Theory and Applications of Categories, Vol. 27, No. 15, 2013, pp. 347–392.

ON ACTIONS AND STRICT ACTIONS IN HOMOLOGICAL CATEGORIES

MANFRED HARTL AND BRUNO LOISEAU

ABSTRACT. Let G be an object of a finitely cocomplete homological category $\mathbb{C}$. We study actions of G on objects A of $\mathbb{C}$ (defined by Bourn and Janelidze as being algebras over a certain monad $\mathbb{T}_G$), with two objectives: investigating to which extent actions can be described in terms of smaller data, called action cores; and to single out those abstract action cores which extend to actions corresponding to semi-direct products of A and G (in a non-exact setting, not every action does). This amounts to exhibiting a subcategory of the category of the actions of G on objects A which is equivalent with the category of points in $\mathbb{C}$ over G, and to describing it in terms of action cores. This notion and its study are based on a preliminary investigation of co-smash products, in which cross-effects of functors in a general categorical context turn out to be a useful tool. The co-smash products also allow us to define higher categorical commutators, different from the ones of Huq, which are not generally expressible in terms of nested binary ones. We use strict action cores to show that any normal subobject of an object E (i.e., the equivalence class of 0 for some equivalence relation on E in $\mathbb{C}$) admits a strict conjugation action of E. If $\mathbb{C}$ is semi-abelian, we show that for subobjects X, Y of some object A, X is proper in the supremum of X and Y if and only if X is stable under the restriction to Y of the conjugation action of A on itself. This also amounts to an alternative proof of Bourn and Janelidze's category equivalence between points over G in $\mathbb{C}$ and actions of G in the semi-abelian context. Finally, we show that the two axioms of an algebra which characterize G-actions are equivalent with three others ones, in terms of action cores. These axioms are commutative squares involving only co-smash products. Two of them are associativity type conditions which generalize the usual properties of an action of one group on another, while the third is kind of a higher coherence condition which is a consequence of the other two in the category of groups, but probably not in general. As an application, we characterize abelian action cores, that is, action cores corresponding to Beck modules; here also the coherence condition follows from the others.

1. Introduction

For two objects G and A of a category $\mathbb{C}$, an *action* of G on A is an algebra over the monad induced by the adjunction between the category of points over G and $\mathbb{C}$ ([Bourn & Janelidze 1998], [Borceux, Janelidze & Kelly 2005]). When the category is semi-abelian, the right adjoint of this adjunction is monadic, hence this induces an equivalence of

Received by the editors 2012-01-31 and, in revised form, 2013-03-07.

Published on 2013-03-13 in the volume of articles from CT2011.

2010 Mathematics Subject Classification: 18A05, 18A20, 18A22.

Key words and phrases: action, semi-direct product, conjugation, normal subobject, ideal, commutator, homological category, semi-abelian category, algebra over monad.

categories between the category of points and the category of actions.

In this paper, we further study this notion of action. We generally work in the context of finitely cocomplete homological categories, which are of special interest in the theory of square ring(oid)s and their modules initiated in [Baues, Hartl & Pirashvili 1997], as, for example, certain categories of filtered objects are of this type, see Example 3.9. Some results, however, are valid in a wider context (notably in Section 2), while others in addition require exactness, i.e. only hold in semi-abelian categories. We show that the essential information of an action ξ of an object G on an object A is contained in the restriction of ξ to the subobject $A \diamond G$ of $T_G(A)$. More precisely, $A \diamond G$ is the kernel of the canonical map from the sum $A + G$ to the product $A \times G$, or equivalently, from $\mathbb{T}_G(A)$ to A (we here adopt the notation from the related (but independent) article [Mantovani & Metere 2010], and extend it to general co-smash products, unlike in [Hartl & Van der Linden 2013] where the latter are denoted by $\otimes$). Therefore, we study the morphisms $\psi\colon A \diamond G \to A$ which can be extended to actions $\xi\colon T_G A \to A$. We call such objects *action cores*. We use them to determine a subcategory of the category of G-actions which is such that the comparison functors restrict to an equivalence between this category and the category of points on G. This problem has been independently studied in [Martins-Ferreira & Sobral 2012] and led to the notion of *strict action*; for this reason, we call *strict action cores* the morphisms $\psi\colon A \diamond G \to A$ which extend to such strict actions, and we characterize them.

Moreover, we construct a *conjugation action (core)* of an object on any normal subobject, which is *strict*, and formalize the fact that the semi-direct product along some action can be viewed as its universal transformation into a conjugation action.

More generally, we define a notion of *one subobject normalizing another one*, in terms of the conjugation action, which is equivalent to the latter being proper in the supremum of both when $\mathbb{C}$ is semi-abelian (Theorem 4.13). This also amounts to a formula for the *normal closure* of a subobject in the join with another one.

These facts lead to many applications: e.g., based on a detailed comparison of action cores with $\mathbb{T}_G$-algebras, they allow to reprove the equivalence between these algebras and points over G when $\mathbb{C}$ is semi-abelian without using Beck's criterion. More applications are given in a thorough study of internal crossed modules in [Hartl & Van der Linden 2013] and in forthcoming further work, and of higher commutators of subobjects as introduced in this paper (Definition 4.7), in [Hartl - in preparation]. For the notion of internal crossed modules, we refer to [Janelidze 2003].

All these applications are based on two observations: the term $A \diamond G$ above comes as the binary case of *co-smash products* of any length originally defined in [Carboni & Janelidze 2003], and co-smash products of different length are interrelated in various ways: on the one hand, we recall the *folding operations* defined in [Hartl & Van der Linden 2013] and introduce similar *compression operations* here; both are crucial in clarifying the relation between actions and action cores. On the other hand, higher co-smash products can be constructed from lower ones by means of *cross-effects of functors*: this concept is fundamental in algebraic topology and was introduced in [Eilenberg & Mac Lane 1954] for functors between abelian categories, and adapted to functors with values in the category of groups in [Baues & Pirashvili 1999], see also [Hartl & Vespa 2011], [Hartl & Van der Linden 2013], [Hartl, Pirashvili & Vespa 2012] and [Hartl - in preparation] for further developments. In this paper we only define binary cross-effects and study some of their basic properties, in the respectively weakest possible contexts.

We also use the co-smash products machinery to cut the axioms of an action core (hence of an action) in three pieces, two of which again look like associativity conditions, on the terms $(A \diamond G) \diamond A$ and $(A \diamond G) \diamond G$, and have nice interpretations in the category of groups: the first one says that G acts by endomorphisms of A, and the second then expresses the usual associativity condition for the action of G on the underlying set of A. Thus the third condition is void in the category of groups, but probably not in general; it involves a ternary co-smash product and is of the type of the coherence conditions which also appeared in the description of internal crossed modules in [Hartl & Van der Linden 2013].

Plan of the paper Section 2 is of a preparatory nature; here we recall and study *co-smash products*. The above-mentioned *operations* between them are constructed in Definitions 2.3 to 2.5. In Proposition 2.7 and Remark 2.8 we show that binary co-smash products give rise to the following decomposition of the sum of two objects as semi direct products: $A+G = ((A \diamond G) \rtimes A) \rtimes G$. We then investigate ternary co-smash products, by observing that they are special cases of *cross-effect* functors (as well as binary ones and, in fact, all n-ary are!). In fact, for a functor F from a pointed category with finite sums to an pointed category with finite limits, its second cross-effect $F(-|-)$ is a bifunctor measuring the difference between the image of a sum and the product of the images. So the co-smash product functor is just the second cross-effect functor of the identity endofunctor. In Proposition 2.12 we show that the ternary co-smash product can be identified with the second cross-effect of another endofunctor (more precisely, the functor $X \diamond - \diamond -$ is the second cross-effect of the functor $X \diamond -$, for a fixed object X). Now Proposition 2.13 states that under suitable conditions, if a functor F *preserves regular epimorphisms* then so do the cross-effect functors $F(A|-)$ and $F(-|A)$, for any object of the category (notably this applies to endofunctors of a homological category with finite sums). As a consequence, in a homological category with finite sums, the *ternary co-smash product functors preserve regular epimorphisms, as do the binary ones* (see Corollary 2.14). Finally, it is observed in Proposition 2.15 that for any split short exact sequence $0 \longrightarrow K \xrightarrow{k} X \underset{p}{\overset{s}{\leftrightarrows}} Y$, any co-smash product $X \diamond Z$ is a quotient of the sum $(K \diamond Y \diamond Z) + (K \diamond Z) + (Y \diamond Z) \to X \diamond Z$.

Section 3 is devoted to a general study of *actions, action cores and semi-direct products* in categories which are at least pointed, finitely complete and finitely cocomplete protomodular. We begin with the observation that a morphism $\xi \colon T_G A \to A$ which satisfies the unit axiom of a T_G-algebra, is uniquely determined by its restriction to $A \diamond G$, which may be called the *core* of ξ. Then we study actions from the view point of their core: we define *strict action cores* as morphisms $\psi \colon A \diamond G \to G$ such that in the following diagram, the morphism l_ψ is a monomorphism:

$$\begin{tikzcd} A \diamond G \arrow[r, "\iota_{A,G}"] \arrow[d, "\psi"'] & A + G \arrow[d, "q_\psi"] \\ A \arrow[r, "l_\psi"'] & Q_\psi \end{tikzcd}$$

where $\iota_{A,G}$ is the inclusion of $A \diamond G$ in $A + G$ defined in Section 2, i_A is the canonical inclusion of A in $A + G$, q_ψ is the coequalizer of $\iota_{A,G}$ and $i_A \circ \psi$, and $l_\psi = q_\psi \circ i_A$. It is then proved that such a *strict action core extends to a T_G-algebra* (i.e. an action, in the sense of [Borceux, Janelidze & Kelly 2005]), and that this extension is strict in the sense of [Martins-Ferreira & Sobral 2012] (in fact, the first version of this paper was written simultaneously but independently from the latter article and [Mantovani & Metere 2010], which explains certain similarities of our work with the cited papers). Moreover, the object Q_ψ in the diagram is the *semi-direct product* of A and G along these actions. If the base category is finitely cocomplete homological, then the category of strict action cores is equivalent to the category of strict actions, and the semi-direct product functor restricts to an equivalence between these categories and the category of points (or of split short exact sequences) (Proposition 3.10). We also define *action cores*, which are those morphisms $\psi\colon A \diamond G \to A$ which extend to actions, but these are only studied in Section 5. Example 3.7 (the category of groups, where action cores and actions are automatically strict, because this category is semi-abelian) shows that action cores actually focus on a different aspect of actions: recall that an action of a group G on a group A is a function $\phi\colon G \times A \to A$ which is a kind of "external conjugation" of G on A, in the sense that when A and G are imbedded in the semi-direct product, $\phi(g, a)$ becomes the "real" conjugate gag^{-1}. The corresponding action core can be seen as a function $\psi : G \times A \to A$ which is a kind of "external commutation" of G on A, in the sense that when A and G are imbedded in the semi-direct product, $\psi(g, a)$ becomes the "real" commutator $gag^{-1}a^{-1}$. Finally, Proposition 3.13 shows how to construct new strict action cores from given ones.

In Section 4, we always work in finitely cocomplete homological categories and are interested in the construction of *conjugation actions*. We use Proposition 3.13 to construct *conjugation action cores* on normal subobjects of any object, which are *strict action cores* (hence induce strict actions) (Proposition 4.1). We introduce the notion of *n-ary Higgins commutators* (binary ones were also independently introduced in [Mantovani & Metere 2010]); these commutators and their relation with actions whose study is started here turned out to also provide a key tool in the study of crossed modules and the "Smith is Huq" condition, see [Hartl & Van der Linden 2013], and also in subsequent work on (co)homology and other subjects, by several authors. We here use them to investigate normality and the normal closure of a subobject in the join with another one, in the case where the category is semi-abelian. This refines the main result in [Mantovani & Metere 2010] where the second subobject is taken to be the whole object.

We exhibit in Section 5 *necessary and sufficient conditions for an arbitrary morphism $\psi\colon A \diamond G \to A$ to be an action core.* More precisely, we first show in Proposition 5.1 that a necessary and sufficient condition for a morphism $\psi\colon A \diamond G \to A$ to extend (uniquely, of course) to a morphism $\xi\colon T_G A \to A$ satisfying the unit axiom of a $\mathbb{T}_G$-algebra is the commutativity of the following diagram:

$$\begin{tikzcd}
(A \diamond G) \diamond A \arrow[r, "C^A_{A,G}"] \arrow[d, "\psi \diamond 1_A"'] & A \diamond G \arrow[d, "\psi"] \\
A \diamond A \arrow[r, "c^A_2"'] & A
\end{tikzcd}$$

where the morphism $C^A_{A,G}$ is one of the compression operations between co-smash products defined in section 2. Then we show in Proposition 5.7 that, given a morphism $\psi\colon A \diamond G \to A$ which satisfies this condition, hence has an extension $\xi\colon T_G A \to A$ satisfying the unit axiom, this extension ξ is a $\mathbb{T}_G$-algebra if and only if the following diagram commutes:

$$\begin{tikzcd}
T_G A \diamond G \arrow[r, "c^{T_G A, A+G}\restriction_G"] \arrow[d, "\xi \diamond 1_G"'] & T_G A \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "\psi"'] & A
\end{tikzcd}$$

(the morphism $c^{T_G A, A+G} \restriction_G$ being (a restriction of) a conjugation core action defined in the former section). But we also show that if moreover the category is semi-abelian, then the commutativity of the latter diagram is equivalent (for a morphism $\psi\colon A \diamond G \to G$ making the former diagram commute), to this ψ being a strict action, hence showing that in a semi-abelian category any action is strict, or equivalently giving an alternative proof of the equivalence between the category of actions on G (i.e. $\mathbb{T}_G$-algebras) and the category of points.

We finally show in Proposition 5.9 that the commutativity of these two diagrams can be translated in terms of the commutativity of three diagrams involving essentially only the morphism ψ, co-smash products of A and G and the folding and compression operations between them introduced in section 2. Two of them are of "associativity condition" type, while the third resembles the "higher coherence conditions" which also came up in the study of internal crossed modules in [Hartl & Van der Linden 2013]. This condition is the most intricate one as it contains a nested cosmash product involving four factors, but is superfluous in the category of groups. Hence it would be an interesting problem to characterize semi-abelian categories where the latter property holds, possibly by relating it to the *Smith-is-Huq* condition studied in [loc.cit.].

At least, the third condition does not appear in the characterization of strict action cores corresponding to *Beck modules* given in Corollary 5.11, which is valid in any finitely cocomplete homological category.

Conventions and recollections When working in a pointed category with finite sums, we denote the canonical inclusion $X_k \to X_1 + \cdots + X_n$ by i_{X_k} or by i_k, and its canonical retraction by r_{X_k} or by r_k; dually, when working in a pointed category with finite products, we denote the canonical projection $X_1 \times \cdots \times X_n \to X_k$ by π_k and its canonical section by σ_k. Identities on objects X of a category $\mathbb{C}$ are denoted by 1_X, while the identity functor on $\mathbb{C}$ is denoted by $\mathrm{Id}_{\mathbb{C}}$. When the category is pointed, the zero morphisms are denoted by 0. Finally, note that in some examples, we denote the unit of a group G by e_G.

In a pointed category with finite limits, a *proper* subobject of an object X is a subobject which is the kernel of some morphism with domain X; a *normal subobject* of an object X is a subobject which is the equivalence class of 0 for some equivalence class R, i.e. a subobject K of X whose inclusion in X can be factored as $K = \operatorname{Ker} r_1 \overset{\ker r_1}{\rightarrowtriangle} R \xrightarrow{r_2} X$ for some equivalence relation (R, r_1, r_2) on X.

Recall that in a finitely complete protomodular category, given a split short exact sequence $0 \longrightarrow A \xrightarrow{l} X \underset{p}{\overset{s}{\leftrightarrows}} G \longrightarrow 0$, then (l, s) is a strongly epimorphic family of morphisms with codomain X, so if moreover the category has finite coproducts the morphism $\langle {l \atop s} \rangle : A + G \to X$ is a strong epimorphism, hence a regular one if moreover the category is exact. We shall often make use of protomodularity in this way.

2. Comparison between sums and products

The product $X \diamond Y$ in the introduction was used in [Mantovani & Metere 2010], in order to characterize proper subobjects in semi-abelian categories. Our present paper and, to a much larger extent, the subsequent article [Hartl & Van der Linden 2013] make essential use of the following facts:

1. the product $\diamond$ comes as the binary case of a whole family of multi-endofunctors called the *co-smash products* [Carboni & Janelidze 2003];

2. the co-smash products give rise to a generalization of the binary *Higgins commutator* (following the terminology in [Mantovani & Metere 2010]) to commutators of any finite family of subobjects of a given object, see section 4 below;

3. the co-smash products of different lengths are *interrelated* in various ways. First of all, we need the *folding operations* from [Hartl & Van der Linden 2013] and similar *compression operations* which we introduce here. Secondly, the n-th co-smash product can be derived from the $(n-1)$-st by taking a *binary cross-effect*, and in fact, can be viewed as the n-th cross-effect of the identity functor. This also means that the product $\diamond$ is just the binary cross-effect of the identity functor, and since analyzing its more subtle properties requires using its own binary cross-effect, these properties involve the ternary co-smash product.

The concept of cross-effects of a functor originally arose in homotopy theory: for functors between abelian categories it is due to Eilenberg and MacLane [Eilenberg & Mac Lane 1954], and later was adapted to functors with values in the category of groups in [Baues & Pirashvili 1999] and further studied in [Hartl & Vespa 2011]. This definition of cross-effects actually works in a wide categorical context, and strong properties arise in the realm of homological and semi-abelian categories, see (the first version on arXiv of) [Hartl & Van der Linden 2013] and [Hartl - in preparation]. As outlined before, these properties of cross-effects can be used to study co-smash products, which themselves turn out to play a key role in the theory of internal crossed modules and of commutators, see [loc. cit.], [Rodelo & Van der Linden 2012] and [Martins-Ferreira & Van der Linden 2012].

Co-smash products We first recall the definition of co-smash products from [Carboni & Janelidze 2003].

2.1. Definition. *In a finitely complete pointed category* $\mathbb{C}$ *with finite sums we call* **co-smash product** $X_1 \diamond \cdots \diamond X_n$ *of objects* $X_1, \ldots, X_n$, $n \geq 2$ *the kernel*

$$X_1 \diamond \cdots \diamond X_n \rightarrowtriangle \coprod_{k=1}^{n} X_k \xrightarrow{r_{X_1,\ldots,X_n}} \prod_{k=1}^{n} \coprod_{j\neq k} X_j$$

where $r_{X_1,\ldots,X_n}$ *is the morphism determined by*

$$\pi_{\coprod_{j\neq m} X_j} \circ r_{X_1,\ldots,X_n} \circ i_{X_l} = \begin{cases} i_{X_l} & \text{if } l \neq m \\ 0 & \text{if } l = m \end{cases}$$

for $l, m \in \{1, \ldots, n\}$*. The kernel morphism is denoted* $\iota_{X_1,\ldots,X_n}$*.*

It should be noted that the product $\diamond$ in general is not associative, nor there is a decomposition like $X \diamond Y \diamond Z = (X \diamond Y) \diamond Z$ of higher co-smash products into nested binary ones.

2.2. Example. Let us make explicit what happens in the lowest-dimensional cases, which are the only ones used in the present article. For objects X, Y, Z of $\mathbb{C}$ we have natural exact sequences

$$0 \longrightarrow X \diamond Y \xrightarrow{\iota_{X,Y}} X + Y \xrightarrow{\left\langle\begin{smallmatrix} 1_X & 0 \\ 0 & 1_Y \end{smallmatrix}\right\rangle} X \times Y$$

for $n = 2$ and

$$0 \longrightarrow X \diamond Y \diamond Z \xrightarrow{\iota_{X,Y,Z}} X + Y + Z \xrightarrow{\left\langle\begin{smallmatrix} i_X & i_X & 0 \\ i_Y & 0 & i_Y \\ 0 & i_Z & i_Z \end{smallmatrix}\right\rangle} (X+Y) \times (X+Z) \times (Y+Z)$$

for $n = 3$.

Co-smash products are interrelated in various ways; in particular, the following *operations* will be used later on. In order to describe them the following notation will be convenient: for an object A of $\mathbb{C}$ and $p \geq 1$ we write $p \cdot A = A + \cdots + A$ and $A^{\diamond p} = A \diamond \cdots \diamond A$ with p summands respectively factors A. Moreover, let $\nabla_A^p \colon p \cdot A \to A$ denote the folding morphism, determined by $\nabla_A^p \circ i_k = 1_A$ for $k = 1, \ldots, p$. If $f \colon A \to B$ is a morphism in $\mathbb{C}$ we write $p \cdot f = f + \cdots + f \colon p \cdot A \to p \cdot B$.

2.3. Definition. *Let X, Y be objects of $\mathbb{C}$ and $p, q \geq 1$. Then the natural* **folding operation**

$$S^{X,Y}_{p,q} \colon X^{\diamond p} \diamond Y^{\diamond q} \to X \diamond Y$$

is the unique morphism such that $\iota_{X,Y} \circ S^{X,Y}_{p,q} = (\nabla^p_X + \nabla^q_Y) \circ \iota_{X,\dots,X,Y,\dots,Y}$.

It is easy to see that $S^{X,Y}_{p,q}$ exists, cf. the cases $(p,q) = (1,2)$ and $(2,1)$ treated in [Hartl & Van der Linden 2013, Notation 2.23]. These morphisms are special cases of the folding operations studied in [Hartl - in preparation]. We also need the following type of operations.

2.4. Definition. *Let $X_1, \dots, X_n$ be objects of $\mathbb{C}$ and $p \geq 1$. Then let*

$$C^{(p)}_{X_1,\dots,X_n} \colon (X_1 \diamond \cdots \diamond X_n) \diamond (\coprod_{k=1}^{n} X_k)^{\diamond p} \to X_1 \diamond \cdots \diamond X_n$$

be the unique morphism rendering the left-hand square of the following diagram commutative where we abbreviate $\Sigma = \coprod_{k=1}^n X_k$, $\Sigma_k = \coprod_{j \neq k} X_j$ and $\iota = \iota_{X_1 \diamond \cdots \diamond X_n, \Sigma, \dots, \Sigma}$:

$$\begin{tikzcd}
(X_1 \diamond \cdots \diamond X_n) \diamond \Sigma^{\diamond p} \arrow[r, "\iota"] \arrow[d, dotted, "C^{(p)}_{X_1,\dots,X_n}"'] & (X_1 \diamond \cdots \diamond X_n) + p \cdot \Sigma \arrow[r, "\left\langle \begin{smallmatrix} 0 \\ p \cdot r_{X_1,\dots,X_n} \end{smallmatrix} \right\rangle"] \arrow[d, "\left\langle \begin{smallmatrix} \iota_{X_1,\dots,X_n} \\ \nabla^p_\Sigma \end{smallmatrix} \right\rangle"'] & p \cdot \prod_{k=1}^n \Sigma_k \arrow[d, "r_{X_1,\dots,X_n} \circ \nabla^p_{\prod_{k=1}^n \Sigma_k}"'] \\
X_1 \diamond \cdots \diamond X_n \arrow[r, tail, "\iota_{X_1,\dots,X_n}"] & \coprod_{k=1}^n X_k \arrow[r, "r_{X_1,\dots,X_n}"] & \prod_{k=1}^n \Sigma_k
\end{tikzcd}$$

Again, it is easy to see that $C^{(p)}_{X_1,\dots,X_n}$ exists since in the above diagram the bottom row is exact, the right-hand square commutes and the composition of the two top arrows is trivial since it factors through $r_{X_1 \diamond \cdots \diamond X_n, \Sigma, \dots, \Sigma}$.

The morphisms $C^{(p)}_{X_1,\dots,X_n}$ induce other operations of which we write out only the simplest family, as it suffices for the needs of this paper.

2.5. Definition. *Let $X_1, \dots, X_n$ be objects of $\mathbb{C}$, $p \geq 1$ and $k_1, \dots, k_p$ a sequence of integers between 1 and n. Then the natural* **compression operation**

$$C^{X_{k_1},\dots,X_{k_p}}_{X_1,\dots,X_n} \colon (X_1 \diamond \cdots \diamond X_n) \diamond X_{k_1} \diamond \cdots \diamond X_{k_p} \to X_1 \diamond \cdots \diamond X_n$$

is defined to be composite morphism $C^{(p)}_{X_1,\dots,X_n} \circ (1_{X_1 \diamond \cdots \diamond X_n} \diamond i_{k_1} \diamond \cdots \diamond i_{k_p})$.

As a first *application* of co-smash products we decompose the functor part of the monad $\mathbb{T}_G$ in $\mathbb{C}$ introduced in [Bourn & Janelidze 1998] (see also section 3), which is crucial for our analysis of actions in the sequel. It also induces a decomposition of the sum which will be used to study normality in section 4.

Let $T_{-}(-)\colon \mathbb{C}^2 \to \mathbb{C}$ be the bifunctor defined on pairs of objects (G, A) by

$$T_G(A) = \mathrm{Ker}(r_G\colon A + G \to G)$$

(with the obvious consequent definition on morphisms); the kernel morphism

$$T_G(A) \rightarrowtail A + G$$

is denoted by $\kappa_{A,G}$. Moreover, the morphism $\eta_{A,G}\colon A \to T_G(A)$ is given by noting that the morphism $i_A\colon A \to A + G$ factors through $\kappa_{A,G}$ since $r_G i_A = 0$.

Notice that $T_G(A)$ is nothing but $G\flat A$, as defined in [Borceux, Janelidze & Kelly 2005], and for fixed G, $\eta_{A,G}$ is nothing but the (value in A of) the unit of the monad $(G\flat -)$. Since, in a semi-direct product, we prefer to denote the kernel part on the left and the cokernel part on the right, hence using the notation $G \rtimes A$ instead of $A \ltimes G$, we avoid the convenient notation $G\flat A$ and prefer keeping $T_G(A)$ as in [Borceux & Bourn 2004]; a possible compromise could be to write $G\flat A = A\natural G$.

2.6. Remark. It can be shown that the natural morphism

$$\nu_A = C^G_{A,G}\colon (A \diamond G) \diamond G \to A \diamond G$$

endows the endofunctor $- \diamond G$ with the structure of a non-unital monad, i.e., ν_A satisfies the associativity axiom of a monad. We do not need this observation here; together with Theorem 5.9, however, it may lead to a generalization of the notion of internal action, as will be pursued elsewhere.

2.7. Proposition. *Using the preceding notations, in a finitely complete pointed category with finite sums one has the following split short exact sequences:*

$$0 \longrightarrow T_G(A) \xrightarrow{\kappa_{A,G}} A + G \underset{r_G}{\overset{i_G}{\leftrightarrows}} G \longrightarrow 0$$

and

$$0 \longrightarrow A \diamond G \xrightarrow{j_{A,G}} T_G(A) \underset{r_A \circ \kappa_{A,G}}{\overset{\eta_{A,G}}{\leftrightarrows}} A \longrightarrow 0$$

Proof. Only the second split exact sequence has to be constructed. It is clear by construction that $\eta_{A,G}$ is a section of $r_A \circ \kappa_{A,G}$. The morphism $j_{A,G}\colon A \diamond G \to T_G(A)$ arises from the facts that $A \diamond G$ is the kernel of the morphism $r_{A,G}\colon A + G \to A \times G$, that $T_G(A)$ is the kernel of $r_G\colon A + G \to G$ and that $r_G = \pi_G \circ r_{A,G}$. The latter observation also implies that $j_{A,G}$ is the kernel of $r_A \circ \kappa_{A,G}$. Note also that all this is related with the fact, observed in [Mantovani & Metere 2010], that $A \diamond G = T_G(A) \wedge T_A(G)$. ∎

2.8. Remark. Whenever we have a split short exact sequence

$$0 \longrightarrow A \longrightarrow B \rightleftarrows C \longrightarrow 0$$

it is convenient to write $B = A \rtimes C$, where the injections of A and C into B are understood. With this notation the split short exact sequences above can be rephrased as:

1. $A + G = T_G(A) \rtimes A$
2. $T_G(A) = (A \diamond G) \rtimes A$

This also implies that

3. $A \diamond G = T_G(A) \wedge T_A(G)$.

Hence in view of (1) and (2), we obtain the following decomposition of the sum which is crucial in our study of normal and proper subobjects in section 4:

2.9. Corollary. *For objects A, G in a finitely complete pointed category with finite sums one has*

$$A + G = ((A \diamond G) \rtimes A) \rtimes G.$$

Cross effects of functors Our main tool in studying co-smash products is the notion of *cross-effects of functors*; for the purpose of this paper, however, it is sufficient to introduce only the second (also called binary) cross-effect, as follows.

Let $F \colon \mathbb{D} \to \mathbb{E}$ be a functor where $\mathbb{D}$ is a pointed category with finite sums and $\mathbb{E}$ is a pointed finitely complete category. For objects X, Y in $\mathbb{D}$ the canonical morphism $\langle F(r_X), F(r_Y)\rangle \colon F(X+Y) \to F(X) \times F(Y)$ is denoted by $r^F_{X,Y}$.

2.10. Definition. *The* **second cross-effect** *of F is defined to be the functor*

$$cr_2(F) \colon \mathbb{D}^2 \to \mathbb{E}$$

given by:

$$cr_2(F)(X,Y) = \operatorname{Ker}\big(r^F_{X,Y} \colon F(X+Y) \to F(X) \times F(Y)\big)\,;$$

The kernel morphism $cr_2(F)(X,Y) \rightarrowtail F(X+Y)$ is denoted by $\iota^F_{X,Y}$. The definition of $cr_2(F)$ on morphisms is immediate.

One often abbreviates $cr_2(F)(X,Y) = F(X|Y)$.

2.11. Proposition.

1. *The bifunctor $cr_2(F)$ is symmetric.*
2. *The functor $cr_2(F)$ is bireduced, i.e. $cr_2(F)(X,Y) = 0$ if $X = 0$ or $Y = 0$.*
3. *If moreover $\mathbb{E}$ is protomodular then the morphism $r^F_{X,Y}$ is a strong epimorphism.*

Proof. Assertion (1) is obvious. For (2), just observe that $cr_2(0, Y) = 0$ since the morphism $\langle F(r_0), F(r_Y)\rangle \colon F(0+Y) \to F(0)\times F(Y)$ admits the second projection followed by $F(i_Y)$ as a retraction.

Now suppose that $\mathbb{E}$ is a finitely complete protomodular category. We have $r^F_{X,Y} \circ F(i_X) = \sigma_{F(X)} \colon F(X) \to F(X) \times F(Y)$; similarly $r^F_{X,Y} \circ F(i_Y) = \sigma_{F(Y)}$. Considering the split short exact sequence $0 \longrightarrow F(X) \xrightarrow{\sigma_{F(X)}} F(X) \times F(Y) \underset{\pi_{F(Y)}}{\overset{\sigma_{F(Y)}}{\leftrightarrows}} F(Y) \longrightarrow 0$ and applying protomodularity one concludes that the family $(\sigma_{F(X)}, \sigma_{F(Y)})$ is strongly epimorphic. Hence by [Borceux & Bourn 2004, Proposition A.4.17, 2], $r^F_{X,Y}$ is a strong epimorphism. ∎

Now we relate co-smash products and cross-effects. Note that for the identity functor $\mathrm{Id}_{\mathbb{C}}$ (which may be defined to be the unary co-smash product) and objects X, Y in $\mathbb{C}$ we have $cr_2(\mathrm{Id}_{\mathbb{C}})(X, Y) = X \diamond Y$. Moreover, $\iota_{X,Y} = \iota^{\mathrm{Id}_{\mathbb{C}}}_{X,Y}$ and $r_{X,Y} = r^{\mathrm{Id}_{\mathbb{C}}}_{X,Y}$. Similarly, the second cross-effect of the binary co-smash product is the ternary co-smash product, as follows.

2.12. Proposition. *Let $\mathbb{C}$ be a finitely complete pointed category with finite sums. Then there is a natural isomorphism*

$$cr_2(X \diamond -)(Y, Z) \;\cong\; X \diamond Y \diamond Z$$

for objects X, Y, Z in $\mathbb{C}$.

Proof. Consider the following commutative diagram of plain arrows:

$$\begin{tikzcd}
(cr_2(X\diamond -))(Y,Z) \arrow[r, dotted, shift left, "\iota_1"] \arrow[d, "\iota^{X\diamond -}_{Y,Z}"'] & X\diamond Y\diamond Z \arrow[l, dotted, shift left, "\iota_3"'] \arrow[d, "\iota_{X,Y,Z}"] \arrow[dl, dotted, "\iota_2"'] & \\
X\diamond(Y+Z) \arrow[r, "\iota_{X,Y+Z}"'] \arrow[d, "\langle 1_X\diamond r_Y, 1_X\diamond r_Z\rangle"'] & X+Y+Z \arrow[r, "r_{X,Y+Z}"] \arrow[d, "r_{X,Y,Z}"'] & X\times(Y+Z) \\
(X\diamond Y)\times(X\diamond Z) \arrow[r, tail, "\langle \iota_{X,Y}\times\iota_{X,Z}, 0\rangle"'] & (X+Y)\times(X+Z)\times(Y+Z) \arrow[ur, "\langle r_X\circ\pi_{X+Y}, \pi_{Y+Z}\rangle"'] &
\end{tikzcd}$$

where the commutativity of the rectangle essentially comes from naturality of $\iota_{-,-}$. Then the factorization ι_1 comes from exactness of the columns and commutativity of the rectangle; ι_2 comes from exactness of the row and commutativity of the triangle; and finally ι_3 comes from the equality $\langle \iota_{X,Y} \times \iota_{X,Z}, 0\rangle \circ \langle 1_X \diamond r_Y, 1_X \diamond r_Z\rangle \circ \iota_2 = r_{X,Y,Z} \circ \iota_{X,Y,Z} = 0$, which implies $\langle 1_X \diamond r_Y, 1_X \diamond r_Z\rangle \circ \iota_2 = 0$ since $\langle \iota_{X,Y} \times \iota_{X,Z}, 0\rangle$ is a monomorphism. Then ι_1 and ι_3 are mutually inverse isomorphisms. ∎

Note that the morphism ι_2 in this proof then is the kernel of $\langle 1_X \diamond r_Y, 1_X \diamond r_Z\rangle$; we shall denote it by $\iota'_{X,Y,Z;2}$ since it refers to a sum in the second variable of the co-smash product. Similarly, there also exists a morphism $\iota'_{X,Y,Z;1} \colon X \diamond Y \diamond Z \to (X + Y) \diamond Z$ which is the kernel of $\langle r_X \diamond 1_Z, r_Y \diamond 1_Z\rangle \colon (X + Y) \diamond Z \to (X \diamond Z) \times (Y \diamond Z)$.

Basic properties of cross-effects and co-smash products The following facts are key tools in handling cross-effects and hence also co-smash products.

2.13. Proposition. *Suppose that $\mathbb{D}$ is a pointed category with finite sums, that $\mathbb{E}$ is homological and that $F\colon \mathbb{D} \to \mathbb{E}$ preserves regular epimorphisms. Then for all objects A in $\mathbb{D}$ the functors $F(A|-)$ and $F(-|A)\colon \mathbb{D} \to \mathbb{E}$ also preserve regular epimorphisms.*

For the co-smash product (i.e. the case when $\mathbb{D} = \mathbb{E}$ is homological with finite coproducts and $F = \mathrm{Id}_{\mathbb{E}}$) this means that the functors $X \diamond -$ and $- \diamond X$ preserve regular epimorphisms. The same result was independently proved for ideal-determined categories in [Mantovani & Metere 2010].

Proof. By symmetry of the bifunctor $F(-|-)$ it is sufficient to prove this for $F(A|-)$. Let $f\colon X \twoheadrightarrow Y$ be a regular epimorphism. Consider the following commutative diagram where k and m are kernels of $F(f)$ and of $F(1+f)$ respectively, and where α is induced by $\langle F(r_X), F(r_Y)\rangle$:

$$\begin{tikzcd}
 & & F(A|X) \arrow[r, "F(1|f)"] \arrow[d, "\iota"] & F(A|Y) \arrow[d, "\iota"] & \\
 & \mathrm{Ker}(F(1+f)) \arrow[r, tail, "m"] \arrow[d, "\alpha"] & F(A+X) \arrow[r, two heads, "F(1+f)"] \arrow[d, "\langle Fr_A, Fr_X\rangle"] & F(A+Y) \arrow[r] \arrow[d, "\langle Fr_A, Fr_Y\rangle"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(F(f)) \arrow[r, tail, "\langle 0,k\rangle"] & F(A)\times F(X) \arrow[r, "1\times F(f)"] & F(A)\times F(Y) &
\end{tikzcd}$$

The columns are exact by definition of the cross-effect, and the rows are exact, too; for the middle row this follows from the hypothesis on F since $1+f$ is a regular epimorphism. Thus the snake lemma provides an exact sequence

$$F(A|X) \xrightarrow{F(1|f)} F(A|Y) \longrightarrow \mathrm{Coker}(\alpha)$$

We claim that $\mathrm{Coker}(\alpha) = 0$: in fact, the morphism $F(i_X)k\colon \mathrm{Ker}(F(f)) \to F(A+X)$ factors through m and thus provides a section s of α, indeed: $F(1+f)\circ F(i_X)\circ k = F(i_Y)\circ F(f)\circ k = 0$, and

$$\langle 0,k\rangle \circ \alpha \circ s = \langle Fr_A, Fr_X\rangle \circ m \circ s = \langle Fr_A, Fr_X\rangle \circ F(i_X)\circ k = \langle 0,1\rangle \circ k = \langle 0,k\rangle$$

whence $\alpha \circ s = 1$ since $\langle 0,k\rangle$ is monic. ∎

2.14. Corollary. *For $k = 1,2,3$ let $f_k\colon X_k \to Y_k$ be a regular epimorphism in a homological category with finite sums. Then the induced morphism $f_1 \diamond f_2 \diamond f_3\colon X_1 \diamond X_2 \diamond X_3 \longrightarrow Y_1 \diamond Y_2 \diamond Y_3$ also is a regular epimorphism.*

Proof. In the decomposition $f_1 \diamond f_2 \diamond f_3 = (f_1 \diamond 1_Y \diamond 1_Z)\circ(1_X \diamond f_2 \diamond 1_Z)\circ(1_X \diamond 1_Y \diamond f_3)$ each of the factors is a regular epimorphism by the symmetry of the ternary co-smash product and Propositions 2.12 and 2.13. ∎

2.15. Proposition. *Suppose that $\mathbb{C}$ is a homological category with finite sums. Let $f\colon X \to Y$ be a morphism in $\mathbb{C}$ with splitting $s\colon Y \to X$, i.e. such that $f \circ s = 1$. Let $k\colon K \to X$ be a kernel of f and let $Z \in Ob(\mathbb{C})$. Then the morphism*

$$\left\langle \begin{smallmatrix} (\langle \begin{smallmatrix} k \\ s \end{smallmatrix} \rangle \diamond 1_Z) \circ \iota'_{K,Y,Z;1} \\ k \diamond 1_Z \\ s \diamond 1_Z \end{smallmatrix} \right\rangle \colon\ (K \diamond Y \diamond Z) + (K \diamond Z) + (Y \diamond Z) \to X \diamond Z$$

is a regular epimorphism.

Proof. This is an immediate consequence of [Hartl & Van der Linden 2013, Proposition 2.24]. ∎

3. General properties of internal object actions

For a pointed finitely complete category $\mathbb{C}$ with finite sums with a fixed object G, one may consider the category $\mathrm{Pt}_G(\mathbb{C})$ of G-points of $\mathbb{C}$, formally defined to be the category $(\mathbb{C}/G)\backslash(1_G)$. It can be more explicitly described as the category of objects E of $\mathbb{C}$ together with a morphism $p\colon E \to G$ (in $\mathbb{C}$) and a section $s\colon G \to E$ of p. A morphism b between two such objects (E, p, s) and (E', p', s') is a morphism $b\colon E \to E'$ in $\mathbb{C}$ making the following diagram commute:

$$\begin{tikzcd} E \arrow[r, shift right, "p"'] \arrow[d, "b"'] & G \arrow[l, shift right, "s"'] \arrow[d, equal] \\ E' \arrow[r, shift right, "p'"'] & G \arrow[l, shift right, "s'"'] \end{tikzcd}$$

This category is obviously equivalent to the category of split extensions of G, whose objects are short split exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "l"] & E \arrow[r, shift right, "p"'] & G \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

(the sequence is exact and s is a splitting of p).

Morphisms between such split extensions are given by pairs of morphisms $a\colon A \to A'$ and $b\colon E \to E'$ (in $\mathbb{C}$) making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "l"] \arrow[d, "a"'] & E \arrow[r, shift right, "p"'] \arrow[d, "b"'] & G \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r, "l'"'] & E' \arrow[r, shift right, "p'"'] & G \arrow[l, shift right, "s'"'] \arrow[r] & 0 \end{tikzcd}$$

The kernel functor $\mathrm{Ker}\colon \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ which associates to a point $\begin{tikzcd} E \arrow[r, shift right, "p"'] & G \arrow[l, shift right, "s"'] \end{tikzcd}$ the kernel of p, has a left adjoint, which sends an object A of $\mathbb{C}$ to the point $\begin{tikzcd} A + G \arrow[r, shift right, "r_G"'] & G \arrow[l, shift right, "i_G"'] \end{tikzcd}$.

So it generates a monad $\mathbb{T}_G$. More precisely, $\mathbb{T}_G = (T_G\colon \mathbb{C} \to \mathbb{C}, \mu_{-,G}\colon T_G \circ T_G \to T_G, \eta_{-,G}\colon \mathrm{Id}_{\mathbb{C}} \to T_G)$ is defined as follows: T_G is the functor defined in section 2; the multiplication $\mu_{A,G}\colon T_G(T_G(A)) \to T_G(A)$ is given such that $\kappa_{A,G} \circ \mu_{A,G} = \left\langle {\kappa_{A,G} \atop i_G} \right\rangle \circ \kappa_{T_G(A),G}$, where $\kappa_{A,G}$ and the unit $\eta_{A,G}\colon A \to T_G(A)$ are defined in Proposition 2.7. The category of (internal object) actions (of an object G on objects of the category) then is the category $\mathbb{C}^{\mathbb{T}_G}$ of Eilenberg-Moore algebras over this monad [Borceux, Janelidze & Kelly 2005]. *Such an algebra* $\xi\colon T_G(A) \to A$ *is called an action of* G *on* A. Note that in [op. cit.], the object $T_G(A)$ is denoted by $G\flat A$, and is considered as a subobject of $G + A$ rather than of $A + G$.

Of course, $\mathrm{Pt}_G(\mathbb{C})$ is a subcategory of the functor category $\mathrm{Pt}(\mathbb{C})$, whose objects are points (on variable objects G), and a morphism between two points (G, E, p, s) and (G', E', p', s') is a pair of morphisms $a\colon G \to G'$ and $b\colon E \to E'$ making the following diagram commute:

$$
\begin{tikzcd}
E \arrow[r, shift left, "p"'] \arrow[d, "b"'] & G \arrow[l, shift left, "s"'] \arrow[d, "a"] \\
E' \arrow[r, shift left, "p'"'] & G' \arrow[l, shift left, "s'"']
\end{tikzcd}
$$

and similarly for the category of split extensions, and for the category of actions.

As usual, one has a comparison functor $\mathcal{J}\colon \mathrm{Pt}_G(\mathbb{C}) \to \mathbb{C}^{\mathbb{T}_G}$, and when the category $\mathbb{C}$ is finitely cocomplete, one also has a *semi-direct product functor* $- \rtimes_- G\colon \mathbb{C}^{\mathbb{T}_G} \to Pt_G(\mathbb{C})$ sending an algebra (A, ξ) to a point $A \rtimes_\xi G \underset{p}{\overset{s}{\leftrightarrows}} G$, giving rise to a comparison adjunction $(- \rtimes_- G, \mathcal{J}, \eta', \epsilon')\colon \mathbb{C}^{\mathbb{T}_G} \to \mathrm{Pt}_G(\mathbb{C})$ [Borceux, Janelidze & Kelly 2005]. It is well-known [Bourn & Janelidze 1998] that when the category is *semi-abelian*, then this is an *equivalence of categories*.

The goal of this paper is two-fold: firstly, we work under a weaker assumption, namely in a *finitely cocomplete homological category* $\mathbb{C}$, and are interested in finding a subcategory of $\mathbb{C}^{\mathbb{T}_G}$ which is such that the above functors again restrict to an equivalence of categories. Secondly, we want to analyze all the information on these actions which is contained in their restriction (along $j_{A,G}$) to $A \diamond G$.

Note that the first results in this section do not need the regularity hypothesis on $\mathbb{C}$.

Action cores and strict action cores The starting point of our discussion is the following observation:

3.1. Lemma. *Let* $\mathbb{C}$ *be a finitely complete pointed protomodular category with finite sums,* A *and* G *objects in it. A morphism* $\xi\colon T_G(A) \to A$ *satisfying the unit axiom (i.e.,* $\xi \circ \eta_{A,G} = 1_A$*) is uniquely determined by its restriction* ψ *(along* $j_{A,G}$*) to* $A \diamond G$.

Proof. Consider two morphisms ξ and $\xi'\colon T_G(A) \to A$ satisfying this unit axiom, i.e. $\xi \circ \eta_{A,G} = \xi' \circ \eta_{A,G} = 1_A$ having same restriction ψ to $A \diamond G$, i.e. $\psi = \xi \circ j_{A,G} = \xi' \circ j_{A,G}$. Then $\xi = \xi'$ since by Corollary 2.7 and by protomodularity the pair $(\eta_{A,G}, j_{A,G})$ is (strongly) epimorphic. ∎

Lemma 3.1 provides an obvious reason why it is reasonable to describe actions in terms of morphisms $A \diamond G \to A$ rather than $T_G(A) \to A$. Less formal reasons are provided by the characterizations of crossed and Beck modules in terms of the former, given in [Hartl & Van der Linden 2013]. This also suggests the following definition:

3.2. Definition. *Let $\xi\colon T_GA \to A$ be an action of an object G on an object A in a finitely complete and cocomplete protomodular category. Then the* **core** *of ξ is the restriction ψ of ξ to $A \diamond G$, i.e. $\psi = \xi \circ j_{A,G}$.*

Recall that when $\xi\colon T_G(A) \to A$ is an action of G on A, the semi-direct product of A and G along ξ is the coequalizer of $\left\langle \begin{smallmatrix} \kappa_{A,G} \\ i_G \end{smallmatrix} \right\rangle$ and $\xi + 1_G\colon T_G(A) + G \to A + G$ (as defined in [Borceux, Janelidze & Kelly 2005], but using our terminology and putting the G's on the right). This definition arises naturally from the general theory of monads, but it is clear that this coequalizer is also the coequalizer of $\kappa_{A,G}$ and $i_A \circ \xi$. The knowledge of the core of ξ to $A \diamond G$ along $j_{G,A}\colon A \diamond G \to T_G(A)$ suffices to determine this coequalizer:

3.3. Proposition. *Let $\mathbb{C}$ be a finitely complete and cocomplete, pointed and protomodular category. Consider a morphism $\xi\colon T_G(A) \to A$ satisfying the unit axiom of a $\mathbb{T}_G$-algebra, and consider $\psi = \xi \circ j_{A,G}\colon A \diamond G \to A$. Then the coequalizer of $\kappa_{A,G}$ and $i_A \circ \xi$ (hence the semi-direct product of A and G along ξ, if moreover ξ is a $\mathbb{T}_G$-algebra, considering the observation here above) is also the coequalizer of $\iota_{A,G}$ $(= \kappa_{A,G} \circ j_{A,G})$ and $i_A \circ \psi$.*

Proof. We have to show that for any morphism $h\colon A+G \to X$ one has: $h\circ\kappa_{A,G} = h\circ i_A\circ\xi$ if and only if $h \circ \kappa_{A,G} \circ j_{A,G} = h \circ i_A \circ \psi = h \circ i_A \circ \xi \circ j_{A,G}$. And of course only the sufficient condition must be proved; but it follows immediately from the fact that the pair $(j_{A,G}, \eta_{A,G})$ is (strongly) epimorphic by Corollary 2.7. ∎

The following proposition underlines the key role of the object $A \diamond G$:

3.4. Proposition. *Let $\mathbb{C}$ be a finitely complete and cocomplete, pointed and protomodular category. Let G and A be objects of $\mathbb{C}$. Consider a morphism $\psi\colon A \diamond G \to A$, and let $q_\psi\colon A + G \to Q_\psi$ be the coequalizer of $\iota_{A,G}$ and $i_A \circ \psi$. Let l_ψ be the composite $q_\psi \circ i_A$. Then:*

1. *r_G coequalizes $\iota_{A,G}$ and $i_A \circ \psi$, giving rise to a unique extension $p_\psi\colon Q_\psi \to G$, such that $p_\psi \circ q_\psi = r_G$;*
2. *The morphism $s_\psi = q_\psi \circ i_G\colon G \to Q_\psi$ is a section of p_ψ;*
3. *If l_ψ is a monomorphism, then ψ extends along $j_{A,G}$ to a $\mathbb{T}_G$-algebra $\xi\colon T_G(A) \to A$ (which is necessarily unique, by Lemma 3.1);*
4. *If moreover $\mathbb{C}$ is regular, hence homological, then the sequence $A \xrightarrow{l_\psi} Q_\psi \xrightarrow{p_\psi} G$ is exact (even if l_ψ is not supposed to be a monomorphism), hence the sequence*

$$0 \longrightarrow A \xrightarrow{l_\psi} Q_\psi \underset{p_\psi}{\overset{s_\psi}{\leftrightarrows}} G \longrightarrow 0$$

is split short exact if and only if l_ψ is a monomorphism.

Proof.

1. One has: $r_G \circ \iota_{A,G} = \pi_G \circ r_{A,G} \circ \iota_{A,G} = 0$ and $r_G \circ i_A \circ \psi = 0 \circ \psi = 0$.

2. $p_\psi \circ s_\psi = p_\psi \circ q_\psi \circ i_G = r_G \circ i_G = 1_G$.

3. Consider the coequalizer $q' \colon T_G(A) \to Q'$ of $j_{A,G}$ and $\eta_{A,G} \circ \psi$. Then since $q_\psi \circ \kappa_{A,G} \circ j_{A,G} = q_\psi \circ \iota_{A,G} = q_\psi \circ i_A \circ \psi = q_\psi \circ \kappa_{A,G} \circ \eta_{A,G} \circ \psi$, one gets a unique $h \colon Q' \to Q_\psi$ such that $h \circ q' = q_\psi \circ \kappa_{A,G}$. In particular, $h \circ q' \circ \eta_{A,G} = q_\psi \circ \kappa_{A,G} \circ \eta_{A,G} = q_\psi \circ i_A = l_\psi$. Now consider the second split short exact sequence of Proposition 2.7:

$$0 \longrightarrow A \diamond G \xrightarrow{j_{A,G}} T_G(A) \underset{r_A \circ \kappa_{A,G}}{\overset{\eta_{A,G}}{\rightleftarrows}} A \longrightarrow 0$$

Its existence implies that the morphism $\left\langle \begin{smallmatrix} j_{A,G} \\ \eta_{A,G} \end{smallmatrix} \right\rangle : (A \diamond G) + A \to T_G(A)$ is a strong epimorphism, by protomodularity of $\mathbb{C}$. Hence since q' is a coequalizer, $q' \circ \left\langle \begin{smallmatrix} j_{A,G} \\ \eta_{A,G} \end{smallmatrix} \right\rangle$ is a strong epimorphism as well. But $q' \circ \left\langle \begin{smallmatrix} j_{A,G} \\ \eta_{A,G} \end{smallmatrix} \right\rangle = q' \circ \eta_{A,G} \circ \left\langle \begin{smallmatrix} \psi \\ 1_A \end{smallmatrix} \right\rangle$. Thus $q' \circ \eta_{A,G} \colon A \to Q'$ also is a strong epimorphism.

   Now suppose that l_ψ is a monomorphism. As $h \circ q' \circ \eta_{A,G} = l_\psi$, the morphism $q' \circ \eta_{A,G}$ is a monomorphism, hence an isomorphism. So let $\xi = (q' \circ \eta_{A,G})^{-1} \circ q' \colon T_G(A) \to A$. Obviously, one has $\xi \circ \eta_{A,G} = 1_A$. We now show that ξ satisfies the second axiom of an algebra, i.e. $\xi \circ \mu_{A,G} = \xi \circ T_G(\xi)$. Since l_ψ is a monomorphism, it suffices to show that $l_\psi \circ \xi \circ \mu_{A,G} = l_\psi \circ \xi \circ T_G(\xi)$. But $l_\psi \circ \xi \circ \mu_{A,G} = q_\psi \circ \kappa_{A,G} \circ \mu_{A,G} = q_\psi \circ \left\langle \begin{smallmatrix} \kappa_{A,G} \\ i_G \end{smallmatrix} \right\rangle \circ \kappa_{T_G A,G}$ by construction of $\mu_{A,G}$. And since q_ψ is also the coequalizer of $i_A \circ \xi$ and $\kappa_{A,G}$ by Proposition 3.3, one has $l_\psi \circ \xi \circ T_G(\xi) = q_\psi \circ \kappa_{A,G} \circ T_G(\xi) = q_\psi \circ (\xi + 1) \circ \kappa_{T_G A,G}$ by construction of $T_G(\xi)$. And finally, $q_\psi \circ (\xi + 1) = q_\psi \circ \left\langle \begin{smallmatrix} \kappa_{A,G} \\ i_G \end{smallmatrix} \right\rangle$.

4. As $p_\psi \circ q_\psi = r_G$, $q_\psi^{-1}\mathrm{Ker}(p_\psi) = \mathrm{Ker}(r_G) = T_G(A)$. As $\mathbb{C}$ is regular, $\mathrm{Ker}(p_\psi) = q_\psi q_\psi^{-1}\mathrm{Ker}(p_\psi) = q_\psi(T_G(A)) = \mathrm{Im}(q_\psi \circ \kappa_{A,G})$. Recalling the constructions at the beginning of the proof of property 3., we have $\mathrm{Im}(q_\psi \circ \kappa_{A,G}) = \mathrm{Im}(h \circ q') = \mathrm{Im}(h) = \mathrm{Im}(h \circ (q' \circ \eta_{A,G})) = \mathrm{Im}(l_\psi)$ since q' and $q' \circ \eta_{A,G}$ are regular epimorphisms, the latter since by regularity of $\mathbb{C}$ any strong epimorphism is a regular epimorphism. ∎

These results suggest the following definition:

3.5. Definition. *Let $\mathbb{C}$ be a finitely complete and cocomplete, pointed protomodular category and A and G be two objects in it.*

1. *An* **action core** *of G on A is a morphism $\psi \colon A \to G$ which has an extension $\xi : T_G A \to A$ that is an Eilenberg-Moore algebra on $\mathbb{T}_G$ (this extension being unique, by Proposition 3.1).*

2. *A* **strict action core** *is a morphism $\psi \colon A \to G$ which is such that the morphism l_ψ, as defined above, is a monomorphism.*

3. *The* **category of strict action cores** *of* $\mathbb{C}$ *is the category* $\mathbb{S}(\mathbb{C})$ *whose objects are triples* (A, G, ψ)*, where* A *and* G *are objects of* $\mathbb{C}$ *and* $\psi : A \diamond G \to A$ *is a strict action core. A morphism* $(A, G, \psi) \to (A', G', \psi')$ *is an ordered pair* (a, g) *where* $a\colon A \to A'$ *and* $g\colon G \to G'$ *are morphisms in* $\mathbb{C}$ *making the following diagram commute:*

$$\begin{tikzcd} A \diamond G \arrow[r, "a \diamond g"] \arrow[d, "\psi"'] & A' \diamond G' \arrow[d, "\psi'"] \\ A \arrow[r, "a"'] & A' \end{tikzcd}$$

Note that by Proposition 3.4, a strict action core is an action core, and by very definition, the core of an action is an action core. Moreover, if ψ is an action core, then by Proposition 3.3 the coequalizer $q_\psi\colon A + G \to Q_\psi$ defined above is nothing but the quotient $A + G \to A \rtimes_\xi G$, where ξ is the unique extension of ψ along $j_{A,G}$ which is a $\mathbb{T}_G$-algebra.

For a fixed object G of $\mathbb{C}$, consider $\mathbb{S}_G$ the fiber of $\mathbb{S}$ on G. The preceding results allow us to compare the category $\mathbb{S}_G$ to the category of Eilenberg-Moore algebras on $\mathbb{T}_G$ (the injectivity on objects being obvious):

3.6. Corollary. *Let* $\mathbb{C}$ *be a finitely complete and cocomplete, pointed protomodular category, and* A *and* G *be objects of it. The extension of the morphisms* $\psi\colon A \diamond G \to A$ *to* $\mathbb{T}_G$*-algebras defined above (for any object* (A, G, ψ) *in* $\mathbb{S}_G$*) gives rise to a full and faithful functor* $\Xi_G\colon \mathbb{S}_G \to \mathbb{C}^{\mathbb{T}_G}$*. Moreover,* Ξ_G *is "injective on objects" so that if* $\mathbb{X}_G$ *is the full subcategory of the objects of* $\mathbb{C}^{\mathbb{T}_G}$ *which are images by* Ξ_G *of objects* ψ *of* $\mathbb{S}_G$*, then* Ξ_G *is an isomorphism of categories between* $\mathbb{S}_G$ *and* $\mathbb{X}_G$*. Finally, if* (A, G, ψ) *is an object of* $\mathbb{S}$*, then the construction of* $A \rtimes_\psi G$ *coincides with the one of* $G \ltimes (A, \Xi_G(\psi))$ *in* [Borceux, Janelidze & Kelly 2005]. ■

It follows from Proposition 3.3 and Corollary 3.6 that the actions ξ in $\mathbb{X}_G$ are exactly those actions $\xi\colon T_G(A) \to A$ for which the composition of the injection of A in $A+G$ and the projection from $A + G$ to $A \rtimes_\xi G$ is a monomorphism, hence are exactly the actions which are called *strict* in [Martins-Ferreira & Sobral 2012]. In view of the isomorphism between $\mathbb{S}_G$ and $\mathbb{X}_G$, morphisms $\psi\colon A \diamond G \to A$ which are strict action cores are exactly the cores of strict actions, which explains the terminology.

Examples

3.7. Example. In the category of groups, it is well known (see for instance [Magnus, Karrass & Solitar 1966]) that for two groups A and G, $A \diamond G$ is the subgroup of $A + G$ generated by commutators $[i_A(a), i_G(g)]$ or equivalently their inverses $[i_G(g), i_A(a)]$ for $a \in A$ and $g \in G$, and that this subgroup is freely generated by the nontrivial commutators. We simplify the notation and denote $i_A(a)$ by a and similarly for g. We prefer to consider that the generators are the $[g, a]$'s. On the other hand, $T_G(A)$ is generated by the $g{\cdot}a{\cdot}g^{-1}$'s. Hence giving a morphism $A \diamond G \to A$ in the category of groups is equivalent to giving a function $G^* \times H^* \to A$ in the category of sets, or equivalently a function $f\colon G \times A \to A$

satisfying $f(g,a) = e_A$ whenever $g = e_G$ or $a = e_A$. So a group morphism $\psi\colon A \diamond G \to A$ can be seen as a function $[\![-,-]\!]\colon G \times A \to A$ such that $[\![h,g]\!] = e_A$ when $a = e_A$ or $g = e_G$. We denote it this way because it is a kind of "external" commutator operation of G on A. Let us define $\phi\colon G \times A \to A$ by $\phi(g,a) = [\![g,a]\!] \cdot a$. Then it is easy to show that l_ψ is a monomorphism if and only if ϕ is an action in the usual sense, and the semi-direct product is of course the classical one.

3.8. Example. Consider the category Nil_2 of 2-step nilpotent groups and group morphisms between them. Recall that a group G is 2-step nilpotent iff its commutators are central, i.e. $[G,[G,G]]$ is trivial, or equivalently if the commutator function $G \times G \to G$ which sends (g,g') to the commutator $[g,g'] = gg'g^{-1}g'^{-1}$ is bilinear. We also write $[G,G] = G'$. We denote the abelianization of a group G by G^{ab}, and the equivalence class under this quotient of $g \in G$ by $\bar{g}$. If A and G are two such groups, then (in the category Nil_2 of course) $A \diamond G = A^{ab} \otimes G^{ab}$; here again for technical reasons, we prefer consider that it is $G^{ab} \otimes A^{ab}$. The sum $A +_2 G$ in Nil_2 is the set $(G^{ab} \otimes A^{ab}) \times A \times G$ with the multiplication $(t,a,g) * (t',a',g') = (t + t' + \bar{g} \otimes \bar{a}', aa', gg')$. So $(\bar{g} \otimes \bar{a}, e_A, e_G)$ is the commutator of $(0, e_A, g)$ and $(0, a, e_G)$ i.e. of $i_G(g)$ and $i_A(a)$ in the 2-step nilpotent group $A +_2 G$, where i_A and i_G are as usual the canonical injections of the groups in their 2-step nilpotent sum. Consider a group morphism $\psi\colon G^{ab} \otimes A^{ab} \to A$. Considering (in $Sets$ again) the composition $G \times A \to G^{ab} \times A^{ab} \to G^{ab} \otimes A^{ab} \to A$, and denoting it by $[\![-,-]\!]$, this means that $[\![-,-]\!]$ is bilinear and that $[\![g,a]\!] = e_A$ when g is a commutator in G or a is in A, and conversely a function $[\![-,-]\!]\colon G \times A \to A$ satisfying these two properties gives rise to a group morphism $\psi\colon G^{ab} \otimes A^{ab} \to A$. Consider such a ψ and define $\phi\colon G \times A \to A$ by $\phi(g,a) = [\![g,a]\!] \cdot a$. Then it can be checked that l_ψ is a monomorphism if and only if ϕ is a group action of H on G, or equivalently if $[\![-,-]\!]$ satisfies two extra properties: it takes values in the center of A and for any $g, g' \in G$ and $a \in A$ one has $[\![g,[\![g',a]\!]]\!] = e_A$. Moreover in this case, $A \rtimes_\psi G$ is the usual semi-direct product $A \rtimes_\phi G$, which happens to be 2-step nilpotent. Conversely, if $\phi\colon G \times A \to A$ is an action in the usual sense, then by defining $[\![h,g]\!] = \phi(g,a) \cdot a^{-1}$ one gets a strict action $\psi\colon G^{ab} \otimes A^{ab} \to A$ if and only if $[\![-,-]\!]$ is bilinear and $[\![g,a]\!] = e_A$ when g is a commutator in G or a is in A; these conditions are also equivalent to the fact that $A \rtimes_\phi G$ is a 2-step nilpotent group.

3.9. Example. Consider now the category of "central pairs" defined as follows. Objects are pairs (G,H) where G is a 2-step nilpotent group, and H is a subgroup satisfying $G' \subset H \subset Z(G)$ (so that H is normal in G, and G/H is abelian). A morphism $f\colon (G,H) \to (A,B)$ is a group morphism $f\colon G \to A$ such that $f(H) \subset B$. Hence it is equivalent to the category of central extensions of groups with abelian codomain and is known to be finitely cocomplete homological, but not semi-abelian [Everaert, Gran & Van der Linden 2008], [Everaert 2012]; it arises in forthcoming work in the realm of non-linear algebra of degree 2 which was introduced in [Baues, Hartl & Pirashvili 1997]. The sum of two objects (A,B) and (G,H) is the pair $((G/H \otimes A/B) \times A \times G, \{0\} \times A \times B)$, with a product defined similarly as in the preceding example, so that in this category $(A,B) \diamond (G,H) = (G/H \otimes A/B, \{0\})$. A morphism $\psi\colon (A,B) \diamond (G,H) \to (A,B)$ in this category is then equivalent to a function

$[\![-,-]\!]\colon G \times A \to A$ which is bilinear and satisfies $[\![g,a]\!] = 1_G$ when $g \in H$ or $a \in B$. Then, similarly as in Nil_2, l_ψ is a monomorphism if and only if it takes values in B and if for any $g, g' \in G$ and $a \in A$ one has $[\![g,[\![g',a]\!]]\!] = 1_A$. Here again, these conditions (once ψ is supposed to be a morphism in the category) are equivalent to the fact that $\phi\colon G \times A \to A$ defined as above is a group action and that $B \cdot H$ contains all commutators in $A \rtimes_\phi G$.

Strict action(cores) in finitely cocomplete homological categories We may now show that if the category is finitely cocomplete and homological, then the strict actions are precisely the ones we were looking for:

3.10. Proposition. *Let $\mathbb{C}$ be finitely cocomplete homological and G an object in it. Then the comparison adjunction $(-\rtimes_- G, \mathcal{J}_G, \eta', \epsilon')\colon \mathbb{C}^{\mathbb{T}_G} \to \mathrm{Pt}_G(\mathbb{C})$ in* [Borceux, Janelidze & Kelly 2005] *restricts to an equivalence between $\mathbb{X}_G$ and $\mathrm{Pt}_G(\mathbb{C})$. Hence composition with Ξ provides an equivalence of categories between $\mathbb{S}$ and $\mathrm{Pt}(\mathbb{C})$, which is compatible with the forgetful functors. Thus $\mathbb{S}$ is a fibration whose fibers are the categories $\mathbb{S}_G$, and the "inverse" functors $\Psi_G\colon \mathrm{Pt}_G(\mathbb{C}) \to \mathbb{S}_G$ are such that $\Xi_G \circ \Psi_G = \mathcal{J}_G$, the comparison functor of the adjunction.*

Proof. First, let us consider a $\mathbb{T}_G$-algebra $\xi\colon T_G(A) \to A$ which is in $\mathbb{X}_G$, i.e. which is the extension of a (unique) $\psi\colon A \diamond G \to G$ in $\mathbb{S}$. We show that η'_ξ is an isomorphism between ξ and $\mathcal{J}_G(F'(\xi))$. Consider the following diagram:

$$\begin{tikzcd}
A \diamond G \arrow[r, "j_{A,G}"] \arrow[dr, "\psi"'] & T_G A \arrow[r, "\kappa_{A,G}"] \arrow[d, "\xi"'] & A + G \arrow[d, "q_\psi"] & \\
& A \arrow[r, "l_\psi"'] \arrow[ur, "i_A"] & A \rtimes_\psi G \arrow[r, shift right, "p_\psi"'] & G \arrow[l, shift right, "s_\psi"']
\end{tikzcd}$$

Then in view of all what proceeds, the point $(A \rtimes_\psi G, p_\psi, s_\psi)$ is the semi-direct product $A \rtimes_\xi G$ in the usual sense, and since l_ψ is the kernel of p_ψ (because ψ is a strict action), $\mathcal{J}_G(A \rtimes_\xi G)$ is the unique arrow h from $T_G A$ to A such that $l_\psi \circ h = q_\psi \circ \kappa_{A,G}$, i.e. it is ξ.

Secondly, we show that $\mathcal{J}_G$ takes values in $\mathbb{X}_G$, i.e. that for any object

$$\begin{tikzcd} X \arrow[r, shift right, "p"'] & G \arrow[l, shift right, "s"'] \end{tikzcd}$$

the algebra $\mathcal{J}_G(X,p,s)$ is (the extension to $T_G(A)$ of) a strict action (A, G, ψ). Moreover we show that it is such that $A \rtimes_\psi G$ is isomorphic to (X, G, p, s), the G-part of the isomorphism being the identity: this in fact shows that $\epsilon'_{(X,p,s)}$ is an isomorphism.

Let $A \rhd\!\xrightarrow{l} X$ be a kernel of p. Consider the following diagram. It is commutative, since $\langle {l \atop s} \rangle = r_G$ (one may check it by composing these morphisms by the canonical injections i_A and i_G):

$$\begin{tikzcd}
A \diamond G \arrow[r, tail, "\iota_{A,G}"] \arrow[d, dotted, "\exists!\psi"'] & A+G \arrow[r, "r_{A,G}"] \arrow[d, "\left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle"'] \arrow[dr, shift left, "r_G"] & A \times G \arrow[d, "\pi_G"] \\
A \arrow[r, tail, "l"'] \arrow[ur, "i_A"] & X \arrow[r, "p"'] & G \arrow[ul, shift left, "i_G"]
\end{tikzcd}$$

Since $\iota_{A,G}$ is the kernel of $r_{A,G}$ and since $\pi_G \circ r_{A,G} = p \circ \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle$, one has: $p \circ \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ \iota_{A,G} = \pi_G \circ r_{A,G} \circ \iota_{A,G} = 0$; so since l is the kernel of p, there exists a unique $\psi\colon A \diamond G \to A$ such that $l \circ \psi = \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ \iota_{A,G}$. We claim that (A, G, ψ) is a strict action core. Since $l = \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ i_A$ is a monomorphism, it suffices to show that $\left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle$, which is known to be a regular epimorphism, is the coequalizer of $\iota_{A,G}$ and $i_A \circ \psi$.

Consider $q_\psi\colon A + G \to Q_\psi$ the coequalizer of $\iota_{A,G}$ and $i_A \circ \psi$, and p_ψ and s_ψ defined as in Proposition 3.4 above. We will show that $q_\psi \circ i_A$ is a monomorphism, thus showing that (A, G, ψ) is an object in $\mathbb{S}$, and that the G-point (Q_ψ, p_ψ, s_ψ), which is nothing but $A \rtimes_\psi G$, is isomorphic to (X, p, s).

First of all, $\left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ i_A \circ \psi = l \circ \psi = \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ \iota_{A,G}$ hence, since q_ψ is the coequalizer of $i_A \circ \psi$ and $\iota_{A,G}$, there exists a unique morphism $e\colon Q_\psi \to X$ such that $q_\psi \circ e = \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle$:

$$\begin{tikzcd}
 & A+G \arrow[d, "\left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle"] \arrow[dd, bend left=60, "q_\psi"] \\
A \arrow[ur, "i_A"] \arrow[r, "l"] \arrow[dr, "q_\psi \circ i_A"'] & X \\
 & Q \arrow[u, "e"']
\end{tikzcd}$$

and of course, then $e \circ q_\psi \circ i_A = \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ i_A = l$, so this diagram commutes. But then, since l is the kernel of p, it is a monomorphism, hence so is $q_\psi \circ i_A$. But then, by Proposition 3.4 above, $q_\psi \circ i_A = \ker p_\psi$.

Then consider the following diagram:

$$\begin{tikzcd}
 & & & A+G \arrow[r, shift right, "r_G"'] \arrow[d, "q_\psi"] \arrow[ddl, bend right, "\left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle"'] & G \arrow[l, shift right, "i_G"'] \arrow[d, equal] \arrow[ddl, equal, bend right] & \\
 & 0 \arrow[r] & A \arrow[r, "q_\psi \circ i_A"'] \arrow[dl, equal] & Q_\psi \arrow[r, shift right, "p_\psi"'] \arrow[dl, "e"] & G \arrow[l, shift right, "s_\psi"'] \arrow[r] \arrow[dl, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "l"'] & X \arrow[r, shift right, "p"'] & G \arrow[l, shift right, "s"'] \arrow[r] & 0 &
\end{tikzcd}$$

The only part of this diagram which has not been shown to commute is the bottom right-hand square. But one has $p_\psi \circ q_\psi = r_G = p \circ \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle = p \circ e \circ q_\psi$, hence $p \circ e = p_\psi$ since q_ψ is a (regular) epimorphism. And $e \circ s_\psi = e \circ q_\psi \circ i_G = \left\langle \begin{smallmatrix} l \\ s \end{smallmatrix} \right\rangle \circ i_G = s$. Hence all conditions of the Short Split Five Lemma are satisfied, so e is an isomorphism, and thus defines an isomorphism $(e, 1_G)$ in $\mathrm{Pt}(\mathbb{C})$.

Note that we then have implicitly constructed (by composition with the inverse of Ξ) a functor $\mathrm{Pt}(\mathbb{C}) \to \mathbb{S}$. We denote it by Ψ (and its restriction to the fibers on G by Ψ_G). For a point $X \underset{p}{\overset{s}{\leftrightarrows}} G$, $\Psi(X, G, p, s)$ is a strict action on $A = \mathrm{Ker}(p)$. It is easy to verify that if $X' \underset{p'}{\overset{s'}{\leftrightarrows}} G'$ is another point and (x, g) is a morphism of points between them, i.e. a pair of morphisms making the following diagram commute

$$\begin{tikzcd} X \arrow[r, shift right, "p"'] \arrow[d, "x"'] & G \arrow[l, shift right, "s"'] \arrow[d, "g"] \\ X' \arrow[r, shift right, "p'"'] & G' \arrow[l, shift right, "s'"'] \end{tikzcd}$$

then $\Psi(x, g)$ is the unique morphism $a \colon A \to A'$ making the following diagram commute

$$\begin{tikzcd} A = \mathrm{Ker}(p) \arrow[r, "\ker(p)"] \arrow[d, "a"'] & X \arrow[d, "x"] \\ A' = \mathrm{Ker}(p') \arrow[r, "\ker(p')"'] & X' \end{tikzcd}$$

which indeed is a morphism of strict actions between $\Psi(X, G, p, s)$ and $\Psi(X', G', p', s')$.

Finally, the very constructions of Ψ_G and Ξ_G ensure that $\Xi_G \Psi_G = \mathcal{J}_G$. ■

3.11. Example. Recall that the conjugation action of an object E of $\mathbb{C}$ on itself is defined in [Bourn & Janelidze 1998], as a split extension on E, to be the short exact sequence $0 \to E \xrightarrow{\sigma_1} E \times E \xrightarrow{\pi_2} E \to 0$ with the splitting $\Delta \colon E \to E \times E$ being the diagonal morphism. By Proposition 3.10, it corresponds to some strict action core $c_2^E \colon E \diamond E \to E$. We claim that this corresponding action core is $\nabla_E^2 \circ \iota_{E,E}$, where $\nabla_E^2 \colon E + E \to E$ is the codiagonal, and that the corresponding algebra $\Xi_E(c_2^E)$ is $\nabla_E^2 \circ \kappa_{E,E}$. Indeed, since $\psi = \Xi_E(\psi) \circ \kappa_{E,E}$, it suffices to prove the second assertion, and by the construction of Ξ_E it suffices to show that the following diagram commutes:

$$\begin{tikzcd} T_E E \arrow[r, "\kappa_{E,E}"] \arrow[d, "\kappa_{E,E}"'] & E + E \arrow[dd, "\left\langle \substack{\sigma_1 \\ \Delta_E} \right\rangle"] \\ E + E \arrow[d, "\nabla_E^2"'] & \\ E \arrow[r, "\sigma_1"] & E \times E \end{tikzcd}$$

Followed by π_1, one gets $\pi_1 \circ \sigma_1 \circ \nabla_E^2 \circ \kappa_{E,E} = \nabla_E^2 \circ \kappa_{E,E}$ on the one hand, and $\pi_1 \circ \left\langle \substack{\sigma_1 \\ \Delta_E} \right\rangle \circ \kappa_{E,E}$ $= \left\langle \substack{\pi_1 \circ \sigma_1 \\ \pi_1 \circ \Delta_E} \right\rangle \circ \kappa_{E,E} = \left\langle \substack{1_E \\ 1_E} \right\rangle \circ \kappa_{E,E} = \nabla_E^2 \circ \kappa_{E,E}$ on the other hand. And followed by π_2, one gets $\pi_2 \circ \sigma_1 \circ \nabla_E^2 \circ \kappa_{E,E} = 0 \circ \nabla_E^2 \circ \kappa_{E,E} = 0$ on the one hand, and $\pi_2 \circ \left\langle \begin{smallmatrix} \sigma_1 \\ \Delta_E \end{smallmatrix} \right\rangle \circ \kappa_{E,E} = \left\langle \begin{smallmatrix} \pi_2 \circ \sigma_1 \\ \pi_1 \circ \Delta_E \end{smallmatrix} \right\rangle \circ \kappa_{E,E} = \left\langle \begin{smallmatrix} 0 \\ 1_E \end{smallmatrix} \right\rangle \circ \kappa_{E,E} = r_2 \circ \kappa_{E,E} = 0$ on the other hand.

Of course, this construction $c_2^{(-)}$ is functorial. More precisely, $c_2^{(-)}$ may be considered as a functor $\mathbb{C} \to \mathbb{S}$, or as a natural transformation $S \to 1_{\mathbb{C}}$, where $S\colon \mathbb{C} \to \mathbb{C}$ is defined by $F(E) = E \diamond E$ and $S(f) = f \diamond f$, both meaning that the following diagram commutes:

$$\begin{tikzcd} E \diamond E \arrow[r, "f \diamond f"] \arrow[d, "c_2^E"'] & F \diamond F \arrow[d, "c_2^F"] \\ E \arrow[r, "f"'] & F \end{tikzcd}$$

which follows immediately from naturality of ι and of ∇^2.

Then the split extension

$$0 \longrightarrow E \xrightarrow{i_1} E \rtimes_{c_2^E} E \xrightarrow{p_2} E \longrightarrow 0$$

is (canonically isomorphic to) the following one:

$$0 \longrightarrow E \xrightarrow{i_1} E \times E \xrightarrow{p_2} E \longrightarrow 0$$

by the very definition of c_2. We will generalize this construction to a (strict) conjugation action core of an object on any normal subobject in section 4.

3.12. Example. The split short exact sequence of Corollary 2.7:

$$\begin{tikzcd} 0 \arrow[r] & A \diamond G \arrow[r, "j_{A,G}"] & T_G(A) \arrow[r, shift right, "r_A \circ \kappa_{A,G}"'] & A \arrow[l, shift right, "\eta_{A,G}"'] \arrow[r] & 0 \end{tikzcd}$$

corresponds to a strict action core ψ of A on $A \diamond G$ such that $T_G(A) = (A \diamond G) \rtimes_\psi A$; it will be explicitly determined in 4.6. Similarly, the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T_G(A) \arrow[r, "\kappa_{A,G}"] & A + G \arrow[r, shift right, "r_G"'] & G \arrow[l, shift right, "i_G"'] \arrow[r] & 0 \end{tikzcd}$$

corresponds to a strict action core ψ' of G on $T_G(A)$ such that $A + G = T_G \rtimes_{\psi'} G$, hence one may write $A + G = ((A \diamond G) \rtimes_\psi A) \rtimes_{\psi'} G$.

The following proposition allows to construct new strict action cores from given ones:

3.13. Proposition. *Let $\psi\colon A \diamond G \to A$ be a strict action core in a finitely cocomplete homological category $\mathbb{C}$, $h\colon H \to G$ be a morphism and $b\colon B \to A$ a monomorphism in $\mathbb{C}$.*

1. *Suppose that B is h-stable under ψ, i.e. the morphism $\psi \circ (b \diamond h)\colon (B \diamond H) \to A$ factors through a morphism $\psi'\colon B \diamond H \to B$ such that $b \circ \psi' = \psi \circ (b \diamond h)$. Then ψ' is a strict action core of H on B.*

2. *If moreover h is a monomorphism, then $b \rtimes h$ is a monomorphism.*

If $b = 1_A$ and h is a monomorphism we call ψ' the **restriction of ψ to H** and denote it by $\psi \restriction_H$.

Proof.

1. Consider the following diagram of solid arrows, where q' is the coequalizer of $\iota_{B,H}$ and $i_B \circ \psi'$; all squares and the bottom triangles are commutative, and the upper triangles are coequalized by q' and $q = q_\psi$ respectively:

$$
\begin{tikzcd}
 & B + H \arrow[rr, "b+h"] \arrow[dd, "q'"] & & A + G \arrow[dd, "q"] \\
B \diamond H \arrow[ur, "\iota_{B,H}"] \arrow[rr, "b \diamond h"] \arrow[dd, "\psi'"'] & & A \diamond G \arrow[ur, "\iota_{A,G}"] \arrow[dd, "\psi"] & \\
 & Q \arrow[rr, dotted, "f"] & & A \rtimes_\psi G \\
B \arrow[uuur, "i_B"] \arrow[ur, "q' \circ i_B"] \arrow[rr, "b"'] & & A \arrow[uuur, "i_A"] \arrow[ur, "l_\psi"'] &
\end{tikzcd}
$$

   We have to show that $q' \circ i_B$ is a monomorphism. Since $q \circ (b + h) \circ i_B \circ \psi' = q \circ i_A \circ b \circ \psi' = q \circ i_A \circ \psi \circ (b \diamond h) = q \circ \iota_{A,G} \circ (b \diamond h) = q \circ (b + h) \circ \iota_{B,H}$, there is a unique $f \colon Q \to A \rtimes_\psi G$ such that $q \circ (b + h) = f \circ q'$. Then $l_\psi \circ b = q \circ i_A \circ b = q \circ (b + h) \circ i_B = f \circ q' \circ i_B$. Hence, since b and l_ψ are monomorphisms, so is $q' \circ i_B$.

2. Note that under these conditions, Q is $B \rtimes_{\psi'} H$ and f is nothing but $b \rtimes h$; if moreover h is a monomorphism, we may complete the diagram with the projections of Q to H and of $A \rtimes_\psi G$ to G and apply [Bourn 2001, Corollary 9] or [Borceux & Bourn 2004, Lemma 4.2.5, 5.] to conclude that f is a monomorphism. ■

3.14. Remark. It is obvious that in the category of points (hence in the equivalent category of short split exact sequences) over a pointed regular category $\mathbb{C}$, kernels and images are computed degreewise, since it is a functor category. So, consider two strict action cores $\psi \colon A \diamond G \to A$ and $\psi' \colon B \diamond H \to B$ and a morphism $(f \colon A \to B, g \colon G \to H)$ between them. Then the equivalence of categories above ensures that there exist strict action cores $\tilde{\psi}$ and $\tilde{\psi}'$ (of $\mathrm{Ker}(g)$ on $\mathrm{Ker}(f)$ and of $\mathrm{Im}(g)$ on $\mathrm{Im}(f)$ respectively) such that

$$\ker(f \rtimes g) = \ker(f) \rtimes \ker(g) \colon \mathrm{Ker}(f) \rtimes_{\tilde{\psi}} \mathrm{Ker}(g) \to A \rtimes_\psi G$$

and

$$\mathrm{im}(f \rtimes g) = \mathrm{im}(f) \rtimes \mathrm{im}(g) \colon \mathrm{Im}(f) \rtimes_{\tilde{\psi}'} \mathrm{Im}(g) \to B \rtimes_{\psi'} H$$

4. Conjugation action core of an object on a normal subobject

In this section and in the following one, the category $\mathbb{C}$ will always at least be finitely cocomplete and homological. We introduce a general notion of (strict) conjugation action core of an object E on any of its normal subobjects[1], and a notion of Higgins commutator. We also explain the link between the two notions.

Conjugation action cores

4.1. Proposition. *Let* $n\colon N \rightarrowtail E$ *be a normal subobject in a finitely cocomplete homological category* $\mathbb{C}$*. Then there is a (necessarily unique) strict action core* $c^{N,E}\colon N\diamond E \to N$ *of* E *on* N *such that* $n\circ c^{N,E} = c_2^E \circ (n\diamond 1)$*. We then call* $c^{N,E}$ *the* **conjugation action core** *of* E *on* N*. It is natural with respect to pair morphisms* $(E,N) \to (E',N')$*, i.e. morphisms* $f\colon E \to E'$ *in* $\mathbb{C}$ *such that* $f(N) \subset N'$ *for a given normal subobject* N' *of* E'*.*

Proof. We first prove the assertion in the case when N is proper in E. Let $\pi\colon E \to G$ be a cokernel of n. Then we have the following diagram of plain arrows which commutes by naturality of commutator morphisms:

$$\begin{tikzcd}
N\diamond E \arrow[r, "n\diamond 1"] \arrow[d, dotted, "c^{N,E}"] & E\diamond E \arrow[r, "\pi\diamond\pi"] \arrow[d, "c_2^E"] & G\diamond G \arrow[d, "c_2^G"] \\
N \arrow[r, "n"] & E \arrow[r, "\pi"] & G
\end{tikzcd}$$

Thus $\pi\circ c_2^E \circ (n\diamond 1) = c_2^G \circ (\pi\diamond\pi)\circ(n\diamond 1) = c_2^G \circ (\pi\circ n\diamond\pi) = 0\diamond\pi = 0$ since the functor $-\diamond-$ is bireduced by Proposition 2.11 (2), whence $c_2^E\circ(n\diamond 1)$ factors through n, thus providing the desired morphism $c^{N,E}$. By Proposition 3.13 it is a strict action core.

Now if N is merely a normal subobject in E, say with associated equivalence relation (R, r_1, r_2) with $s\colon E \to R$ as a common section of both r_1 and r_2, then the inclusion n of N in E is the composition $N \xrightarrow{k} R \xrightarrow{r_2} E$, where k is a kernel of r_1. Then N is proper in R, hence, as proved above, the conjugation action core of R on N is defined, which provides the dotted arrow on the left side in the bottom of the following commutative diagram. But E is a subobject of R via s, so by Proposition 3.13 and commutativity of the outer pentagon the morphism $c^{N,E} = c^{N,R}\circ(1_N\diamond s)$ is the desired strict action core of E on N.

$$\begin{tikzcd}
& N\diamond E \arrow[dl, "1_N\diamond s"'] \arrow[dr, "n\diamond 1_E"] & \\
N\diamond R \arrow[r, "k\diamond 1"] \arrow[d, dotted, "c^{N,R}"'] & R\diamond R \arrow[r, "r_2\diamond r_2"] \arrow[d, "c_2^R"] & E\diamond E \arrow[d, "c_2^E"] \\
N \arrow[r, "k"] \arrow[rr, bend right, "n"'] & R \arrow[r, "r_2"] & E
\end{tikzcd}$$

Naturality of this strict action core is immediate by naturality of c_2^E. ∎

[1]In a former version of this paper, we only quoted the conjugation action core on proper subobjects. We thank Tim Van der Linden for having observed this generalization to normal subobjects.

4.2. Example. The restriction to $X = A$ or $X = G$ of the conjugation action core of $A+G$ on $A \diamond G$ is nothing but the corresponding compression operation: we have

$$c^{A\diamond G, A+G} \upharpoonright_X = C^X_{A,G} \quad \text{and hence also} \quad c^{A\diamond G, T_G(A)} \upharpoonright_A = C^A_{A,G},$$

cf. Definition 2.5. To see this, consider the following commutative diagram

$$\begin{tikzcd}
(A\diamond G)\diamond X \arrow[r, "1_{A\diamond G}\diamond i_X"'] \arrow[d, "j_{A,G}\diamond 1_X"] \arrow[rrr, bend left, "\iota_{A,G}\circ C^X_{A,G}"] & (A\diamond G)\diamond(A+G) \arrow[r, "\iota_{A\diamond G, A+G}"'] \arrow[d, "\iota_{A,G}\diamond 1_{A+G}"] & (A\diamond G)+(A+G) \arrow[r, "\left\langle {\iota_{A,G} \atop 1_{A+G}} \right\rangle"'] \arrow[d, "\iota_{A,G}+1_{A+G}"] & A+G \arrow[d, equal] \\
T_G(A)\diamond X \arrow[r, "\kappa_{A,G}\diamond i_X"] \arrow[rrr, bend right, "\kappa_{A,G}\circ c^{T_G(A),A+X}\upharpoonright_X"'] & (A+G)\diamond(A+G) \arrow[r, "\iota_{A+G,A+G}"] & (A+G)+(A+G) \arrow[r, "\nabla^2_{A+G}"] & A+G
\end{tikzcd}$$

Then the assertion follows from the fact that $j_{A,G}\diamond 1_X$ composed with the lower composite morphism is $\iota_{A,G}\circ c^{A\diamond G, A+G}\upharpoonright_X$.

4.3. Example. Consider the (non exact) category of central pairs of Example 3.9. We compute the conjugation action core on any normal subobject of any object. Let $\mathcal{G} = (G,H)$ be an object in this category. An equivalence relation on (G,H) in $\mathbb{C}$ is (the inclusion into $(G\times G, H\times H)$ of) a pair (R,S), where R is a congruence on G, S is a congruence on H, $S \subseteq R$, and S contains all commutators in R (that it is central in R is automatic). A normal subobject of (G,H) is "the equivalence class of 0 for such an equivalence relation", i.e. the inverse image of R by the inclusion σ_1 (or equivalently σ_2) of (G,H) in $(G,H)\times(G,H)$. So it is the pair $\mathcal{N} = ([e]_R, [e]_S)$, where e is the unit of G and $[e]_R$ its equivalence class (in G) for R, and $[e]_S$ in H for S. Recall that the object $\mathcal{G}\diamond\mathcal{G}$ is the pair $(G/H\otimes G/H, \{0\})$, the inclusion in the sum sending $\bar{g}\otimes\bar{g}'$ on the commutator of $(0,1,g)$ and $(0,g',1)$ in the sum $\mathcal{G}+\mathcal{G}$, while the conjugation action core $c\colon \mathcal{G}\diamond\mathcal{G}\to\mathcal{G}$ sends $\bar{g}\otimes\bar{g}'$ to the commutator of g' and g in G. The diagram that has to be filled up to get the result is the following:

$$\begin{tikzcd}
\mathcal{N}\diamond\mathcal{G} = ((G/H\otimes [e]_R/[e]_S),\{0\}) \arrow[r] \arrow[d, dotted] & \mathcal{G}\diamond\mathcal{G} = (G/H\otimes G/H,\{0\}) \arrow[d, "c_2^{\mathcal{G}}"] \\
\mathcal{N} = ([e]_R,[e]_S) \arrow[r] & \mathcal{G} = (G,H)
\end{tikzcd}$$

To get this morphism, it suffices to show that the morphism $G\times[e]_R \to G, (g,g')\mapsto [g',g]$ is bilinear (which is immediate by the very fact that G is 2-step nilpotent), takes values in $[e]_R$ and takes the value e if $g\in H$ or $g'\in[e]_S$, which follow immediately from the hypotheses on H and S.

We now give two properties which literally generalize certain standard facts in the theory of groups or Lie algebras.

First we quote a *universal property* of the semi-direct product:

4.4. Proposition. *Let* $\psi\colon A\diamond G\to A$ *be a strict action core in* $\mathbb{C}$*, and let*

$$A \xrightarrow{f} X \xleftarrow{g} G$$

be morphisms in $\mathbb{C}$*. Then there exists a morphism* $h=\overline{\left\langle \begin{smallmatrix} f\\ g\end{smallmatrix}\right\rangle}\colon A\rtimes_\psi G\to X$ *such that* $h\circ l_\psi=f$ *and* $h\circ s_\psi=g$ *iff the following square commutes:*

$$\begin{tikzcd} A\diamond G \arrow[r, "\psi"] \arrow[d, "f\diamond g"'] & A \arrow[d, "f"] \\ X\diamond X \arrow[r, "c_2^X"'] & X \end{tikzcd} \tag{1}$$

Moreover, if h *exists it is unique.*

Proof. Uniqueness follows from the fact that (l_ψ,s_ψ) is a (strongly) epimorphic pair. To prove existence, by the definition of $A\rtimes_\psi G$ as a quotient of $A+G$, we must show that commutativity of diagram (1) is equivalent with the relation $\left\langle \begin{smallmatrix} f\\ g\end{smallmatrix}\right\rangle\circ\iota_{A,G}=\left\langle \begin{smallmatrix} f\\ g\end{smallmatrix}\right\rangle\circ i_A\circ\psi$. But $\left\langle \begin{smallmatrix} f\\ g\end{smallmatrix}\right\rangle\circ\iota_{A,G}=\nabla_X^2\circ(f+g)\circ\iota_{A,G}=\nabla_X^2\circ\iota_{X,X}\circ(f\diamond g)=c_2^X\circ(f\diamond g)$. On the other hand, $\left\langle \begin{smallmatrix} f\\ g\end{smallmatrix}\right\rangle\circ i_A\circ\psi=f\circ\psi$, whence the assertion. ∎

In particular, Proposition 4.4 shows that the semi-direct product can be viewed as a universal transformation of an abstract strict action core into a conjugation action core:

4.5. Corollary. *A strict action core* $\psi\colon A\diamond G\to A$ *in* $\mathbb{C}$ *coincides with the restriction to* G *of the conjugation action core of* $A\rtimes_\psi G$ *on* A*, or formally,* $c^{A,A\rtimes_\psi G}\circ(1_A\diamond s_\psi)=\psi$.

Proof. In Proposition 4.4, take $X=A\rtimes_\psi G$, $f=l_\psi$, $g=s_\psi$ so that $h=1_{A\rtimes_\psi G}$. We get $l_\psi\circ\psi=c_2^X(l_\psi\diamond s_\psi)=c_2^X\circ(l_\psi\diamond 1_G)\circ(1_A\diamond s_\psi)=l_\psi\circ c^{A,X}\circ(1_A\diamond s_\psi)$, whence the assertion since l_ψ is a monomorphism. ∎

4.6. Corollary. *We have* $T_G(A)=(A\diamond G)\rtimes_{C^A_{A,G}}A$*, cf. Definition 2.5.*

This is an immediate consequence of Corollary 4.5 and Example 4.2. ∎

Higgins Commutators To make further progress we define *Higgins commutators of subobjects* in terms of co-smash products; this actually is the starting point of a new approach to categorical commutator calculus which is further developed in [Hartl & Van der Linden 2013] and [Hartl - in preparation], and serves as a fundamental tool in [Rodelo & Van der Linden 2012] and [Martins-Ferreira & Van der Linden 2012].

4.7. Definition. *The* n**-fold commutator morphism** *of an object* X *of* $\mathbb{C}$ *is the natural composite morphism*

$$c_n^X\colon X\diamond\cdots\diamond X \overset{\iota_{X\ldots X}}{\rightarrowtail} X+\cdots+X \xrightarrow{\nabla_X^n} X$$

Moreover, if $x_i\colon X_i\hookrightarrow X$ *are subobjects of* X*, define their* **Higgins commutator** *to be the following subobject of* X*:*

$$[X_1,\ldots,X_n]=\mathrm{Im}(\,X_1\diamond\cdots\diamond X_n \xrightarrow{x_1\diamond\cdots\diamond x_n} X\diamond\cdots\diamond X \xrightarrow{c_n^X} X\,).$$

4.8. Remark. This generalizes the definition of a binary Higgins commutator which was independently given in [Mantovani & Metere 2010]. With the notations of definition 4.7, the Huq commutator of X_1, X_2 is the proper closure of the Higgins commutator $[X_1, X_2]$ defined above (where the proper closure of a subobject is defined to be the kernel of its cokernel).

Recall that the Huq commutator $[x_1, x_2]^{\mathrm{Huq}} \colon [X_1, X_2]^{\mathrm{Huq}} \to X$ of two (mono)morphisms $x_1 \colon X_1 \to X$ and $x_2 \colon X_2 \to X$ is the smallest normal subobject of X that should be divided out to make x_1 and x_2 commute – so that they do commute if and only if $[X_1, X_2]^{\mathrm{Huq}} = 0$ [Huq 1968]. It is the kernel of the regular (hence normal) epimorphism $q \colon X \to Q$, where Q is the colimit (via the dotted arrows) of the four plain arrows in the following diagram:

$$\begin{tikzcd}
& X_1 \arrow[dl, "\sigma_1"'] \arrow[d, dotted, "q_1"] \arrow[dr, "x_1"] & & \\
X_1 \times X_2 \arrow[r, dotted, "q_{12}"] & Q & X \arrow[l, dotted, two heads, "q"'] & {[X_1, X_2]^{\mathrm{Huq}}} \arrow[l, dashed, tail, "{[x_1,x_2]^{\mathrm{Huq}}}"'] \\
& X_2 \arrow[ul, "\sigma_2"] \arrow[u, dotted, "q_2"'] \arrow[ur, "x_2"'] & &
\end{tikzcd}$$

Since q is normal, to show that $[x_1, x_2]^{\mathrm{Huq}}$ is the proper closure of the Higgins commutator $[x_1, x_2]^{\mathrm{Hig}} \colon [X_1, X_2]^{\mathrm{Hig}} \to X$ defined above, it suffices to show that the morphism q above is the cokernel of $[x_1, x_2]^{\mathrm{Hig}}$. So one has to show that if a morphism $f \colon X \to A$ is such that $f \circ [x_1, x_2]^{\mathrm{Huq}} = 0$ then one can complete a cocone

$$\{f \colon X \to A, f_1 \colon X_1 \to A, f_2 \colon X_2 \to A, f_{12} \colon X_1 \times X_2 \to A\}$$

on this diagram of four arrows. Of course, it suffices to put $f_i = f \circ x_i$ for $i = 1, 2$. So it remains to get the morphism $f_{12} \colon X_1 \times X_2 \to A$. Recall that $r_{X_1,X_2} \colon X_1 + X_2 \to X_1 \times X_2$ is a strong epimorphism (this is well known and is a special case of Proposition 2.11), hence it is a cokernel, hence the cokernel of its kernel, which by very definition is $X_1 \diamond X_2$, or more precisely ι_{X_1,X_2}. Therefore, we show that the morphism $f \circ \langle \begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \rangle \colon X_1 + X_2 \to C$ is such that $f \circ \langle \begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \rangle \circ \iota_{X_1,X_2} = 0$. Indeed

$$\begin{aligned}
f \circ \langle \begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \rangle \circ \iota_{X_1,X_2} &= f \circ \nabla_X^2 \circ (x_1 + x_2) \circ \iota_{X_1,X_2} \\
&= f \circ \nabla_X^2 \circ \iota_{X,X} \circ (x_1 \diamond x_2) \\
&= f \circ [x_1, x_2]^{\mathrm{Hig}} \circ q_{\mathrm{Hig}} \\
&= 0
\end{aligned}$$

$$\begin{tikzcd}
 & & X_1+X_2 \arrow[ddll, bend right, "r_{X_1,X_2}"'] \arrow[dd, "\left\langle \substack{x_1 \\ x_2} \right\rangle"'] & & \\
 & X_1 \arrow[dl, "\sigma_1"'] \arrow[d, "f_1=f\circ x_1"'] \arrow[dr, "x_1"] & & & \\
X_1\times X_2 \arrow[r, dotted, "f_{12}"] & A & X \arrow[l, "f"'] & {[X_1,X_2]^{\mathrm{Hig}}} \arrow[l, tail, "{[x_1,x_2]^{\mathrm{Hig}}}"'] & X_1\diamond X_2 \arrow[l, two heads, "q_{\mathrm{Hig}}"] \arrow[uull, "\iota_{X_1,X_2}=\ker(r_{X_1,X_2})"'] \arrow[d, "x_1\diamond x_2"] \\
 & X_2 \arrow[ul, "\sigma_2"] \arrow[u, "f_2=f\circ x_2"] \arrow[ur, "x_2"'] & X+X \arrow[u, "\nabla_X"] & & X\diamond X \arrow[ll, "\iota_{X,X}=\ker(r_{X,X})"]
\end{tikzcd}$$

Hence, there exists a unique morphism $f_{12}\colon X_1\times X_2 \to A$ such that $f_{12}\circ r_{X_1,X_2} = f\circ \left\langle \substack{x_1 \\ x_2} \right\rangle$. Then of course $f_{12}\circ\sigma_i = f_i$ $(i=1,2)$ as required.

Thus, if X_1, X_2 generate X (i.e. if the morphism $[x_1,x_2]\colon X_1+X_2\to X$ is a regular epimorphism), then $[X_1,X_2]$ coincides with the commutators of X_1, X_2 of Huq and of Smith; the Smith case is due to Everaert and Gran (see [Everaert & Van der Linden 2012]).

The link with the subject of this paper is that stability under the conjugation action core can be expressed in terms of commutators:

4.9. Lemma. *Let* $X \xrightarrow{x} A \xleftarrow{y} Y$ *be subobjects of an object of* $\mathbb{C}$. *Then* X *is* Y*-stable under the conjugation action (core) of* A *on itself (see Proposition 3.13) iff* $[X,Y]\le X$, *i.e. the injection of* $[X,Y]$ *into* A *factors through* x.

Proof. Consider the following commutative diagram of solid arrows:

$$\begin{tikzcd}
X\diamond Y \arrow[rr, "x\diamond y"] \arrow[dr, "q"] \arrow[dd, dotted, "\psi"'] & & A\diamond A \arrow[dd, "c_2^A"] \\
 & {[X,Y]} \arrow[dl, dotted, "u"'] \arrow[dr, hook, "i"] & \\
X \arrow[rr, tail, "x"] & & A
\end{tikzcd}$$

Then by the respective definitions, X is Y-stable iff a morphism ψ as indicated exists and renders the diagram commutative; and $[X,Y]\subset X$ iff a morphism u as indicated exists and renders the diagram commutative. But these conditions are equivalent since q is a regular epimorphism hence a strong one, and x is monic (see for instance [Borceux & Bourn 2004, Appendix A.4.]). ∎

4.10. Definition. *Let* $X \xrightarrow{x} A \xleftarrow{y} Y$ *be subobjects of an object of* $\mathbb{C}$. *We say that* Y **normalizes** X *if* X *is* Y*-stable under the conjugation action core of* A *on itself (i.e. if* $[X,Y]\le X$, *in view of Lemma 4.9).*

4.11. Proposition. *The following properties are equivalent for subobjects X, Y of A as above:*

1. *X is a normal subobject of $X \vee Y$, the subobject generated by X and Y (i.e. the image of the morphism $\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle : X + Y \to A$).*

2. *$X \vee Y$ acts strictly on X by conjugation.*

3. *Y normalizes X.*

In particular, when $Y = A$, one gets:
X is a normal subobject in A if and only if A acts strictly on X by conjugation.

Proof. (1) implies (2) by Proposition 4.1.

(2) implies (3): Let us denote by x', y', z the inclusions of X and Y in $X \vee Y$ and the inclusion of $X \vee Y$ in A, respectively. (2) means the existence of a morphism $\psi' \colon X \diamond (X \vee Y) \to X$ making the left-hand bottom square in the following diagram commute; (3) means the existence of a morphism $\psi \colon X \diamond Y \to X$ making the outer square commute; so it suffices obviously to take $\psi = \psi' \circ (1_X \diamond y')$.

$$\begin{tikzcd}
X \diamond Y \arrow[d, "1_X \diamond y'"] \arrow[rrr, bend left, "x \diamond y"] \arrow[dd, dotted, bend right=60, "\psi"'] & & & \\
X \diamond (X \vee Y) \arrow[d, "\psi'"] \arrow[rr, "x' \diamond 1_{X \vee Y}"] & & (X \vee Y) \diamond (X \vee Y) \arrow[d, "c_2^{(X \vee Y) \diamond (X \vee Y)}"] \arrow[r, "z \diamond z"] & A \diamond A \arrow[d, "c_2^A"] \\
X \arrow[rr, "x'"'] \arrow[rrr, bend right, "x"'] & & X \vee Y \arrow[r, "z"'] & A
\end{tikzcd}$$

(3) implies (1): by Proposition 3.13 we obtain a strict action core ψ of Y on X, and the universal property (Proposition 4.4) of the semi-direct product implies that there is a morphism $\overline{\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle} : X \rtimes_\psi Y \to A$ such that $\overline{\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle} \circ l_\psi = x$ and $\overline{\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle} \circ s_\psi = y$. Now $\mathrm{Im}(\overline{\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle}) = \mathrm{Im}(\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle) = X \vee Y$ since by construction $X \rtimes_\psi Y$ is a quotient of $X + Y$. But $X = \mathrm{Im}(l_\psi)$ is proper in $X \rtimes_\psi Y$, whence $\mathrm{Im}(\overline{\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle} \circ l_\psi) = \mathrm{Im}(x) = X$ is the image (in $\mathrm{Im}(\overline{\langle\begin{smallmatrix}x\\y\end{smallmatrix}\rangle} = X \vee Y)$ of a proper subobject of $X \rtimes_\psi Y$ by a regular epimorphism, hence it is a normal subobject, since proper subobjects are normal and since normal subobjects are stable under direct images along regular epimorphisms. (See for instance [Borceux & Bourn 2004, Propositions 3.2.7 and 4.3.2]. Notice however that the definition of a normal subobject in this book is more general than the one used here, but they show in Proposition 3.2.12 that they are equivalent in a protomodular pointed category.) ∎

Note also that the notions of normal and proper subobjects, and other related ones, have been studied more deeply in [Janelidze, Márki & Ursini 2007], [Janelidze, Márki & Ursini 2009] and [Mantovani & Metere 2010], where it is proved that conversely, in a pointed regular Mal'tsev category, a direct image of a kernel is a normal subobject.

4.12. Corollary. *A finitely cocomplete homological category $\mathbb{C}$ is semi-abelian (i.e. Barr exact) iff the following condition holds:*

(P) A subobject X of an object A of $\mathbb{C}$ is proper in A iff it is stable under the conjugation action core of A, i.e. iff $[X, A] \leq X$.

The result that property (P) holds in semi-abelian categories was recently independently obtained by Mantovani and Metere [Mantovani & Metere 2010].

Proof. Immediate since a finitely cocomplete homological category is semi-abelian if and only if it has the property that every normal subobject is a proper subobject. ∎

The following is an extremely useful criterion of normality in semi-abelian categories, as will be shown in the sequel and in subsequent work.

4.13. Theorem. *Suppose that $\mathbb{C}$ is semi-abelian. Then the following properties are equivalent for subobjects X, Y of A as above:*

1. *Y normalizes X.*
2. *X is a proper subobject of $X \vee Y$.*
3. *The object $X \wedge Y$ is proper in Y and the sequence*

$$0 \to X \xrightarrow{x'} X \vee Y \xrightarrow{q \circ r_Y} Y/(X \wedge Y) \to 0$$

is short exact where x' is the factorization of x through $X \vee Y$ and $q\colon Y \to Y/(X \wedge Y)$ is the projection.

If these conditions (1)-(3) are satisfied we write $X \vee Y = [X]Y$, or even $X \cdot Y$ when no ambiguity can occur.

We note that the implication (1) $\Rightarrow$ (3) is a crucial ingredient in the commutator theory in [Hartl - in preparation].

Proof. The equivalence of (2) and (3) is immediate using the second Noether isomorphism theorem, and the equivalence of (2) and (1) follows from Proposition 4.11 and the exactness of $\mathbb{C}$. ∎

Of course, the notation $[X]Y$ comes from the fact that in the category of groups, when the conditions are verified the join $X \vee Y$ indeed is the group $X \cdot Y$ of products $x \cdot y$ with $x \in X$ and $y \in Y$.

4.14. Proposition. *Let $\mathbb{C}$ be a finitely cocomplete homological category. Consider two subobjects X, Y of an object A as above. Then:*

1. *$[X, Y]$ is a normal subobject in $[X, Y] \vee X$;*
2. *$[X, Y] \vee X$ is a normal subobject in $X \vee Y$ and is the smallest normal subobject of $X \vee Y$ containing X;*
3. *If moreover $\mathbb{C}$ is semi-abelian, then one may write:*

$$X \vee Y = ([X, Y] \cdot X) \cdot Y$$

and

$$\lhd X \rhd_{X \vee Y} = [X, Y] \cdot X$$

where $\lhd U \rhd_V$ denotes the proper closure of a subobject U in an object V. In particular,

$$\lhd X \rhd_A = [X, A] \cdot A$$

Proof. By Lemma 2.7, one has the decomposition $X + Y = ((X \diamond Y) \rtimes X) \rtimes Y$, hence in particular $X \diamond Y$ is a proper subobject of $(X \diamond Y) \rtimes X$, and $(X \diamond Y) \rtimes X$ is a proper subobject of $X + Y = ((X \diamond Y) \rtimes X) \rtimes Y$. All of them are subobjects of $A + A$. Taking the images of these subobjects by the folding morphism $\nabla^2_A \colon A + A \to A$, we get that $\nabla^2_A(X \diamond Y) = [X, Y]$ is a normal subobject of $\nabla^2_A((X \diamond Y) \rtimes X)$ and $\nabla^2_A((X \diamond Y) \rtimes X)$ is a normal subobject of $\nabla^2_A(X + Y) = X \vee Y$. But $\nabla^2_A((X \diamond Y) \rtimes X) = [X, Y] \vee X$, so indeed $[X, Y]$ is a normal subobject in $[X, Y] \vee X$ and $[X, Y] \vee X$ is a normal subobject in $X \vee Y$. So $[X, Y] \vee X$ is the smallest normal subobject of $X \vee Y$ containing X, because by Lemma 4.9 and Proposition 4.11, any normal subobject of $X \vee Y$ containing X must contain $[X, X \vee Y]$, hence must contain $[X, Y]$. So (1) and (2) are proved, and (3) is an immediate consequence of them and of the preceding results. ∎

We conclude this section with the following result, which is useful in [Hartl & Van der Linden 2013]:

4.15. Proposition. *Let $\mathbb{C}$ be a finitely cocomplete homological category. Consider a morphism $f \colon X \to Y$. Then the image $\mathrm{Im}(f)$ is a normal subobject of Y if and only if the composite morphism*

$$X \diamond Y \xrightarrow{f \diamond 1_Y} Y \diamond Y \xrightarrow{c_2^Y} Y$$

factors through it.

In particular, if $\mathbb{C}$ is semi-abelian, then these two equivalent conditions are equivalent to the fact that f is proper (i.e. $\mathrm{Im}(f)$ is proper in Y).

PROOF. Let $X \xrightarrow{q} \mathrm{Im}(f) \xrightarrow{m} Y$ be the regular epi-mono decomposition of f. By Proposition 4.11, $\mathrm{Im}(f)$ is a normal subobject of Y if and only Y acts on it by conjugation, i.e. if the composite $c_2^Y \circ (m \diamond 1_Y)$ factors through m by some morphism $c^{\mathrm{Im}(f),Y}$. Then of course $c_2^Y \circ (f \diamond 1_Y)$ factors through m by $c^{\mathrm{Im}(f),Y} \circ (q \diamond 1_Y)$. Conversely, suppose that $c_2^Y \circ (f \diamond 1_Y)$ factors through m by some morphism h. Then since $q \diamond 1_Y$ is a regular epimorphism by Proposition 2.13, hence a strong one, and since m is a monomorphism, $c_2^Y \circ (m \diamond 1_Y)$ factors through m, so $\mathrm{Im}(f)$ is a normal subobject in Y.

$$\begin{tikzcd} X \diamond Y \arrow[r, "q \diamond 1_Y"] \arrow[dr, dotted, "h"'] & \mathrm{Im}(f) \diamond Y \arrow[r, "m \diamond 1_Y"] \arrow[d, dotted, "c^{\mathrm{Im}(f),Y}"] & Y \diamond Y \arrow[d, "c_2^Y"] \\ & \mathrm{Im}(f) \arrow[r, tail, "m"'] & Y \end{tikzcd}$$

■

5. Other characterizations of (strict) action cores

In this paragraph $\mathbb{C}$ remains a finitely cocomplete homological category, even if, at some places, we shall pay special attention to the semi-abelian case. The functor Ξ_G of Corollary 3.6 associates to any object (A, G, ψ) of $\mathbb{S}$ on another object A an algebra (A, ξ) over the monad $\mathbb{T}_G$, which moreover makes the following diagram commute.

$$\begin{tikzcd} A \diamond G \arrow[rr, "j_{A,G}"] \arrow[dr, "\psi"'] & & T_G(A) \arrow[dl, "\xi"] \\ & A & \end{tikzcd}$$

Using our preceding results, we here give necessary and sufficient conditions for an arbitrary morphism $\psi \colon A \diamond G \to A$ to have such an extension to an algebra over $\mathbb{T}_G$, so they are necessary for ψ to be a strict action core. Hence if the category $\mathbb{C}$ is such that the comparison functor $\mathcal{J}_G$ is an equivalence of categories between $\mathrm{Pt}_G(\mathbb{C})$ and $\mathbb{C}^{\mathbb{T}_G}$ (in particular if $\mathbb{C}$ is semi-abelian) they are also sufficient, but we also give a direct proof of this fact in a semi-abelian category, providing an alternative proof (without using Beck's criterion) of the fact that $\mathcal{J}_G$ is an equivalence of categories in this case. Based on a result of [Hartl & Van der Linden 2013] we also provide a characterization of action cores corresponding to Beck modules in the semi-abelian setting.

5.1. PROPOSITION. *Let $\psi \colon A \diamond G \to A$ be a morphism in $\mathbb{C}$. Then the following conditions are equivalent:*

1. ψ can be extended along $j_{A,G}$ to a morphism $\xi \colon T_G(A) \to A$ satisfying the unit axiom of a $\mathbb{T}_G$-algebra, i.e. $\xi \circ \eta_{A,G} = 1_A$.

2. The following diagram commutes:

$$\begin{tikzcd} (A\diamond G)\diamond A \arrow[r, "C^A_{A,G}"] \arrow[d, "\psi\diamond 1_A"'] & A\diamond G \arrow[d, "\psi"] \\ A\diamond A \arrow[r, "c_2^A"'] & A \end{tikzcd} \tag{2}$$

Moreover under these conditions, the resulting morphism ξ also satisfies $l_\psi \circ \xi = q_\psi \circ \kappa_{A,G}$, where $q_\psi \colon A+G \to Q_\psi$ is the coequalizer of $\iota_{A,G}$ and $i_A \circ \psi$ and l_ψ is the composite $q_\psi \circ i_A$ as in Proposition 3.4.

Proof. That some ξ extends ψ along $j_{A,G}$ with $\xi \circ \eta_{A,G} = 1_A$ can be translated into the following diagram:

$$\begin{tikzcd} A\diamond G \arrow[r, "j_{A,G}"] \arrow[dr, "\psi"'] & T_G(A) \arrow[d, "\xi"] & A \arrow[l, "\eta_{A,G}"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

so by Proposition 4.4 and Example 4.2, it is immediate that such an extension exists if and only diagram (2) commutes. Moreover, since the pair $(j_{A,G}, \eta_{A,G})$ is a (strongly) epimorphic family, in order to prove that $l_\psi \circ \xi = q_\psi \circ \kappa_{A,G}$, it suffices to verify that $l_\psi \circ \xi \circ j_{A,G} = q_\psi \circ \kappa_{A,G} \circ j_{A,G}$ and $l_\psi \circ \xi \circ \eta_{A,G} = q_\psi \circ \kappa_{A,G} \circ \eta_{A,G}$. The first equality amounts to $q_\psi \circ i_A \circ \psi = q_\psi \circ \iota_{A,G}$, which is true because q_ψ is the coequalizer of these two morphisms, and the second to $q_\psi \circ i_A \circ \xi \circ \eta_{A,G} = q_\psi \circ \kappa_{A,G} \circ \eta_{A,G}$, which is true because $\xi \circ \eta_{A,G} = 1_A$ and $\kappa_{A,G} \circ \eta_{A,G} = i_A$. ■

5.2. Example. In the category of groups, consider a homomorphism $\psi\colon A\diamond G \to A$. As seen above, ψ is the extension to $A \diamond G$ of a unique function $[\![-,-]\!]\colon G\times A \to A$ such that $[\![e_G, a]\!] = [\![g, e_A]\!] = e_A$. In Example 3.7 we defined $\phi(g,a) = [\![g,a]\!]\cdot a$ and saw that ψ is a strict action core if and only if ϕ is an action in the usual sense. More generally, and rather starting out from ϕ, one may determine the condition on ϕ which insures that ψ satisfies the equivalent conditions of Proposition 5.1. In view of the proof of this proposition, it is more convenient to apply the classical universal property of the semi-direct product than condition 2. Considering that the strict action core ψ_0 of Proposition 5.1 corresponds to a classical action ϕ_0, the property that is needed to insure the existence of ξ is: for any $x \in A\diamond G, \psi(\phi_0(a,x)) = a\cdot\psi(x)\cdot a^{-1}$ (note that we indeed work with ψ on the one hand, and with ϕ_0 on the other hand). But since $\phi_0(a,-)$ and conjugation by a both are group morphisms, it suffices to prove this for an x of the form $[g,a']$. And $\phi_0(a,[g,a']) = a\cdot[g,a']\cdot a^{-1}$ (in $T_G(A)$, or in $A+G$). But, working in $A+G$, using the well-known formula $x\cdot[y,z] = [x,y]\cdot[y,xz]\cdot x$ (or equivalently $[x,[y,z]] = [x,y]\cdot[y,xz]\cdot[z,y]$), one gets: $a\cdot[g,a']\cdot a^{-1} = [a,g]\cdot[g,aa']\cdot a\cdot a^{-1} = [g,a]^{-1}\cdot[g,aa']$. So since ψ is a morphism and since $\psi([g,a]) = \phi(g,a)\cdot a^{-1}$, one gets $\psi(\phi_0(a,[g,a'])) = \psi([g,a])^{-1}\cdot\psi([g,aa']) = a\cdot(\phi(g,a))^{-1}\cdot\phi(g,aa')\cdot(aa')^{-1}$ and

$a\cdot\psi[g,a']\cdot a^{-1} = a\cdot\phi(g,a')\cdot a'^{-1}\cdot a^{-1} = a\cdot\phi(g,a')\cdot(aa')^{-1}$. So these terms are equal if and only if $(\phi(g,a))^{-1}\cdot\phi(g,aa') = \phi(g,a')$, or $\phi(g,aa') = \phi(g,a)\cdot\phi(g,a')$, i.e. $\phi(g,-)$ is an endomorphism of A. So we see that ψ extends to a unital morphism ξ iff the corresponding function $\phi\colon G\times A\to A$ satisfies part (2) of the classical definition on an action of G on the group A, which says that (1) ϕ is an action on the set A and (2) ϕ acts through endomorphisms of A.

We now give a characterization of strict action cores in terms of their extensions to $T_G(A)$:

5.3. Proposition. *Let $\psi\colon A\diamond G\to A$ be any morphism. Then, using the same notations as above, the following properties are equivalent:*
1) ψ is a strict action core, i.e. l_ψ is a monomorphism;
2) ψ satisfies the equivalent conditions of Proposition 5.1 and the induced $\xi\colon T_G(A)\to A$ is such that $\mathrm{Ker}(\xi)$ *(or more precisely: the morphism $\kappa_{A,G}\circ\ker(\xi)$) is proper in $A+G$.*

Moreover, when these two equivalent conditions are verified, $\mathrm{Ker}(\xi)$ *is the kernel of $q_\psi\colon A+G\to Q_\psi$ (or more precisely: the morphism $\kappa_{A,G}\circ\ker(\xi)$ is the kernel of q_ψ).*

Proof. We first prove that 1) $\Rightarrow$ 2). By definition, the action core ψ induces the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r, dotted] & \mathrm{Ker}(\xi) \arrow[r, dotted] \arrow[d, "\ker(\xi)"'] & \mathrm{Ker}(q_\psi) \arrow[r] \arrow[d, "\ker(q_\psi)"] & 0 \arrow[d] & \\
0 \arrow[r] & T_G A \arrow[r, "\kappa_{A,G}"] \arrow[d, "\xi"'] \arrow[dr, phantom, "(*)"] & A+G \arrow[r, "p_G"] \arrow[d, "q_\psi"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dashed] & A \arrow[r, dashed, "l_\psi"'] \arrow[d] & Q_\psi \arrow[r, "p_\psi"'] \arrow[d] & G \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

of which the columns and the middle row are short exact. If now ψ is strict, then by Proposition 3.4 the bottom row is also exact, so the top row is exact by the 3×3 lemma. In particular, the dotted arrow between the kernels is an isomorphism, so that $\kappa_{A,G}\circ\ker(\xi)$ is not only proper, but even a kernel of q_ψ.
Conversely, if ψ satisfies the equivalent conditions of Proposition 5.1, then $\eta_{A,G}$ is a section of the induced map $\xi\colon T_G(A)\to A$. If moreover ξ is such that the morphism $\kappa_{A,G}\circ\ker(\xi)$ is proper in $A+G$, then this subobject needs to be the kernel of q_ψ. Indeed, since it is proper, it suffices to show that its cokernel is q_ψ. But by Proposition 3.3 q_ψ is the coequalizer of $\kappa_{A,G}$ and $i_A\circ\xi$. So it suffices to show that if $h : A+G\to H$ is such that $h\circ\kappa_{A,G}\circ\ker\xi = 0$, then also $h\circ\kappa_{A,G} = h\circ i_A\circ\xi$. Since ξ has a section, it is the cokernel of its kernel, hence there exists a unique $\bar h\colon A\to H$ such that $h\circ\kappa_{A,G} = \bar h\circ\xi$. But then $\bar h = \bar h\circ\xi\circ\eta_{A,G} = h\circ\kappa_{A,G}\circ\eta_{A,G} = h\circ i_A$ since $\kappa_{A,G}\circ\eta_{A,G} = i_A$. So

$h \circ \kappa_{A,G} = h \circ i_A \circ \xi$ as required. Hence we know that in the previous diagram, the dotted arrow is an isomorphism, so the diagram's bottom row is exact by the 3×3 lemma. It follows that ψ is a strict action core. ■

5.4. Lemma. *Consider $\mu_{A,G}\colon T_G(T_GA) \to T_GA$ the multiplication of the monad and $j_{T_GA,G}$ the inclusion of $T_GA \diamond G$ in $T_G(T_GA)$. Then the composite $\mu_{A,G} \circ j_{T_GA,G}$ is the restriction to G of the conjugation action core of $A+G$ on T_GA (which exists by Proposition 4.1 since T_GA is a proper subobject of $A+G$). In formal terms, $\mu_{A,G} \circ j_{T_GA,G} = c^{T_GA,A+G} \restriction_G$.*

Proof. By Proposition 3.13, this restriction is the unique morphism $c\colon T_GA \diamond G \to T_GA$ making the following diagram commute:

$$\begin{tikzcd}
T_GA \diamond G \arrow[r, "\kappa_{A,G} \diamond i_G"] \arrow[d, "c"'] & (A+G) \diamond (A+G) \arrow[d, "c^2_{A+G}"] \\
T_GA \arrow[r, "\kappa_{A,G}"'] & A+G
\end{tikzcd}$$

so we have to show that $c = \mu_{A,G} \circ j_{T_GA,G}$ makes it commute. This follows from the commutativity of all components of the following diagram:

$$\begin{tikzcd}
T_GA \diamond G \arrow[rr, "\kappa_{A,G} \diamond i_G"] \arrow[d, "j_{T_GA,G}"'] \arrow[dr, "\iota_{T_GA,G}"] & & (A+G) \diamond (A+G) \arrow[d, "\iota_{A+G,A+G}"'] \arrow[dd, bend left=60, "c^2_{A+G}"] \\
T_G(T_GA) \arrow[r, "\kappa_{T_GA,G}"'] \arrow[d, "\mu_{A,G}"'] & (T_GA)+G \arrow[r, "\kappa_{A,G}+i_G"] \arrow[dr, "\left\langle \begin{smallmatrix} \kappa_{A,G} \\ i_G \end{smallmatrix} \right\rangle"'] & (A+G)+(A+G) \arrow[d, "\nabla^2_{A+G}"] \\
T_GA \arrow[rr, "\kappa_{A,G}"'] & & A+G
\end{tikzcd}$$

■

To appreciate the meaning of this lemma recall that for any monad $\mathbb{T}$ on $\mathbb{C}$ and an object X of $\mathbb{C}$ the multiplication μ_X is a $\mathbb{T}$-algebra over X. Hence $\mu_{A,G}$ is a G-action on T_GA, hence a strict one if $\mathbb{C}$ is semi-abelian; then $\mu_{A,G} \circ j_{T_GA,G}$ is a strict action core. Lemma 5.4 generalizes this fact to the non-exact case.

It is worth noting that $c^{T_GA,A+G} \restriction_G$ factors through $A \diamond G$:

5.5. Proposition. *The morphism $c^{T_GA,A+G} \restriction_G\colon T_GA \diamond G \to T_GA$ factors through the inclusion $j_{A,G}$ of $A \diamond G$ in T_GA.*

Proof. Since $j_{A,G}$ is the kernel of $r_A \circ \kappa_{A,G}$ by Proposition 2.7, it suffices to show that $r_A \circ \kappa_{A,G} \circ \mu_{A,G} \circ j_{T_GA,G} = 0$. But by definition of $\mu_{A,G}$, the central square in the following diagram commutes, and it is easy to check that the right hand square also commutes:

$$\begin{tikzcd}
T_GA \diamond A \arrow[r, "j_{T_GA,G}"] \arrow[ddd, dotted] & T_GT_GA \arrow[r, "\kappa_{T_GA,G}"] \arrow[ddd, "\mu_{A,G}"] & T_GA+G \arrow[r, "r_{T_GA}"] \arrow[ddd, "\left\langle {\kappa_{A,G} \atop i_G} \right\rangle"'] & T_GA \arrow[d, "\kappa_{A,G}"] \\
 & & & A+G \arrow[dd, "r_A"] \\
 & & & \\
A \diamond G \arrow[r, "j_{A,G}"'] & T_GA \arrow[r, "\kappa_{A,G}"'] & A+G \arrow[r, "r_A"'] & A
\end{tikzcd}$$

so $r_A \circ \kappa_{A,G} \circ \mu_{A,G} \circ j_{T_GA,G} = r_A \circ \kappa_{A,G} \circ r_{T_GA} \circ \kappa_{T_GA,G} \circ j_{T_GA,G} = 0$, since by Proposition 2.7 again $r_{T_GA} \circ \kappa_{T_GA,G} \circ j_{T_GA,G} = 0$. ■

5.6. Example. In the category of groups, an element of T_GA is an element of $A+G$ of the form $(\prod_i [g_i, a_i]) \cdot a$. We denote by $[-,-]$ the commutators in $A+G$, and by $[\![-,-]\!]$ the commutators in $(A+G)+G$. Hence a generator of $T_GA \diamond G$ has the form $[\![g, (\prod_i [g_i, a_i]) \cdot a]\!]$ (with g in the second copy of G, and g_i in the first one), and $c^{T_GA,A+G} \restriction_G ([\![g, (\prod_i [g_i, a_i]) \cdot a]\!])$ $= [g, (\prod_i [g_i \cdot a_i]) \cdot a]$ where both commutators are considered in $A+G$. Proposition 5.5 is illustrated by the fact that this element is in $A \diamond G$.

5.7. Proposition. *A morphism $\xi \colon T_GA \to A$ which satisfies the unit axiom of a $\mathbb{T}_G$-algebra also satisfies the associativity axiom if and only if the following diagram commutes:*

$$\begin{tikzcd}
T_GA \diamond G \arrow[r, "c^{T_GA,A+G}\restriction_G"] \arrow[d, "\xi \diamond 1_G"'] & T_GA \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "\xi \circ j_{A,G}"'] & A
\end{tikzcd}$$

Hence if a morphism $\psi \colon A \diamond G \to A$ satisfies the conditions of Proposition 5.1, with extension ξ to T_GA, then (A, ξ) is a $\mathbb{T}$-algebra if and only if the following diagram commutes:

$$\begin{tikzcd}
T_GA \diamond G \arrow[r, "c^{T_GA,A+G}\restriction_G"] \arrow[d, "\xi \diamond 1_G"'] & T_GA \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "\psi"'] & A
\end{tikzcd} \tag{3}$$

Proof. Recall that the associativity condition for ξ is commutativity of the following diagram:

$$\begin{tikzcd}
T_G(T_GA) \arrow[r, "\mu_{A,G}"] \arrow[d, "T_G\xi"'] & T_GA \arrow[d, "\xi"] \\
T_GA \arrow[r, "\xi"'] & A
\end{tikzcd}$$

Since the pair $\{j_{T_GA,G} \colon T_GA \diamond G \to T_G(T_GA), \eta_{T_GA,G} \colon T_GA \to T_G(T_GA)\}$ is (strongly) epimorphic this condition is equivalent to the system of two equations $\xi \circ \mu_{A,G} \circ \eta_{T_GA,G} =$

$\xi \circ T_G\xi \circ \eta_{T_GA,G}$ and $\xi \circ \mu_{A,G} \circ j_{T_GA,G} = \xi \circ T_G\xi \circ j_{T_GA,G}$. The first one is automatically satisfied, because $\xi \circ T_G\xi \circ \eta_{T_GA,G} = \xi \circ \eta_{A,G} \circ \xi = \xi$ since η is a natural transformation and ξ satisfies the unit axiom; and $\xi \circ \mu_{A,G} \circ \eta_{T_GA,G} = \xi$ since $\mu_{A,G} \circ \eta_{T_GA,G} = 1_{T_GA}$. Considering that in the following diagram the left-hand square commutes, the second condition is equivalent to the commutativity of the outer rectangle; but in view of Lemma 5.4 this amounts to the required commutativity.

$$\begin{tikzcd}
T_GA \diamond G \arrow[r, "j_{T_GA,G}"] \arrow[d, "\xi \diamond 1_G"'] & T_G(T_GA) \arrow[r, "\mu_{A,G}"] \arrow[d, "T_G\xi"'] & T_GA \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "j_{A,G}"'] & T_GA \arrow[r, "\xi"'] & A
\end{tikzcd}$$

■

The following result shows that in a semi-abelian category, the conditions of Proposition 5.7 are also sufficient for some $\psi\colon A \diamond G \to A$ to be a (strict) action core. This provides an alternative proof of the fact that in any semi-abelian category, all $\mathbb{T}_G$-algebras are strict actions so that the category of points is equivalent to the category of $\mathbb{T}_G$-algebras. We point out that this proof is based on Proposition 5.3 and the characterization of proper subobjects by stability under the conjugation action core in Corollary 4.12, instead of Beck's criterion as in [Bourn & Janelidze 1998].

5.8. Proposition. *Let $\mathbb{C}$ be semi-abelian. Let $\psi\colon A \diamond G \to A$ be a morphism satisfying the equivalent conditions of Proposition 5.1. Then ψ is a strict action core if and only if the diagram (3) of Proposition 5.7 commutes, i.e.*

$$\begin{tikzcd}
T_GA \diamond G \arrow[r, "c^{T_GA,A+G}\restriction_G"] \arrow[d, "\xi \diamond 1_G"'] & T_GA \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "\psi = \xi \circ j_{A,G}"'] & A
\end{tikzcd}$$

Proof. The condition is necessary in view of Proposition 5.7 (even if the category is not semi-abelian), since if ψ is a strict action core then its extension ξ is an algebra by Proposition 3.4, 3. Conversely, suppose $\mathbb{C}$ is semi-abelian and suppose that ψ satisfies the conditions of Propositions 5.1 and 5.7. To show that ψ is a strict action core, it suffices by Proposition 5.3 to show that $\mathrm{Ker}(\xi)$ is proper in $A+G$. Since $\mathbb{C}$ is semi-abelian, it suffices by Corollary 4.12 to show that $\mathrm{Ker}(\xi)$ is stable under the conjugation action core of $A+G$, i.e. that there exists a morphism $c''\colon \mathrm{Ker}(\xi) \diamond (A+G) \to \mathrm{Ker}(\xi)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Ker}(\xi) \diamond (A+G) \arrow[r, "(\kappa_{A,G} \circ \ker(\xi)) \diamond 1_{A+G}"] \arrow[d, dashed, "c''"'] & (A+G) \diamond (A+G) \arrow[d, "c_2^{A+G}"] \\
\mathrm{Ker}(\xi) \arrow[r, "\kappa_{A,G} \circ \ker(\xi)"'] & A+G
\end{tikzcd}$$

But since T_GA is proper in $A+G$ one has the conjugation action core $c^{T_GA,A+G}$ of $A+G$ over T_GA which makes the following diagram commute:

$$\begin{tikzcd}
T_GA \diamond (A+G) \arrow[r, "\kappa_{A,G}\diamond 1_{A+G}"] \arrow[d, "c^{T_GA,A+G}"'] & (A+G)\diamond(A+G) \arrow[d, "c_2^{A+G}"] \\
T_GA \arrow[r, "\kappa_{A,G}"'] & A+G
\end{tikzcd}$$

so that one gets the result if one can find a morphism c'' such that

$$\begin{tikzcd}
\mathrm{Ker}(\xi)\diamond(A+G) \arrow[r, "\ker(\xi)\diamond 1_{A+G}"] \arrow[d, dotted, "c''"'] & T_GA\diamond(A+G) \arrow[d, "c^{T_GA,A+G}"] \\
\mathrm{Ker}(\xi) \arrow[r, "\ker(\xi)"'] & T_GA
\end{tikzcd}$$

commutes, i.e. if $\xi\circ c^{T_GA,A+G}\circ(\ker(\xi)\diamond 1_{A+G})=0$.

Now by [Hartl & Van der Linden 2013, Lemma 2.12] the morphisms $1_{\mathrm{Ker}(\xi)}\diamond i_A\colon \mathrm{Ker}(\xi)\diamond A\to \mathrm{Ker}(\xi)\diamond(A+G)$, $1_{\mathrm{Ker}(\xi)}\diamond i_G\colon \mathrm{Ker}(\xi)\diamond G\to \mathrm{Ker}(\xi)\diamond(A+G)$ and $\iota'_{T_GA,A,G;2}\colon \mathrm{Ker}(\xi)\diamond A\diamond G\to \mathrm{Ker}(\xi)\diamond(A+G)$ are jointly epimorphic, hence it suffices to check that the latter identity holds after composing both sides with each of these three morphisms.

We first show that $\xi\circ c^{T_GA,A+G}\circ(\ker(\xi)\diamond i_A)=0$. Indeed,

$$\begin{aligned}
c^{T_GA,A+G}\circ(\ker(\xi)\diamond i_A) &= c^{T_GA,A+G}\circ(1_{T_GA}\diamond\kappa_{A,G})\circ(\ker(\xi)\diamond 1_{T_GA})\circ(1_{\mathrm{Ker}(\xi)}\diamond\eta_{A,G})\\
&= c^{T_GA,T_GA}\circ(\ker(\xi)\diamond 1_{T_GA})\circ(1_{\mathrm{Ker}(\xi)}\diamond\eta_{A,G}) \quad \text{by naturality of } c\\
&= \ker(\xi)\circ c^{\mathrm{Ker}(\xi),T_GA}\circ(1_{\mathrm{Ker}(\xi)}\diamond\eta_{A,G}),
\end{aligned}$$

whence the assertion.

Next we check that $\xi\circ c^{T_GA,A+G}\circ(\ker(\xi)\diamond i_G)=0$. Indeed,

$$\begin{aligned}
\xi\circ c^{T_GA,A+G}\circ(\ker(\xi)\diamond i_G) &= \xi\circ c^{T_GA,A+G}\restriction_G\circ(\ker(\xi)\diamond 1_G)\\
&= \psi\circ(\xi\diamond 1_G)\circ(\ker(\xi)\diamond 1_G) \quad \text{by hypothesis}\\
&= \psi\circ((\xi\circ\ker(\xi))\diamond 1_G)\\
&= \psi\circ(0\diamond 1_G)\\
&= 0 \quad \text{by Proposition 2.11.(2)}
\end{aligned}$$

Finally, we show that $\xi\circ c^{T_GA,A+G}\circ(\ker(\xi)\diamond 1_{A+G})\circ\iota'_{T_GA,A,G;2}=0$. Consider the following commutative diagram.

$$\begin{tikzcd}
T_GA \diamond A \diamond G \arrow[r, "\iota'_{T_GA,A,G;2}"] \arrow[rr, bend left, "\iota_{T_GA,A,G}"] \arrow[d, "1_{T_GA}\diamond\eta_{A,G}\diamond 1_G"] & T_GA \diamond (A+G) \arrow[r, "\iota_{T_GA,A+G}"] & T_GA + A + G \arrow[dl, "1_{T_GA}+\eta_{A,G}+1_G"'] \arrow[d, "\left\langle\begin{smallmatrix}\kappa_{A,G}\\ i_A\\ i_G\end{smallmatrix}\right\rangle"] \\
T_GA \diamond T_GA \diamond G \arrow[r, "\iota_{T_GA,T_GA,G}"] \arrow[d, "S_{2,1}^{T_GA,G}"] & T_GA + T_GA + G \arrow[r, "\left\langle\begin{smallmatrix}\kappa_{A,G}\\ \kappa_{A,G}\\ i_G\end{smallmatrix}\right\rangle"] \arrow[d, "\nabla^2_{T_GA}+1_G"] & A+G \arrow[d, equal] \\
T_GA \diamond G \arrow[r, "\iota_{T_GA,G}"] \arrow[rr, bend right, "\kappa_{A,G}\circ c^{T_GA,A+G}\restriction_G"'] & T_GA + G \arrow[r, "\left\langle\begin{smallmatrix}\kappa_{A,G}\\ i_G\end{smallmatrix}\right\rangle"] & A+G
\end{tikzcd}$$

We deduce that

$$\begin{aligned}
\kappa_{A,G} \circ c^{T_GA,A+G} \circ \iota'_{T_GA,A,G;2} &= \left\langle\begin{smallmatrix}\kappa_{A,G}\\ 1_{A+G}\end{smallmatrix}\right\rangle \circ \iota_{T_GA,A+G} \circ \iota'_{T_GA,A,G;2} \\
&= \kappa_{A,G} \circ c^{T_GA,A+G}\restriction_G \circ S_{2,1}^{T_GA,G} \circ (1_{T_GA} \diamond \eta_{A,G} \diamond 1_G)
\end{aligned}$$

and hence

$$c^{T_GA,A+G} \circ (\ker\xi \diamond 1_{A+G}) \circ \iota'_{\mathrm{Ker}\xi,A,G;2} = c^{T_GA,A+G}\restriction_G \circ S_{2,1}^{T_GA,G} \circ (\ker\xi \diamond \eta_{A,G} \diamond 1_G)$$

by naturality of ι'. It follows that

$$\begin{aligned}
\xi \circ c^{T_GA,A+G} \circ (\ker\xi \diamond 1_{A+G}) \circ \iota'_{\mathrm{Ker}\xi,A,G;2} &= \xi \circ c^{T_GA,A+G}\restriction_G \circ S_{2,1}^{T_GA,G} \circ (\ker\xi \diamond \eta_{A,G} \diamond 1_G) \\
&= \xi \circ j_{A,G} \circ (\xi \diamond 1_G) \circ S_{2,1}^{T_GA,G} \circ (\ker\xi \diamond \eta_{A,G} \diamond 1_G) \\
&\quad \text{by hypothesis} \\
&= \xi \circ j_{A,G} \circ S_{2,1}^{A,A} \circ (\xi \diamond \xi \diamond 1_G) \circ (\ker\xi \diamond \eta_{A,G} \diamond 1_G) \\
&\quad \text{by naturality of } S_{2,1} \\
&= \xi \circ j_{A,G} \circ S_{2,1}^{A,A} \circ ((\xi \circ \ker\xi) \diamond (\xi \circ \eta_{A,G}) \diamond 1_G) \\
&= \xi \circ j_{A,G} \circ S_{2,1}^{A,A} \circ (0 \diamond 1_A \diamond 1_G) \\
&= \xi \circ j_{A,G} \circ S_{2,1}^{A,A} \circ 0 \quad \text{by Proposition 2.11.(2).} \\
&= 0,
\end{aligned}$$

as desired. ∎

We finally give a characterization of the morphisms $\psi\colon A \diamond G \to A$ which are action cores, i.e. which extend to actions, hence to strict ones if the category is semi-abelian. Instead of the one unit and one associativity diagram which constitute the definition of a $\mathbb{T}_G$-algebra, it consists of two "associativity-type" diagrams and one kind of coherence condition, and thus is more complicated than the former; on the other hand, our diagrams only involve co-smash products of A and G and the morphism ψ, not its extension ξ. Note that the second diagram (5) expresses the fact that ψ is an algebra structure over the non-unital monad $-\diamond G$, cf. Remark 2.6. While the first two diagrams involve only (nested) binary co-smash products, the third one makes use of a ternary co-smash product. It turns out, however, that in the category of groups the third condition may be skipped because in this category the two first conditions happen to be sufficient for a morphism ψ to be an action core, see Example 5.10.

5.9. Theorem. *Let $\mathbb{C}$ be finitely cocomplete homological, and let $\psi\colon A\diamond G\to A$ be a morphism in it. Then ψ can be extended to a $\mathbb{T}_G$-algebra if and only if the following three diagrams commute:*

$$\begin{tikzcd} (A\diamond G)\diamond A \arrow[r, "C^A_{A,G}"] \arrow[d, "\psi\diamond 1_A"'] & A\diamond G \arrow[d, "\psi"] \\ A\diamond A \arrow[r, "c^A_2"'] & A \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} (A\diamond G)\diamond G \arrow[r, "C^G_{A,G}"] \arrow[d, "\psi\diamond 1_G"'] & A\diamond G \arrow[d, "\psi"] \\ A\diamond G \arrow[r, "\psi"'] & A \end{tikzcd} \tag{5}$$

$$\begin{tikzcd} (A\diamond G)\diamond A\diamond G \arrow[rr, "C^{A,G}_{A,G}"] \arrow[d, "\psi\diamond 1_A\diamond 1_G"'] & & A\diamond G \arrow[d, "\psi"] \\ A\diamond A\diamond G \arrow[r, "S^{A,G}_{2,1}"] & A\diamond G \arrow[r, "\psi"] & A \end{tikzcd} \tag{6}$$

Proof. Since the first diagram is (2) in Proposition 5.1 we know that it commutes iff ψ has an extension ξ which satisfies the unit axiom for an algebra. Supposing that this holds, it remains to show that the associativity axiom then is equivalent to the commutativity of the two latter diagrams. But we know that under these conditions, the associativity axiom is equivalent to the commutativity of diagram (3) in Proposition 5.7. We apply Proposition 2.15 to $Z=G$, $X=T_GA$, $K=A\diamond G$, $Y=A$, $k=j_{A,G}$, $f=r_A\circ\kappa_{A,G}$, $s=\eta_{A,G}$. Then the family $(\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle\circ\iota'_{A\diamond G,A,G;1}, j_{A,G}\diamond 1_G, \eta_{A,G}\diamond 1_G)$ of morphisms with codomain $T_GA\diamond G$ is jointly epimorphic, so this diagram (3) commutes iff its compositions with these three morphisms commute.

First we compose diagram (3) of Proposition 5.7 with $\eta_{A,G} \diamond 1_G$:

$$\begin{tikzcd}
A \diamond G \arrow[d, "\eta_{A,G} \diamond 1_G"] \arrow[dr, dotted, "j_{A,G}"] \arrow[dd, bend right=60, equal, "(\xi \circ \eta_{A,G}) \diamond 1_G = 1_{A \diamond G}"'] & \\
T_G A \diamond G \arrow[r, "c^{T_G A, A+G} \restriction_G"] \arrow[d, "\xi \diamond 1_G"] & T_G A \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "\psi"'] & A
\end{tikzcd}$$

If one shows that the upper triangle commutes, i.e. $c^{T_G A, A+G} \restriction_G \circ (\eta_{A,G} \diamond 1_G) = j_{A,G}$ then one may conclude that the whole outer diagram commutes, so that the resulting condition is void here. In fact, since $\kappa_{A,G}$ is a monomorphism, it suffices to show that $\kappa_{A,G} \circ c^{T_G A, A+G} \restriction_G \circ (\eta_{A,G} \diamond 1_G) = \kappa_{A,G} \circ j_{A,G} = \iota_{A,G}$. But by Proposition 3.13, one has $\kappa_{A,G} \circ c^{T_G A, A+G} \restriction_G = c_2^{A+G} \circ (\kappa_{A,G} \diamond i_G)$, so one has:

$$\begin{aligned}
\kappa_{A,G} \circ c^{T_G A, A+G} \restriction_G \circ (\eta_{A,G} \diamond 1_G) &= c_2^{A+G} \circ (\kappa_{A,G} \diamond i_G) \circ (\eta_{A,G} \diamond 1_G) \\
&= c_2^{A+G} \circ ((\kappa_{A,G} \circ \eta_{A,G}) \diamond i_G) \\
&= c_2^{A+G} \circ (i_A \diamond i_G) \\
&= \iota_{A,G}
\end{aligned}$$

because the following diagram commutes:

$$\begin{tikzcd}
A \diamond G \arrow[r, "i_A \diamond i_G"] \arrow[d, "\iota_{A,G}"'] & (A+G) \diamond (A+G) \arrow[d, "\iota_{A+G,A+G}"] \\
A + G \arrow[r, "i_A + i_G"] \arrow[dr, equal] & (A+G) + (A+G) \arrow[d, "\nabla^2_{A+G}"] \\
& A + G
\end{tikzcd}$$

Secondly, we compose diagram (3) of Proposition 5.7 with $j_{A,G} \diamond 1_G$:

$$\begin{tikzcd}
(A \diamond G) \diamond G \arrow[r, "c^{A \diamond G, A+G} \restriction_G"] \arrow[d, "j_{A,G} \diamond 1_G"] \arrow[dd, bend right=60, "(\xi \circ j_{A,G}) \diamond 1_G = \psi \diamond 1_G"'] & A \diamond G \arrow[d, "j_{A,G}"] \arrow[dd, bend left=60, "\psi"] \\
T_G A \diamond G \arrow[r, "c^{T_G A, A+G} \restriction_G"] \arrow[d, "\xi \diamond 1_G"] & T_G A \arrow[d, "\xi"] \\
A \diamond G \arrow[r, "\psi"'] & A
\end{tikzcd}$$

All parts of this diagram except the bottom square are known to commute, so diagram (3) composed with $j_{A,G} \diamond 1_G$ commutes iff diagram (5) commutes.

Finally, we compose diagram (3) with $(\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle \diamond 1_G)\circ \iota'_{A\diamond G,A,G;1}$. To compute the composition of the latter morphism with $\psi\circ c^{T_GA,A+G}\restriction_G$ consider the following diagram:

$$\begin{tikzcd}
(A\diamond G)\diamond A\diamond G \arrow[rrr, "C^{A,G}_{A,G}"] \arrow[dr, "\iota_{(A\diamond G),A,G}"] \arrow[dd, "\iota'_{A\diamond G,A,G;1}"'] & & & A\diamond G \arrow[ddl, "\iota_{A,G}"'] \arrow[ddddl, "j_{A,G}"] \arrow[dddd, "\psi"] \\
& (A\diamond G)+A+G \arrow[dr, "\left\langle\begin{smallmatrix} \iota_{A,G}\\ i_A\\ i_G\end{smallmatrix}\right\rangle"] \arrow[dd, "\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle+1_G"'] & & \\
((A\diamond G)+A)\diamond G \arrow[ur, "\iota_{A\diamond G+A,G}"] \arrow[dd, "\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle\diamond 1_G"'] & & A+G & \\
& T_GA+G \arrow[ur, "\left\langle\begin{smallmatrix} \kappa_{A,G}\\ i_G\end{smallmatrix}\right\rangle"] & & \\
T_GA\diamond G \arrow[ur, "\iota_{T_GA,G}"] \arrow[rr, "c^{T_GA,A+G}\restriction_G"] & & T_GA \arrow[uu, tail, "\kappa_{A,G}"] \arrow[r, "\xi"] & A
\end{tikzcd}$$

We deduce that $c^{T_GA,A+G}\restriction_G \circ(\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle \diamond 1_G)\circ \iota'_{A\diamond G,A,G;1} = j_{A,G}\circ C^{A,G}_{A,G}$ since $j_{A,G}$ is monic. Hence the composition of the former morphism with ξ equals $\psi\circ C^{A,G}_{A,G}$.

On the other hand, the composition of $(\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle \diamond 1_G)\circ \iota'_{A\diamond G,A,G;1}$ with $\psi\circ(\xi\diamond 1_G)$ is computed by means of the following commutative diagram.

$$\begin{tikzcd}
& & (A\diamond G)\diamond A\diamond G \arrow[r, "\psi\diamond 1_A\diamond 1_G"] \arrow[dl, "\iota'_{A\diamond G,A,G;1}"'] \arrow[d, "\iota_{A\diamond G,A,G}"] & A\diamond A\diamond G \arrow[d, tail, "\iota_{A,A,G}"'] \arrow[ddr, "S^{A,G}_{2,1}"] & \\
& ((A\diamond G)+A)\diamond G \arrow[r, "\iota_{A\diamond G+A,G}"] \arrow[dl, "\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle\diamond 1_G"'] \arrow[d, "\left\langle\begin{smallmatrix} \psi\\ 1_A\end{smallmatrix}\right\rangle\diamond 1_G"] & (A\diamond G)+A+G \arrow[r, "\psi+1_A+1_G"] \arrow[d, "\left\langle\begin{smallmatrix} \psi\\ 1_A\end{smallmatrix}\right\rangle+1_G"] & A+A+G \arrow[d, "\nabla^2_A+1_G"] & \\
T_GA\diamond G \arrow[r, "\xi\diamond 1_G"] & A\diamond G \arrow[r, tail, "\iota_{A,G}"] & A+G \arrow[r, equal] & A+G & A\diamond G \arrow[l, tail, "\iota_{A,G}"']
\end{tikzcd}$$

Thus $\psi\circ(\xi\diamond 1_G)\circ(\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle \diamond 1_G)\circ \iota'_{A\diamond G,A,G;1} = \psi\circ S^{A,G}_{2,1}\circ(\psi\diamond 1_A\diamond 1_G)$. Hence diagram (3) composed with $(\left\langle\begin{smallmatrix} j_{A,G}\\ \eta_{A,G}\end{smallmatrix}\right\rangle \diamond 1_G)\circ \iota'_{A\diamond G,A,G;1}$ commutes iff diagram (6) commutes, which achieves the proof. ■

5.10. Example. In the category of groups, consider a morphism $\psi\colon A\diamond G\to A$ and the corresponding[2] $\phi\colon G{\times}A\to G$. We know that the first axiom in Theorem 5.9 expresses that

[2]Note that if one defines ϕ from ψ, one can put $\phi(g,a)=\psi([g,a])\cdot a$, even if $[g,a]$ is the unit, i.e. g is the unit of G or a is the unit of A: this insures that $\phi(e_G,a)=a$ and $\phi(g,e_A)=e_A$, and of course $\psi[g,a]=\phi(g,a)\cdot a^{-1}$. So there is no need for a special discussion for the case when $gg_1=e_G$ in what follows.

the $\phi(g,-)$ are group endomorphisms of A. We now examine the second diagram. Considering that $X\diamond Y$ is the subgroup of $X+Y$ generated by the $[y,x]$'s (with x, y different from the units), a generator of $(A\diamond G)\diamond G$ has the form $[\![g,\prod_{i=1}^n [g_i,a_i]^{z_i}]\!]$, where the outer commutator $[\![\ldots]\!]$ is considered in $(A+G)+G$, while the inner one $[\ldots]$ is in $A+G$. Consider the case $n=1$ with $z_1=1$. For a generator $[\![g,[g_1,a_1]]\!]$ one has $c^{A\diamond G,A+G}\restriction_G([\![g,[g_1,a_1]]\!]) = [g,[g_1,a_1]] = [gg_1,a_1]\cdot[a_1,g]\cdot[a_1,g_1] = [gg_1,a_1]\cdot[g,a_1]^{-1}\cdot[g_1,a_1]^{-1}$, the second equality arising from the formula $[x,[y,z]]=[xy,z]\cdot[z,x]\cdot[z,y]$. So one gets $\psi(c^{A\diamond G,A+G}\restriction_G([\![g,[g_1,a_1]]\!]) = \psi([gg_1,a_1]\cdot[g,a_1]^{-1}\cdot[g_1,a_1]^{-1}) = \psi([gg_1,a_1])\cdot(\psi([g,a_1])^{-1}\cdot\psi([g_1,a_1])^{-1} = \phi(gg_1,a_1)\cdot a_1^{-1}\cdot(\phi(g,a_1)a_1^{-1})^{-1}\cdot(\phi(g_1,a_1)\cdot a_1^{-1})^{-1} = \phi(gg_1,a_1)\cdot(\phi(g,a_1))^{-1}\cdot a_1\cdot(\phi(g_1,a_1))^{-1}$ on the one hand. On the other hand, one gets $\psi((\psi|1_G)([\![g,[g_1,a_1]]\!])) = \psi([g,\psi([g_1,a_1]))$ $=\psi([g,\phi(g,a_1)\cdot a_1^{-1}]) = \phi(g,\phi(g,a_1)\cdot a_1^{-1})\cdot(\phi(g,a_1)\cdot a_1^{-1})^{-1} = \phi(g,\phi(g,a_1)\cdot a_1^{-1})\cdot a_1\cdot(\phi(g,a_1))^{-1}$. So the equation $\psi(c^{A\diamond G,A+G}\restriction_G([\![g,[g_1,a_1]]\!]) = \psi((\psi|1_G)([\![g,[g_1,a_1]]\!]))$ is verified if and only if so is the equation $\phi(gg_1,a_1).(\phi(g,a_1))^{-1} = \phi(g,\phi(g_1,a_1)\cdot a_1^{-1})$. Now if one assumes moreover that the first diagram commutes, i.e. that the $\phi(g,-)$'s are endomorphisms of A, then this amounts to the relation $\phi(gg_1,a_1)\cdot(\phi(g,a_1))^{-1} = \phi(g,\phi(g_1,a_1))\cdot(\phi(g,a_1))^{-1}$, i.e. $\phi(gg_1,a_1)=\phi(g,\phi(g_1,a_1))$. So, assuming the commutativity of the first diagram in Theorem 5.9, the commutativity of the second one implies that ϕ is a group action in the usual sense; and of course, conversely, if ϕ is an action in the usual sense, then the corresponding ψ is a strict action, hence the three diagrams of Theorem 5.9 commute.

We conclude by providing a "minimalistic" description of *Beck modules* in a semi-abelian category $\mathbb{C}$ which might be useful in the study of cohomology with non-trivial coefficients. Recall that a Beck module over an object G of $\mathbb{C}$ is an abelian group object in the category of points $\mathrm{Pt}_G(\mathbb{C})$. It is well-known that if a given point admits an abelian group structure the latter is unique. In section 6 of [Hartl & Van der Linden 2013] a characterization of strict action cores corresponding to Beck modules $A\rtimes_\psi G \underset{p_\psi}{\overset{s_\psi}{\rightleftarrows}} G$ is given. We combine it with Theorem 5.9 to obtain the following result.

5.11. Corollary. *Suppose that the category $\mathbb{C}$ is semi-abelian. Let A and G be objects in $\mathbb{C}$ and $\psi\colon A\diamond G\to A$ be a morphism. Then ψ is a strict action core such that the point $A\rtimes_\psi G \underset{p_\psi}{\overset{s_\psi}{\rightleftarrows}} G$ is a Beck module iff A is abelian, the diagrams (4) and (5) commute and the composite morphism $\psi_{2,1}=\psi\circ S_{2,1}^{A,G}$ vanishes.*

Proof. By Theorem 5.9 we know that ψ is a strict action core iff the diagrams (4), (5) and (6) commute. Moreover, by [Hartl & Van der Linden 2013, Theorem 6.2] the point $A\rtimes_\psi G \underset{p_\psi}{\overset{s_\psi}{\rightleftarrows}} G$ is a Beck module iff A is abelian and $\psi_{2,1}=0$. So it suffices to show that under the latter condition diagram (6) automatically commutes. To see this consider the following commutative diagram of plain arrows where τ and τ' denote the flip of the first two summands resp. factors:

$$
\begin{tikzcd}
(A\diamond G)\diamond A\diamond G \arrow[r, tail, "\iota_{(A\diamond G),A,G}"] \arrow[dd, dotted, "\epsilon"] & (A\diamond G)+A+G \arrow[r, "r_{A\diamond G,A,G}"] \arrow[d, "\tau+1_G"] & (A+G)\times((A\diamond G)+G)\times((A\diamond G)+A) \arrow[d, "\tau'\times 1_{A\diamond G+A}"] \\
& A+(A\diamond G)+G \arrow[d, "1_A+\left\langle {\iota_{A,G} \atop i_G} \right\rangle"] & ((A\diamond G)+G)\times(A+G)\times((A\diamond G)+A) \arrow[d, "\left\langle {\iota_{A,G} \atop i_G} \right\rangle\times 1_{A+G}\times\left\langle {0 \atop i_1} \right\rangle"] \\
A\diamond A\diamond G \arrow[r, tail, "\iota_{A,A,G}"] & A+A+G \arrow[r, "r_{A,A,G}"] & (A+G)\times(A+G)\times(A+A)
\end{tikzcd}
$$

Exactness of the rows then implies the existence of the indicated morphism ϵ. Now consider the following commutative diagram.

$$
\begin{tikzcd}
(A\diamond G)\diamond A\diamond G \arrow[rr, "\epsilon"] \arrow[d, "\iota_{(A\diamond G),A,G}"] \arrow[dd, bend right, "C^{A,G}_{A,G}"'] & & A\diamond A\diamond G \arrow[d, "\iota_{A,A,G}"] \arrow[dd, bend left, "S^{A,G}_{2,1}"] \\
(A\diamond G)+A+G \arrow[r, "\tau+1_G"] \arrow[dr, "\left\langle \begin{smallmatrix} \iota_{A,G} \\ i_A \\ i_G \end{smallmatrix} \right\rangle"] & A+(A\diamond G)+G \arrow[r, "1_A+\left\langle {\iota_{A,G} \atop i_G} \right\rangle"] & A+A+G \arrow[dl, "\nabla^2_A+1_G=\left\langle {1_A \atop 1_{A+G}} \right\rangle"] \\
A\diamond G \arrow[r, tail, "\iota_{A,G}"] & A+G & A\diamond G \arrow[l, tail, "\iota_{A,G}"']
\end{tikzcd}
$$

It shows that $\psi\circ C^{A,G}_{A,G}=\psi_{2,1}\circ\epsilon$, so if $\psi_{2,1}=0$ then diagram (6) plainly commutes since both compositions are trivial. ■

Acknowledgement

The authors are indebted to the referee for his many detailed suggestions which lead to a considerable improvement of the original version of the paper. In particular, we owe to him the much shorter proof of Proposition 5.3.

References

[Baues, Hartl & Pirashvili 1997] H.J. Baues, M. Hartl, T. Pirashvili, Quadratic categories and square rings. *J. Pure Appl. Algebra*, 122:1–40, 1997.

[Baues & Pirashvili 1999] H.J. Baues, T. Pirashvili, Quadratic endofunctors of the category of groups. *Adv. Math.* 141(1):167–206, 1999.

[Borceux & Bourn 2004] F. Borceux, D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566 of *Math. Appl.* Kluwer Acad. Publ., 2004.

[Borceux, Janelidze & Kelly 2005] F. Borceux, G. Janelidze, M. Kelly, Internal object actions. *Comment. Math. Univ. Carolinae*, 46(2):235–255, 2005.

[Bourn 1991] D. Bourn, Normalization equivalence, kernel equivalence and affine categories. In A. Carboni, M. C. Pedicchio and G. Rosolini, editors, *Category Theory, Proceedings Como 1990*, volume 1448 of *Lecture Notes in Mathematics*, pages 43-62, Springer, 1991.

[Bourn 2001] D. Bourn, 3×3 Lemma and protomodularity. *J. Algebra*, 236:778–795, 2001.

[Bourn & Janelidze 1998] D. Bourn, G. Janelidze, Protomodularity, descent and semidirect products. *Theory Appl. Categ.*, 4(2):37–46, 1998.

[Carboni & Janelidze 2003] A. Carboni, G. Janelidze, Smash product of pointed objects in lextensive categories. *J. Pure Appl. Algebra*183:27–43, 2003.

[Eilenberg & Mac Lane 1954] S. Eilenberg, S. Mac Lane, On the groups $H(\pi, n)$ II. *Ann. of Math.*, 60:49–139, 1954.

[Everaert 2012] T. Everaert, Effective descent morphisms of regular epimorphisms. *J. Pure Appl. Algebra*, 216 (8-9)1896–1904, 2012.

[Everaert, Gran & Van der Linden 2008] T. Everaert, M. Gran and T. Van der Linden, Higher Hopf formulae for homology via Galois theory. *Adv. Math.*, 217(5):2231–2267, 2008

[Everaert & Van der Linden 2012] T. Everaert, T. Van der Linden, Relative commutator theory in semi-abelian categories. *J. Pure Appl. Algebra*, 216(8–9):1791–1806, 2012.

[Hartl - in preparation] M. Hartl, Algebraic functor calculus and commutator theory, in preparation.

[Hartl, Pirashvili & Vespa 2012] M. Hartl, T. Pirashvili, C. Vespa, Polynomial functors from algebras over a set-operad and non-linear Mackey functors, submitted, `arxiv:1209.1607`.

[Hartl & Van der Linden 2013] M. Hartl, T. Van der Linden, The ternary commutator obstruction for internal crossed modules. *Adv. Math.*, 232(1):571–607, 2013.

[Hartl & Vespa 2011] M. Hartl, C. Vespa, Quadratic functors on pointed categories. *Adv. Math.* 226(5):3927–4010, (2011).

[Huq 1968] S. A. Huq, Commutator, nilpotency and solvability in categories. *Quart. J. Math. Oxford*, 19(2):363–389, 1968.

[Janelidze 2003] G. Janelidze, Internal crossed modules. *Georgian Math. J.*,10(1):99–114, 2003.

[Janelidze, Márki & Ursini 2007] G. Janelidze, L. Márki, A. Ursini, Ideals and clots in universal algebra and in semi-abelian categories. *J. Algebra*, 307:191–208, 2007.

[Janelidze, Márki & Ursini 2009] G. Janelidze, L. Márki, A. Ursini, Ideals and clots in pointed regular categories. *Appl. Categor. Struct.*, 17(4):345–350, 2009.

[Magnus, Karrass & Solitar 1966] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Wiley Interscience, 1966.

[Mantovani & Metere 2010] S. Mantovani, G. Metere, Normalities and Commutators. *J. Algebra*, 324(9):2568–2588, 2010.

[Martins-Ferreira & Sobral 2012] N. Martins-Ferreira, M. Sobral, On categories with semi-direct products. *J. Pure Appl. Algebra* 216 1968–1975, 2012.

[Martins-Ferreira & Van der Linden 2012] N. Martins-Ferreira and T. Van der Linden, Further remarks on the "Smith is Huq" condition. Submitted, *Pré-Publicações DMUC*, 12(34):1–18, 2012.

[Rodelo & Van der Linden 2012] D. Rodelo and T. Van der Linden, Higher central extensions via commutators. *Theory Appl. Categ.* 27 (9):189–209, 2012.

Université de Lille–Nord de France, Laboratoire de Mathématiques et de leurs Applications de Valenciennes and FR CNRS 2956
UVHC, Le Mont Houy, F-59313 Valenciennes Cedex 9, France.
Email: `manfred.hartl@univ-valenciennes.fr`
`bruno.loiseau@lille.iufm.fr`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/27/15/27-15.{dvi,ps,pdf}`