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ON THE VALUE SHARING OF SHIFT MONOMIAL OF MEROMORPHIC FUNCTIONS

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Abstract. We employ the notion of weighted and truncated sharing to study the uniqueness problems of generalized shift monomial sharing the same 1-points. The corollary deducted from our main results will improve a number of results of recent time. As an application of the main result we will also improve a recent result under the periphery of a more generalized shift operator. Some examples have been exhibited by us relevant to the content of the paper.

1 Introduction and Results

We adopt the standard notations of value distribution theory (see [8]) and by meromorphic functions we shall always mean meromorphic functions in the complex plane. For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic function of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. The order of f is defined by

$$\rho(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r,f)}{\log r}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let a be any complex constant. We say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share a IM (ignoring multiplicities) if we do not consider the multiplicities.

Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{k)}(a;f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a;f)$=$E_{\infty)}(a;g)$ we say that f, g share the value a CM.

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The shift and the difference operator of a meromorphic function are respectively represented by $f(z+c)$ and $\Delta f = f(z+c) - f(z)$, $c \in \mathbb{C}\backslash\{0\}$.

For a transcendental meromorphic function, it is an interesting feature among researchers to investigate the value distributions of $f^n f^{'}$.

In this respect, the first attempt was made by Hayman. 1959, Hayman (see [7], Corollary of Theorem 9) obtained the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n \in \mathbb{N}$ such that $n \geq 3$. Then $f^n f' = 1$ has infinitely many solutions.*

In 1979, Mues [18] settled the case for $n = 2$.

Laine-Yang [12] converted the above investigation into the value distribution of difference products of entire functions as follows.

Theorem B. *[12] Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Afterwards, many researchers like Liu-Yang [14], Zhang [25], Chen-Chen [5] further extended the results of Laine-Yang [12].

Meanwhile, there have been an increasing interest among the researchers to investigate the uniqueness problem of entire or meromorphic functions and their shift or difference operators. In this respect, a handful number of elegant results have been appeared in the literature.

Around 2001, the notion of weighted sharing was appeared in the literature: [10], [11]. It indicates the gradual change of shared values from CM to IM. Below we recall the definition.

Definition 1. *[11] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2010, motivated by the results of Yang-Hua [22], Qi-Yang-Liu [19] studied the uniqueness of the difference polynomials of entire functions and obtained the following result.

Theorem C. *[19] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $c \in \mathbb{C} \setminus \{0\}$; let $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share $(1, \infty)$, then $f(z)g(z) \equiv t_1$ or $f(z) \equiv t_2 g(z)$ for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2012, Wang-Han-Wen [20] relaxed the nature of sharing in the following manner.

Theorem D. *[20] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $c \in \mathbb{C}\setminus\{0\}$. Let $n \in \mathbb{N}$ such that $E_{k)}(1, f^n(z)f(z+c)) = E_{k)}(1, g^n(z)g(z+c))$.*

(i) If $k = 3$ and $n \geq 6$ or

(ii) if $k = 2$ and $n \geq 7$ or

(iii) if $k = 1$ and $n \geq 10$,
then conclusion of Theorem C holds.

In the mean time, in 2011, Liu-Liu-Cao [15] extended Theorem C for meromorphic function as follows.

Theorem E. *[15] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order and $c \in \mathbb{C}\setminus\{0\}$. Let $n \in \mathbb{N}$ such that $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share $(1, l)$.*

(i) If $l = \infty$ and $n \geq 14$ or

(ii) if $l = 0$ and $n \geq 26$,
then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$ for some constants t that satisfy $t^{n+1} = 1$.

In 2015, for meromorphic function, Liu-Wang-Liu [16] improved Theorem C in the direction of Theorem D. Their result was as follows:

Theorem F. *[16] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order and $c \in \mathbb{C}\setminus\{0\}$. Let $n \in \mathbb{N}$ such that $E_{k)}(1, f^n(z)f(z+c)) = E_{k)}(1, g^n(z)g(z+c))$.*

(i) If $k \geq 3$ and $n \geq 14$ or

(ii) if $k = 2$ and $n \geq 16$ or

(iii) if $k = 1$ and $n \geq 22$,
then conclusion of Theorem C holds.

We would like to point out that though for a meromorphic function h, all the previous authors mentioned $h^n(z)h(z+c)$ as difference polynomial, but there was no presence of difference operator, in the expression. So to make the definition more pragmatic, it will be reasonable to call $h^n(z)h(z+c)$ as shift monomial. In fact, in the paper we shall deal with the more generalized shift monomial.

Let us assume that $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial, where $a_0, a_1, \ldots, a_n \neq 0$ be complex constants and $n(\geq 1)$ is an integer. Also let $\mu_j(\geq 0)(j = 1, 2, \ldots, s)$ are positive integers such that $\sigma = \sum\limits_{j=1}^{s} \mu_j$. For a meromorphic function h and for finite complex constants $c_j(j=1,2,\ldots,s)$ we say $P(h)(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}$ as the generalized shift monomial.

The purpose of this paper is to improve all the Theorems C-F for the most generalized shift polynomial under different relaxed sharing environment. We have also relaxed the CM sharing results to sharing of weight 2.

Throughout the paper for sake of convenience we assume that $\Gamma_0=m_1+m_2$ and $\Gamma_1=m_1+2m_2$, where m_1, m_2 respectively be the number of simple and multiple zeros of $P(z)$.

We now present the following four theorems which are the main results of the paper.

Theorem 2. *Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of finite order, $c_j(j=1,2,\ldots,s)$ be finite complex constants. Suppose also that $F=P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}$ and $G=P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$ share $(1,l)$. Now*

(i) if $l\geq 2$ and $n>2\Gamma_1+5\sigma+4$ or

(ii) if $l=1$ and $n>2\Gamma_1+\frac{1}{2}\Gamma_0+5\sigma+s+\frac{9}{2}$ or

(iii) if $l=0$ and $n>2\Gamma_1+3\Gamma_0+5\sigma+6s+7$,
then either

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\equiv 1$$

or

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

If in particular, for an integer $n(\geq 1)$ we take $P(f)=f^n$ and

(i) if $l\geq 2$ and $n>8+5\sigma$ or

(ii) if $l=1$ and $n>9+5\sigma+s$ or

(iii) if $l=0$ and $n>14+5\sigma+6s$,
then either $f\equiv tg$, or $fg\equiv t$, for some constant t such that $t^{n+\sigma}=1$.

From Theorem 2, putting $\mu_1=1=s$, we can easily deduce the following corollary.

Corollary 3. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order, and $c(\neq 0)$ be a finite complex constant. Suppose $n(\geq 1)$ be an integer such that $f^n(z)(f(z+c))$ and $g^n(z)(g(z+c))$ share $(1,l)$. Now*

(i) if $l\geq 2$ and $n\geq 14$ or

(ii) if $l=1$ and $n\geq 16$ or

(iii) if $l=0$ and $n\geq 26$,
then either $f\equiv tg$ or $fg\equiv t$, for some constant t such that $t^{n+1}=1$.

The following examples show that both conclusions of Theorem 2 actually holds when $P(f)=f^n$ and $c_j=c$, where $c(\neq 0)$ be a finite complex constant, for all $j=1,2,\dots,s$.

Example 4. Let $f=\frac{e^{\frac{z\log\omega}{c}}}{e^{\frac{2\pi i z}{c}}-1}$ and $g=\omega f$, where ω is the $n+\sigma$-th root of unity. Then $f^n(z)\prod_{j=1}^{s}(f(z+c))^{\mu_j}$ and $g^n(z)\prod_{j=1}^{s}(g(z+c))^{\mu_j}$ share $(1,\infty)$.

Example 5. Let $f=\frac{e^{\frac{2\pi i z}{c}}-1}{e^{\frac{2\pi i z}{c}}+1}$ and $g=\omega\frac{e^{\frac{2\pi i z}{c}}+1}{e^{\frac{2\pi i z}{c}}-1}$, where ω is the $n+\sigma$-th root of unity. Then $f^n(z)\prod_{j=1}^{s}(f(z+c))^{\mu_j}$ and $g^n(z)\prod_{j=1}^{s}(g(z+c))^{\mu_j}$ share $(1,\infty)$. Here $fg\equiv\omega$.
The following example shows that Corollary 3 is not true for infinite ordered meromorphic function.

Example 6. Let $f=\frac{e^{e^{\frac{z\log(-n)}{c}}}}{e^{e^{\frac{z\log(-n)}{c}}}+1}$ and $g=\frac{1}{e^{e^{\frac{z\log(-n)}{c}}}+1}$, Then it is easy to verify that $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share $(1,\infty)$, but neither $f\equiv tg$ or $fg\equiv t$, for some constant t such that where $t^{n+1}=1$.

Theorem 7. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order, $m(\geq 1)$, $r(\geq 1)$ be integers. Suppose also that $P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}$ and $P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$ share $(1,l)$. Now*

(i) if $l\geq 2$ and $n>2\Gamma_1+\sigma$ or

(ii) if $l=1$ and $n>2\Gamma_1+\frac{1}{2}\Gamma_0+\sigma+\frac{s}{2}$ or

(iii) if $l=0$ and $n>2\Gamma_1+3\Gamma_0+\sigma+3s$,
then the following cases hold:

(I) when $P(z)$ is not of the form of $z^r(z^m-1)$, $m(\geq 1)$ or $z^r(z-1)^m$, $m(\geq 2)$, one of the following two cases holds:

(IA) $f(z)\equiv tg(z)$ for a constant t such that $t^\lambda=1$, where λ is the GCD of the elements of J, $J=\{\sigma+k\in I: a_k\neq 0\}$ and $I=\{\sigma,\sigma+1,\dots,\sigma+n\}$. In particular $P(z)=a_nz^n$, $f\equiv tg$ for a constant t such that $t^{n+\sigma}=1$.

(IB) $P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$;

(II) *when* $P(z)=z^r(z^m-1)$ *and* $n>m+\sigma+2s+2$, *then* $f(z)\equiv tg(z)$, *for some constant* t *such that* $t^m=t^{r+\sigma}=1$;

(III) *when* $P(z)=z^r(z-1)^m$ $(m\geq 2)$, *one of the following two cases holds:*

(IIIA) $f(z)\equiv g(z)$,

(IIIB) $f^r(z)(f(z)-1)^m\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv g^r(z)(g(z)-1)^m\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$.

(IV) $f(z)=e^{\alpha(z)}$ *and* $g(z)=\zeta\, e^{-\alpha(z)}$, *where* α *is a non-constant polynomial and* ζ *is a complex constant satisfying* $a_n^2\zeta^{n+\sigma}\equiv 1$.
If in particular, for an integer $n(\geq 1)$, *we take* $P(f)=f^n$ *and one of the following holds*

(i) $l\geq 2$ *and* $n>4+\sigma$ *or*

(ii) $l=1$ *and* $n>4\frac{1}{2}+\sigma+\frac{s}{2}$ *or*

(iii) $l=0$ *and* $n>7+\sigma+3s$,
then either $f\equiv tg$ *or* $fg\equiv t$, *for some constant* t *such that* $t^{n+\sigma}=1$.

Corollary 8. *Let* $f(z)$, $g(z)$ *be two transcendental entire functions of finite order, and* $c(\neq 0)$ *be a finite complex constant. Suppose* $n(\geq 1)$ *be an integer such that* $f^n(z)(f(z+c))$ *and* $g^n(z)(g(z+c))$ *share* $(1,l)$. *Now*

(i) *if* $l\geq 2$ *and* $n\geq 6$ *or*

(ii) *if* $l=1$ *and* $n\geq 7$ *or*

(iii) *if* $l=0$ *and* $n\geq 12$,
then either $f\equiv tg$, *or* $fg\equiv t$, *for some constant* t *such that* $t^{n+1}=1$.

The following examples show that both conclusions of Theorem 7 actually holds when $P(f)=f^n$ and $c_j=c$, where $c(\neq 0)$ be a finite complex constant, for all $j=1,2,\ldots,s$.

Example 9. Let $f=e^{\frac{2\pi i z}{c}}$ and $g=\omega f$, where ω is the $n+\sigma$-th root of unity. Then $f^n(z)\prod_{j=1}^{s}(f(z+c))^{\mu_j}$ and $g^n(z)\prod_{j=1}^{s}(g(z+c))^{\mu_j}$ share $(1,\infty)$.

Example 10. Let $f=e^{\frac{2\pi i z}{c}}$ and $g=\frac{\omega}{e^{\frac{2\pi i z}{c}}}$, where ω is the $n+\sigma$-th root of unity. Then $f^n(z)\prod_{j=1}^{s}(f(z+c))^{\mu_j}$ and $g^n(z)\prod_{j=1}^{s}(g(z+c))^{\mu_j}$ share $(1,\infty)$. Here $fg\equiv\omega$.

The following example shows that Corollary 8 is not true for infinite ordered entire function.

Example 11. Let $f = e^{e^{\frac{z\log(-n)}{c}}} + 1$ and $g = \frac{1}{e^{e^{\frac{z\log(-n)}{c}}}} + 1$, Then it is easy to verify that $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share $(1,\infty)$, but neither $f \equiv tg$ or $fg \equiv t$, for some constant t such that where $t^{n+1} = 1$.

Theorem 12. *Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of finite order and $k(\geq 1)$ be an integer such that $E_{k)}(1, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}) = E_{k)}(1, P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j})$. Now*

(i) if $k \geq 3$ and $n > 2\Gamma_1 + 5\sigma + 4$ or

(ii) if $k = 2$ and $n > 2\Gamma_1 + \frac{1}{2}\Gamma_0 + 5\sigma + s + \frac{9}{2}$ or

(iii) if $k = 1$ and $n > 2\Gamma_1 + 2\Gamma_0 + 5\sigma + 4s + 6$,
then either

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j} \equiv 1$$

or

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j} \equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

If in particular, for an integer $n(\geq 1)$ we take $P(f) = f^n$ and

(i) if $k \geq 3$ and $n > 8 + 5\sigma$ or

(ii) if $k = 2$ and $n > 9 + 5\sigma + s$ or

(iii) if $k = 1$ and $n > 12 + 5\sigma + 4s$,
then either $f \equiv tg$ or $fg \equiv t$, for some constant t such that $t^{n+\sigma} = 1$.

From Theorem 12, putting $\mu_1 = 1 = s$, we have the following corollary.

Corollary 13. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order, and $c(\neq 0)$ be a finite complex constant. Suppose $E_{k)}(1, f^n(z)(f(z+c)))$ and $E_{k)}(1, g^n(z)(g(z+c)))$. Now*

(i) if $k \geq 3$ and $n \geq 14$ or

(ii) if $k = 2$ and $n \geq 16$ or

(iii) if $k = 1$ and $n \geq 22$,
then either $f \equiv tg$ or $fg \equiv t$, for some constant t such that $t^{n+1} = 1$.

Theorem 14. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order and*
$E_{k)}(1, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}) = E_{k)}(1, P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$. *Now*

(i) if $k \geq 3$ and $n > 2\Gamma_1 + \sigma$ or

(ii) if $k = 2$ and $n > 2\Gamma_1 + \frac{1}{2}\Gamma_0 + \sigma + \frac{s}{2}$ or

(iii) if $k = 1$ and $n > 2\Gamma_1 + 2\Gamma_0 + \sigma + 2s$,
then one of the conclusions (I), (II), (III) or (IV) of Theorem 7 holds.

If in particular, $P(f) = f^n$ and one of the following holds

(i) $k \geq 3$ and $n > 4 + \sigma$ or

(ii) $k = 2$ and $n > 4\frac{1}{2} + \sigma + \frac{s}{2}$ or

(iii) $k = 1$ and $n > 6 + \sigma + 2s$,
then either $f \equiv tg$ or $fg \equiv t$, for some constant t such that $t^{n+\sigma} = 1$.

Corollary 15. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order and $c(\neq 0)$ be a finite complex constant. Suppose $n(\geq 1)$ be an integer such that $E_{k)}(1, f^n(z)(f(z+c))) = E_{k)}(1, g^n(z)(g(z+c)))$. Now*

(i) if $k \geq 3$ and $n \geq 6$ or

(ii) if $k = 2$ and $n \geq 7$ or

(iii) if $k = 1$ and $n \geq 10$,
then either $f \equiv tg$ or $fg \equiv t$, for some constant t such that $t^{n+1} = 1$.

2 Auxiliary Definitions

Throughout the paper we have used the following definitions and notations.

Definition 16. *[9] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid = 1)$ the counting function of simple a points of f. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid \leq p)$ we denote the corresponding reduced counting function.*

In an analogous manner we can define $N(r, a; f \mid \geq p)$ and $\overline{N}(r, a; f \mid \geq p)$.

Definition 17. *[11] Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then $N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \ldots + \overline{N}(r, a; f \mid \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.*

Definition 18. *[23] Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$. Let z_0 be an a-point of f with multiplicity p, an a-point* *of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the reduced counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$, by $\overline{N}_E^{(2}(r,a;f)$ the reduced counting function of those a-points of f and g where $p=q\geq 2$. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a\in\mathbb{C}\cup\{\infty\}$.*

When f and g share (a,m), $m\geq 1$, then $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.

Definition 19. *[10, 11] Let f, g share a value $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 20. *Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_{k)}(a;f)=E_{k)}(a;g)$ where $a\in\mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity $p>0$, an a-point of g with multiplicity $q>0$. We denote by $\overline{N}_L^{(k+1}(r,a;f)$ ($\overline{N}_L^{(k+1}(r,a;g)$) the counting function of those common a-points of f and g where $p>q\geq k+1$ ($q>p\geq k+1$), each a-point is counted only once.*

Definition 21. *Let k be a positive integer. Also let z_0 be a zero of $f(z)-a$ of multiplicity p and a zero of $g(z)-a$ of multiplicity q. We denote by $\overline{N}_{f\geq k+1}(r,a;f\mid g\neq a)$ ($\overline{N}_{g\geq k+1}(r,a;g\mid f\neq a)$) the reduced counting functions of those a-points of f and g for which $p\geq k+1$ and $q=0$ ($q\geq k+1$ and $p=0$).*

Definition 22. *For $E_{k)}(a;f)=E_{k)}(a;g)$ we define $\overline{N}_\otimes(r,a;f,g)$ as follows*

$$\begin{aligned}
&\overline{N}_\otimes(r,a;f,g)\\
=\;&\overline{N}_L^{(k+1}(r,a;f)+\overline{N}_L^{(k+1}(r,a;g)+\overline{N}_{f\geq k+1}(r,a;f\mid g\neq a)\\
&+\overline{N}_{g\geq k+1}(r,a;g\mid f\neq a)\\
\leq\;&\overline{N}(r,a;f\mid\geq k+1)+\overline{N}(r,a;g\mid\geq k+1).
\end{aligned}$$

3 Lemmas

Henceforth for two non-constant meromorphic functions F and G, H represents the following function

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{3.1}$$

Lemma 23. *[21] Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots,$ $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 24. *[4] Let $f(z)$ be a meromorphic function of finite order ρ and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\varepsilon}).$$

The following lemma has little modifications of the original version Theorem 2.1 of [4])

Lemma 25. *[6] Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r, f),\ N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f),$$

$$\overline{N}(r, 0; f(z+c)) \leq \overline{N}(r, 0; f(z)) + S(r, f),\ \overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 26. *[24] Let F, G be two non-constant meromorphic functions sharing $(1, 0)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r, 1; F) = N_E^{1)}(r, 1; G) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 27. *[13] For $E_{k)}(1; F) = E_{k)}(1; G)$ and $H \not\equiv 0$.*

$$N(r, 1; F \mid= 1) = N(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 28. *If two non-constant meromorphic functions F and G share $(1, 0)$ and $H \not\equiv 0$ then*

$$\begin{aligned}\overline{N}(r, \infty; H) \leq\ & \overline{N}(r, 0; F \mid\geq 2) + \overline{N}(r, 0; G \mid\geq 2) + \overline{N}(r, \infty; F \mid\geq 2) \\ & + \overline{N}(r, \infty; G \mid\geq 2) + \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'),\end{aligned}$$

where by $\overline{N}_0(r, 0; F')$ we mean the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

Proof. We omit the proof as the same can be carried out in the line of proof of Lemma 2 [11]. □

Lemma 29. *Let F and G be two non-constant meromorphic functions such that $E_{k)}(1;F)=E_{k)}(1;G)$ and $H\not\equiv 0$ then*

$$\begin{aligned}\overline{N}(r,\infty;H) &\leq \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,\infty;F\mid\geq 2)\\ &+\overline{N}(r,\infty;G\mid\geq 2)+\overline{N}_{\otimes}(r,1;F,G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}),\end{aligned}$$

where by $\overline{N}_0(r,0;F^{'})$ and $N_0(r,0;G^{'})$ are defined as in Lemma 28.

Proof. The proof can be carried out in the line of proof of Lemma 28. So we omit the details. □

Lemma 30. *[3] Let f, g be two non-constant meromorphic functions sharing $(1,l)$, where*
$0\leq l<\infty$. Then

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N_E^{1)}(r,1;f)+\left(l-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\leq\frac{1}{2}[N(r,1;f)+N(r,1;g)].$$

Lemma 31. *Let f and g be two non-constant meromorphic functions such that $E_{k)}(1;f)=E_{k)}(1;g)$, where $1\leq k<\infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)\leq\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]\\ &-\left(\frac{k}{2}-\frac{1}{2}\right)\left\{\overline{N}_{\otimes}(r,1;f,g)\right\}.\end{aligned}$$

Proof. We omit the proof as it can be carried out in the line of proof of Lemma 2.3 [2]. □

Lemma 32. *Let f and g be any two meromorphic function and they share $(1,l)$. Then*

$$\begin{aligned}\overline{N}_*(r,1;f,g) &\leq \frac{1}{l+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right]\\ &+S(r,f)+S(r,g).\end{aligned}$$

Proof. In view of Definition 19, the lemma follows from Lemma 2.14 of [1]. □

Lemma 33. *Let f and g be two non-constant meromorphic functions such that $E_{k)}(1;F)=E_{k)}(1;G)$, where $1\leq k<\infty$. Then*

$$\begin{aligned}\overline{N}_{\otimes}(r,1;f,g) &\leq \frac{1}{k}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right]\\ &+S(r,f)+S(r,g).\end{aligned}$$

Proof. The Lemma can be proved in the line of proof of Lemma 32. So we omit it. □

Lemma 34. *Let $f(z)$ be a transcendental entire function of finite order. Then for ε, we have*

$$T(r, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}) = (n+\sigma)T(r,f) + S(r,f).$$

Proof. By Lemmas 23 and 24 we have

$$\begin{aligned}
& T(r, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}) \\
= \;& m(r, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}) \\
\leq \;& m(r, P(f)(z)(f(z))^{\sigma}) + m\left(r, \frac{\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}}{(f(z))^{\sigma}}\right) \\
\leq \;& m(r, P(f)(z)(f(z))^{\sigma}) + O(r^{\rho-1+\varepsilon}) \\
= \;& T(r, P(f)(z)(f(z))^{\sigma}) + O(r^{\rho-1+\varepsilon}) \\
\leq \;& (n+\sigma)T(r,f) + O(r^{\rho-1+\varepsilon})
\end{aligned}$$

Also we have

$$\begin{aligned}
& (n+\sigma)T(r,f) = T(r, f^{\sigma}(z)P(f)(z)) + O(1) \\
= \;& m(r, P(f)(z)(f(z))^{\sigma}) + O(1) \\
\leq \;& m\left(r, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right) + m\left(r, \frac{(f(z))^{\sigma}}{\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}}\right) \\
\leq \;& m\left(r, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right) + O(r^{\rho-1+\varepsilon}) \\
\leq \;& T\left(r, P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right) + O(r^{\rho-1+\varepsilon}).
\end{aligned}$$

Therefore the lemma is proved. □

Lemma 35. *Let $f(z)$ be a transcendental meromorphic function of finite orders. Let F be given as in Theorem 2. Then for $n > \sigma$ we have*

$$(n-\sigma)\,T(r,f) \leq T(r,F) + S(r,f) \text{ and } T(r,F) \leq (n+\sigma)T(r,f) + S(r,f).$$

Proof. By Lemmas 23 and 25 we get

$$
\begin{aligned}
& (n+\sigma)T(r,f) = T(r,P(f)f^{\sigma})+O(1) = m(r,P(f)f^{\sigma})+N(r,P(f)f^{\sigma})+O(1) \\
\leq\ & m(r,F)+m\left(r,\frac{f^{\sigma}}{\prod\limits_{j=1}^{s}(f(z+c_j))^{\mu_j}}\right)+N(r,F)+N\left(r,\frac{f^{\sigma}}{\prod\limits_{j=1}^{s}(f(z+c_j))^{\mu_j}}\right) \\
& +S(r,f) \\
\leq\ & T(r,F)+\sum_{j=1}^{s}\mu_j\, m\left(r,\frac{f(z)}{f(z+c_j)}\right)+\sum_{j=1}^{s}N\left(r,\frac{f(z)}{f(z+c_j)}\right)^{\mu_j}+S(r,f) \\
\leq\ & T(r,F)+2\sigma T(r,f)+S(r,f).
\end{aligned}
$$

So,

$$
(n-\sigma)\,T(r,f)\ \leq\ T(r,F)+S(r,f).
$$

Again in view of Lemmas 24 and 25 we get

$$
\begin{aligned}
& T(r,F(z)) = m(r,F(z))+N(r,F(z)) \\
\leq\ & m(r,P(f))+m\left(r,f^{\sigma}\prod_{j=1}^{s}\frac{(f(z+c_j))^{\mu_j}}{f^{\mu_j}}\right)+N(r,F(z)) \\
\leq\ & nm(r,f)+\sigma m(r,f)+\sum_{j=1}^{s}\mu_j\, m\left(r,\frac{f(z+c_j)}{f(z)}\right)+nN(r,f) \\
& +N\left(r,\prod_{j=1}^{s}f(z+c_j)\right)+S(r,f) \\
\leq\ & (n+\sigma)T(r,f)+S(r,f).
\end{aligned}
$$

This completes the proof of the lemma. □

Note 36. From Lemma 35 we see that $S(r,F)$ can be replaced by $S(r,f)$.

Remark 37. The inequalities in Lemma 35 can not further be improved for the case $c_j = c$, $j=1,2,\ldots,s$ can easily be verified from the following two examples.

Let $P(z) = a_n z^n + a_{n-1}z^{n-1}+\ldots+a_{\sigma}z^{\sigma}$ and for a non-zero complex number d_1, $f(z) = tan\frac{\pi z}{2d_1}$. Then

$$
P(f)f^{\sigma}(z+d_1) = (-1)^{\sigma}\left[a_n tan^{n-\sigma}\frac{\pi z}{2d_1}+a_{n-1}tan^{n-1-\sigma}\frac{\pi z}{2d_1}+\ldots+a_{\sigma}\right]
$$

and so $T(r,P(f)f^{\sigma}(z+d_1)) = (n-\sigma)T(r,f)+S(r,f)$.

Again for a non-zero complex number d_2, choose $f(z)=tan\frac{\pi z}{2d_2}$. Then

$$P(f)f(z+d_2)=\left[a_n tan^{n+\sigma}\frac{\pi z}{2d_2}+a_{n-1}tan^{n-1+\sigma}\frac{\pi z}{2d_2}+\ldots+a_\sigma tan^{2\sigma}\frac{\pi z}{2d_2}\right]$$

and so $T(r,P(f)f^{\sigma}(z+d_2))=(n+\sigma)T(r,f)+S(r,f)$.

Lemma 38. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order and $n>2\Gamma_1+5\sigma+4$. Also let F and G be given as in Theorem 2. If $H\equiv 0$, then F and G share $(1,\infty)$ and either*

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\equiv 1$$

or

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

In particular, if $P(f)=f^n$, then either $f\equiv tg$, or $fg\equiv t$, for some constant t such that where $t^{n+\sigma}=1$.

Proof. Since $H\equiv 0$, on integration we get

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{3.2}$$

where $a(\neq 0)$, b are constants. From (3.2) it is clear that F and G share $(1,\infty)$. We now consider the following cases:
Case 1. Let $b\neq 0$ and $a\neq b$. If $b=-1$, then from (3.2) we have

$$F\equiv\frac{-a}{G-a-1}.$$

From Lemma 25 we see that

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)\leq(s+1)\overline{N}(r,\infty;f).$$

So in view of Lemma 35 using the second fundamental theorem we get

$$\begin{aligned}&(n-\sigma)\,T(r,g)\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,a+1;G)+S(r,g)\\ \leq\;&(m_1+m_2)T(r,g)+s\overline{N}(r,0;g(z+c_j))+(s+1)\{\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)\}\\ &+S(r,g)\\ \leq\;&(m_1+m_2+2s+1)T(r,g)+(s+1)T(r,f)+S(r,g).\end{aligned}$$

As F and G are symmetric, in a similar manner we can get

$$(n-\sigma)\, T(r,f) \leq (m_1+m_2+2s+1)T(r,f)+(s+1)T(r,g)+S(r,f).$$

Combining the above two we can get

$$(n-\sigma)\{T(r,f)+T(r,g)\} \leq (\Gamma_0+3s+2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

a contradiction for $n > 2\Gamma_1+5\sigma+4$.

If $b \neq -1$, from (3.2) we obtain that

$$F-\left(1+\frac{1}{b}\right) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So

$$\overline{N}(r,\frac{(b-a)}{b};G)=\overline{N}(r,\infty;F).$$

Using Lemma 35 and with the same argument as used in the case for $b=-1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a=b$. If $b=-1$, then from (3.2) we have

$$FG \equiv 1,$$

i.e.,

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j} \equiv 1.$$

In particular, when $P(f)=f^n$, choose $M(z)=f(z)g(z)$. When $M(z)$ is non-constant, we get from above

$$M^n(z) \equiv \frac{1}{\prod\limits_{j=1}^{s}(M(z+c_j))^{\mu_j}}.$$

So using first fundamental theorem we get

$$nT(r,M)=\sum_{j=1}^{s}\mu_j T(r,M(z+c_j))+O(1)=\sigma T(r,M)+S(r,M),$$

a contradiction. So $M(z)$ must be a constant and so $M(z)^{n+\sigma} \equiv 1$, which implies $fg \equiv t$, where $t^{n+\sigma}=1$.

If $b \neq -1$, from (3.2) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore
$$\overline{N}\left(r,\frac{1}{1+b};G\right)=\overline{N}(r,0;F).$$
So in view of Lemma 35 using the second fundamental theorem we get
$$\begin{aligned}&(n-\sigma)\,T(r,g)\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,\frac{1}{1+b};G)+S(r,g)\\ \leq\ &(m_1+m_2+2s+1)T(r,g)+(m_1+m_2+s)T(r,f)+S(r,g).\end{aligned}$$
As F and G are symmetric, in a similar manner we can get
$$(n-\sigma)\,T(r,f)\leq(m_1+m_2+2s+1)T(r,f)+(m_1+m_2+s)T(r,g)+S(r,f).$$
Combining the above two we can get
$$(n-\sigma)\{T(r,f)+T(r,g)\}\leq(2\Gamma_0+3s+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$
a contradiction for $n>2\Gamma_1+5\sigma+4$.
Case 3. Let $b=0$. From (3.2) we obtain
$$F\equiv\frac{G+a-1}{a}. \tag{3.3}$$
If $a\neq 1$ then from (3.3) we obtain
$$\overline{N}(r,1-a;G)=\overline{N}(r,0;F).$$
So using the same argument as done in Case 2, for $b\neq-1$, we can similarly deduce a contradiction. Therefore $a=1$ and from (3.3) we obtain $F\equiv G$, i.e.,
$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$
In particular, when $P(f)=f^n$, choose $H(z)=\frac{f(z)}{g(z)}$. Now proceeding in the same way as for the case $b=-1$, in Case 2, we can show that $H(z)$ must be a constant and $f\equiv tg$, where $t^{n+\sigma}=1$.

This completes the proof. □

Lemma 39. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order and $n>2\Gamma_1+\sigma$. Also let F and G be given as in Theorem 2. If $H\equiv 0$, then F and G share $(1,\infty)$ and either*
$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\equiv 1$$
or
$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

Proof. Using Lemma 34, instead of Lemma 35 the proof can be carried out in the line of proof of Lemma 38. So we omit the details. □

Lemma 40. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order and $n(\geq 1)$, $m(\geq 1)$, $r(\geq 1)$, be integers. Suppose also*

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j} \equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

(I) When $P(z)$ is not of the form of $z^r(z^m-1)$ or $z^r(z-1)^m$, when $m(\geq 2)$, one of the following two cases holds:

(IA) $f(z) \equiv tg(z)$ for a constant t such that $t^\lambda = 1$, where λ is the GCD of the elements of J, $J = \{\sigma + k \in I : a_k \neq 0\}$ and $I = \{\sigma, \sigma+1, \ldots, \sigma+n\}$. In particular $P(z) = a_n z^n$, $f \equiv tg$ for a constant t such that $t^{n+\sigma} = 1$.

(IB) $P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j} \equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$;

(II) when $P(z) = z^r(z^m-1)$ and $n > m+\sigma+2s+2$, then $f(z) \equiv tg(z)$ for some constant t such that $t^m = t^{r+\sigma} = 1$;

(III) when $P(z) = z^r(z-1)^m(m \geq 2)$, one of the following two cases holds:

(IIIA) $f(z) \equiv g(z)$,

(IIIB) $f^r(z)(f(z)-1)^m \prod_{j=1}^{s}(f(z+c_j))^{\mu_j} \equiv g^r(z)(g(z)-1)^m \prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$.

Proof. Suppose

$$P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j} \equiv P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}. \tag{3.4}$$

Case 1. Suppose $P(z)$ is not of the form of $z^r(z^m-1)$ or $z^r(z-1)^m$, when $m(\geq 2)$. Let $h = \frac{f}{g}$ and h is a constant. Putting $f = hg$ in (3.4) we get

$$a_n g^n(z)(h^{n+\sigma}-1)+a_{n-1}g^{n-1}(z)(h^{n-1+\sigma}-1)+\ldots+a_1 g(z)(h^{\sigma+1}-1)+a_0(h^\sigma-1) \equiv 0.$$

We shall prove that $h^\lambda = 1$, where λ is the GCD of the elements of J, $J = \{k+\sigma \in I : a_k \neq 0\}$ and $I = \{\sigma, \sigma+1, \ldots, n+\sigma\}$. In particular if $P(z) = a_n z^n$, then from above we get $h^{n+\sigma} = 1$. Thus $f \equiv tg$ for a constant t such that $t^{n+\sigma} = 1$. Suppose there exist at least one non-zero coefficient a_k, $k \neq n$, then if $h^\lambda \neq 1$, from (3.4) we get $T(r,g) = S(r,g)$, a contradiction to the fact that g is transcendental. So $h^\lambda = 1$, where λ is the GCD of the elements of J, $J = \{k+\sigma \in I : a_k \neq 0\}$ and

$I=\{\sigma,\sigma+1,\ldots,n+\sigma\}$.
Next suppose that h is not a constant. Then we get (3.4).
Case 2. Let $P(z)=z^r(z^m-1)$. Clearly $n=r+m$.

Then from (3.4) we have

$$f^r(z)(f^m(z)-1)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv g^r(z)(g^m(z)-1)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}. \tag{3.5}$$

Let $h=\frac{f}{g}$. From (3.5) we get

$$g^m(z)[h^{r+m}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1]\equiv h^r(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1. \tag{3.6}$$

First we suppose that h is non-constant. We assert that both $h^{r+m}(z)\prod\limits_{j=1}^{s}(h(z+c_j))^{\mu_j}$ and $h^r(z)\prod\limits_{j=1}^{s}(h(z+c_j))^{\mu_j}$ are non-constant. Suppose on the contrary

$$h^{r+m}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}\equiv d\in\mathbb{C}\setminus\{0\}$$

. Then we have $h^{r+m}(z)\equiv\frac{d}{\prod\limits_{j=1}^{s}(h(z+c_j))^{\mu_j}}$.

Now by *Lemmas 23, 24* and *25* we get

$$\begin{aligned}
(r+m)\,T(r,h) &= T(r,h^{r+m})+S(r,h)\\
&= T\left(r,\frac{d}{\prod\limits_{j=1}^{s}(h(z+c_j))^{\mu_j}}\right)+S(r,h)\\
&\leq \sum_{j=1}^{s}\mu_jN(r,0;h(z+c_j))+\sum_{j=1}^{s}\mu_j\;m\left(r,\frac{1}{h(z+c_j)}\right)+S(r,h)\\
&\leq \sum_{j=1}^{s}\mu_j\;N(r,0;h(z))+\sum_{j=1}^{s}\mu_j\;m(r,\frac{1}{h(z)})+S(r,h)\\
&\leq \sigma\,T(r,h)+S(r,h),
\end{aligned}$$

which is a contradiction. Similarly we can prove that $h^n(z)\prod\limits_{j=1}^{s}(h(z+c_j))^{\mu_j}$ is non-constant.

Thus from (3.6) we have

$$f^m(z) \equiv h^m(z) \frac{h^r(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j} - 1}{h^{r+m}(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j} - 1}$$

and

$$g^m(z) \equiv \frac{h^r(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j} - 1}{h^{r+m}(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j} - 1}.$$

Let z_0 be a zero of $h^{r+m}(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j} - 1$. Since g is an entire function, it follows that z_0 is also a zero of $h^r(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j} - 1$. Then clearly $h^m(z_0) - 1 = 0$ and so

$$\overline{N}(r, 1; h^{r+m} \prod_{j=1}^{s}(h(z+c_j))^{\mu_j}) \leq \overline{N}(r, 1; h^m) \leq m\, T(r,h) + O(1),$$

So in view of Lemmas 23, 25, 35 and the second fundamental theorem we get

$$\begin{aligned}
& (n-\sigma)\, T(r,h) \\
= \; & (r+m-\sigma)\, T(r,h) \\
= \; & T(r, h^{r+m}(z) \prod_{j=1}^{s}(h(z+c_j))^{\mu_j}) + S(r,h) \\
\leq \; & \overline{N}(r, 0; h^{r+m} \prod_{j=1}^{s}(h(z+c_j))^{\mu_j}) + \overline{N}(r, \infty; h^{r+m} \prod_{j=1}^{s}(h(z+c_j))^{\mu_j}) \\
& + \overline{N}(r, 1; h^{r+m} \prod_{j=1}^{s}(h(z+c_j))^{\mu_j}) + S(r,h) \\
\leq \; & \overline{N}(r, 0; h) + \sum_{j=1}^{s} \overline{N}(r, 0; h(z+c_j)) + \overline{N}(r, \infty; h) \\
& + \sum_{j=1}^{s} \overline{N}(r, \infty; h(z+c_j)) + m\, T(r,h) + S(r,h) \\
\leq \; & N(r, 0; h) + \sum_{j=1}^{s} N(r, 0; h(z)) + N(r, \infty; h) + \sum_{j=1}^{s} N(r, \infty; h(z)) + m\, T(r,h) \\
& + S(r,h) \\
\leq \; & (m + 2s + 2)\, T(r,h) + S(r,h),
\end{aligned}$$

which contradicts with $n > m + \sigma + 2s + 2$.

Hence h is a constant. Since g is transcendental entire function, from (3.6) we have

$$h^{r+m}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\equiv 0 \Longleftrightarrow h^{r}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\equiv 0$$

and so $h^m(z)=1$, $h^{r+\sigma}=1$. Thus $f(z)\equiv tg(z)$ for a constant t such that $t^m=t^{r+\sigma}=1$.

Case 3. Let $P(z)=z^r(z-1)^m$, $m(\geq 2)$.

Then from (3.6) we have

$$f^r(z)(f(z)-1)^m\prod_{j}^{s}(f(z+c_j))^{\mu_j}\equiv g^r(z)(g(z)-1)^m\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}. \tag{3.7}$$

Let $h=\frac{f}{g}$. First we suppose that h is a constant.

Then from (3.7) we get

$$\begin{aligned} & f^r(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\ \sum_{i=0}^{m}(-1)^{i}\ {}^{m}C_{m-i}\ f^{m-i}(z) \\ \equiv\ & g^r(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\ \sum_{i=0}^{m}(-1)^{i}\ {}^{m}C_{m-i}g^{m-i}(z). \end{aligned} \tag{3.8}$$

Now substituting $f=gh$ in (3.8) we get

$$\sum_{i=0}^{m}(-1)^{i}\ {}^{m}C_{m-i}\ g^{m-i}(z)(h^{r+m+\sigma-i}(z)-1)\equiv 0,$$

which implies that $h=1$. Hence $f(z)\equiv g(z)$.

Next we suppose that h is non-constant.

Then from (3.7) we can say that

$$f^r(z)(f(z)-1)^m\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv g^r(z)(g(z)-1)^m\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

This completes the proof of the lemma. □

Lemma 41. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order. If*

$$\left[P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right]\left[P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\right]\equiv 1,$$

then $f(z)=e^{\alpha(z)}$ and $g(z)=\zeta\ e^{-\alpha(z)}$, where α is a non-constant polynomial and ζ be a complex constant satisfying $a_n^2\zeta^{n+\sigma}\equiv 1$.

Proof. Suppose

$$\left[P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right]\left[P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\right]\equiv 1. \tag{3.9}$$

Noting that f and g are transcendental entire function, we see that $P(f)\neq 0$, $P(g)\neq 0$. As f and g can have only one finite Picard exceptional value, we must have $P(f)=a_n(f-a)^n$, $P(g)=a_n(g-a)^n$, for some complex constant a. So (3.9) reduces to

$$\left[a_n(f-a)^n\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right]\left[a_n(g-a)^n\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\right]\equiv 1. \tag{3.10}$$

Also we know that $f-a$ and $g-a$ do not have zeros. It follows that $f(z)=e^{\alpha(z)}+a$, $g(z)=e^{\beta(z)}+a$, where $\alpha(z)$ and $\beta(z)$ be two non-constant polynomials. As $f(z+c_j)\neq 0$, $g(z+c_j)\neq 0$, for $j=1,2,\ldots,s$, we see that $e^{\alpha(z+c_j)}+a\neq 0$, $e^{\beta(z+c_j)}+a\neq 0$, which implies $a=0$. So $f(z)=e^{\alpha(z)}$, $g(z)=e^{\beta(z)}$. Hence from (3.10) we get

$$a_n^2 e^{\left\{n\{\alpha(z)+\beta(z)\}+\sum\limits_{j=1}^{s}\mu_j[(\alpha(z+c_j))+\beta(z+c_j)]\right\}}\equiv 1. \tag{3.11}$$

From (3.10) we get $\alpha(z)+\beta(z)=\xi$, for some constant ξ. Finally we get $f(z)=e^{\alpha(z)}$ and $g(z)=\zeta\ e^{-\alpha(z)}$, where α is a non-constant polynomial and ζ be a complex constant satisfying $a_n^2\zeta^{n+\sigma}\equiv 1$. So the proof of the lemma is complete. □

4 Proofs of the Theorems

Proof of Theorem 2. Let

$$F(z)=P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j};\ \ \textit{and}\ \ G(z)=P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}.$$

Then F and G share $(1,l)$.

Case-1 Let $H\not\equiv 0$. By the second fundamental theorem we get

$$T(r,F)\ \leq\ \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-\overline{N}_0(r,0;F^{'})+S(r,f). \tag{4.1}$$

Similarly we have

$$T(r,G)\ \leq\ \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-\overline{N}_0(r,0;G^{'})+S(r,g). \tag{4.2}$$

Combining (4.1) and (4.2) with the help of Lemmas 26, 28, 30 and 32 we have

$$\begin{aligned}
[T(r,F)+T(r,G)] &\leq [\overline{N}(r,0;F)+\overline{N}(r,0;G)]+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)] \quad (4.3)\\
&+[\overline{N}(r,1;F)+\overline{N}(r,1;G)]-[\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})]\\
&+S(r,f)+S(r,g)\\
&\leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\
&+\frac{1}{2}[T(r,F)+T(r,G)]-\left(l-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,g)\\
&\leq +\frac{1}{2}[T(r,F)+T(r,G)]+N_2(r,0;F)+N_2(r,0;G)\\
&+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{(3-2l)}{2(l+1)}[\overline{N}(r,0;F)\\
&+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)]+S(r,f)+S(r,g).
\end{aligned}$$

Subcase 1.1. While $l \geq 2$, in view of Lemma 25 and Lemma 35, from (4.3) we get

$$\begin{aligned}
&\frac{(n-\sigma)}{2}[T(r,f)+T(r,g)] \quad (4.4)\\
\leq\ & (m_1+2m_2)T(r,f)+\sum_{j=1}^{s}\mu_j N(r,0;f(z+c_j))+(m_1+2m_2)T(r,g)\\
&+\sum_{j=1}^{s}\mu_j N(r,0;g(z+c_j))+2\overline{N}(r,\infty;f)+\sum_{j=1}^{s}\mu_j N(r,\infty;f(z+c_j))\\
&+2\overline{N}(r,\infty;g)+\sum_{j=1}^{s}\mu_j N(r,\infty;g(z+c_j))]+S(r,f)+S(r,g)\\
\leq\ & [m_1+2m_2+2\sigma+2]\left\{T(r,f)+T(r,g)\right\}+S(r,f)+S(r,g).
\end{aligned}$$

From (4.4) it follows that

$$\begin{aligned}
&(n-\sigma)[T(r,f)+T(r,g)]\\
\leq\ & [2\Gamma_1+4\sigma+4]\left\{T(r,f)++T(r,g)\right\}+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction for $n > 2\Gamma_1+5\sigma+4$.

Subcase 1.2. While $l = 1$, using 25 and Lemma 35, from (4.3) we get

$$\begin{aligned}
&\frac{(n-\sigma)}{2}[T(r,f)+T(r,g)] \qquad (4.5)\\
\leq\ & (m_1+2m_2)T(r,f)+\sum_{j=1}^{s}\mu_j N(r,0;f(z+c_j))+(m_1+2m_2)T(r,g)\\
&+\sum_{j=1}^{s}\mu_j N(r,0;g(z+c_j))+2\overline{N}(r,\infty;f)+\sum_{j=1}^{s}\mu_j N(r,\infty;f(z+c_j))\\
&+2\overline{N}(r,\infty;g)+\sum_{j=1}^{s}\mu_j N(r,\infty;g(z+c_j))]+\left(\frac{1}{4}\right)[(m_1+m_2)T(r,f)\\
&+sN(r,0;f(z+c_j))+(s+1)\overline{N}(r,\infty;f)+(m_1+m_2)T(r,g)\\
&+sN(r,0;g(z+c_j))+(s+1)\overline{N}(r,\infty;g)]+S(r,f)+S(r,g).\\
\leq\ & \left[m_1+2m_2+2\sigma+2+\frac{1}{4}(m_1+m_2+2s+1)\right]\{T(r,f)+T(r,g)\}\\
&+S(r,f)+S(r,g).
\end{aligned}$$

From (4.5) it follows that

$$\begin{aligned}
&(n-\sigma)[T(r,f)+T(r,g)]\\
\leq\ & \left[2\Gamma_1+4\sigma+4+\frac{1}{2}(\Gamma_0+2s+1)\right]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction for $n > 2\Gamma_1+\frac{1}{2}\Gamma_0+5\sigma+s+\frac{9}{2}$.

Subcase 1.3. Next let $l = 0$. Again using Lemma 25 and Lemma 35, from (4.3) we get

$$\begin{aligned}
&\frac{(n-\sigma)}{2}[T(r,f)+T(r,g)] \qquad (4.6)\\
\leq\ & \left[m_1+2m_2+2\sigma+2+\frac{3}{2}(m_1+m_2+2s+1)\right]\{T(r,f)+T(r,g)\}\\
&+S(r,f)+S(r,g).
\end{aligned}$$

From (4.6) we get

$$\begin{aligned}
&(n-\sigma)[T(r,f)+T(r,g)]\\
\leq\ & [2\Gamma_1+4\sigma+4+3(\Gamma_0+2s+1)]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction for $n > 2\Gamma_1+3\Gamma_0+5\sigma+6s+7$.

Case-2 Let $H \equiv 0$.

The theorem follows from Lemma 38. This competes the proof of the theorem. □

Proof of Theorem 7. In this case we have to proceed in the same manner as done in the proof of Theorem 2. Only difference is that in the case $H \not\equiv 0$ we will use Lemma 34 instead of Lemma 35. Again when $H \equiv 0$, the theorem follows from Lemmas 39, 40 and 41. $\square$

Proof of Theorem 12. Let F and G be given as in Theorem 2. Combining (4.1) and (4.2) with the help of Lemmas 27, 29, 31 and 33 we have

$$\begin{aligned}[T(r,F)+T(r,G)] \leq & \frac{1}{2}[T(r,F)+T(r,G)] + N_2(r,0;F) + N_2(r,0;G) \\ & + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{(3-k)}{2k}[\overline{N}(r,0:F) \\ & + \overline{N}(r,\infty;F) + \overline{N}(r,0:G) + \overline{N}(r,\infty;G)] \\ & + S(r,f) + S(r,g). \end{aligned} \tag{4.7}$$

The rest can be proved easily. So we omit that. $\square$

Proof of Theorem 14. Theorem 14 can be proved in the line of proof of Theorem 7. So we omit the detail. $\square$

5 Applications

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial. We say that $f(z)$ and $g(z)$ share $p(z)$ with weight l if $f(z) - p(z)$ and $g(z) - p(z)$ share $(0, l)$.

Recently Majumder [17] improved Theorem E by replacing the value sharing to a non-zero polynomial sharing as follows.

Theorem G. *[17] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$. Let $p(z)$ be a non zero polynomial such that $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, l)$.*

(i) If $l = 2$ and $n \geq 14$, or

(ii) if $l = 1$ and $n \geq 16$ or if

(iii) If $l = 0$ and $n \geq 26$ and $deg(p) < \frac{n-1}{2}$
then either $f(z) \equiv tg(z)$ for some constant t such that $t^{n+1} = 1$ or $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant d, say, and t is a constant such that $t^{n+1} = d^2$.

As an application of Theorem 2 we can significantly improve Theorem G in the following manner:

Theorem 42. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$. Let $c_j (j = 1, 2, \ldots, s)$ be finite complex constants and $p(z)$ be a non-zero polynomial such that $f^n(z) \prod_{j=1}^{s} (f(z+c_j))^{\mu_j} - p(z)$ and $g^n(z) \prod_{j=1}^{s} (g(z+c_j))^{\mu_j} - p(z)$ share $(0, l)$.*

(i) If $l \geq 2$ and $n > 8 + 5\sigma$ or

(ii) if $l = 1$ and $n > 9 + 5\sigma + s$ or

(iii) if $l = 0$ and $n > 14 + 5\sigma + 6s$
and deg$(p) < \frac{n-\sigma}{2}$ then either $f(z) \equiv tg(z)$ for some constant t such that $t^{n+\sigma} = 1$ or $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant d, say and t is a constant such that $t^{n+\sigma} = d^2$. In particular, when f and g are transcendental entire functions and $\deg(p) < \frac{n+\sigma}{2}$, then either $f(z) \equiv tg(z)$ for some constant t such that $t^{n+\sigma} = 1$ or $f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, $Q(z)$ is a non-constant polynomial, t is a constant such that $t^{n+\sigma} = d^2$, where $p(z)$ reduces to a non-zero constant d.

Proof. Let $F(z) = \frac{f^n(z)f(z+c)}{p(z)}$ and $G(z) = \frac{g^n(z)g(z+c)}{p(z)}$. Also F, G share $(1, l)$ except for zeros of $p(z)$. Noting that f and g are transcendental the theorem can be proved in the line of proof of Theorem 2. The only difference is that when we will use *Case 2* of Lemma 38, we will get

$$f^n(z) \prod_{j=1}^{s} (f(z+c_j))^{\mu_j} g^n(z) \prod_{j=1}^{s} (g(z+c_j))^{\mu_j} \equiv p^2. \tag{5.1}$$

Let $h_1 = fg$. Then from (5.1) we have

$$M_1^n(z) \equiv \frac{p^2(z)}{\prod_{j=1}^{s} (M_1(z+c_j))^{\mu_j}}. \tag{5.2}$$

First we suppose that $M_1(z)$ is a non-constant meromorphic function. We now consider following two cases.
Case 1. Let $M_1(z)$ be a transcendental meromorphic function.

Now by Lemmas 23, 24 and 25 we get

$$\begin{aligned} nT(r,M_1) &= T(r,M_1^n)+S(r,M_1)\\ &= T\left(r,\frac{p^2}{\prod\limits_{j=1}^{s}(M_1(z+c_j))^{\mu_j}}\right)+S(r,M_1)\\ &\leq \sum_{j=1}^{s}\mu_j N(r,0;M_1(z+c_j))+m\left(r,\frac{1}{\prod\limits_{j=1}^{s}(M_1(z+c_j))^{\mu_j}}\right)+S(r,M_1)\\ &\leq \sigma N(r,0;M_1(z))+\sigma m\left(r,\frac{1}{M_1(z)}\right)+S(r,M_1)\\ &\leq \sigma T(r,M_1)+S(r,M_1), \end{aligned}$$

which is a contradiction.

Case 2. Let $M_1(z)$ be a rational function.

Let

$$M_1=\frac{M_2}{M_3}, \tag{5.3}$$

where M_2 and M_3 are two nonzero relatively prime polynomials. From (5.3) we have

$$T(r,M_1)=\max\{\deg(M_2),\deg(M_3)\}\log r+O(1). \tag{5.4}$$

Now from (5.2), (5.3) and (5.4) we have

$$\begin{aligned} & n\ \max\{\deg(M_2),\deg(M_3)\}\log r \qquad (5.5)\\ =\ & T(r,M_1^n)+O(1)\\ \leq\ & T\left(r,\prod_{j=1}^{s}(M_1(z+c_j))^{\mu_j}\right)+2\,T(r,p)+S(r,f)\\ =\ & \sigma\max\{\deg(M_2),\deg(M_3)\}\ \log r+2\ \deg(p)\ \log r+S(r,f). \end{aligned}$$

We see that $\max\{\deg(M_2),\deg(M_3)\}\geq 1$. Now from (5.5) we deduce that $n-\sigma\leq 2\ \deg(p)$, which contradicts our assumption that $2\deg(p)<n-\sigma$.

Hence $M_1(z)$ is a non-zero constant. Let

$$M_1=t\in\mathbb{C}\setminus\{0\}. \tag{5.6}$$

Therefore in this case $p(z)$ reduces to a non-zero constant. Let $p(z)=d\in\mathbb{C}\setminus\{0\}$. So from (5.6) we see that

$$M_1^{n+\sigma}\equiv d^2,\ \ i.e.,\ \ t^{n+\sigma}\equiv d^2.$$

Therefore $f(z)g(z) \equiv t$, where t is a constant such that $t^{n+\sigma} = d^2$.

In particular, when $f(z)$ and $g(z)$ are transcendental entire function, from (5.1) it is easy to see that $f(z)$ and $g(z)$ have no zeros that is to say 0 is a Picard exceptional value of both $f(z)$ and $g(z)$.

Let $M_1 = fg$. First we suppose that M_1 is non-constant.

From Case 1, one can easily say that M_1 can not be a transcendental entire function. Hence M_1 is a non-constant polynomial. Since $2\deg(p) < n + \sigma$, from (5.1), we arrive at a contradiction. Hence M_1 is a non-zero constant, say t. Therefore in this case $p(z)$ reduces to a non-zero constant. Let $p(z) = d \in \mathbb{C} \setminus \{0\}$.

Consequently, $f(z)$ and $g(z)$ take the forms $f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, $Q(z)$ being a non-constant polynomial and t is a constant such that $t^{n+\sigma} = d^2$. This completes the proof. □

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