(\widetilde{\mathcal{A}})=1$ and $p_g(\widetilde{\mathcal{A}})=2$. This concludes the proof of the theorem. □

3 The Hénon-Heiles and a 5-dimensional system

Consider the system [17, 26, 27, 32] :

$$\begin{aligned}\dot{y}_1&=x_1, &\dot{x}_1=-Ay_1-2y_1y_2,\\ \dot{y}_2&=x_2, &\dot{x}_2=-By_2-y_1^2-\varepsilon y_2^2,\end{aligned}\tag{3.1}$$

corresponding to a generalized Hénon-Heiles Hamiltonian

$$H=\frac{1}{2}(x_1^2+x_2^2)+\frac{1}{2}(Ay_1^2+By_2^2)+y_1^2y_2+\frac{\varepsilon}{3}y_2^3,\tag{3.2}$$

where A, B, ε are constant parameters and y_1,y_2,x_1,x_2 are canonical coordinates and momenta, respectively. First studied as a mathematical model to describe the chaotic motion of a test star in an axisymmetric galactic mean gravitational field [17], this system is widely explored in other branches of physics. It well-known from applications in stellar dynamics, statistical mechanics and quantum mechanics. It provides a model for the oscillations of atoms in a three-atomic molecule [9].

Usually, the Hénon-Heiles system is not integrable and represents a classical example of chaotic behavior. Nevertheless at some special values of the parameters it is integrable; to be precise, there are known three integrable cases :

(i) $\varepsilon = 6$, A and B arbitrary. The second integral of motion is

$$H_2 = y_1^4 + 4y_1^2y_2^2 - 4x_1^2y_2 + 4x_1x_2y_1 + 4Ay_1^2y_2 + (4A - B)x_1^2 + A(4A - B)y_1^2. \quad (3.3)$$

(ii) $\varepsilon = 1$, $A = B$. The second integral of motion is

$$H_2 = x_1x_2 + \frac{1}{3}y_1^3 + y_1y_2^2 + Ay_1y_2.$$

(iii) $\varepsilon = 16$, $B = 16A$. The second integral of motion is

$$H_2 = 3x_1^4 + 6Ax_1^2y_1^2 + 12x_1^2y_1^2y_2 - 4x_1x_2y_1^3 - 4Ay_1^4y_2 - 4y_1^4y_2^2 + 3A^2y_1^4 - \frac{2}{3}y_1^6.$$

In the two cases (i) and (ii), the system (3.1) has been integrated by making use of genus one and genus two theta functions. For the case (i), the system separates in translated parabolic coordinates. Solving the problem in case (ii) is not difficult (this case trivially separates in cartesian coordinates). In the case (iii), the system can also be integrated [38] by making use of elliptic functions. The general solutions of the equations of motion for Hamiltonian (3.2) , for the case (i) and (ii), have the Painlevé propriety, i.e., that they admit only poles in the complex time variable. This section deals with the case (iii). When one examines all possible singularities, one finds that it possible for the variable y_1 to contain square root terms of the type $\sqrt{t}$, which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing some new variables $z_1, \ldots, z_5$, which restores the Painlevé property to the system. And reasoning as above, we obtain a new algebraically completely integrable system. The system (3.1) for case (iii), i.e.,

$$\begin{array}{rcll} \dot{y}_1 & = & x_1, & \dot{x}_1 = -Ay_1 - 2y_1y_2, \\ \dot{y}_2 & = & x_2, & \dot{x}_2 = -16Ay_2 - y_1^2 - 16y_2^2, \end{array} \quad (3.4)$$

can be written in the form

$$\dot{u} = J\frac{\partial H}{\partial u}, \quad u = (y_1, y_2, x_1, x_2)^\top,$$

where

$$H \equiv H_1 = \frac{1}{2}(x_1^2 + x_2^2) + \frac{A}{2}(y_1^2 + 16y_2^2) + y_1^2y_2 + \frac{16}{3}y_2^3,$$

and
$$\frac{\partial H}{\partial z}=\left(\frac{\partial H}{\partial y_1},\frac{\partial H}{\partial y_2},\frac{\partial H}{\partial x_1},\frac{\partial H}{\partial x_2}\right)^{\intercal},\qquad J=\begin{pmatrix} O & I\\ -I & O\end{pmatrix}.$$
The functions H_1 and

$$H_2=3x_1^4+6Ax_1^2y_1^2+12x_1^2y_1^2y_2-4x_1x_2y_1^3-4Ay_1^4y_2-4y_1^4y_2^2+3A^2y_1^4-\frac{2}{3}y_1^6,$$

commute :
$$\{H_1,H_2\}=\sum_{k=1}^{2}\left(\frac{\partial H_1}{\partial x_k}\frac{\partial H_2}{\partial y_k}-\frac{\partial H_1}{\partial y_k}\frac{\partial H_2}{\partial x_k}\right)=0.$$
The second flow commuting with the first is regulated by the equations :

$$\dot{u}=J\frac{\partial H_2}{\partial u},\quad u=(y_1,y_2,x_1,x_2)^{\intercal},$$

and is written explicitly as

$$\begin{aligned}
\dot{y}_1 &= -24Ax_1-8x_1y_2+4x_2y_1,\\
\dot{y}_2 &= 4x_1y_1,\\
\dot{x}_1 &= 24A^2y_1-4x_1x_2-8Ay_1y_2-8y_1y_2^2-4y_1^3,\\
\dot{x}_2 &= 4x_1^2-4Ay_1^2-8y_1^2y_2,
\end{aligned}$$

The system (3.4) admits Laurent solutions in $\sqrt{t}$, depending on three free parameters: α, β, γ and they are explicitly given as follows

$$\begin{aligned}
y_1 &= \frac{\alpha}{\sqrt{t}}+\beta t\sqrt{t}-\frac{\alpha}{18}t^2\sqrt{t}+\frac{\alpha A_1^2}{10}t^3\sqrt{t}-\frac{\alpha^2\beta}{18}t^4\sqrt{t}+\cdots,\\
y_2 &= -\frac{3}{8t^2}-\frac{A_1}{2}+\frac{\alpha^2}{12}t-\frac{2A_1^2}{5}t^2+\frac{\alpha\beta}{3}t^3-\gamma t^4+\cdots,\\
x_1 &= -\frac{1}{2}\frac{\alpha}{t\sqrt{t}}+\frac{3}{2}\beta\sqrt{t}-\frac{5}{36}\alpha t\sqrt{t}+\frac{7}{20}\alpha A_1^2t^2\sqrt{t}-\frac{1}{4}\alpha^2\beta t^3\sqrt{t}+\cdots,\\
x_2 &= \frac{3}{4t^3}+\frac{1}{12}\alpha^2-\frac{4}{5}A_1^2t+\alpha\beta t^2-4\gamma t^3+\cdots
\end{aligned}\tag{3.5}$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1=b_1$ and $H_2=b_2$, i.e.,

$$\begin{aligned}
H_1 &= \frac{1}{9}\alpha^2-\frac{21}{4}\gamma+\frac{13}{288}\alpha^4+\frac{4}{3}A^3=b_1,\\
H_2 &= -144\alpha\beta^3+\frac{294}{5}\alpha^3\beta A^2+\frac{8}{9}\alpha^6-33\gamma\alpha^4=b_2,
\end{aligned}$$

one eliminates the parameter γ linearly, leading to an equation connecting the two remaining parameters α and β :

$$144\alpha\beta^3 - \frac{294A^2}{5}\alpha^3\beta + \frac{143}{504}\alpha^8 - \frac{4}{21}\alpha^6 + \frac{44}{21}\left(4A^3 - 3b_1\right)\alpha^4 + b_2 = 0.$$

This is the equation of an algebraic curve $\mathcal{D}$ along which $u(t)$ blow up. To be more precise $\mathcal{D}$ is the closure of the continuous components of the set Laurent series solutions $u(t)$ such that $H_k(u(t)) = b_k$, $1 \le k \le 2$. The invariant variety

$$\mathcal{A} = \bigcap_{k=1}^{2}\{z \in \mathbb{C}^4 : H_k(z) = b_k\}, \tag{3.6}$$

is a smooth affine surface for generic $(b_1, b_2) \in \mathbb{C}^2$. The Laurent solutions restricted to the surface $\mathcal{A}$ are parameterized by the curve $\mathcal{D}$. We show that the system (3.4) is part of a new system of differential equations in five unknowns having two cubic and one quartic invariants (constants of motion). By inspection of the expansions (3.5) , we look for polynomials in (y_1, y_2, x_1, x_2) without fractional exponents. Let

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C}^5, \quad (y_1, y_2, x_1, x_2) \longmapsto (z_1, z_2, z_3, z_4, z_5), \tag{3.7}$$

be a morphism on the affine variety $\mathcal{A}$ (3.6) where $z_1, \ldots, z_5$ are defined as

$$z_1 = y_1^2, \quad z_2 = y_2, \quad z_3 = x_2, \quad z_4 = y_1x_1, \quad z_5 = 3x_1^2 + 2y_1^2y_2.$$

Using the two first integrals H_1, H_2 and differential equations (3.4) , we obtain a system of differential equations in five unknowns,

$$\begin{aligned} \dot{z}_1 &= 2z_4, \qquad \dot{z}_3 = -z_1 - 16A_1z_2 - 16z_2^2, \\ \dot{z}_2 &= z_3, \qquad \dot{z}_4 = -A_1z_1 + \frac{1}{3}z_5 - \frac{8}{3}z_1z_2, \\ \dot{z}_5 &= -6A_1z_4 + 2z_1z_3 - 8z_2z_4, \end{aligned} \tag{3.8}$$

having two cubic and one quartic invariants (constants of motion),

$$\begin{aligned} F_1 &= \frac{1}{2}A_1z_1 + \frac{1}{6}z_5 + 8A_1z_2^2 + \frac{1}{2}z_3^2 + \frac{2}{3}z_1z_2 + \frac{16}{3}z_2^3, \\ F_2 &= 9A_1^2z_1^2 + z_5^2 + 6A_1z_1z_5 - 2z_1^3 - 24A_1z_1^2z_2 - 12z_1z_3z_4 + 24z_2z_4^2 - 16z_1^2z_2^2, \\ F_3 &= z_1z_5 - 3z_4^2 - 2z_1^2z_2. \end{aligned}$$

This new system is completely integrable and the Hamiltonian structure is defined by the Poisson bracket

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^{5} J_{kl}\frac{\partial F}{\partial z_k}\frac{\partial H}{\partial z_l},$$

where $\frac{\partial H}{\partial z}=\left(\frac{\partial H}{\partial z_1},\frac{\partial H}{\partial z_2},\frac{\partial H}{\partial z_3},\frac{\partial H}{\partial z_4},\frac{\partial H}{\partial z_5}\right)^\top$, and

$$J=\begin{pmatrix} 0 & 0 & 0 & 2z_1 & 12z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2z_1 \\ -2z_1 & 0 & 0 & 0 & -8z_1z_2+2z_5 \\ -12z_4 & 0 & 2z_1 & 8z_1z_2-2z_5 & 0 \end{pmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The system (3.8) can be written as

$$\dot{z}=J\frac{\partial H}{\partial z},\quad z=(z_1,z_2,z_3,z_4,z_5)^\top,$$

where $H=F_1$. The second flow commuting with the first is regulated by the equations $\dot{z}=J\frac{\partial F_2}{\partial z}$, $\quad z=(z_1,z_2,z_3,z_4,z_5)^\top$, and is written explicitly as

$$\begin{aligned}
\dot{z}_1 &= 24z_4z_5-24z_1^2z_3+96z_1z_2z_4+72Az_1z_4,\\
\dot{z}_2 &= -12z_1z_4,\\
\dot{z}_3 &= 12Az_1^2-12z_1z_5+48z_1^2z_2,\\
\dot{z}_4 &= 4z_5^2-36A^2z_1^2+12z_1^3-16z_1z_2z_5+48Az_1^2z_2+24z_1z_3z_4+64z_1^2z_2^2,\\
\dot{z}_5 &= -96z_2z_4z_5+768z_1z_2^2z_4-72Az_4z_5+576Az_1z_2z_4+144z_3z_4^2\\
&\quad -216A^2z_1z_4+48z_1^2z_4-96z_1^2z_2z_3+24z_1z_3z_5.
\end{aligned}$$

These vector fields are in involution, i.e., $\{F_1,F_2\}=\left\langle\frac{\partial F_1}{\partial z},J\frac{\partial F_2}{\partial z}\right\rangle=0$, and the remaining one is Casimir, i.e., $J\frac{\partial F_3}{\partial z}=0$. Consequently, the system (3.8) is integrable in the sense of Liouville.

The invariant variety

$$\mathcal{B}=\bigcap_{k=1}^{3}\{z\in\mathbb{C}^5:F_k(z)=c_k\},\tag{3.9}$$

is a smooth affine surface for generic values of c_1,c_2 and c_3. The system (3.8) possesses Laurent series solutions which depend on four free parameters. These meromorphic solutions restricted to the surface $\mathcal{B}$ (3.9) can be read off from (3.5) and the change of variable (3.7) . Following the methods previously used, one find the compactification of $\mathcal{B}$ into an Abelian surface $\widetilde{B}$, the system of differential equations (3.8) is algebraic complete integrable and the corresponding flows evolve on $\widetilde{\mathcal{B}}$. Also, we show that the invariant surface $\mathcal{A}$ (3.6) can be completed as a cyclic double cover $\overline{\mathcal{A}}$ of an Abelian surface $\widetilde{\mathcal{B}}$. The system (3.4) is algebraic complete integrable in the generalized sense. Moreover, $\overline{\mathcal{A}}$ is smooth except at the point lying over the singularity of type A_3 and the resolution $\widetilde{\mathcal{A}}$ of $\overline{\mathcal{A}}$ is a surface of general type. We have shown that the morphism φ (3.7) maps the vector field (3.4) into an algebraic completely integrable system (3.8) in five unknowns and the affine variety $\mathcal{A}$ (3.6) onto the affine part $\mathcal{B}$ (3.9) of an Abelian variety $\widetilde{\mathcal{B}}$. This explains (among other) why the asymptotic solutions to the differential equations (3.4) contain fractional powers. All this is summarized as follows :

Theorem 10. *The system* (3.4) *admits Laurent solutions with fractional powers (i.e., contain square root terms of the type $\sqrt{t}$ which are strictly not allowed by the Painlevé test) depending on three free parameters and is algebraic complete integrable in the generalized sense. The morphism φ* (3.7) *(which restores the Painlevé property) maps this system into a new algebraic completely integrable system* (3.8) *in five unknowns.*

4 The RDG potential and a 5-dimensional system

Consider the Ramani Dorizzi Grammaticos (RDG) system [37],

$$\begin{aligned} \ddot{q}_1 - q_1\left(q_1^2 + 3q_2^2\right) &= 0, \\ \ddot{q}_2 - q_2\left(3q_1^2 + 8q_2^2\right) &= 0, \end{aligned} \tag{4.1}$$

corresponding to the Hamiltonian

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2) - \frac{3}{2}q_1^2 q_2^2 - \frac{1}{4}q_1^4 - 2q_2^4. \tag{4.2}$$

This system is integrable in the sense of Liouville, the second first integral (of degree 8) being

$$H_2 = p_1^4 - 6q_1^2 q_2^2 p_1^2 + q_1^4 q_2^4 - q_1^4 p_1^2 + q_1^6 q_2^2 + 4q_1^3 q_2 p_1 p_2 - q_1^4 p_2^2 + \frac{1}{4}q_1^8. \tag{4.3}$$

The first integrals H_1 and H_2 are in involution, i.e., $\{H_1, H_2\} = 0$. The system (4.1) is weight-homogeneous with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that H_1 (4.2) and H_2 (4.3) have weight 4 and 8 respectively.

When one examines all possible singularities, one finds that it possible for the variable q_1 to contain square root terms of the type $\sqrt{t}$, which are strictly not allowed by the Painlevé test. However, we will see later that these terms are trivially removed by introducing the variables $z_1, \ldots, z_5$ (4.8) which restores the Painlevé property to the system.

Let $\mathcal{B}$ be the affine variety defined by

$$\mathcal{B} = \bigcap_{k=1}^{2} \left\{ z \in \mathbb{C}^4 : H_k(z) = b_k \right\}, \tag{4.4}$$

for generic $(b_1, b_2) \in \mathbb{C}^2$.

Theorem 11. *a) The system* (4.1) *admits Laurent solutions ,*

$$(q_1, q_2, p_1, p_2) = (t^{-1/2}, t^{-1}, t^{-3/2}, t^{-2}) \times \text{ a Taylor series in } t,$$

depending on three free parameters : u, v and w. These solutions restricted to the surface $\mathcal{B}$ (4.4) *are parameterized by two copies Γ_1 and Γ_{-1} of the Riemann surface Γ* (4.6) *of genus* 16.

b) The system (4.1) *is algebraic complete integrable in the generalized sense and extends to a new system* (4.9) *of five differential equations algebraically completely integrable with three quartics invariants* (4.10)*. Generically, the invariant manifold $\mathcal{A}$* (4.11) *defined by the intersection of these quartics form the affine part of an Abelian surface $\widetilde{\mathcal{A}}$. The reduced divisor at infinity $\widetilde{\mathcal{A}} \setminus \mathcal{A} = \mathcal{C}_1 + \mathcal{C}_{-1}$, is very ample and consists of two components $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ of a genus 7 curve $\mathcal{C}$* (4.13)*. In addition, the invariant surface $\mathcal{B}$ can be completed as a cyclic double cover $\overline{\mathcal{B}}$ of the Abelian surface $\widetilde{\mathcal{A}}$, ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. Moreover, $\overline{\mathcal{B}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ and the resolution $\widetilde{\mathcal{B}}$ of $\overline{\mathcal{B}}$ is a surface of general type with invariants : $\mathcal{X}(\widetilde{\mathcal{B}}) = 1$ and $p_g(\widetilde{\mathcal{B}}) = 2$.*

Proof. *a*) The system (4.1) possesses 3-dimensional family of Laurent solutions (principal balances) depending on three free parameters u, v and w. There are precisely two such families, labelled by $\varepsilon = \pm 1$, and they are explicitly given as follows

$$\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}(u - \frac{1}{4}u^3 t + vt^2 - \frac{5}{128}u^7 t^3 + \frac{1}{8}u(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w)t^4 + \cdots),\\
q_2 &= \frac{1}{t}(\frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon u^2 t + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{4}\varepsilon u(\frac{1}{32}u^5 - 3v)t^3 + wt^4 + \cdots),\\
p_1 &= \frac{1}{2t\sqrt{t}}(-u - \frac{1}{4}u^3 t + 3vt^2 - \frac{25}{128}t^3 u^7 +\\
&\qquad \frac{7}{8}u(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w)t^4 + \cdots),\\
p_2 &= \frac{1}{t^2}(-\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{2}\varepsilon u(\frac{1}{32}u^5 - 3v)t^3 + 3wt^4 + \cdots).
\end{aligned} \tag{4.5}$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter w linearly, leading to an equation connecting the two remaining parameters u and v :

$$\begin{aligned}
\Gamma : \quad & \frac{65}{4}uv^3 + \frac{93}{64}u^6 v^2 + \frac{3}{8192}\left(-9829u^8 + 26112H_1\right)u^3 v\\
& -\frac{10299}{65536}u^{16} - \frac{123}{256}H_1 u^8 + H_2 + \frac{15362\,98731}{52} = 0.
\end{aligned} \tag{4.6}$$

According to Hurwitz's formula, this defines a Riemann surface Γ of genus 16. The Laurent solutions restricted to the affine surface $\mathcal{B}$ (4.4) are thus parameterized by two copies Γ_{-1} and Γ_1 of the same Riemann surface Γ.

b) Let

$$\varphi : \mathcal{B} \longrightarrow \mathbb{C}^5, \quad (q_1, q_2, p_1, p_2) \longmapsto (z_1, z_2, z_3, z_4, z_5), \tag{4.7}$$

be the morphism defined on the affine variety $\mathcal{B}$ (4.4) by

$$z_1 = q_1^2, \quad z_2 = q_2, \quad z_3 = p_2, \quad z_4 = q_1 p_1, \quad z_5 = p_1^2 - q_1^2 q_2^2. \tag{4.8}$$

These variables are easily obtained by simple inspection of the series (4.5). By using the variables (4.8) and differential equations (4.1), one obtains

$$\begin{aligned} \dot{z}_1 &= 2z_4, \qquad \dot{z}_3 = z_2(3z_1 + 8z_2^2), \\ \dot{z}_2 &= z_3, \qquad \dot{z}_4 = z_1^2 + 4z_1 z_2^2 + z_5, \\ \dot{z}_5 &= 2z_1 z_4 + 4z_2^2 z_4 - 2z_1 z_2 z_3. \end{aligned} \tag{4.9}$$

This new system on $\mathbb{C}^5$ admits the following three first integrals

$$\begin{aligned} F_1 &= \frac{1}{2} z_5 - z_1 z_2^2 + \frac{1}{2} z_3^2 - \frac{1}{4} z_1^2 - 2z_2^4, \\ F_2 &= z_5^2 - z_1^2 z_5 + 4z_1 z_2 z_3 z_4 - z_1^2 z_3^2 + \frac{1}{4} z_1^4 - 4z_2^2 z_4^2, \\ F_3 &= z_1 z_5 + z_1^2 z_2^2 - z_4^2. \end{aligned} \tag{4.10}$$

The first integrals F_1 and F_2 are in involution , while F_3 is trivial (Casimir function). The invariant variety $\mathcal{A}$ defined by

$$\mathcal{A} = \bigcap_{k=1}^{2} \{z : F_k(z) = c_k\} \subset \mathbb{C}^5, \tag{4.11}$$

is a smooth affine surface for generic values of $(c_1, c_2, c_3) \in \mathbb{C}^3$. The system (4.9) is completely integrable and possesses Laurent series solutions which depend on four free parameters α, β, γ et θ :

$$\begin{aligned} z_1 &= \frac{1}{t}\alpha - \frac{1}{2}\alpha^2 + \beta t - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right) t^2 + \gamma t^3 + \cdots, \\ z_2 &= \frac{1}{2t}\varepsilon - \frac{1}{4}\varepsilon\alpha + \frac{1}{8}\varepsilon\alpha^2 t - \frac{1}{32}\varepsilon\left(-\alpha^3 + 12\beta\right) t^2 + \theta t^3 + \cdots, \\ z_3 &= -\frac{1}{2t^2}\varepsilon + \frac{1}{8}\varepsilon\alpha^2 - \frac{1}{16}\varepsilon\left(-\alpha^3 + 12\beta\right) t + 3\theta t^2 + \cdots, \\ z_4 &= -\frac{1}{2t^2}\alpha + \frac{1}{2}\beta - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right) t + \frac{3}{2}\gamma t^2 + \cdots, \\ z_5 &= \frac{1}{2t^2}\alpha^2 - \frac{1}{4t}\left(\alpha^3 + 4\beta\right) + \frac{1}{4}\alpha\left(\alpha^3 + 2\beta\right) - \left(\alpha^2\beta - 2\gamma + 4\varepsilon\theta\alpha\right) t + \cdots, \end{aligned} \tag{4.12}$$

where $\varepsilon = \pm 1$. The convergence of these series is guaranteed by the majorant method. Substituting these developments in equations (4.10), one obtains three polynomial relations between α, β, γ and θ. Eliminating γ and θ from these equations, leads to an equation connecting the two remaining parameters α and β :

$$\begin{aligned}\mathcal{C}: \quad & 64\beta^3 - 16\alpha^3\beta^2 - 4\left(\alpha^6 - 32\alpha^2 c_1 - 16c_3\right)\beta \\ & +\alpha\left(32c_2 - 32\alpha^4 c_1 + \alpha^8 - 16\alpha^2 c_3\right) = 0.\end{aligned} \tag{4.13}$$

The Laurent solutions restricted to the surface $\mathcal{A}$ (4.11) are thus parameterized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface $\mathcal{C}$ (4.13). According to the Riemann-Hurwitz formula, the genus of $\mathcal{C}$ is 7. Applying the method used in the previous problems, we embed these curves in a hyperplane of $\mathbb{P}^{15}(\mathbb{C})$ using the sixteen functions :

$$1, \quad z_1, \quad z_2, \quad 2z_5 - z_1^2, \quad z_3 + 2\varepsilon z_2^2, \quad z_4 + \varepsilon z_1 z_2, \quad W(f_1, f_2),$$

$$f_1(f_1 + 2\varepsilon f_4), \quad f_2(f_1 + 2\varepsilon f_4), \quad z_4(f_3 + 2\varepsilon f_6), \quad z_5(f_3 + 2\varepsilon f_6),$$

$$f_5(f_1 + 2\varepsilon f_4), \quad f_1 f_2(f_3 + 2\varepsilon f_6), \quad f_4 f_5 + W(f_1, f_4),$$

$$W(f_1, f_3) + 2\varepsilon W(f_1, f_6), \quad f_3 - 2z_5 + 4f_4^2,$$

where $W(s_j, s_k) \equiv \dot{s}_j s_k - s_j \dot{s}_k$ (Wronskian) and we show that these curves have two points in common in which $\mathcal{C}_1$ is tangent to $\mathcal{C}_{-1}$. The system (4.1) is algebraic complete integrable in the generalized sense. The invariant surface $\mathcal{B}$ (4.4) can be completed as a cyclic double cover $\overline{\mathcal{B}}$ of the Abelian surface $\widetilde{\mathcal{A}}$, ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. Moreover, $\overline{\mathcal{B}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ (double points of intersection of the curves $\mathcal{C}_1$ and $\mathcal{C}_{-1}$) and the resolution $\widetilde{\mathcal{B}}$ of $\overline{\mathcal{B}}$ is a surface of general type. We shall resume with more detail (already used previously in other similar problems) the proof of these results. Observe that the morphism φ (4.7) is an unramified cover. The Riemann surface Γ (4.6) play an important role in the construction of a compactification $\overline{\mathcal{B}}$ of $\mathcal{B}$. Let G be a cyclic group of two elements $\{-1, 1\}$ on $V_\varepsilon^j = U_\varepsilon^j \times \{\tau \in \mathbb{C} : 0 < |\tau| < \delta\}$, where $\tau = t^{1/2}$ and U_ε^j is an affine chart of Γ_ε for which the Laurent solutions are defined. The action of G is defined by $(-1) \circ (u, v, \tau) = (-u, -v, -\tau)$ and is without fixed points in V_ε^j. So we can identify the quotient V_ε^j / G with the image of the smooth map $h_\varepsilon^j : V_\varepsilon^j \longrightarrow \mathcal{B}$ defined by the expansions (4.5). As before, we have $(-1, 1).(u, v, \tau) = (-u, -v, \tau)$ and $(1, -1).(u, v, \tau) = (u, v, -\tau)$, i.e., $G \times G$ acts separately on each coordinate. Thus, identifying V_ε^j / G^2 with the image of $\varphi \circ h_\varepsilon^j$ in $\mathcal{A}$. Note that $\mathcal{B}_\varepsilon^j = V_\varepsilon^j / G$ is smooth (except for a finite number of points) and the coherence of the $\mathcal{B}_\varepsilon^j$ follows from the coherence of V_ε^j and the action of G. By taking $\mathcal{B}$ and by gluing on various varieties $\mathcal{B}_\varepsilon^j \backslash \{\text{some points}\}$, we obtain a smooth complex manifold $\widehat{\mathcal{B}}$ which is a double cover of the Abelian variety $\widetilde{\mathcal{A}}$ ramified along $\mathcal{C}_1 + \mathcal{C}_{-1}$, and can be completed to an algebraic cyclic cover of $\widetilde{\mathcal{A}}$. To see what happens to the missing points, we must investigate the image of $\Gamma \times \{0\}$ in $\cup \mathcal{B}_\varepsilon^j$. The quotient

$\Gamma \times \{0\}/G$ is birationally equivalent to the Riemann surface Υ of genus 7 of affine equation :

$$\Upsilon : \frac{65}{4}y^3 + \frac{93}{64}x^3y^2 + \frac{3}{8192}\left(-9829x^4 + 26112b_1\right)x^2y$$
$$+x\left(-\frac{10299}{65536}x^8 - \frac{123}{256}b_1x^4 + b_2 + \frac{1536298731}{52}\right) = 0,$$

where $y = uv$, $x = u^2$. The Riemann surface Υ is birationally equivalent to $\mathcal{C}$. The only points of Υ fixed under $(u,v) \longmapsto (-u,-v)$ are the points at ∞, which correspond to the ramification points of the map

$$\Gamma \times \{0\} \xrightarrow{2-1} \Upsilon : (u,v) \longmapsto (x,y),$$

and coincides with the points at ∞ of the Riemann surface $\mathcal{C}$. Then the variety $\widehat{\mathcal{B}}$ constructed above is birationally equivalent to the compactification $\overline{\mathcal{B}}$ of the generic invariant surface $\mathcal{B}$. So $\overline{\mathcal{B}}$ is a cyclic double cover of the Abelian surface $\widetilde{\mathcal{A}}$ ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$, where $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ have two points in commune at which they are tangent to each other. It follows that The system (4.1) is algebraic complete integrable in the generalized sense. Moreover, $\overline{\mathcal{B}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$. In term of an appropriate local holomorphic coordinate system (X, Y, Z), the local analytic equation about this singularity is $X^4 + Y^2 + Z^2 = 0$. Now, let $\widetilde{\mathcal{B}}$ be the resolution of singularities of $\overline{\mathcal{B}}$, $\mathcal{X}(\widetilde{\mathcal{B}})$ be the Euler characteristic of $\widetilde{\mathcal{B}}$ and $p_g(\widetilde{\mathcal{B}})$ the geometric genus of $\widetilde{\mathcal{B}}$. Then $\widetilde{\mathcal{B}}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{\mathcal{B}}) = 1$ and $p_g(\widetilde{\mathcal{B}}) = 2$. This ends the proof of the theorem. □

Consider on the Abelian variety $\widetilde{\mathcal{A}}$ the holomorphic 1-forms dt_1 and dt_2 defined by $dt_i(X_{F_j}) = \delta_{ij}$, where X_{F_1} and X_{F_2} are the vector fields generated respectively by F_1 and F_2. Taking the differentials of $\zeta = 1/z_2$ and $\xi = \dfrac{z_1}{z_2}$ viewed as functions of t_1 and t_2, using the vector fields and the Laurent series (4.12) and solving linearly for dt_1 and dt_2, we obtain the holomorphic differentials

$$\begin{aligned}
\omega_1 &= dt_1|_{\mathcal{C}_\varepsilon} = \frac{1}{\Delta}\left(\frac{\partial\xi}{\partial t_2}d\zeta - \frac{\partial\zeta}{\partial t_2}d\xi\right)|_{\mathcal{C}_\varepsilon} = \frac{8}{\alpha\left(-4\beta + \alpha^3\right)}d\alpha, \\
\omega_2 &= dt_2|_{\mathcal{C}_\varepsilon} = \frac{1}{\Delta}\left(\frac{-\partial\xi}{\partial t_1}d\zeta - \frac{\partial\zeta}{\partial t_1}d\xi\right)|_{\mathcal{C}_\varepsilon} = \frac{2}{\left(-4\beta + \alpha^3\right)^2}d\alpha,
\end{aligned}$$

with

$$\Delta \equiv \frac{\partial\zeta}{\partial t_1}\frac{\partial\xi}{\partial t_2} - \frac{\partial\zeta}{\partial t_2}\frac{\partial\xi}{\partial t_1}.$$

The zeroes of ω_2 provide the points of tangency of the vector field X_{F_1} to $\mathcal{C}_\varepsilon$. We have

$$\frac{\omega_1}{\omega_2} = \frac{4}{\alpha}\left(-4\beta + \alpha^3\right),$$

and X_{F_1} is tangent to $\mathcal{H}_\varepsilon$ at the point covering $\alpha = \infty$. The reflection σ on $\mathcal{A}$ amounts to the flip

$$\sigma : (z_1, z_2, z_3, z_4, z_5) \longmapsto (z_1, -z_2, z_3, -z_4, z_5),$$

changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1, t_2) \mapsto (-t_1, -t_2)$ on $\widetilde{\mathcal{A}}$, where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2}. The involution σ acts on the parameters of the Laurent solution (4.12) as follows

$$\sigma : (t, \alpha, \beta, \gamma, \theta) \longmapsto (-t, -\alpha, -\beta, -\gamma, \theta),$$

interchanges the curves $\mathcal{C}_\varepsilon$ and the linear space $\mathcal{L}$ can be split into a direct sum of even and odd functions. Geometrically, this involution interchanges $\mathcal{C}_1$ and $\mathcal{C}_{-1}$, i.e., $\mathcal{C}_{-1} = \sigma\mathcal{C}_1$.

5 The Goryachev-Chaplygin top and a 7-dimensional system

The Goryachev-Chaplygin top is a rigid body rotating about a fixed point for which the principal moments of inertia I_1, I_2, I_3 satisfying the relation: $I_1 = I_2 = 4I_3$ and the center of mass lying in the equatorial plane through the fixed point and the principal angular momentum is perpendicular to the direction of gravity. The equations of the motion can be written in the form

$$\begin{array}{rclrcl} \dot{m}_1 &=& 3m_2m_3, & \dot{\gamma}_1 &=& 4m_3\gamma_2 - m_2\gamma_3, \\ \dot{m}_2 &=& -3m_1m_3 - 4\gamma_3, & \dot{\gamma}_2 &=& m_1\gamma_3 - 4m_3\gamma_1, \\ \dot{m}_3 &=& 4\gamma_2, & \dot{\gamma}_3 &=& m_2\gamma_1 - m_1\gamma_2, \end{array} \tag{5.1}$$

where $m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3$ are the coordinates of the phase space. The following four quadrics are constants of motion for this system

$$\begin{aligned} H_1 &= m_1^2 + m_2^2 + 4m_3^2 - 8\gamma_1 = 6b_1, \\ H_2 &= (m_1^2 + m_2^2)m_3 + 4m_1\gamma_3 = 2b_2, \\ H_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = b_3, \\ H_4 &= m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = 0, \end{aligned}$$

for generic $b_1, b_2, b_3 \in \mathbb{C}$.

This system is completely integrable, H_1 (energy) and H_4 are in involution while H_2, H_3 are Casimir invariants. The Goryachev-Chaplygin system has asymptotic solutions which are meromorphic in $\sqrt{t}$ depending on four free parameters b_1, b_2, b_3 and u, v, namely (where $\varepsilon \equiv \pm i$) :

$$
\begin{aligned}
m_1 &= \frac{u}{t^{3/2}} - \frac{3uv}{t^{1/2}} + \frac{3\varepsilon v + 3b_1u^2 - 15u^2v^2}{2u} t^{1/2} + o(t^{3/2}), \\
m_2 &= -\frac{\varepsilon u}{t^{3/2}} - \frac{3uv}{t^{1/2}} - \frac{\varepsilon(3b_1u^2 - 15u^2v^2) + u}{2u} t^{1/2} + o(t^{3/2}), \\
m_3 &= -\frac{\varepsilon}{2t} + v + \varepsilon(b_1 - 2v^2)t - \frac{16b_3u^4 + 5v^2}{4\varepsilon u^2} t^2 + \cdots, \\
\gamma_1 &= -\frac{1}{8t^2} - \frac{b_1 - 2v^2}{4} - \frac{16b_3u^4 - 3v^2}{8u^2} t + \cdots, \\
\gamma_2 &= \frac{\varepsilon}{8t^2} + \frac{\varepsilon(b_1 - 2v^2)}{4} - \frac{16b_3u^4 + 5v^2}{8\varepsilon u^2} t + \cdots, \\
\gamma_3 &= -\frac{v}{2ut^{1/2}} + \frac{3v^2}{2\varepsilon u} t^{1/2} - \frac{\varepsilon(b_1v - 11v^3) + 16b_3u^2}{4\varepsilon u} t^{3/2} + \cdots
\end{aligned}
$$

Let $\mathcal{A}$ the affine variety defined by

$$
\mathcal{A} = \{x : H_1(x) = 6b_1, H_2(x) = 2b_2, H_3(x) = b_3, H_4(x) = 0\}, \tag{5.2}
$$

where $x = (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)$. These solutions restricted to $\mathcal{A}$ are parameterized by two copies $\mathcal{C}_{\varepsilon=+i}$ and $\mathcal{C}_{\varepsilon=-i}$ of the curve $\mathcal{C}$ of genus 4 :

$$
\mathcal{C} : 16b_3u^4 + \varepsilon u^2(b_2 + 6b_1v - 16v^3) - v^2 = 0.
$$

We have seen that the asymptotic solutions of the system (5.1) contain fractional powers, i.e., contain square root terms of the type $\sqrt{t}$, which are strictly not allowed by the Painlevé test but the new variables $(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ defined as

$$
z_1 = m_1^2 + m_2^2, \quad z_2 = m_3, \quad z_3 = \gamma_3^2, \quad z_4 = \gamma_1,
$$

$$
z_5 = \gamma_2, \quad z_6 = m_1\gamma_3, \quad z_7 = m_2\gamma_3,
$$

restores the Painlevé property to the system. Let

$$
\varphi : \mathcal{A} \longrightarrow \mathbb{C}^7, (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) \longmapsto (z_1, z_2, z_3, z_4, z_5, z_6, z_7), \tag{5.3}
$$

be a morphism on the affine variety $\mathcal{A}$. These affine variables were originally used in [6] without any discussion of their origin and algebraic properties. By using the first terms of the Laurent series, these variables are easily obtained and the morphism (5.3) maps the vector field (5.1) into the system [6] :

$$
\begin{aligned}
\dot{z}_1 &= -8z_7, & \dot{z}_4 &= 4z_2z_5 - z_7, \\
\dot{z}_2 &= 4z_5, & \dot{z}_5 &= z_6 - 4z_2z_4, \\
\dot{z}_3 &= 2(z_4z_7 - z_5z_6), & \dot{z}_6 &= -z_1z_5 + 2z_2z_7, \\
\dot{z}_7 &= z_1z_4 - 2z_2z_6 - 4z_3,
\end{aligned} \tag{5.4}
$$

in seven unknowns having five quadrics invariants

$$\begin{aligned} F_1 &= z_1 - 8z_4 + 4z_2^2 = 6c_1, \qquad F_3 = z_3 + z_4^2 + z_5^2 = c_3,\\ F_2 &= z_1z_2 + 4z_6 = 2c_2, \qquad F_4 = z_2z_3 + z_4z_6 + z_5z_7 = c_4,\\ F_5 &= z_6^2 + z_7^2 - z_1z_3 = c_5, \end{aligned}$$

where c_1, c_2, c_3, c_4, c_5 are generic constants. To obtain these invariants, just use the first integrals H_1, H_2, H_3, H_4 and differential equations (5.1). This system is completely integrable and the symplectic structure is defined by the Poisson bracket $\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z}\right\rangle$, where

$$J = \begin{pmatrix} 0 & 0 & -A & z_7 & -z_6 & B & -C\\ 0 & 0 & 0 & -\frac{1}{2}z_5 & \frac{1}{4}z_4 & -\frac{1}{2}z_7 & \frac{1}{2}z_6\\ A & 0 & 0 & 0 & 0 & -z_3z_5 & z_3z_4\\ -z_7 & \frac{1}{2}z_5 & 0 & 0 & 0 & 0 & -\frac{1}{2}z_3\\ z_6 & -\frac{1}{2}z_4 & 0 & 0 & 0 & \frac{1}{2}z_3 & 0\\ -B & \frac{1}{2}z_7 & z_3z_5 & 0 & -\frac{1}{2}z_3 & 0 & -z_2z_3\\ C & -\frac{1}{2}z_6 & -z_3z_4 & \frac{1}{2}z_3 & 0 & z_2z_3 & 0 \end{pmatrix},$$

$$A \equiv 2z_4z_7 - 2z_5z_6, \quad B \equiv z_1z_5 + 2z_2z_7, \quad C \equiv z_1z_4 + 2z_2z_6,$$

is a skew-symmetric matrix whose elements polynomial satisfy the Jacobi identity. The system (5.4) can be written as

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad H = F_1, \quad x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^\top.$$

The two first integrals F_1 and F_2 are in involution : $\{F_1, F_2\} = 0$, while F_3, F_4 et F_5 are Casimir invariants : $J\frac{\partial F_k}{\partial x} = 0$, $k = 3, 4, 5$. The system (5.4) admits Laurent solutions depending on six free parameters :

$$\begin{aligned} z_1 &= -\frac{2\varepsilon\alpha}{t} + 2\alpha^2 - 2\varepsilon(\alpha(\alpha^2 - 2c_1) + \zeta)t - (2\xi + \zeta)\alpha t^2 + o(t^3),\\ z_2 &= -\frac{\varepsilon}{2t} - \frac{\alpha}{2} - \frac{\varepsilon}{2}(\alpha^2 - 2c_1)t - \frac{1}{4}(2\xi + \zeta)\alpha t^2 + o(t^3),\\ z_3 &= \frac{\varepsilon}{8t}(\xi + \zeta) + \frac{3\alpha}{8}(\xi + \zeta) - \frac{\varepsilon}{8}((5\alpha^2 - c_1)(\xi + \zeta) - 8(2c_3\alpha + c_4))t + o(t^2),\\ z_4 &= -\frac{1}{8t^2} + \frac{1}{8}(\alpha^2 - 2c_1) + \frac{\varepsilon}{8}(2\xi + \zeta)t + o(t^2),\\ z_5 &= \frac{\varepsilon}{8t^2} - \frac{\varepsilon}{8}(\alpha^2 - 2c_1) - \frac{1}{8}(2\xi + 3\zeta)t + o(t^2),\\ z_6 &= \frac{\alpha}{4t^2} + \frac{1}{4}(2\xi - (\alpha^2 - 2c_1)\alpha + \zeta) - \frac{\varepsilon\alpha}{4}(2\xi + 3\zeta)t + o(t^2),\\ z_7 &= -\frac{\varepsilon\alpha}{4t^2} + \frac{\varepsilon}{4}(\alpha(\alpha^2 - 2c_1) + \zeta) + \frac{\alpha}{4}(2\xi + \zeta)t + o(t^2), \end{aligned} \tag{5.5}$$

where $\varepsilon = \pm i$, $\xi(\alpha) = 2\alpha^3 - 3c_1\alpha + c_2$ and the parameters α, ζ belong to a genus 2 hyperelliptic curve,

$$\mathcal{H} : \zeta^2 = (2\alpha^3 - 3c_1\alpha + c_2)^2 - 4(4c_3\alpha^2 + 4c_4\alpha + c_5). \tag{5.6}$$

Using the majorant method, we can show that the formal Laurent series solutions are convergent. Consequently, the Laurent solutions are parameterized by two copies $\mathcal{H}_{+i}$ and $\mathcal{H}_{-i}$ of the genus 2 hyperelliptic curve $\mathcal{H}$ for $\varepsilon = \pm 1$. Using similar reasoning to that done previously, we show that the embedding $\mathcal{D}$ of $\mathcal{H}_i + \mathcal{H}_{-i}$ into $\mathbb{P}^{15}(\mathbb{C})$ is done via the functions of the space

$$\mathcal{L}(2(\mathcal{H}_i + \mathcal{H}_{-i})) = \{1, z_1, z_2, z_3, z_4, z_6, z_8, z_9, z_{10}, z_{14}\} \oplus \{z_5, z_7, z_{11}, z_{12}, z_{13}, z_{15}\},$$

where z_1, z_2, z_3, z_4, z_5, z_6, z_7, are given above and

$$z_8 \equiv z_2 z_3 = \frac{1}{16t^2}(\xi + \zeta) - \frac{\varepsilon}{4t}\alpha(\xi + \zeta) + o(1),$$

$$z_9 \equiv z_1 z_3 = \frac{1}{4t^2}\alpha(\xi + \zeta) - \frac{\varepsilon}{2t}\alpha^2(\xi + \zeta) + o(1),$$

$$z_{10} \equiv z_1 z_4 + 2z_2 z_6 = -\frac{1}{2t^2}\alpha^2 - \frac{\varepsilon}{2t}(\xi + \zeta) + o(1),$$

$$z_{11} \equiv \frac{1}{4}W(x_1, x_2) = z_1 z_5 + 2z_2 z_7 = \frac{\varepsilon}{2t^2}\alpha^2 + \frac{\zeta}{2t} + o(1),$$

$$z_{12} \equiv \frac{1}{2}\dot{x}_3 = -z_5 z_6 + z_4 z_7 = -\frac{\varepsilon}{16t^2}(\xi + \zeta) + o(1),$$

$$z_{13} \equiv \frac{1}{2}W(x_2, x_3) = z_2 z_{12} - 2z_3 z_5 = -\frac{\varepsilon}{16t^2}\alpha(\xi + \zeta) + o(\frac{1}{t}),$$

$$z_{14} \equiv z_3^2 = -\frac{1}{64t^2}(\xi + \zeta)^2 + \frac{3\varepsilon}{32t}\alpha(\xi + \zeta)^2 + o(1),$$

$$z_{15} \equiv \frac{1}{2}W(z_1, z_3) = z_1 z_{12} + 4z_3 z_7 = -\frac{\varepsilon}{2t^2}\alpha^2(\xi + \zeta) + o(\frac{1}{t}),$$

where $W(s_j, s_k) = \dot{s}_j s_k - s_j \dot{s}_k$ is the Wronskian of s_j and s_k. Using the functions $1, z_1, ..., z_{15}$, one embeds the curves $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ into a hyperplane of $\mathbb{P}^{15}(\mathbb{C})$. Thus embedded, these curves have one point in common at which they are tangent to each other. In the neighborhood of $\alpha = \infty$ the curve $\mathcal{H}$ has two points at which $\xi + \zeta$ behaves as follows :

$$\xi + \zeta = 4\alpha^3 + o(\alpha),$$

picking the + sign for ζ and

$$\xi + \zeta = \frac{4c_3\alpha^2 + 4c_4\alpha + c_5}{\alpha^3} + \text{lower order terms},$$

picking the - sign for ζ. Therefore, choosing the + sign for ζ and dividing the vector $(1, z_1, ..., z_{15})$ by $z_{14} = z_3^2$, the corresponding point is sent to the point $[0 : \cdots : 1 : 0] \in \mathbb{P}^{15}(\mathbb{C})$, which is independent of ε. The choice of the sign - for ζ conducts to two different points, taking into account the sign of ε. Therefore, the divisor $\mathcal{D}$ obtained in this way has genus 5 and thus $2\mathcal{D}$ has genus 17, satisfying the relation : geometric genus of $2\mathcal{D} = N+2$, i.e., $2\mathcal{D} \subset \mathbb{P}^{15}(\mathbb{C}) = \mathbb{P}^{g-2}(\mathbb{C})$. Following the method explained and used repeatedly in previous problems, we show that the affine surface

$$\mathcal{B} = \bigcap_{k=1}^{5} \{z = (z_1, z_2, z_3, z_4, z_5) : F_k(z) = c_k\} \subset \mathbb{C}^7, \tag{5.7}$$

can be completed into an Abelian surface $\widetilde{\mathcal{B}}$ by adjoining at infinity the divisor $\mathcal{D} = \mathcal{H}_i + \mathcal{H}_{-i}$. The variety $\widetilde{\mathcal{B}}$ is equipped with two commuting, linearly independent vector fields. Let dt_1 and dt_2 be two holomorphic 1-forms on $\widetilde{\mathcal{B}}$ corresponding respectively to the vectors fields X_{F_1} and X_{F_2}. Letting $y_1 = \frac{x_1}{x_2}$, $y_2 = \frac{1}{x_2}$, one obtains the differential forms

$$\omega_1 = dt_1|_{\mathcal{H}_\varepsilon} = \frac{1}{\Delta}\left(\frac{\partial y_1}{\partial t_2}dy_2 - \frac{\partial y_2}{\partial t_2}dy_1\right) = \frac{a\alpha}{\zeta}d\alpha,$$

$$\omega_2 = dt_2|_{\mathcal{H}_\varepsilon} = -\frac{1}{\Delta}\left(\frac{\partial y_1}{\partial t_1}dy_2 - \frac{\partial y_2}{\partial t_1}dy_1\right) = \frac{b}{\zeta}d\alpha,$$

where a and b are constants and $\Delta = \frac{\partial y_2}{\partial t_1}\frac{\partial y_1}{\partial t_2} - \frac{\partial y_2}{\partial t_2}\frac{\partial y_1}{\partial t_1}$. The points where the vector field X_{F_1} is tangent to the curves $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ on $\widetilde{\mathcal{B}}$ are provided by the zeros of the form ω_2. Note that X_{F_1} is tangent to $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ at the point where both curves touch; this point correspond to $\alpha = \infty$. The involution

$$\sigma : (z_1, z_2, z_3, z_4, z_5, z_6, z_7) \longmapsto (z_1, z_2, z_3, z_4, -z_5, z_6, -z_7),$$

on $\mathcal{B}$ acts on the free parameters as follows

$$\sigma : (t, \alpha, \zeta, \varepsilon, c_1, c_2, c_3, c_4, c_5) \longmapsto (-t, \alpha, \zeta, -\varepsilon, c_1, c_2, c_3, c_4, c_5).$$

Hence $\mathcal{H}_i = \sigma\mathcal{H}_{-i}$ and geometrically this means that $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are deduced from one another by a translation in the Abelian variety $\widetilde{\mathcal{B}}$. Therefore, we have the following result [6, 36]:

Theorem 12. *a) The differential system* (5.4) *is algebraically completely integrable. The Laurent solution* (5.5) *depend on six free parameters. The affine surface* $\mathcal{B}$ (5.7) *completes into an Abelian variety* $\widetilde{\mathcal{B}}$ *by adjoining a divisor* $\mathcal{H}_i + \mathcal{H}_{-i}$ *where* $\mathcal{H}_{+i}$ *and* $\mathcal{H}_{-i}$ *are two copies of the same genus* 2 *hyperelliptic curve* $\mathcal{H}$ (5.6) *for* $\varepsilon = \pm 1$, *that intersect each other in a tacnode belonging to* $\mathcal{H}_i + \mathcal{H}_{-i}$.

b) The invariant variety $\mathcal{A}$ (5.2) can be compactified as a cyclic double cover $\overline{\mathcal{A}}$ of the Jacobian of a genus two curve, ramified along the divisor $\mathcal{H}_i+\mathcal{H}_{-i}$. Moreover, $\overline{\mathcal{A}}$ is smooth except at the point (tacnode) lying over the singularity ((of type A_3) of $\mathcal{H}_i+\mathcal{H}_{-i}$ and the resolution $\widetilde{\mathcal{A}}$ of $\overline{\mathcal{A}}$ is a surface of general type with invariants : Euler characteristic of $\widetilde{\mathcal{A}} = \mathcal{X}(\widetilde{\mathcal{A}}) = 1$ and geometric genus of $\widetilde{\mathcal{A}} = p_g(\widetilde{\mathcal{A}}) = 2$. Consequently, the system (5.1) is algebraic completely integrable in the generalized sense.

Note also that the extended system (5.4) include some others known integrable systems. In particular, it is shown [6] that the system (5.1) is rationally related to the three-body Toda system.

6 The Lagrange top

The Lagrange top is a symmetric top with a constant vertical gravitational force acting on its center of mass and leaving the base point of its body symmetry axis fixed. We will show that the differential equations governing the motion of the Lagrange top form an algebraic completely integrable system in the generalized sense. The equations of this problem are explicitly written in the form

$$\begin{aligned}
\lambda_1\dot{m}_1 &= \lambda_1(\lambda_3-\lambda_1)m_2m_3-\gamma_2, &\quad \dot{\gamma}_1 &= \lambda_3m_3\gamma_2-\lambda_1m_2\gamma_3,\\
\lambda_1\dot{m}_2 &= \lambda_1(\lambda_1-\lambda_3)m_1m_3+\gamma_1, &\quad \dot{\gamma}_2 &= \lambda_1m_1\gamma_3-\lambda_3m_3\gamma_1,\\
\dot{m}_3 &= 0, &\quad \dot{\gamma}_3 &= \lambda_1(m_2\gamma_1-m_1\gamma_2).
\end{aligned}$$

This system admits the following four first integrals :

$$\begin{aligned}
H_1 &= \frac{\lambda_1^2}{2}(m_1^2+m_2^2)+\frac{\lambda_1\lambda_3}{2}m_3^2-\gamma_3, &\quad H_3 &= \lambda_1(m_1\gamma_1+m_2\gamma_2+m_3\gamma_3),\\
H_2 &= \gamma_1^2+\gamma_2^2+\gamma_3^2, &\quad H_4 &= \lambda_3m_3.
\end{aligned}$$

and forms an integrable Hamiltonian vector field in the sense of Liouville. The Poisson structure is given by $\{m_i,m_j\}=-\epsilon_{ijk}m_k$, $\{m_i,\gamma_j\}=-\epsilon_{ijk}\gamma_k$, $\{\gamma_i,\gamma_j\}=0$, where $1\le i,j,k\le 3$ and ϵ_{ijk} is the total antisymmetric tensor for which $\epsilon_{ijk}=1$. Let

$$\mathcal{M}_c=\{(m_1,m_2,m_3,\gamma_1,\gamma_2,\gamma_3)\in\mathbb{C}^6 : H_1=c_1, H_2=1, H_3=c_3, H_4=c_4\},$$

be the affine variety defined by the intersection of the constants of the motion and let $\mathbb{C}^*\sim\mathbb{C}/2\pi i\mathbb{Z}$ be the group of rotations defined by the flow of the vector field generated by H_4, i.e.,

$$\dot{m}_1=m_2,\qquad \dot{m}_2=-m_1,\qquad \dot{m}_3=0,$$

$$\dot{\gamma}_1=\gamma_2,\qquad \dot{\gamma}_2=-\gamma_1,\qquad \dot{\gamma}_3=0.$$

We know that the quotient $\mathcal{M}_c/\mathbb{C}^*$ is an elliptic curve. We show that the algebraic variety $\mathcal{M}_c$ is not isomorphic to the direct product of the curve $\mathcal{M}_c/\mathbb{C}^*$ and $\mathbb{C}^*$. For generic constants c_j, the complex invariant manifold $\mathcal{M}_c$ is biholomorphic to an affine subset of $\mathbb{C}^2/\Lambda$ where $\Lambda \subset \mathbb{C}^2$ is a lattice of rank 3,

$$\Lambda = \mathbb{Z}\left(\begin{array}{c} 2\pi i \\ 0 \end{array}\right) \oplus \mathbb{Z}\left(\begin{array}{c} 0 \\ 2\pi i \end{array}\right) \oplus \mathbb{Z}\left(\begin{array}{c} \tau_1 \\ \tau_2 \end{array}\right), \quad \mathrm{Re}(\tau_1) < 0.$$

Hence, $\mathbb{C}^2/\Lambda$ is an non-compact algebraic group and can be considered as a non-trivial extension of the elliptic curve $\mathbb{C}/\{2\pi i\mathbb{Z} \oplus \tau_1\mathbb{Z}\}$ by $\mathbb{C}^* \sim \mathbb{C}/2\pi i\mathbb{Z}$,

$$0 \longrightarrow \mathbb{C}/2\pi i\mathbb{Z} \longrightarrow \mathbb{C}^2/\Lambda \xrightarrow{\varphi} \mathbb{C}/\{2\pi i\mathbb{Z} \oplus \tau_1\mathbb{Z}\} \longrightarrow 0, \quad \varphi(z_1, z_2) = z_1.$$

The algebraic group $\mathbb{C}/2\pi i\mathbb{Z}$ is the generalized Jacobian of an elliptic curve with two points identified at infinity. We have the following result [12] :

Theorem 13. *The differential system governing the Lagrange top form an algebraic completely integrable system in the generalized sense.*

7 The Yang-Mills system and cyclic covering of Abelian varieties

We consider the Hamiltonian,

$$H = \frac{1}{2}(x_1^2 + x_2^2 + a_1 y_1^2 + a_2 y_2^2) + \frac{1}{4}y_1^4 + \frac{1}{4}a_3 y_2^4 + \frac{1}{2}a_4 y_1^2 y_2^2.$$

It has been shown in [21] that if $a_2 = 4a_1 \equiv 4a$, $a_3 = 16$, $a_4 = 6$, i.e.,

$$H_1 \equiv H = \frac{1}{2}(x_1^2 + x_2^2) + \frac{a}{2}(y_1^2 + 4y_2^2) + \frac{1}{4}y_1^4 + 4y_2^4 + 3y_1^2 y_2^2, \tag{7.1}$$

the corresponding system, i.e.,

$$\begin{aligned} \dot{y}_1 &= x_1, \qquad \dot{x}_1 = -(a + y_1^2 + 6y_2^2)y_1, \\ \dot{y}_2 &= x_2, \qquad \dot{x}_2 = -2(2a + 3y_1^2 + 8y_2^2)y_2, \end{aligned} \tag{7.2}$$

is integrable, the second integral is

$$H_2 = a_1 y_1^2 y_2 + y_1 y_2\left(y_1^3 + 2y_1 y_2^2\right) + x_1\left(x_2 y_1 - x_1 y_2\right) \tag{7.3}$$

but no description of solutions is given. We solve the system (7.2) in terms of genus two hyperelliptic functions. When one examines all possible singularities of the system (7.2), one finds that it possible for the variable y_1 to contain square root terms of the type $\sqrt{t}$, which are strictly not allowed by the Painlevé test (the general solutions should have no movable singularities other than poles in the complex plane). Let $\mathcal{A}$ be the affine variety defined by

$$\mathcal{A} = \bigcap_{k=1}^{2} \left\{ z \in \mathbb{C}^4 : H_k(z) = b_k \right\}, \tag{7.4}$$

where $(b_1, b_2) \in \mathbb{C}^2$. Since $\mathcal{A}$ is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ over $(b_1, b_2) \in \mathbb{C}^2$, for almost all b_1, b_2, therefore $\mathcal{A}$ is a smooth affine surface. We show that the system (7.2) admits Laurent solutions in $\sqrt{t}$, depending on three free parameters: u, v and w. These pole solutions restricted to the surface $\mathcal{A}$ (7.4) are parameterized by two smooth curves $\mathcal{C}_{\varepsilon=\pm i}$ (7.6) of genus 4. Applying the method explained in Piovan [36], we show that the invariant variety $\mathcal{A}$ (7.4) can be completed as a cyclic double cover $\overline{\mathcal{A}}$ of the Jacobian of a genus curve, ramified along a divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ where $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are two isomorphic hyperelliptic curves (7.12) of genus 2 that intersect in only one point at which they are tangent to each other. Moreover, $\overline{A}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$ and the resolution $\widetilde{A}$ of $\overline{A}$ is a surface of general type with invariants : Euler characteristic of $\widetilde{A} = \mathcal{X}(\widetilde{A}) = 1$ and geometric genus of $\widetilde{A} = p_g(\widetilde{A}) = 2$. Consequently, the system (7.2) is algebraic completely integrable in the generalized sense.

The system (7.2) is weight-homogeneous with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that H_1 and H_2 have weight 4 and 5 respectively.

Theorem 14. *The system (7.2) admits Laurent solutions in $\sqrt{t}$, depending on three free parameters: u, v and w. These solutions restricted to the surface $\mathcal{A}$ (7.4) are parameterized by two smooth curves $\mathcal{C}_{\varepsilon=\pm i}$ (7.6) of genus 4.*

Proof. The system (7.2) possesses 3-dimensional family of Laurent solutions (principal balances) depending on three free parameters u, v and w. There are precisely two such families, labelled by $\varepsilon = \pm i$, and they are explicitly given as follows

$$\begin{aligned}
y_1 &= \frac{1}{\sqrt{t}}\Big(u - \frac{1}{2}u^3 t + vt^2 + u^2\big(-\frac{11}{16}u^5 + \frac{1}{3}au + v\big)t^3 \\
&\quad + \frac{u}{4}\big(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3 v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w\big)t^4 + \cdots\Big), \qquad (7.5)\\
y_2 &= \frac{\varepsilon\sqrt{2}}{4t}\Big(1 + u^2 t + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{8}u(24v - u^5)t^3 - 2\varepsilon\sqrt{2}wt^4 + \cdots\Big),\\
x_1 &= \frac{1}{t\sqrt{t}}\Big(-\frac{1}{2}u - \frac{1}{4}u^3 t + \frac{3}{2}vt^2 + \frac{5}{2}u^2\big(-\frac{11}{16}u^5 + \frac{1}{3}au + v\big)t^3 \\
&\quad + \frac{7u}{8}\big(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3 v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w\big)t^4 + \cdots\Big),\\
x_2 &= \frac{\varepsilon\sqrt{2}}{4t^2}\Big(-1 + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{4}u(24v - u^5)t^3 - 6\varepsilon\sqrt{2}wt^4 + \cdots\Big).
\end{aligned}$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter w linearly, leading to an equation connecting the two remaining parameters u and v :

$$2v^2 + \frac{1}{6}(15u^4 - 8a)uv - \frac{39}{32}u^{10} + \frac{7}{6}au^6 + \frac{2}{9}(a^2 + 9b_1)u^2 - \varepsilon\sqrt{2}b_2 = 0. \tag{7.6}$$

According to Hurwitz' formula, this defines two smooth curves $\mathcal{C}_\varepsilon$ ($\varepsilon = \pm i$) of genus 4, which establishes the theorem. □

Theorem 15. *The system of differential equations (7.2) can be integrated in terms of genus 2 hyperelliptic functions.*

Proof. We set

$$\begin{aligned} y_2 &= s_1 + s_2, \\ y_1^2 &= -4s_1s_2, \\ x_2 &= \dot{s}_1 + \dot{s}_2, \\ y_1x_1 &= -2(\dot{s}_1s_2 + s_1\dot{s}_2). \end{aligned}$$

The latter equation together with the second implies that

$$x_1^2 = -\frac{(\dot{s}_1s_2 + s_1\dot{s}_2)^2}{s_1s_2}.$$

In term of these new variables, equations (7.1) and (7.3) take the following form

$$\begin{aligned} &(s_1 - s_2)\left(s_2(\ldots_1)^2 - s_1(\ldots_2)^2\right) \\ &+4s_1s_2\left(2s_1^4 + 2s_1^3s_2 + 2s_1^2s_2^2 + 2s_1s_2^3 + 2s_2^4 + as_1^2 + as_1s_2 + as_2^2\right) \\ &-2b_1s_1s_2 = 0, \\ &(s_1 - s_2)\left(s_2^2(\dot{s}_1)^2 - s_1^2(\dot{s}_2)^2\right) \\ &+4s_1^2s_2^2\left(s_1 + s_2\right)\left(a + 2s_1^2 + 2s_2^2\right) + b_2s_1s_2 = 0. \end{aligned}$$

These equations are solved linearly for $(\dot{s}_1)^2$ and $(\dot{s}_2)^2$ as

$$\begin{aligned} (\dot{s}_1)^2 &= \frac{s_1(-8s_1^5 - 4as_1^3 + 2b_1s_1 + b_2)}{(s_1 - s_2)^2}, \\ (\dot{s}_2)^2 &= \frac{s_2(-8s_2^5 - 4as_2^3 + 2b_1s_2 + b_2)}{(s_1 - s_2)^2}, \end{aligned}$$

which leads immediately to

$$\begin{aligned} \frac{ds_1}{\sqrt{P_6(s_1)}} - \frac{ds_2}{\sqrt{P_6(s_2)}} &= 0, \\ \frac{s_1ds_1}{\sqrt{P_6(s_1)}} - \frac{s_2ds_2}{\sqrt{P_6(s_2)}} &= dt, \end{aligned}$$

where

$$P_6(s) = s(-8s^5 - 4as^3 + 2b_1 s + b_2).$$

The solution of these equations is done using the Abel transformation $\mathcal{H} \longrightarrow Jac(\mathcal{H})$, where the hyperelliptic curve $\mathcal{H}$ of genus 2 is given by an equation : $\zeta^2 = P_6(s)$. Consequently, the differential equations (7.2) are integrated in terms of genus 2 hyperelliptic functions. This ends the proof of the theorem. □

We have seen that it possible for the variables y_1 and x_1 to contain square root terms of the type $\sqrt{t}$, which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing some new variables $z_1, \ldots, z_5$, which restores the Painlevé property to the system. Indeed, let

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C}^5, (y_1, y_2, x_1, x_2) \longmapsto (z_1, z_2, z_3, z_4, z_5), \tag{7.7}$$

be a morphism on the affine variety $\mathcal{A}$ (7.4) where $z_1, \ldots, z_5$ are defined as

$$z_1 = y_1^2, \quad z_2 = y_2, \quad z_3 = x_2, \quad z_4 = y_1 x_1, \quad z_5 = 2y_1^2 y_2^2 + x_1^2.$$

The morphism (7.7) maps the vector field (7.2) into the system

$$\begin{array}{rcl} \dot{z}_1 & = & 2z_4, \qquad \dot{z}_3 = -4az_2 - 6z_1 z_2 - 16z_2^3, \\ \dot{z}_2 & = & z_3, \qquad \dot{z}_4 = -az_1 - z_1^2 - 8z_1 z_2^2 + z_5, \\ \dot{z}_5 & = & -8z_2^2 z_4 - 2az_4 - 2z_1 z_4 + 4z_1 z_2 z_3, \end{array} \tag{7.8}$$

in five unknowns having three quartic invariants

$$\begin{array}{rcl} F_1 & = & \frac{1}{2}z_5 + 2z_1 z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}az_1 + 2az_2^2 + \frac{1}{4}z_1^2 + 4z_2^4, \\ F_2 & = & az_1 z_2 + z_1^2 z_2 + 4z_1 z_2^3 - z_2 z_5 + z_3 z_4, \\ F_3 & = & z_1 z_5 - 2z_1^2 z_2^2 - z_4^2. \end{array} \tag{7.9}$$

This system is completely integrable and the Hamiltonian structure is defined by the Poisson bracket

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^{5} J_{kl} \frac{\partial F}{\partial z_k} \frac{\partial H}{\partial z_l},$$

where $\frac{\partial H}{\partial z} = \left(\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3}, \frac{\partial H}{\partial z_4}, \frac{\partial H}{\partial z_5}\right)^\top$, and

$$J = \begin{pmatrix} 0 & 0 & 0 & 2z_1 & 4z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4z_1 z_2 \\ -2z_1 & 0 & 0 & 0 & 2z_5 - 8z_1 z_2^2 \\ -4z_4 & 0 & 4z_1 z_2 & -2z_5 + 8z_1 z_2^2 & 0 \end{pmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The system (7.9) can be written as

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

where $H = F_1$. The second flow commuting with the first is regulated by the equations $\dot{z} = J\frac{\partial F_2}{\partial z}$, $z = (z_1, z_2, z_3, z_4, z_5)^\top$, i.e.,

$$\begin{aligned}
\dot{z}_1 &= 2z_1z_3 - 4z_2z_4, \qquad \dot{z}_3 = z_5 - 8z_1z_2^2 - az_1 - z_1^2,\\
\dot{z}_2 &= z_4, \qquad\qquad\qquad \dot{z}_4 = -2az_1z_2 - 4z_1^2z_2 - 2z_2z_5,\\
\dot{z}_5 &= -4az_2z_4 - 4z_1z_2z_4 - 16z_2^3z_4 - 2z_3z_5 + 8z_1z_2^2z_3.
\end{aligned}$$

These vector fields are in involution, i.e.,

$$\{F_1, F_2\} = \left\langle \frac{\partial F_1}{\partial z}, J\frac{\partial F_2}{\partial z} \right\rangle = 0,$$

and the remaining one is Casimir, i.e., $J\frac{\partial F_3}{\partial z} = 0$. Let $\mathcal{B}$ be the complex affine variety defined (for generic $(c_1, c_2, c_3) \in \mathbb{C}^3$) by

$$\mathcal{B} = \bigcap_{k=1}^{2} \{z : F_k(z) = c_k\} \subset \mathbb{C}^5, \tag{7.10}$$

The system (7.9) possesses Laurent series solutions which depend on four free parameters : α, β, γ and θ :

$$\begin{aligned}
z_1 &= \frac{1}{t}(\alpha - \alpha^2 t + \beta t^2 + \frac{1}{6}\alpha(3\beta - 9\alpha^3 + 4a\alpha)t^3 + \gamma t^4 + \cdots),\\
z_2 &= \frac{\varepsilon\sqrt{2}}{4t}(1 + \alpha t + \frac{1}{3}(-3\alpha^2 + 2a)t^2 + \frac{1}{2}(3\beta - \alpha^3)t^3 - 2\varepsilon\sqrt{2}\theta t^4 + \cdots),\\
z_3 &= \frac{\varepsilon\sqrt{2}}{4t^2}(-1 + \frac{1}{3}(-3\alpha^2 + 2a)t^2 + (3\beta - \alpha^3)t^3 - 6\varepsilon\sqrt{2}\theta t^4 + \cdots),\\
z_4 &= \frac{1}{2t^2}(-\alpha + \beta t^2 + \frac{1}{3}\alpha(3\beta - 9\alpha^3 + 4a\alpha)t^3 + 3\gamma t^4 + \cdots),\\
z_5 &= \frac{1}{t}(-\frac{1}{3}a\alpha + \alpha^3 - \beta + (3\alpha^4 - a\alpha^2 - 3\alpha\beta)t\\
&\quad + (4\varepsilon\sqrt{2}\alpha\theta + 2\gamma + \frac{8}{3}a\alpha^3 - \frac{1}{3}a\beta - \alpha^2\beta - 3\alpha^5 - \frac{4}{9}a^2\alpha)t^2 + \cdots),
\end{aligned} \tag{7.11}$$

with $\varepsilon = \pm i$. These meromorphic solutions restricted to the surface $\mathcal{B}$ (7.10) are parameterized by two isomorphic smooth hyperelliptic curves $\mathcal{H}_{\varepsilon=\pm i}$ of genus 2 :

$$\beta^2 + \frac{2}{3}(3\alpha^2 - 2a)\alpha\beta - 3\alpha^6 + \frac{8}{3}a\alpha^4 + \frac{4}{9}(a^2 + 9c_1)\alpha^2 - 2\varepsilon\sqrt{2}c_2\alpha + c_3 = 0, \tag{7.12}$$

The variety $\mathcal{B}$ (7.10) is embedded in $\mathbb{P}^{15}(\mathbb{C})$ and generically is the affine part of an Abelian surface $\widetilde{\mathcal{B}}$, more precisely the Jacobian of a genus 2 curve. The reduced divisor at infinity $\widetilde{\mathcal{B}}\backslash\mathcal{B} = \mathcal{H}_i+\mathcal{H}_{-i}$, consists of two smooth isomorphic genus 2 curves $\mathcal{H}_\varepsilon$ (7.12), that intersect in only one point at which they are tangent to each other. The system of differential equations (7.9) is algebraic complete integrable and the corresponding flows evolve on $\widetilde{\mathcal{B}}$.

Observe that the reflection σ on the affine variety $\mathcal{B}$ amounts to the flip

$$\sigma : (z_1, z_2, z_3, z_4, z_5) \longmapsto (z_1, z_2, -z_3, -z_4, z_5),$$

changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1, t_2) \longmapsto (-t_1, -t_2)$ on $\widetilde{\mathcal{B}}$, where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2}. By inspection, we can see that the involution σ acts on the parameters of the Laurent solution (7.11) as follows

$$\sigma : (t, \alpha, \beta, \gamma, \theta, \varepsilon) \longmapsto (-t, -\alpha, -\beta, -\gamma, -\theta, -\varepsilon),$$

interchanges the curves $\mathcal{H}_{\varepsilon=\pm i}$ (7.12). Geometrically, this involution interchanges $\mathcal{H}_i$ and $\mathcal{H}_{-i}$, i.e., $\mathcal{H}_{-i} = \sigma\mathcal{H}_i$.

The asymptotic solution (7.5) can be read off from (7.11) and the change of variable : $q_1 = \sqrt{z_1}$, $q_2 = z_2$, $p_1 = z_4/q_1$, $p_2 = z_3$. The function z_1 has a simple pole along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ and a double zero along a hyperelliptic curve of genus 2 defining a double cover of $\widetilde{\mathcal{B}}$ ramified along $\mathcal{H}_i + \mathcal{H}_{-i}$. Like before, we have the following result :

Theorem 16. *The invariant surface $\mathcal{A}$ (7.4) can be completed as a cyclic double cover $\overline{\mathcal{A}}$ of the Abelian surface $\widetilde{\mathcal{B}}$ (the Jacobian of a genus 2 curve), ramified along the divisor $\mathcal{H}_i+\mathcal{H}_{-i}$. The system (7.2) is algebraic complete integrable in the generalized sense. Moreover, $\overline{\mathcal{A}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i+\mathcal{H}_{-i}$ and the resolution $\widetilde{\mathcal{A}}$ of $\overline{\mathcal{A}}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{\mathcal{A}}) = 1$ and $p_g(\widetilde{\mathcal{A}}) = 2$.*

Proof. We have shown that the morphism φ (7.7) maps the vector field (49) into an algebraic completely integrable system (7.9) in five unknowns and the affine variety $\mathcal{A}$ (7.4) onto the affine part $\mathcal{B}$ (7.10) of an Abelian variety $\widetilde{\mathcal{B}}$ (the Jacobian of a genus 2 curve with $\widetilde{\mathcal{B}}\backslash\mathcal{B} = \mathcal{H}_i + \mathcal{H}_{-i}$). Observe that $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an unramified cover. The curves $\mathcal{C}_\varepsilon$ (7.6) play an important role in the construction of a compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$. Let G be a cyclic group of two elements $\{-1, 1\}$ on $V_\varepsilon^j = U_\varepsilon^j \times \{\tau \in \mathbb{C} : 0 < |\tau| < \delta\}$, where $\tau = \sqrt{t}$ and U_ε^j is an affine chart of $\mathcal{C}_\varepsilon$ for which the Laurent solutions (7.5) are defined. The action of G is defined by $(-1) \circ (u, v, \tau) = (-u, -v, -\tau)$ and is without fixed points in V_ε^j. So we can identify the quotient V_ε^j/G with the image of the smooth map $h_\varepsilon^j : V_\varepsilon^j \to \mathcal{A}$ defined by the expansions (7.5). We have

$$(-1, 1).(u, v, \tau) = (-u, -v, \tau), \quad (1, -1).(u, v, \tau) = (u, v, -\tau),$$

i.e., $G \times G$ acts separately on each coordinate. Thus, identifying V_ε^j/G^2 with the image of $\varphi \circ h_\varepsilon^j$ in B. Note that $\mathcal{A}_\varepsilon^j = V_\varepsilon^j/G$ is smooth (except for a finite number of points) and the coherence of the $\mathcal{A}_\varepsilon^j$ follows from the coherence of V_ε^j and the action of G. Now by taking $\mathcal{A}$ and by gluing on various varieties $A_\varepsilon^j \backslash \{\text{some points}\}$, we obtain a smooth complex manifold $\widehat{\mathcal{A}}$ which is a double cover of the Abelian variety $\widetilde{\mathcal{B}}$ ramified along $\mathcal{H}_i + \mathcal{H}_{-i}$, and therefore can be completed to an algebraic cyclic cover of $\widetilde{\mathcal{B}}$. To see what happens to the missing points, we must investigate the image of $\mathcal{C}_\varepsilon \times \{0\}$ in $\cup \mathcal{A}_\varepsilon^j$. The quotient $\mathcal{C}_\varepsilon \times \{0\}/G$ is birationally equivalent to the smooth hyperelliptic curve Γ_ε of genus 2 :

$$2w^2 + \frac{1}{6}(15z^2 - 8a)zw + z(-\frac{39}{32}z^5 + \frac{7}{6}az^3 + \frac{2}{9}(a^2 + 9b_1)z - \varepsilon\sqrt{2}b_2) = 0,$$

where $w = uv, z = u^2$. The curve Γ_ε is birationally equivalent to $\mathcal{H}_\varepsilon$. The only points of $\mathcal{C}_\varepsilon$ fixed under $(u, v) \longmapsto (-u, -v)$ are the two points at ∞, which correspond to the ramification points of the map

$$\mathcal{C}_\varepsilon \times \{0\} \xrightarrow{2-1} \Gamma_\varepsilon : (u, v) \longmapsto (z, w),$$

and coincides with the points at ∞ of the curve $\mathcal{H}_\varepsilon$. Then the variety $\widehat{\mathcal{A}}$ constructed above is birationally equivalent to the compactification $\overline{\mathcal{A}}$ of the generic invariant surface A. So $\overline{\mathcal{A}}$ is a cyclic double cover of the Abelian surface $\widetilde{\mathcal{B}}$ (the Jacobian of a genus 2 curve) ramified along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$, where $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ intersect each other in a tacnode. It follows that the system (7.2) is algebraic complete integrable in the generalized sense. Moreover, $\overline{\mathcal{A}}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$. In term of an appropriate local holomorphic coordinate system (x, y, z), the local analytic equation about this singularity is $x^4 + y^2 + z^2 = 0$. Now, let $\widetilde{\mathcal{A}}$ be the resolution of singularities of $\overline{\mathcal{A}}$, $\mathcal{X}(\widetilde{\mathcal{A}})$ be the Euler characteristic of $\widetilde{\mathcal{A}}$ and $p_g(\widetilde{A})$ the geometric genus of $\widetilde{\mathcal{A}}$. Then $\widetilde{\mathcal{A}}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{\mathcal{A}}) = 1$ and $p_g(\widetilde{\mathcal{A}}) = 2$. This concludes the proof of the theorem. □

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