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# A DOUBLE CATEGORICAL MODEL OF WEAK 2-CATEGORIES

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ABSTRACT. We introduce the notion of weakly globular double categories, a particular class of strict double categories, as a way to model weak 2-categories. We show that this model is suitably equivalent to bicategories and give an explicit description of the functors involved in this biequivalence. As an application we show that groupoidal weakly globular double categories model homotopy 2-types.

## 1. Introduction

Category theory has seen the development of several types of higher dimensional structures, which find applications to diverse areas such as homotopy theory, mathematical physics, algebraic geometry, and computer science. The structures for higher dimensional categories include three main classes; namely, strict $n$-categories, $n$-fold categories and weak $n$-categories.

The notion of *strict $n$-categories* is the simplest of the higher structures and is obtained by iterated enrichment: a strict $n$-category is a category enriched in strict $(n-1)$-categories with respect to the cartesian monoidal structure. Although widely studied and well-understood, strict $n$-categories are often too simple to capture the complexity needed for applications. For instance, strict $n$-groupoids are not enough to model $n$-types of topological spaces.

A much wider type of higher dimensional categorical structure is formed by the *$n$-fold categories*, obtained by iterated internalization: an $n$-fold category is a category internal to $(n-1)$-fold categories. The notion of a double category, and the general idea of considering internal categories inside another category was introduced by Charles Ehresmann in 1963 [14]. The foundational properties of $n$-fold categories (at that time called *multiple categories*) were studied in a series of papers by Andrée and Charles Ehresmann [1, 13, 12, 11]. A homotopical treatment of $n$-fold categories, in the context of Quillen model structures, was given in [15] and [16]. The case $n = 2$, where they are called double categories, has been studied extensively. In their full generality, double categories give rise to surprisingly complex phenomena, such as the non-composability of certain tiling diagrams of double cells [9] and [8].

Another very important class of higher dimensional structures is that of *weak $n$-*

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*categories*. This comprises several different models (see for instance [21] and [22]); in the case $n = 2$, these are all suitably equivalent to the classical notion of bicategory [2].

The fundamental question we are addressing in this work concerns the relations between these three classes of higher dimensional structures. Some of these relations are well known: it is easily seen that there are full and faithful embeddings of strict $n$-categories into $n$-fold categories; likewise, most definitions of weak $n$-categories contain strict $n$-categories as a special case. However, the relation between $n$-fold categories and weak $n$-categories is less understood.

A hint in this direction comes from homotopy theory. Blanc and the first author showed in [3] and [4] that a suitable subcategory of $n$-fold groupoids, called *weakly globular $n$-fold groupoids*, model $n$-types of topological spaces. Likewise, in the path-connected case, the first author showed [24] that weakly globular $\mathrm{cat}^{n-1}$-groups model path-connected $n$-types.

These results motivate us to search for a suitable subcategory of $n$-fold categories which models weak $n$-categories.

In this paper we present a solution to this problem for the case $n = 2$. This low dimensional case presents a special interest in this context because it links with the vast existing literature on double categories, see for instance, [14], [6], [17], [30] and [15]. The extension of this work to general $n$ is the subject of a subsequent paper [26].

The elements of our double categorical model of weak 2-categories are called *weakly globular double categories*. This model is based on the idea of weak globularity. In a strict 2-category, the 0-cells have a discrete structure, that is, they form a set. This is called the globularity condition, as it determines the globular shape of the 2-cells. In a weakly globular double category we relax the globularity condition into weak globularity: the 0-cells have a categorical structure of their own which is not discrete but it is equivalent to a discrete one. The double categorical context gives the possibility to encode this extra structure precisely. We refer the reader to Section 2 for a longer discussion of the intuition behind the notion of weakly globular double category.

Our main result is that weakly globular double categories, with the appropriate classes of morphisms and 2-cells, are suitably equivalent to bicategories: more precisely, there is a biequivalence of 2-categories:

$$\mathbf{Bic} : \mathsf{WGDbl}_{\mathrm{ps},v} \simeq \mathsf{Bicat}_{\mathrm{icon}} : \mathbf{Dbl}. \tag{1}$$

The proof of this result goes via a comparison with another existing model of weak 2-categories due to Tamsamani [33], which was proved by Lack and the first author to be suitably equivalent to bicategories [20]. Thus, we obtain the biequivalence (1) from biequivalences

$$\mathsf{WGDbl}_{\mathrm{ps},v} \simeq (\mathsf{Ta}_2)_{\mathrm{ps}} \simeq \mathsf{Bicat}_{\mathrm{icon}}$$

The comparison functors between weakly globular double categories and the Tamsamani model factors through a subcategory of pseudo-functors $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$. We refer the reader to Section 2 for an overview of our methodology.

The biequivalence (1) gives a new way of rigidifying a bicategory into a strict double structure, which differs from the classical strictification of bicategories into strict 2-categories. An explicit description of both functors involved in this biequivalence reveals some interesting details about how weakly globular double categories model the weak higher structure. Specifically, we see that the weakly globular double category associated to a bicategory is obtained by local rigidifications of the bicategory which are glued together in a suitable sense by vertical isomorphisms, whereas the bicategory associated to a weakly globular double category is obtained as a kind of quotient.

We end this paper by highlighting some features of our model which one would expect from a model of weak 2-categories. Namely, we show that suitably groupoidal weakly globular double categories model homotopy 2-types and we show that for weakly globular double categories, pinwheel-tilings do form composable pasting diagrams. A complete proof establishing that every tiling diagram in a weakly globular double category is composable is forthcoming [27].

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## 2. Overview of the main results

This section provides a summary of the main ideas and results of this paper. This paper presents a new notion of a weak 2-category via a subcategory of strict double categories, which we call weakly globular double categories.

While several different models of weak 2-categories exist in the literature (see for instance [21]), the prototype is the classical notion of bicategory [2]. The main result of this paper (Theorem 7.10) asserts that there is a biequivalence between the 2-category of weakly globular double categories with pseudo-functors and vertical transformations and the 2-category of bicategories with homomorphisms and icons:

$$\mathbf{Bic} : \mathsf{WGDbl}_{\mathrm{ps},v} \simeq \mathsf{Bicat}_{\mathrm{icon}} \ : \mathbf{Dbl}. \tag{2}$$

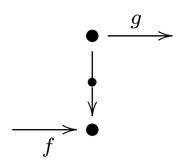
Weakly globular double categories also form an intermediate structure between strict 2-categories and (strict) double categories in the following way:

$$\text{2-}\mathsf{Cat} \hookrightarrow \mathsf{WGDbl} \hookrightarrow \mathsf{DblCat},$$

where the first inclusion considers a 2-category as a double category with only identities as vertical arrows, i.e., the category of objects and vertical morphisms is discrete. This discreteness condition is also called the *globularity condition*, as it determines the globular shape of the 2-cells in a strict 2-category.

The first step in the process to obtain the notion of weakly globular double category (Definition 6.7) is to replace the globularity condition by *weak globularity*: the category of objects and vertical arrows in a weakly globular double category $\mathbb{X}$ is no longer discrete but merely equivalent to a discrete category, and therefore a posetal groupoid (which can also be viewed as a set with an equivalence relation). The set of connected components of this posetal groupoid is the set of objects of the corresponding bicategory $\mathbf{Bic}\mathbb{X}$.

The second step in obtaining the correct definition of a weakly globular double category is to realize that arrows in $\mathbf{Bic}\mathbb{X}$ correspond to horizontal arrows in $\mathbb{X}$. So we need to provide extra structure allowing one for instance to compose in $\mathbf{Bic}\mathbb{X}$ pairs of horizontal arrows in $\mathbb{X}$ whose target and source may not be identical, but which are in the same vertical connected component:



(3)

The general version of this extra condition asserts that for each $n \geq 2$, the equivalence of categories $\gamma : \mathbb{X}_0 \to \mathbb{X}_0^d$ between the posetal groupoid $\mathbb{X}_0$ and its discretization induces equivalences of categories

$$\mathbb{X}_1 \times_{\mathbb{X}_0} \overset{n}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1 \simeq \mathbb{X}_1 \times_{\mathbb{X}_0^d} \overset{n}{\cdots} \times_{\mathbb{X}_0^d} \mathbb{X}_1 \ . \tag{4}$$

This implies in particular that the maps $f$ and $g$ in the configuration (3) can be lifted to a composable pair of horizontal arrows in $\mathbb{X}$ (see Section 8.1 for details).

The methodology to establish the comparison result (2) makes use of another model of weak 2-categories, due to Tamsamani, which we review in Section 4. This is a simplicial model based on the functor category $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ of simplicial objects in $\mathsf{Cat}$ and of the notion of Segal maps, i.e, the maps $\mathbb{X}_n \to \mathbb{X}_1 \times_{\mathbb{X}_0} \overset{n}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1$ that are part of the simplicial structure. The details are described in Section 4.1.

Notice that this setting is natural in our context as both strict 2-categories and double categories can be described in this simplicial language: a double category is the same as an object $\mathbb{X} \in [\Delta^{\mathrm{op}}, \mathsf{Cat}]$ such that the Segal maps are isomorphisms, and if we further require $\mathbb{X}_0$ to be discrete we obtain a strict 2-category.

It was shown in [20] that there is a biequivalence of 2-categories

$$N : (\mathsf{Ta}_2)_{\mathrm{ps}} \simeq \mathsf{Bicat}_{\mathrm{icon}} : G \tag{5}$$

where $(\mathsf{Ta}_2)_{\mathrm{ps}}$ is the 2-category of Tamsamani weak 2-categories with pseudo-functors. In this paper we show (Theorem 7.6) that there is a biequivalence of 2-categories

$$D\colon \mathsf{WGDbl}_{\mathrm{ps},v} \simeq (\mathsf{Ta}_2)_{\mathrm{ps}} : Q \ . \tag{6}$$

Hence the main result (2) follows by composition of (5) and (6); that is, $\mathbf{Bic} = ND$ and $\mathbf{Dbl} = QG$.

The functors $D$ and $Q$ are called *discretization* and *rigidification* respectively. The functor $D$ replaces the posetal groupoid of vertical arrows in the horizontal nerve $N_h\mathbb{X}$ of a weakly globular double category $\mathbb{X}$ by its equivalent discrete category. This produces an object of $[\Delta^{op}, \mathsf{Cat}]$ which is discrete at level zero, so we recover the globularity condition. This however comes at the expense of the strictness of the Segal maps, which from being isomorphisms in $N_h\mathbb{X}$ now become mere equivalences of categories. We therefore obtain an object $D\mathbb{X} \in \mathsf{Ta}_2$, which is suitably equivalent to $\mathbb{X}$ (see Section 7.3 for further details).

The functor $Q$ associates to a Tamsamani object $X \in \mathsf{Ta}_2$, in which composition is only weakly associative and unital, a weakly globular double category $QX$ in which all compositions are strictly associative and unital, but in which we no longer have the globularity condition. Thus, we think of $QX$ as a rigidification of $X$. We note that this is however completely different from the classical strictification of a bicategory into a strict 2-category [23].

Unlike the functor $D$, the construction of the rigidification functor $Q$ is rather elaborate and requires an intermediate passage, via the category $\mathrm{Ps}[\Delta^{op}, \mathsf{Cat}]$ of pseudo-functors from $\Delta^{op}$ to $\mathsf{Cat}$.

We show in Section 5 how to associate an object of $\mathrm{Ps}[\Delta^{op}, \mathsf{Cat}]$ to a Tamsamani weak 2-category in a functorial way, using a standard categorical technique known as transport of structure along an adjunction. For $X \in \mathsf{Ta}_2 \in [\Delta^{op}, \mathsf{Cat}]$ this allows us to replace each category $X_n$ for $n \geq 2$ with the equivalent category $X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$ and equip the resulting object with a pseudo simplicial structure, so we have a functor

$$S : \mathsf{Ta}_2 \to \mathrm{Ps}[\Delta^{op}, \mathsf{Cat}] \;.$$

The pseudo-functors in the image of $S$ are of a particular type. We write $\widetilde{\mathrm{Ps}}[\Delta^{op}, \mathsf{Cat}]$ for the subcategory of pseudo-functors $H\colon \Delta^{op} \to \mathsf{Cat}$, called *Segalic*, such that $H_0$ is a discrete category and $H_n \cong H_1 \times_{H_0} \overset{n}{\cdots} \times_{H_0} H_1$ for each $n \geq 2$. We see that the functor $S$ in fact lands in the subcategory of Segalic pseudo-functors,

$$S : \mathsf{Ta}_2 \to \widetilde{\mathrm{Ps}}[\Delta^{op}, \mathsf{Cat}] \;.$$

On the other hand, pseudo-functors can be strictified to strict functors (see [28]); that is, there is a strictification functor

$$St : \mathrm{Ps}[\Delta^{op}, \mathsf{Cat}] \to [\Delta^{op}, \mathsf{Cat}]$$

together with, for each $X \in \mathrm{Ps}[\Delta^{op}, \mathsf{Cat}]$, a pseudo-natural transformation $St\,X \to X$, which is a levelwise equivalence of categories.

We show (see Sections 6 and 7) that the strictification functor $St$, when restricted to the subcategory $\widetilde{\mathrm{Ps}}[\Delta^{op}, \mathsf{Cat}]$ of Segalic pseudo-functors, lands in the subcategory $N_h(\mathsf{WGDbl})$ of $[\Delta^{op}, \mathsf{Cat}]$ whose objects are horizontal nerves of weakly globular double categories:

$$St : \widetilde{\mathrm{Ps}}[\Delta^{op}, \mathsf{Cat}] \to N_h(\mathsf{WGDbl}) \;.$$

Since, on the other hand, $N_h(\mathsf{WGDbl})$ is isomorphic to $\mathsf{WGDbl}$ itself, we finally obtain by composition the rigidification functor

$$Q : (\mathsf{Ta}_2)_{\mathrm{ps}} \xrightarrow{S} \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \xrightarrow{St} N_h(\mathsf{WGDbl}) \xrightarrow{P} \mathsf{WGDbl}_{\mathrm{ps},v}$$

together with a pseudo-functor $N_hQX \to X$ for every $X \in (\mathsf{Ta}_2)_{\mathrm{ps}}$ which is a levelwise equivalence of categories.

It is instructive to give an explicit description of the functors **Bic** and **Dbl** in the biequivalence (2). The associated double category $\mathbf{Dbl}\mathcal{B}$ of a bicategory $\mathcal{B}$ has an interesting description in terms of marked paths. The horizontal arrows in this double category consist of paths in the original bicategory with two marked objects. One can think of this arrow as corresponding to a particular chosen composite of the path between the two marked objects. This is then a local rigidification of the bicategory in the sense that it only rigidifies or strictifies the composition with other arrows along the path (which may extend beyond the marked objects). The vertical arrow structure allows us to glue all these local rigidifications together into a strict double category. For details on how this works, see Section 8.2.

Section 8.1 contains the description of $\mathbf{Bic}\mathbb{X}$ for a weakly globular double category $\mathbb{X}$, which can be viewed as a fundamental bicategory of $\mathbb{X}$. Its objects correspond to vertical isomorphism classes of the objects in $\mathbb{X}$. However, for the arrows and double cells we do not take equivalences classes and the second condition in the definition of weakly globular double categories, involving the equivalences (4), plays a central role in the definition of horizontal composition of the arrows and 2-cells of $\mathbf{Bic}\mathbb{X}$. This will also be called the *induced Segal maps condition*.

This condition plays a crucial role in Section 9, where we show that the pathological double categorical pasting diagrams known as *pinwheels* can be factored and then composed when working in a weakly globular double category. This is in some sense an expected feature, as each weakly globular double category models a bicategory.

Another expected feature concerns the modelling of 2-types for those weakly globular double categories satisfying suitable invertibility conditions. We establish this in Section 10, where an appropriate subcategory of groupoidal weakly globular double categories is shown to correspond to Tamsamani weak 2-groupoids, and therefore to model 2-types of topological spaces.

## 3. Double categories

Since we will introduce weakly globular double categories as a special kind of double categories, we use this section to review the definition of a double category and some of the basic concepts related to double categories, such as morphisms and transformations. In particular, we consider two types of morphisms between double categories, strict functors and pseudo-functors. We inherit these notions from the category of simplicial sets via the adjoint to the nerve functor.

Strict functors correspond to strict transformations of simplicial sets and are what a category theorist would expect them to be, but the notion of pseudo-functor, which is inherited from pseudo-natural transformations between the nerve functors, turns out to be a bit weaker than what one usually assumes for double categories (as in [10], for instance). We introduce horizontal and vertical transformations and note that for pseudo-functors it only makes sense to consider vertical transformations.

### 3.1. Double categories.

A double category $\mathbb{X}$ is an internal category in $\mathsf{Cat}$, i.e., a diagram of the form

$$\mathbb{X} = \left( \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \xrightarrow{\;m\;} \mathbb{X}_1 \underset{d_1}{\overset{d_0}{\underset{\xleftarrow{\;s\;}}{\rightrightarrows}}} \mathbb{X}_0 \right). \tag{7}$$

The elements of $\mathbb{X}_{00}$, i.e., the objects of the category $\mathbb{X}_0$, are the *objects* of the double category. The elements of $\mathbb{X}_{01}$, i.e., the arrows of the category $\mathbb{X}_0$, are the *vertical arrows* of the double category. Their domains, codomains, identities, and composition in the double category $\mathbb{X}$ are as in the category $\mathbb{X}_0$. For objects $A, B \in \mathbb{X}_{00}$, we write

$$\mathbb{X}_v(A, B) = \mathbb{X}_0(A, B)$$

for the set of vertical arrows from $A$ to $B$. We denote a vertical identity arrow by $1_A$ and write $\cdot$ for vertical composition. The elements of $\mathbb{X}_{10}$, i.e., the objects of the category $\mathbb{X}_1$, are the *horizontal arrows* of the double category, and their domain, codomain, identities and composition are determined by $d_0$, $d_1$, $s$, and $m$ in (7). For objects $A, B \in \mathbb{X}_{00}$, we write

$$\mathbb{X}_h(A, B) = \{f \in \mathbb{X}_{10} |\, d_0(f) = A \text{ and } d_1(f) = B\}.$$

We will also use $\circ$ for the composition of a horizontal arrows, $g \circ f = m(g, f)$. In order to make a notational distinction between horizontal and vertical arrows, we denote the vertical arrows by $\xrightarrow{\bullet}$ and the horizontal arrows by $\longrightarrow$. The elements of $\mathbb{X}_{11}$, i.e., the arrows of the category $\mathbb{X}_1$, are the *double cells* of the double category. An element $\alpha \in \mathbb{X}_{11}$ has a vertical domain and codomain in $\mathbb{X}_{10}$ (since $\mathbb{X}_1$ is a category), which are horizontal arrows, say $h$ and $k$ respectively. The cell $\alpha$ also has a horizontal domain, $d_0(\alpha)$, and a horizontal codomain, $d_1(\alpha)$. The arrows $d_0(\alpha)$ and $d_1(\alpha)$ are vertical arrows. Furthermore, the horizontal and vertical domains and codomains of these arrows match up in such a way that all this data fits together in a diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\;h\;} & \cdot \\ {\scriptstyle d_0(\alpha)}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle d_1(\alpha)} \\ \cdot & \xrightarrow[\;k\;]{} & \cdot \end{array}.$$

These double cells can be composed vertically by composition in $\mathbb{X}_1$ (again written as $\cdot$) and horizontally by using $m$, and written as $\alpha_1 \circ \alpha_2 = m(\alpha_1, \alpha_2)$. The identities in $\mathbb{X}_1$

give us vertical identity cells, denoted by

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\big\downarrow & 1_f & \big\downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B \end{array} \quad \text{or} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \Vert & 1_f & \Vert \\ A & \xrightarrow[f]{} & B \end{array}.$$

The image of $s$ gives us horizontal identity cells, denoted by

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle v}\big\downarrow & \mathrm{id}_v & \big\downarrow{\scriptstyle v} \\ B & \xrightarrow[\mathrm{Id}_B]{} & B \end{array} \quad \text{or} \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle v}\big\downarrow & \mathrm{id}_v & \big\downarrow{\scriptstyle v} \\ B & = & B \end{array},$$

where $\mathrm{Id}_A = s(A)$ and $\mathrm{id}_v = s(v)$. Composition of squares satisfies horizontal and vertical associativity laws and the middle four interchange law. Further, $\mathrm{id}_{1_A} = 1_{\mathrm{Id}_A}$ and we will denote this cell by $\iota_A$,

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle 1_A}\big\downarrow & \iota_A & \big\downarrow{\scriptstyle 1_A} \\ A & \xrightarrow[\mathrm{Id}_A]{} & A \end{array}.$$

For any double category $\mathbb{X}$, the *horizontal nerve* $N_h\mathbb{X}$ is defined to be the functor $N_h\mathbb{X}\colon \Delta^{\mathrm{op}} \to \mathsf{Cat}$ such that $(N_h\mathbb{X})_0 = \mathbb{X}_0$, $(N_h\mathbb{X})_1 = \mathbb{X}_1$ and $(N_h\mathbb{X})_k = \mathbb{X}_1 \times_{\mathbb{X}_0} \overset{k}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1$ for $k \geq 2$. So $N_h\mathbb{X}$ is given by the diagram

$$\cdots \quad \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \;\substack{\xrightarrow{\pi_1} \\ \xrightarrow{m} \\ \xrightarrow[\pi_2]{}}\; \mathbb{X}_1 \;\substack{\xrightarrow{d_0} \\ \xleftarrow{s} \\ \xrightarrow[d_1]{}}\; \mathbb{X}_0$$

3.2. Pseudo-functors and strict functors.
As maps between double categories we consider those functors that correspond to natural transformations between their horizontal nerves.

3.3. Definition.

1. A *strict functor* $F\colon \mathbb{X} \to \mathbb{Y}$ between double categories is given by a (strict) natural transformation
$$F\colon N_h\mathbb{X} \Rightarrow N_h\mathbb{Y}\colon \Delta^{\mathrm{op}} \to \mathsf{Cat}.$$
2. A *pseudo-functor* $F\colon \mathbb{X} \to \mathbb{Y}$ between double categories is given by a pseudo-natural transformation $F\colon N_h\mathbb{X} \Rightarrow N_h\mathbb{Y}\colon \Delta^{\mathrm{op}} \to \mathsf{Cat}$.

So strict functors send objects to objects, horizontal arrows to horizontal arrows, vertical arrows to vertical arrows, and double cells to double cells, and preserve domains, codomains, identities and horizontal and vertical composition strictly. Pseudo-functors will preserve all this structure only up to appropriate vertical isomorphisms. Note that pseudo-functors between double categories as defined here are weak in a different way from what is described in [10], for instance. A pseudo-functor $(F, \varphi, \sigma, \mu)\colon \mathbb{X} \to \mathbb{Y}$ in our definition consists of functors $F_0\colon \mathbb{X}_0 \to \mathbb{Y}_0$, $F_1\colon \mathbb{X}_1 \to \mathbb{Y}_1$ and $F_k\colon \mathbb{X}_1 \times_{\mathbb{X}_0} \overset{k}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1 \to \mathbb{Y}_1 \times_{\mathbb{Y}_0} \overset{k}{\cdots} \times_{\mathbb{Y}_0} \mathbb{Y}_1$ for $k \geq 2$, together with invertible natural transformations,

$$\begin{array}{ccc} \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1 \\ {\scriptstyle d_i}\downarrow & \varphi_i & \downarrow{\scriptstyle d_i} \\ \mathbb{X}_0 & \xrightarrow[F_0]{} & \mathbb{Y}_0 \end{array} \qquad \begin{array}{ccc} \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0 \\ {\scriptstyle s}\downarrow & \sigma & \downarrow{\scriptstyle s} \\ \mathbb{X}_1 & \xrightarrow[F_1]{} & \mathbb{Y}_1 \end{array}$$

$$\begin{array}{ccc} \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & \xrightarrow{F_2} & \mathbb{Y}_1 \times_{\mathbb{Y}_0} \mathbb{Y}_1 \\ {\scriptstyle m}\downarrow & \mu & \downarrow{\scriptstyle m} \\ \mathbb{X}_1 & \xrightarrow[F_1]{} & \mathbb{Y}_1 \end{array} \qquad \begin{array}{ccc} \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & \xrightarrow{F_2} & \mathbb{Y}_1 \times_{\mathbb{Y}_0} \mathbb{Y}_1 \\ {\scriptstyle \pi_i}\downarrow & \theta_i & \downarrow{\scriptstyle \pi_i} \\ \mathbb{X}_1 & \xrightarrow[F_1]{} & \mathbb{Y}_1\,, \end{array}$$

(where $i = 1, 2$), and analogously for $F_k$ with $k \geq 2$. These satisfy the usual naturality and coherence conditions, that can be derived from their (pseudo) simplicial description.

Note that vertical composition and domains and codomains are preserved strictly, but horizontal domains and codomains are only preserved up to a vertical isomorphism. (Compare this to the notion of pseudo-morphism in [10] which requires that domains and codomains are preserved strictly, but horizontal composition and units are only preserved up to coherent isomorphisms. So the pseudo-morphisms considered there form a strict subclass of the ones considered in this paper.)

3.4. Vertical and horizontal transformations.

Since double categories have two types of arrows, there are two possible choices for types of transformations between maps of double functors: vertical and horizontal transformations. Vertical transformations correspond to modifications between natural transformations of functors from $\Delta^{\mathrm{op}}$ into $\mathsf{Cat}$, so these are the ones that are relevant to our study of the rigidification functor from Tamsamani weak 2-categories to weakly globular double categories.

3.5. Definition. A *vertical transformation* $\gamma\colon F \Rightarrow G\colon \mathbb{X} \rightrightarrows \mathbb{Y}$ between strict double functors has components vertical arrows $\gamma_A\colon FA \to\!\!\!\!\bullet\; GA$ indexed by the objects of $\mathbb{X}$ and for each horizontal arrow $A \xrightarrow{h} B$ in $\mathbb{X}$, a double cell

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ {\scriptstyle \gamma_A}\downarrow\bullet & \gamma_h & \bullet\downarrow{\scriptstyle \gamma_B} \\ GA & \xrightarrow[Gh]{} & GB, \end{array}$$

such that $\gamma$ is strictly functorial in the horizontal direction, i.e., $\gamma_{h_2 \circ h_1} = \gamma_{h_2} \circ \gamma_{h_1}$, and natural in the vertical direction, i.e.,

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ {\scriptstyle Fv}\downarrow & F\zeta & \downarrow{\scriptstyle Fw} \\ FC & \xrightarrow[Fk]{} & FD \\ {\scriptstyle \gamma_C}\downarrow & \gamma_k & \downarrow{\scriptstyle \gamma_D} \\ GC & \xrightarrow[Gk]{} & GD \end{array} \quad \equiv \quad \begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ {\scriptstyle \gamma_A}\downarrow & \gamma_h & \downarrow{\scriptstyle Gw} \\ GA & \xrightarrow[Gh]{} & GB \\ {\scriptstyle Gv}\downarrow & G\zeta & \downarrow{\scriptstyle Gw} \\ GC & \xrightarrow[Gk]{} & GD \end{array}, \tag{8}$$

for any double cell $\zeta$ in $\mathbb{X}$.

To give a vertical transformation between pseudo-functors of double categories, we need to require that the data above fits together with the structure cells of the pseudo-transformations. We spell out part of the details and will leave the rest for the reader.

3.6. Definition. A *vertical transformation* $\gamma\colon F \Rightarrow G\colon \mathbb{X} \rightrightarrows \mathbb{Y}$ between pseudo-functors has components vertical arrows $\gamma_A\colon FA \xrightarrow{\bullet} GA$ indexed by the objects of $\mathbb{X}$ and for each horizontal arrow $A \xrightarrow{h} B$ in $\mathbb{X}$, a double cell

$$\begin{array}{ccc} d_0Fh & \xrightarrow{Fh} & d_1Fh \\ {\scriptstyle d_0\gamma_h}\downarrow & \gamma_h & \downarrow{\scriptstyle d_1\gamma_h} \\ d_0Gh & \xrightarrow[Gh]{} & d_1Gh, \end{array}$$

such that the following squares of vertical arrows commute:

$$\begin{array}{ccc} F_0A & \xrightarrow{\bullet} & d_0Fh \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle d_0\gamma_h} \\ G_0A & \xrightarrow{\bullet} & d_0G_1h \end{array} \qquad \begin{array}{ccc} F_0B & \xrightarrow{\bullet} & d_1F_1h \\ {\scriptstyle \gamma_B}\downarrow & & \downarrow{\scriptstyle d_1\gamma_h} \\ G_0B & \xrightarrow{\bullet} & d_1G_1h \end{array}$$

where the unlabelled arrows are the structure isomorphisms corresponding to $F$ and $G$.

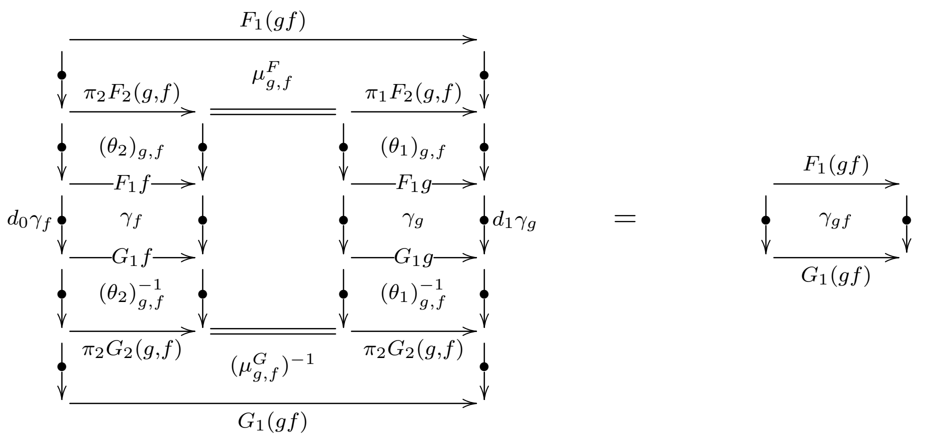
We require that $\gamma$ is natural in the vertical direction, in the sense that the following square of vertical arrows commutes for a vertical arrow $A \xrightarrow{v} B$ in $\mathbb{X}$,

$$\begin{array}{ccc} F_0A & \xrightarrow{F_1v} & F_0B \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\ G_0A & \xrightarrow[G_1v]{} & G_0B \end{array}$$

and furthermore, for any double cell $\zeta$ in $\mathbb{X}$,

$$\begin{array}{ccc} d_0F_1h & \xrightarrow{Fh} & d_1F_1h \\ {\scriptstyle d_0F_1\zeta}\downarrow & F_1\zeta & \downarrow{\scriptstyle d_1F_1\zeta} \\ d_0F_1k & \xrightarrow[Fk]{} & d_1F_1k \\ {\scriptstyle d_0\gamma_k}\downarrow & \gamma_k & \downarrow{\scriptstyle d_1\gamma_k} \\ d_0G_1k & \xrightarrow[Gk]{} & d_1G_1k \end{array} \equiv \begin{array}{ccc} d_0F_1h & \xrightarrow{Fh} & d_1F_1h \\ {\scriptstyle d_0\gamma_h}\downarrow & \gamma_h & \downarrow{\scriptstyle d_1\gamma_h} \\ d_0G_1h & \xrightarrow[Gh]{} & d_1G_1h \\ {\scriptstyle d_0G_1\zeta}\downarrow & G_1\zeta & \downarrow{\scriptstyle d_1G_1\zeta} \\ d_0G_1k & \xrightarrow[Gk]{} & d_1G_1k\,. \end{array}$$

In the horizontal direction we require pseudo-functoriality, which means that




For pseudo-functors, it only makes sense to consider vertical transformations. (Note that since the pseudo-aspect of these transformations is completely determined by their domain and codomain pseudo-functors, we will speak of vertical transformations without the adjective strict or pseudo.) For further details on pseudo-functors and vertical transformations between them, see the sequel [25] to this paper where we will spell out some of the coherence and naturality conditions and use them in the study of the weakly globular double category of fractions.

For strict double functors there is also the dual notion of *horizontal transformation*. Although this notion of 2-cell is not mentioned in the equivalence with the 2-category of bicategories, horizontal transformations do play an important role in the category of weakly globular double categories. We will illustrate this in a sequel to this paper [25] in which we define and study the construction of a weakly globular double category of fractions. The fact that horizontal transformations play a role next to the vertical transformations should not surprise us since vertical transformations correspond to a very special class of 2-cells in the category of bicategories, namely the icons.

The definition of a horizontal transformation is dual to that of a vertical transformation in that all mentions of vertical and horizontal have been exchanged.

3.7. Definition. A *horizontal transformation* $a\colon G \Rightarrow K\colon \mathbb{D} \rightrightarrows \mathbb{E}$ between functors of double categories has components horizontal arrows $GX \xrightarrow{a_X} KX$ indexed by the objects of $\mathbb{D}$ and for each vertical arrow $X \overset{v}{\longrightarrow\!\!\!\!\!\bullet\;\;} Y$ in $\mathbb{D}$, a double cell

$$\begin{array}{ccc} GX & \xrightarrow{a_X} & KX \\ {\scriptstyle Gv}\downarrow\bullet & a_v & \bullet\downarrow{\scriptstyle Kv} \\ GY & \xrightarrow[a_Y]{} & KY, \end{array}$$

such that $a$ is strictly functorial in the vertical direction, i.e., $a_{v_1 \cdot v_2} = a_{v_1} \cdot a_{v_2}$ and natural in the horizontal direction, i.e., the composition of

$$\begin{array}{ccccc} GX & \xrightarrow{Gf} & GX' & \xrightarrow{a_{X'}} & KX' \\ {\scriptstyle Gv}\downarrow\bullet & G\zeta & \bullet\downarrow{\scriptstyle Gv'} & a_{v'} & \bullet\downarrow{\scriptstyle Kv'} \\ GY & \xrightarrow[Gg]{} & GY' & \xrightarrow[a_{Y'}]{} & KY' \end{array}$$

is equal to the composition of

$$\begin{array}{ccccc} GX & \xrightarrow{a_X} & KX & \xrightarrow{Kf} & KX' \\ {\scriptstyle Gv}\downarrow\bullet & a_v & \bullet\downarrow{\scriptstyle Kv} & K\zeta & \bullet\downarrow{\scriptstyle Kv'} \\ GY & \xrightarrow[a_Y]{} & KY & \xrightarrow[Kg]{} & KY' \end{array}$$

for any double cell $\zeta$ in $\mathbb{D}$.

We write $\mathsf{DblCat}_{\mathrm{st},v}$, respectively $\mathsf{DblCat}_{\mathrm{st},h}$, for the 2-categories of weakly globular double categories, strict functors, and vertical transformations, respectively horizontal transformations. We will write $\mathsf{DblCat}_{\mathrm{ps},v}$ for the 2-category of double categories, pseudo-functors, and vertical transformations of pseudo-functors.

There is an inclusion functor $H\colon \mathsf{Cat} \to \mathsf{DblCat}_{\mathrm{st},h}$ that sends a category $\mathbf{C}$ to a double category with the category $\mathbf{C}$ as horizontal arrows and only identity arrows (on objects of $\mathbf{C}$) as vertical arrows and vertical identity cells as squares. Adjoint to this functor there is $h\colon \mathsf{DblCat}_{\mathrm{st},h} \to \mathsf{Cat}$ sending a double category to its category of horizontal arrows. Analogously, there is a functor $V\colon \mathsf{Cat} \to \mathsf{DblCat}_{\mathrm{st},v}$ that sends a category $\mathbf{C}$ to a double category with the arrows of $\mathbf{C}$ in the vertical direction (and a discrete horizontal category) with adjoint $v\colon \mathsf{DblCat}_{\mathrm{st},v} \to \mathsf{Cat}$.

## 4. Tamsamani weak 2-categories

We recall some background on the Tamsamani model of weak 2-categories, see also [29], [33] and [20]. This is a simplicial model based on the concept of Segal maps. The latter can be used to describe strict 2-categories as simplicial objects in $\mathsf{Cat}$ which are discrete at level zero and such that the Segal maps are isomorphisms. The Segal map isomorphisms give the associativity and unit laws in a strict 2-category, as part of the simplicial identities.

The idea of a Tamsamani weak 2-category is to relax the structure by requiring that the Segal maps are no longer isomorphisms but merely equivalences of categories. The associativity and unit laws do not hold strictly but only up to coherent isomorphism, so one obtains an associated bicategory. Conversely, there is a 2-nerve construction that associates to a bicategory a Tamsamani weak 2-categories, and an appropriate equivalence between the two notions, as recalled below.

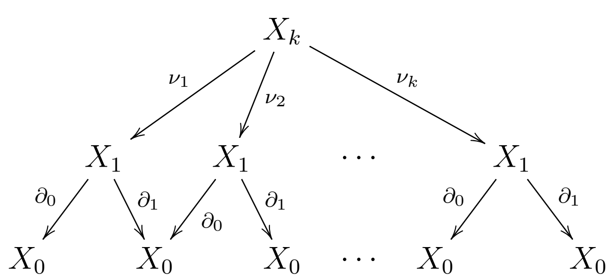
4.1. Tamsamani weak 2-categories.
We first recall the notion of Segal map. Let $\mathbf{C}$ be a category with pullbacks and let $X \in [\Delta^{\mathrm{op}}, \mathbf{C}]$, the category of simplicial objects in $\mathbf{C}$, that is, strict functors from $\Delta^{\mathrm{op}}$ to $\mathbf{C}$. For each $k \geq 2$, we denote by

$$X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$$

the limit of the diagram

$$X_1 \xrightarrow{\partial_1} X_0 \xleftarrow{\partial_0} \cdots \xrightarrow{\partial_1} X_0 \xleftarrow{\partial_0} X_1 \; .$$

For each $1 \leq j \leq k$, let $\nu_j : X_k \to X_1$ be induced by the map $[1] \to [k]$ in $\Delta$ sending 0 to $j-1$ and 1 to $j$. Then the following diagram commutes:



By definition of limit, there is a unique map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$$

such that $\mathrm{pr}_j \, \eta_k = \nu_j$, where $\mathrm{pr}_j$ is the $j^{th}$ projection.

The maps $\eta_k$ are called Segal maps and they play an important role in Tamsamani's model of weak 2-categories. They can also be used to characterize nerves of internal categories: a simplicial object in $\mathbf{C}$ is the nerve of an internal category in $\mathbf{C}$ if and only if the Segal maps are isomorphisms for all $k \geq 2$.

4.2. Definition. [8] The category $\mathsf{Ta}_2$ of *Tamsamani weak* 2*-categories* is the full subcategory of $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ whose objects $X$ are such that $X_0$ is discrete and, for all $k \geq 2$, the Segal map $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ is a categorical equivalence.

Let $\pi_0 : \mathsf{Cat} \to \mathsf{Set}$ associate to a category the set of isomorphism classes of its objects. The functor $\pi_0$ induces a functor $\pi_0^* : [\Delta^{\mathrm{op}}, \mathsf{Cat}] \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$, $(\pi_0^* X)_n = \pi_0 X_n$. If $X \in \mathsf{Ta}_2$, then $\pi_0^* X$ is the nerve of a category. In fact, since $\pi_0$ sends categorical equivalences to isomorphisms and preserves fibered products over discrete objects, for all $k \geq 2$, we have

$$\pi_0 X_k \cong \pi_0(X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1) \cong \pi_0 X_1 \times_{\pi_0 X_0} \overset{k}{\cdots} \times_{\pi_0 X_0} \pi_0 X_1 .$$

We write $\Pi_0 X$ for the category whose nerve is $\pi_0^* X$. This defines a functor

$$\Pi_0 : \mathsf{Ta}_2 \to \mathsf{Cat} .$$

Given $X \in \mathsf{Ta}_2$ and $a, b \in X_0$ let $X_{(a,b)}$ be the full subcategory of $X_1$ whose objects $z$ are such that $d_0 z = a$ and $d_1 z = b$. By considering the functor $(d_0, d_1) : X_1 \to X_0 \times X_0$, since $X_0$ is discrete, we obtain a coproduct decomposition $X_1 = \coprod_{a,b \in X_0} X_{(a,b)}$.

4.3. Definition. A morphisms $F : X \to Y$ in $\mathsf{Ta}_2$ is a *2-equivalence* if, for all $a, b \in X_0$, $F_{(a,b)} : X_{(a,b)} \to Y_{(Fa,Fb)}$ and $\Pi_0 F$ are categorical equivalences.

4.4. Remark. Notice that if a morphism in $\mathsf{Ta}_2$ is a levelwise equivalence of categories, it is in particular a 2-equivalence.

4.5. Bicategories and Tamsamani weak 2-categories.
In this section we recall some results from [20]. We consider the 2-category $\mathsf{Bicat}_{\text{icon}}$ whose objects are bicategories, whose morphisms are normal homomorphisms and whose 2-cells are icons; the latter are oplax natural transformations with identity components. The fully faithful inclusion $J : \Delta \to \mathsf{Bicat}_{\text{icon}}$ gives rise to a 2-nerve functor

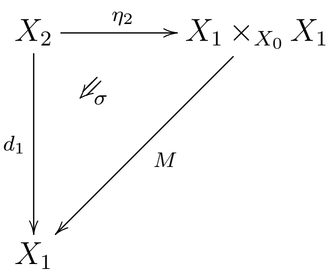
$$N : \mathsf{Bicat}_{\text{icon}} \to [\Delta^{\text{op}}, \mathsf{Cat}] ,$$

$$(N\mathcal{B})_n = \mathsf{Bicat}_{\text{icon}}([n], \mathcal{B}) .$$

It is shown in [20] that $N$ is fully faithful and that the 2-nerve of a bicategory is in fact a Tamsamani weak 2-category. Given a bicategory $\mathcal{B}$, $(N\mathcal{B})_0$ is the discrete category with objects the objects of $\mathcal{B}$. An object of $(N\mathcal{B})_1$ is a morphism of $\mathcal{B}$ while a morphism in $(N\mathcal{B})_1$ is a 2-cell in $\mathcal{B}$. A complete characterization of 2-functors $X : \Delta^{\text{op}} \to \mathsf{Cat}$ which are 2-nerves of bicategories is given in [20, Theorem 7.1].

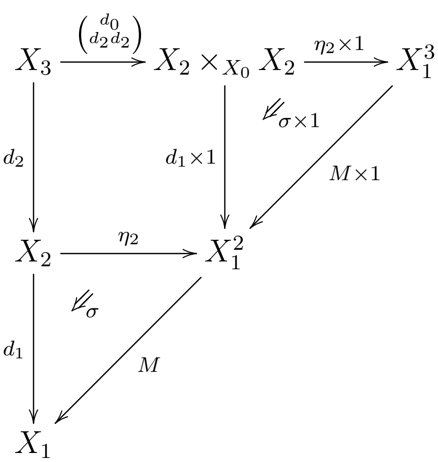
The 2-nerve functor $N$ has a left 2-adjoint $G$, which was defined in [33]. Given a Tamsamani weak 2-category $X$, the objects of $GX$ are the element of $X_0$, the 1- and 2-cells are the objects and morphisms of $X_1$ and the vertical composition of 2-cells is the composition in $X_1$.

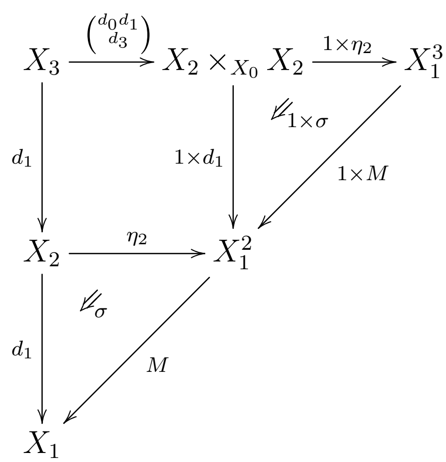
Since the Segal map $\eta_2 : X_2 \to X_1 \times_{X_0} X_1$ is an equivalence, we can choose a functor $M : X_1 \times_{X_0} X_1 \to X_1$ and an isomorphism $\sigma : d_1 \cong M\eta_2$ as follows:




This gives the composition of 1-cells and the horizontal composition of 2-cells.

The identity isomorphisms are $\sigma s_0$, $\sigma s_1$ (where $s_0, s_1 : X_1 \to X_2$ are the degeneracy maps). For the associativity isomorphisms, one needs to consider the following pasting diagrams, where we denote $X_1^k = X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$, for $k = 2, 3$.




Since the left hand composites of the diagrams are equal and the top composites are both equal to the equivalence $\eta_3 : X_3 \to X_1^3$, there is a unique invertible cell $M(M \times 1) \cong M(1 \times M)$ which pasted onto the left diagram gives the right diagram. The proof of the coherence laws uses the fact that $\eta_4$ is an equivalence, see [20] for details.

The relation between the functors $N$ and $G$ is summarized in the following theorem.

4.6. Theorem. [6, Theorem 7.2]. *The 2-nerve 2-functor $N : \mathsf{Bicat}_{\text{icon}} \to \mathsf{Ta}_2$, seen as landing in the 2-category $\mathsf{Ta}_2$ of Tamsamani weak 2-categories, has a left 2-adjoint given by $G$. Since $N$ is fully faithful, the counit $GN \to 1$ is invertible. Each component $u : X \to NGX$ of the unit is a pointwise equivalence, and $u_0$ and $u_1$ are identities.*

A morphism $f : X \to Y$ in $\mathsf{Ta}_2$ is a 2-equivalence if and only if $Gf$ is a biequivalence of bicategories. It is not hard to see that inverting these maps in $\mathsf{Bicat}_{\text{icon}}$ and in $\mathsf{Ta}_2$ gives equivalent categories. Another approach consists in enlarging the class of morphisms in $\mathsf{Ta}_2$ to include pseudo-natural transformations. One then obtains:

4.7. Theorem. [6, Theorem 7.3]. *The 2-nerve 2-functor $N : \mathsf{Bicat}_{\text{icon}} \to (\mathsf{Ta}_2)_{\text{ps}}$ is a biequivalence of 2-categories with pseudo-inverse $G$.*

## 5. From Tamsamani weak 2-categories to pseudo-functors

In this section we associate functorially to a Tamsamani weak 2-category a pseudo-functor from $\Delta^{\mathrm{op}}$ to $\mathsf{Cat}$. The idea of this construction is, given $H \in \mathsf{Ta}_2$, to replace $H_n$ with its equivalent category $H_1 \times_{H_0} \overset{n}{\cdots} \times_{H_0} H_1$ when $n \geq 2$. The resulting structure is no longer simplicial but pseudo-simplicial, that is an object of $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$. The proof of this is based on a general fact (Lemma 5.3) which is essentially known, and which is an instance of transport of structure along an adjunction (Theorem 5.2).

The pseudo-functors $H$ constructed from a Tamsamani weak 2-category are such that $H_0$ is discrete and $H_n \cong H_1 \times_{H_0} \overset{n}{\cdots} \times_{H_0} H_1$ for $n \geq 2$. We call such pseudo-functors *Segalic*. In the next section, we will see that the strictification of a Segalic pseudo-functor yields the horizontal nerve of a weakly globular double category.

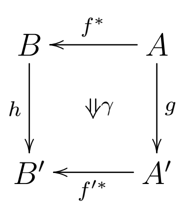
5.1. Transport of structure along an adjunction.
We now recall a general categorical property, known as transport of structure along an adjunction, with one of its applications.

5.2. Theorem. [19, Theorem 6.1] *Given an equivalence* $\eta,\ \varepsilon : f \dashv f^* : A \to B$ *in the complete and locally small 2-category* $\mathcal{A}$*, and an algebra* $(A, a)$ *for the monad* $T = (T, i, m)$ *on* $\mathcal{A}$*, the equivalence enriches to an equivalence*

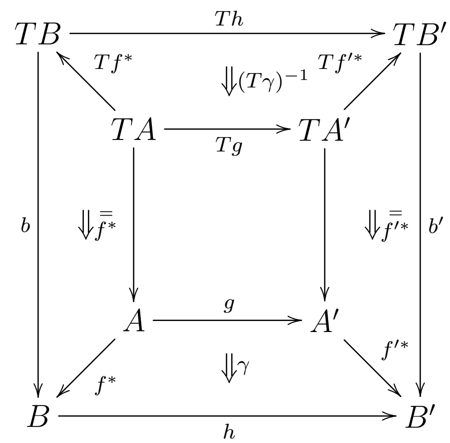
$$\eta, \varepsilon : (f, \overline{\overline{f}}) \vdash (f^*, \overline{\overline{f^*}}) : (A, a) \to (B, b, \hat{b}, \overline{b})$$

*in* $\mathsf{Ps}$-$T$-$\mathsf{alg}$*, where* $\hat{b} = \eta$*,* $\overline{b} = f^* a \cdot T\varepsilon \cdot Ta \cdot T^2 f$*,* $\overline{\overline{f}} = \varepsilon^{-1} a \cdot Tf$*, and* $\overline{\overline{f^*}} = f^* a \cdot T\varepsilon$*.*

Let $\eta', \varepsilon' : f' \vdash f'^* : A' \to B'$ be another equivalence in $\mathcal{A}$ and let $(B', b', \hat{b'}, \overline{b'})$ be the corresponding pseudo-$T$-algebra as in Theorem 5.2. Suppose $g : (A, a) \to (A', a')$ is a morphism in $\mathcal{A}$ and $\gamma$ is an invertible 2-cell in $\mathcal{A}$



Let $\overline{\gamma}$ be the invertible 2-cell given by the following pasting:




Then it is not difficult to show that $(h, \overline{\gamma}) : (B, b, \hat{b}, \overline{b}) \to (B', b', \hat{b}', \overline{b'})$ is a pseudo-$T$-algebra morphism.

The following fact is essentially known and, as sketched in the proof below, it is an instance of Theorem 5.2.

5.3. Lemma. *Let $\mathcal{C}$ be a small 2-category, $F, F' : \mathcal{C} \to \mathsf{Cat}$ be 2-functors, $\alpha : F \to F'$ a 2-natural transformation. Suppose that, for all objects $C$ of $\mathcal{C}$, the following conditions hold:*

*i) $G(C)$, $G'(C)$ are objects of $\mathsf{Cat}$ and there are adjoint equivalences of categories $\mu_C \vdash \eta_C$, $\mu'_C \vdash \eta'_C$,*

$$\mu_C : G(C) \rightleftarrows F(C) : \eta_C \qquad\qquad \mu'_C : G'(C) \rightleftarrows F'(C) : \eta'_C,$$

*ii) there are functors $\beta_C : G(C) \to G'(C)$,*

*iii) there is an invertible 2-cell*

$$\gamma_C : \beta_C \eta_C \Rightarrow \eta'_C \alpha_C.$$

*Then*

*a) There exists a pseudo-functor $G : \mathcal{C} \to \mathsf{Cat}$ given on objects by $G(C)$, and pseudo-natural transformations $\eta : F \to G$, $\mu : G \to F$ with $\eta(C) = \eta_C$, $\mu(C) = \mu_C$; these are part of an adjoint equivalence $\mu \vdash \eta$ in the 2-category* $\mathrm{Ps}[\mathcal{C}, \mathsf{Cat}]$.

*b) There is a pseudo-natural transformation $\beta : G \to G'$ with $\beta(C) = \beta_C$ and an invertible 2-cell in* $\mathrm{Ps}[\mathcal{C}, \mathsf{Cat}]$, *$\gamma : \beta\eta \Rightarrow \eta\alpha$ with $\gamma(C) = \gamma_C$.*

Proof. Recall [28] that the functor 2-category $[\mathcal{C}, \mathsf{Cat}]$ is 2-monadic over $[|\mathcal{C}|, \mathsf{Cat}]$, where $|C|$ is the set of objects in $\mathcal{C}$. Let $T$ be the 2-monad; then the pseudo-$T$-algebras are precisely the pseudo-functors from $\mathcal{C}$ to $\mathsf{Cat}$. Let

$$\mathcal{U} : \mathsf{Ps}\text{-}T\text{-}\mathsf{alg} \equiv \mathrm{Ps}[\mathcal{C}, \mathsf{Cat}] \to [|\mathcal{C}|, \mathsf{Cat}]$$

be the forgetful functor.

Then the adjoint equivalences $\mu_C \vdash \eta_C$ amount precisely to an adjoint equivalence in $[|\mathcal{C}|, \mathsf{Cat}]$, $\mu_0 \vdash \eta_0$, $\mu_0 : G_0 \rightleftarrows \mathcal{U}F : \eta_0$ where $G_0(C) = G(C)$ for all $C \in |\mathcal{C}|$. By Theorem 5.2, this equivalence enriches to an adjoint equivalence $\mu \vdash \eta$ in $\mathrm{Ps}[\mathcal{C}, \mathsf{Cat}]$

$$\mu : G \rightleftarrows F : \eta$$

between $F$ and a pseudo-functor $G$; it is $\mathcal{U}G = G_0$, $\mathcal{U}\eta = \eta_0$, $\mathcal{U}\mu = \mu_0$; hence on objects $G$ is given by $G(C)$, and $\eta(C) = \mathcal{U}\eta(C) = \eta_C$, $\mu(C) = \mathcal{U}\mu(C) = \mu_C$.

Let $\nu_C : \mathrm{id}_{G(C)} \Rightarrow \eta_C\mu_C$ and $\varepsilon_C : \mu_C\eta_C \Rightarrow \mathrm{id}_{F(C)}$ be the unit and counit of the adjunction $\mu_C \vdash \eta_C$. From Theorem 5.2, given a morphism $f : C \to D$ in $\mathcal{C}$, it is

$$G(f) = \eta_D F(f)\mu_C$$

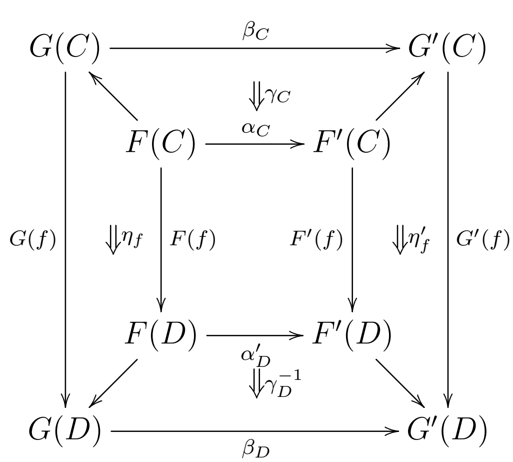
and we have natural isomorphisms:

$$\eta_f : G(f)\eta_C = \eta_D F(f)\mu_C\eta_C \xRightarrow{\eta_D F(f)\varepsilon_C} \eta_D F(f)$$
$$\mu_f : F(f)\mu_C \xRightarrow{\nu_{F(f)}\mu_C} \mu_D\eta_D F(f)\mu_C = \mu_D G(f).$$

Also, the natural isomorphism

$$\beta_f : G'(f)\beta_C \Rightarrow \beta_D G(f)$$

is the result of the following pasting




■

5.4. From Tamsamani weak 2-categories to pseudofunctors.
We now use the results of the previous section to associate to a Tamsamani weak 2-category a pseudo-functor.

5.5. Proposition. *There is a functor*

$$S : \mathsf{Ta}_2 \to \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}].$$

*which associates to an object $X$ of $\mathsf{Ta}_2$ a pseudo-functor $SX \in \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ with*

$$(SX)_n = \begin{cases} X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1 & n \geq 2, \\ X_1 & n = 1, \\ X_0 & n = 0. \end{cases}$$

*To a morphism $F : X \to X'$, $S$ associates a pseudo-natural transformation $\beta(F) : SX \to SX'$ with*

$$\beta(F)_n = \begin{cases} (F_1, \dots, F_1) & n \geq 2, \\ F_1 & n = 1, \\ F_0 & n = 0. \end{cases}$$

*Further, there is a pseudo-natural transformation $\alpha : SX \to X$ which is a levelwise categorical equivalence.*

Proof. We apply Lemma 5.3 to the case where $\mathcal{C} = \Delta^{\mathrm{op}}$, considered as a 2-category with identity 2-cells; let $X : \Delta^{\mathrm{op}} \to \mathsf{Cat}$ be an object of $\mathsf{Ta}_2$. By definition, for each $n \geq 2$, there is an equivalence of categories $X_n \simeq X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$. We can always choose this equivalence to be an adjoint equivalence; thus let $\eta_n : X_n \to X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$ be the Segal map and $\mu_n$ its left adjoint. By Lemma 5.3, we deduce that there is a pseudo-functor $SX \in \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ with

$$(SX)_n = \begin{cases} X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1 & n \geq 2, \\ X_1 & n = 1, \\ X_0 & n = 0. \end{cases}$$

Suppose $F : X \to X'$ is a morphism in $\mathsf{Ta}_2$ and let $\beta_n : (SX)_n \to (SX')_n$ be

$$\beta_n = \begin{cases} (F_1, \dots, F_1) & n \geq 2, \\ F_1 & n = 1, \\ F_0 & n = 0. \end{cases}$$

It is immediate to check from the definition of Segal map that the following diagram commutes for all $n \geq 0$,

$$\begin{array}{ccc} X_n & \xrightarrow{F_n} & X'_n \\ {\scriptstyle \eta_n}\downarrow & & \downarrow{\scriptstyle \eta'_n} \\ SX_n & \xrightarrow[\beta_n]{} & SX'_n. \end{array}$$

Thus the condition in the hypothesis of Lemma 5.3 is satisfied, with $\gamma_n$ the identity 2-cell. It follows from Lemma 5.3 that there is a pseudo-natural transformation $\beta : SX \to SX'$ with $\beta(F)_n = \beta_n$.

Suppose that $X \xrightarrow{F} X' \xrightarrow{F'} X''$ is a pair of composable morphisms in $\mathsf{Ta}_2$; then, for each $n \geq 2$,

$$\beta(F'F)_n = ((F'F)_1, \dots, (F'F)_1) = (F'_1, \dots, F'_1)(F_1, \dots, F_1) = \beta(F')_n \beta(F)_n.$$

Therefore, $\beta(F'F) = \beta(F')\beta(F)$.

Finally, the existence of a pseudo-natural transformation $\alpha : SX \to X$ follows immediately by Lemma 5.3, taking $\alpha_i = \text{id}$ for $i = 0, 1$ and $\alpha_n = \mu_n$ for $n \geq 2$. ■

We see from Proposition 5.5 that the pseudo-functors arising from Tamsamani weak 2-categories have a special form. We call them Segalic pseudo-functors, as in the following definition.

5.6. Definition. *The category* $\widetilde{\text{Ps}}[\Delta^{\text{op}}, \mathsf{Cat}]$ *of* Segalic pseudo-functors *from* $\Delta^{\text{op}}$ *to* $\mathsf{Cat}$ *is the full subcategory of* $\text{Ps}[\Delta^{\text{op}}, \mathsf{Cat}]$ *whose objects* $H$ *are such that* $H_n \cong H_1 \times_{H_0} \overset{n}{\cdots} \times_{H_0} H_1$ *for each* $n \geq 2$ *and* $H_0$ *is discrete.*

Then Proposition 5.5 immediately implies

5.7. Corollary. *There is a functor*

$$S : \mathsf{Ta}_2 \to \widetilde{\text{Ps}}[\Delta^{\text{op}}, \mathsf{Cat}],$$

*and a pseudo-natural transformation* $\alpha : SX \to X$ *which is a levelwise categorical equivalence.*

## 6. Strictification of Segalic pseudo-functors

In this section we apply the strictification method of Power [28] to Segalic pseudo-functors (Definition 5.6). We show in Theorem 6.5 that the resulting strict functor from $\Delta^{\text{op}}$ to $\mathsf{Cat}$ is the horizontal nerve of a double category satisfying additional properties. We identify these double categories as our central notion of weakly globular double categories (Definition 6.7), and hence conclude in Corollary 6.9 that the strictification of a Segalic pseudo-functor yields the horizontal nerve of a weakly globular double category. In the next section we will use these results to build a rigidification functor from Tamsamani weak 2-categories to weakly globular double categories.

We start by recalling the construction of the strictification functor from [28],

$$St : \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \to [\Delta^{\mathrm{op}}, \mathsf{Cat}].$$

6.1. Strictification of pseudo-functors.
As explained in [28, 4.2], the functor 2-category $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ is 2-monadic over $[\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$, where $|\Delta^{\mathrm{op}}|$ is the set of objects of $\Delta^{\mathrm{op}}$.

Let $U : [\Delta^{\mathrm{op}}, \mathsf{Cat}] \to [\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$ be the forgetful functor, $(UX)_n = X_n$ for all $[n] \in \Delta^{\mathrm{op}}, X \in [\Delta^{\mathrm{op}}, \mathsf{Cat}]$. Then its left adjoint $F$ is given on objects by

$$(FH)_n = \coprod_{[m]\in|\Delta^{\mathrm{op}}|} \Delta^{\mathrm{op}}([m],[n]) \times H_m,$$

for $H \in [\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$, $[n] \in |\Delta^{\mathrm{op}}|$. If $T$ is the monad corresponding to the adjunction $F \dashv U$ then

$$(TH)_n = \coprod_{[m]\in|\Delta^{\mathrm{op}}|} \Delta^{\mathrm{op}}([m],[n]) \times H_m,$$

for $H \in [\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$, and $[n] \in |\Delta^{\mathrm{op}}|$.

A pseudo-$T$-algebra is given by $H \in [\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$, functors

$$h_n : \coprod_{[m]\in|\Delta^{\mathrm{op}}|} \Delta^{\mathrm{op}}([m],[n]) \times H_m \to H_n$$

and additional data as described in [28, 4.2]. This amounts precisely to a pseudo-functor from $\Delta^{\mathrm{op}}$ to $\mathsf{Cat}$ and the 2-category $\mathsf{Ps}\text{-}T\text{-}\mathsf{alg}$ of pseudo-$T$-algebras corresponds to the 2-category $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ of pseudo-functors, pseudo-natural transformations and modifications.

The general strictification result proved in [28, 3.4], when applied to this case, yields that every pseudo-functor from $\Delta^{\mathrm{op}}$ to Cat is equivalent, in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$, to a 2-functor.

The construction given in [28] is as follows. Given a pseudo-$T$-algebra as above, factor $h : TH \to H$ as $TH \xrightarrow{r} L \xrightarrow{g} H$ with $r_n$ bijective on objects and $g_n$ fully faithful for each $[n] \in \Delta^{\mathrm{op}}$. Then it is shown in [28] that it is possible to give a strict $T$-algebra structure $TL \to L$ such that $(g, Tg)$ is an equivalence of pseudo-$T$-algebras.

6.2. Remark. Since $(g, Tg)$ is an equivalence of pseudo-$T$-algebras, $g_n$ is an equivalence of categories for every $[n] \in \Delta^{\mathrm{op}}$. In fact, by definition there is a map $(g', Tg')$ with invertible 2-cells $\alpha : (g, Tg)(g', Tg') \Longrightarrow \mathrm{id}$ and $\beta : (g', Tg')(g, Tg) \Longrightarrow \mathrm{id}$. A 2-cell in $\mathsf{Ps}\text{-}T\text{-}\mathsf{alg}$ amounts to a 2-cell in $[\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$ satisfying the condition of [28, 2.6]. Since the 2-cells in $[\,|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$ are modifications, this implies that for each $[n] \in \Delta^{\mathrm{op}}$ there are natural transformations $\mathrm{id} \cong g_n g'_n$, and $g'_n g_n \cong \mathrm{id}$; that is, $g_n$ is an equivalence of categories for each $n$.

6.3. A special case of strictification.
We now apply the strictification technique of Section 6.1 to the class of Segalic pseudo-functors.

We first need a preliminary lemma that establishes a more explicit description of $TUH$ and of the structure map $h : TUH \to H$ for a Segalic pseudo-functor $H$.

6.4. Lemma. *Let $U : \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \to [|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$ be the forgetful functor, $(UH)_n = H_n$ for all $n \geq 0$, and let $T$ be the monad on $[|\Delta^{\mathrm{op}}|, \mathsf{Cat}]$ as in Section 6.1. Then, if $H \in \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$,*

*a) The pseudo-$T$-algebra corresponding to $H$ has structure map $h : TUH \to UH$ given as follows: for each $k \geq 0$,*

$$(TUH)_k = \coprod_{[n]\in\Delta} \Delta([k],[n]) \times H_n = \coprod_{[n]\in\Delta} \; \coprod_{\Delta([k],[n])} H_n.$$

*For $n \geq 0$ and $f \in \Delta([k],[n])$, let $i_n : \coprod_{\Delta([k],[n])} H_n \to (TUH)_k$ and $j_f : H_n \to \coprod_{\Delta([k],[n])} H_n$ be the coproduct injections. Then $h_k i_n j_f = H(f)$.*

*b) There are functors $\partial'_0, \partial'_1 : (TUH)_1 \rightrightarrows (TUH)_0$ making the following diagram commute:*

$$\begin{array}{ccc} (TUH)_1 & \xrightarrow{h_1} & H_1 \\ {\scriptstyle \partial'_0}\downarrow\;\downarrow{\scriptstyle \partial'_1} & & {\scriptstyle \partial_0}\downarrow\;\downarrow{\scriptstyle \partial_1} \\ (TUH)_0 & \xrightarrow{h_0} & H_0 \end{array} \tag{9}$$

*that is, $\partial_i h_1 = h_0 \partial'_i$, for $i = 0, 1$.*

*c) For each $k \geq 2$, $(TUH)_k \cong (TUH)_1 \times_{(TUH)_0} \overset{k}{\cdots} \times_{(TUH)_0} (TUH)_1$.*

*d) For each $k \geq 2$, the morphism $h_k : (TUH)_k \to H_k$ is $h_k = (h_1, \ldots, h_1)$.*

Proof.

a) From the general correspondence between pseudo-$T$-algebras and pseudo-functors, the pseudo-$T$-algebra corresponding to $H$ has structure map $h : TUH \to UH$ as stated. Recalling that, if $X$ is a set and $\mathbf{C}$ is a category, $X \times \mathbf{C} \cong \coprod_X \mathbf{C}$, we have

$$(TUH)_k = \coprod_{[n]\in\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}([n],[k]) \times H_n = \coprod_{[n]\in\Delta} \Delta([k],[n]) \times H_n \cong \coprod_{[n]\in\Delta} \; \coprod_{\Delta([k],[n])} H_n.$$

b) Let $\delta_i : [0] \to [1]$, $\delta_i(0) = 0$, $\delta_i(1) = i$, for $i = 0, 1$. For $f \in \Delta([1],[n])$ let $j_f : H_n \to \coprod\limits_{\Delta([1],[n])} H_n$ and $i_n \colon \coprod\limits_{\Delta([1],[n])} H_n \to \coprod\limits_{[n]\in\Delta} \coprod\limits_{\Delta([1],[n])} H_n$ be the coproduct injections. Let $\partial'_i : (TUH)_1 \to (TUH)_0$ be the functors determined by

$$\partial'_i i_n j_f = i_n j_{f\delta_i}. \tag{10}$$

From a), we have

$$h_0 \partial'_i i_n j_f = h_0 i_n j_{f\delta_i} = H(f\delta_i) \quad \text{and} \quad \partial_i h_1 i_n j_f = H(\delta_i) H(f)\,.$$

Since $H \in \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ and $H_0$ is discrete, it follows that $H(f\delta_i) = H(\delta_i)H(f)$, so that, from above, $h_0\partial'_i i_n j_f = \partial_i h_1 i_n j_f$ for each $[n] \in \Delta$, $f \in \Delta([1],[n])$. We conclude that $h_0 \partial'_i = \partial_i h_1$.

c) For each $k \geq 2$, $[k]$ is the colimit in $\Delta$ of the diagram

$$\begin{array}{ccccccccc} [1] & & & & [1] & & & & [1] \\ & \nwarrow^{0} & & \nearrow^{1} & & \nwarrow^{0} & & \nearrow^{1} & \\ & & [0] & & & [0] & \cdots & [0] & \end{array}$$

that is, $[k] = [1] \coprod\limits_{[0]} \overset{k}{\cdots} \coprod\limits_{[0]} [1]$. In fact, it is easy to check by direct computation that there is a pushout in $\Delta$:

$$\begin{array}{ccc} [0] & \xrightarrow{\;0\;} & [k-1] \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle p} \\ [1] & \xrightarrow[\;q\;]{} & [k] \end{array}$$

where $q(i) = i$, for $i = 0, 1$, and $p(t) = t + 1$, for $t = 0, \dots, k-1$. In particular, $[2] = [1] \coprod\limits_{[0]} [1]$. Inductively, if $[k-1] = [1] \coprod\limits_{[0]} \overset{k-1}{\cdots} \coprod\limits_{[0]} [1]$ then, from the above pushout, $[k] = [k-1] \coprod\limits_{[0]} [1] = [1] \coprod\limits_{[0]} \overset{k}{\cdots} \coprod\limits_{[0]} [1]$, as claimed. It follows that there is a bijection, for $k \geq 2$

$$\Delta([k],[n]) \cong \Delta([1],[n]) \times_{\Delta([0],[n])} \overset{k}{\cdots} \times_{\Delta([0],[n])} \Delta([1],[n]).$$

From the proof of b), the functors $\partial_i : (TUH)_1 \to (TUH)_0$ are determined by the functors $(\overline{\delta}_i, \mathrm{id}) : \Delta([1],[n]) \times H_n \to \Delta([0],[n]) \times H_n$ where $\overline{\delta}_i(f) = f\delta_i$ for

$f \in \Delta([1],[n])$. Hence, from above, we obtain

$$
\begin{aligned}
&(TUH)_1 \times_{(TUH)_0} \overset{k}{\cdots} \times_{(TUH)_0} (TUH)_1 = \\
&= \coprod_{[n]\in\Delta} (\Delta([1],[n]) \times H_n) \times_{\coprod_{[n]\in\Delta}(\Delta([0],[n])\times H_n)} \overset{k}{\cdots} \times_{\coprod_{[n]\in\Delta}(\Delta([0],[n])\times H_n)} \coprod_{[n]\in\Delta} (\Delta([1],[n]) \times H_n) \\
&\cong \coprod_{[n]\in\Delta} (\Delta([1],[n]) \times H_n) \times_{(\Delta([0],[n])\times H_n)} \overset{k}{\cdots} \times_{(\Delta([0],[n])\times H_n)} (\Delta([1],[n]) \times H_n) \\
&= \coprod_{[n]\in\Delta} (\Delta([1],[n]) \times_{\Delta([0],[n])} \overset{k}{\cdots} \times_{\Delta([0],[n])} \Delta([1],[n])) \times (H_n \times_{H_n} \overset{k}{\cdots} \times_{H_n} H_n) \\
&= \coprod_{[n]\in\Delta} \Delta([k],[n]) \times H_n = (TUH)_k.
\end{aligned}
$$

d) From a), $h_k i_n j_f = H(f)$ for $f \in \Delta([k],[n])$, $n > 0$. Let $f$ correspond to $(\delta_1, \dots, \delta_k)$ in the isomorphism $\Delta([k],[n]) \cong \Delta([1,[n])\times_{\Delta([0,[n])} \overset{k}{\cdots} \times_{\Delta([0,[n])} \Delta([1,[n])$. Then $j_f = (j_{\delta_1}, \dots, j_{\delta_k})$. Since $H_k \cong H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1$, $H(f)$ corresponds to $(H(\delta_1), \dots, H(\delta_k))$ with $p_i H(f) = H(\delta_i)$. Then, for all $f \in \Delta([k],[n])$, $n > 0$ we have

$$
\begin{aligned}
h_k i_n j_f &= H(f) = (H(\delta_1), \dots, H(\delta_k)) = (h_1 i_n j_{\delta_1}, \dots, h_1 i_n j_{\delta_k}) = \\
&= (h_1, \dots, h_1) i_n (j_{\delta_1}, \dots, j_{\delta_k}) = (h_1, \dots, h_1) i_n j_f .
\end{aligned}
$$

We conclude that $h_k = (h_1, \dots, h_1)$

■

6.5. Theorem. *Let $H \in \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ and $L = StH$ be the strictification of $H$ as in Section 6.1. Then*

*a) There is a morphism $g : L \to H$ in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ such that, for each $k \geq 0$, $g_k$ is an equivalence of categories.*

*b) $L_k \cong L_1 \times_{L_0} \overset{k}{\cdots} \times_{L_0} L_1$ for all $k \geq 2$.*

*c) The functor $g_0 : L_0 \to H_0$ induces equivalences of categories $L_1 \times_{L_0} \overset{k}{\cdots} \times_{L_0} L_1 \simeq L_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} L_1$ for all $k \geq 2$.*

PROOF.

a) This follows directly from [28] (see Remark 6.2).

b) Let $h : TUH \to UH$ be as in Lemma 6.4. As described in Section 6.1, factor $h = gr$ so that, for each $i \geq 0$, $h_i$ factors as $(TUH)_i \xrightarrow{r_i} L_i \xrightarrow{g_i} H_i$ with $r_i$ bijective on objects and $g_i$ fully faithful. As explained in [28], the $g_i$ are in fact equivalences.

Since the bijective on objects and fully faithful functors form a factorization system in $\mathsf{Cat}$, the commutativity of (9) implies that there are functors $d_0, d_1 : L_1 \to L_0$ such that the following commutes:

$$\begin{array}{ccccc} (TUH)_1 & \xrightarrow{r_1} & L_1 & \xrightarrow{g_1} & H_1 \\ \partial'_0 \downarrow \downarrow \partial'_1 & & d_0 \downarrow \downarrow d_1 & & \partial_0 \downarrow \downarrow \partial_1 \\ (TUH)_0 & \xrightarrow[r_0]{} & L_0 & \xrightarrow[g_0]{} & H_0 \end{array}$$

that is, $\partial_i r_1 = r_0 \partial'_i$, $\partial_i g_1 = g_0 d_i$ for $i = 0, 1$. By Lemma 6.4 d) , $h_k$ factors as

$$\begin{aligned} (TUH)_k = (TUH)_1 \times_{(TUH)_0} \overset{k}{\cdots} \times_{(TUH)_0} (TUH)_1 \xrightarrow{(r_1,\dots,r_1)} L_1 \times_{L_0} \overset{k}{\cdots} \times_{L_0} L_1 \\ \xrightarrow{(g_1,\dots,g_1)} H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \cong H_k. \end{aligned} \tag{11}$$

Since $r_0, r_1$ are bijective on objects, so is $(r_1, \dots, r_1)$. Since $g_0, g_1$ are fully faithful, so is $(g_1, \dots, g_1)$. Hence (11) is the factorization of $h_k$ and we conclude that $L_k \cong L_1 \times_{L_0} \overset{k}{\cdots} \times_{L_0} L_1$.

c) Since $H_1 \simeq L_1$ and $H_0$ is discrete, $H_k \cong H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \simeq L_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} L_1$. On the other hand, $H_k \simeq L_k \cong L_1 \times_{L_0} \overset{k}{\cdots} \times_{L_0} L_1$. In conclusion, $L_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} L_1$ $\simeq L_1 \times_{L_0} \overset{k}{\cdots} \times_{L_0} L_1$.

■

6.6. WEAKLY GLOBULAR DOUBLE CATEGORIES.
We see from Theorem 6.5 b) that $L$ is a simplicial object in $\mathsf{Cat}$ whose Segal maps are isomorphisms: it is therefore the horizontal nerve of a double category. Further, parts a) and c) of Theorem 6.5 show that this double category satisfies additional conditions. We identify such double categories as our central notion of weakly globular double category, as in the following definition:

6.7. DEFINITION. A *weakly globular double category* $\mathbb{X}$ is a double category which satisfies the following two conditions

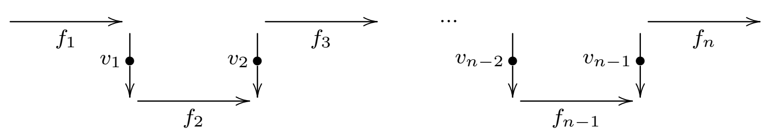
- (*the weak globularity condition*) there is an equivalence of categories $\gamma : \mathbb{X}_0 \to \mathbb{X}_0^d$, where $\mathbb{X}_0^d$ is the discrete category of the path components of $\mathbb{X}_0$;

- (*the induced Segal maps condition*) $\gamma$ induces an equivalence of categories, for all $n \geq 2$,

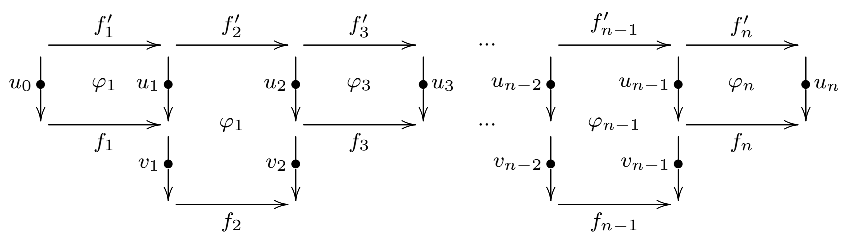
$$\mathbb{X}_1 \times_{\mathbb{X}_0} \overset{n}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1 \simeq \mathbb{X}_1 \times_{\mathbb{X}_0^d} \overset{n}{\cdots} \times_{\mathbb{X}_0^d} \mathbb{X}_1 \,. \tag{12}$$

Analogously to the notation introduced above we write $\mathsf{WGDbl}_{\mathrm{ps},v}$ for the 2-category of weakly globular double categories, pseudo-functors, and vertical transformations between them. We also write $\mathsf{WGDbl}_{\mathrm{st},v}$ and $\mathsf{WGDbl}_{\mathrm{st},h}$ for the 2-categories with strict functors and vertical, respectively horizontal, transformations.

6.8. Remark. *For each $n > 0$, the equivalence (12) in the induced Segal maps condition gives us for each 'stair-case-path' of alternating horizontal and vertical arrows,*



*a corresponding horizontal path with vertically invertible double cells,*



If we denote by $N_h(\mathsf{WGDbl})$ the subcategory of $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ of horizontal nerves of weakly globular double categories, we finally obtain:

6.9. Corollary. *The strictification functor $St : \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \to [\Delta^{\mathrm{op}}, \mathsf{Cat}]$ when restricted to Segalic pseudo-functors gives rise to a functor*

$$St : \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \to N_h(\mathsf{WGDbl}) \,.$$

*Further, for any $H \in \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ there is a morphism $g : St\, H \to H$ in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ such that, for each $k \geq 0$, $g_k$ is an equivalence of categories.*

Proof. It follows immediately from Theorem 6.5 and Definition 6.7. ∎

We end this section with a remark which will be used in Section 8.2 in giving the explicit description of the weakly globular double category associated to a bicategory.

6.10. Remark. Recall that the factorization of any functor $F : \mathbf{C} \to \mathbf{D}$ as the composite $\mathbf{C} \xrightarrow{S} \mathbf{E} \xrightarrow{T} \mathbf{D}$ with $S$ bijective on objects and $T$ fully faithful is constructed as follows.

Consider the pullbacks of sets

$$\begin{array}{ccc} \mathbf{E}_1 & \xrightarrow{(\tilde{d}_0,\tilde{d}_1)} & \mathbf{C}_0 \times \mathbf{C}_0 \\ {\scriptstyle T_1}\downarrow & & \downarrow{\scriptstyle F_0\times F_0} \\ \mathbf{D}_1 & \xrightarrow[(d_0,d_1)]{} & \mathbf{D}_0 \times \mathbf{D}_0 \end{array}$$

where $d_0, d_1$ are the source and target map in the category $\mathbf{D}$. It is easy to see that there is a category $\mathbf{E}$ with objects $\mathbf{E}_0 = \mathbf{C}_0$ and $\mathbf{E}_1$ and source and target maps $\tilde{d}_0$ and $\tilde{d}_1$ as in the pullback diagram above. Further, there are functors $S : \mathbf{C} \to \mathbf{E}$, $T : \mathbf{E} \to \mathbf{D}$ with $S_0 = \mathrm{id}$, $T_0 = F_0$, and $S_1$ determined by $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ and $(d'_0, d'_1) : \mathbf{C}_1 \to \mathbf{C}_0 \times \mathbf{C}_0$. Hence, in the notation of Theorem 6.5, we have

$$L_{00} = (TUH)_{00} = \coprod_{[n]\in\Delta} \Delta([0],[n]) \times H_{n0}$$

$$L_{10} = (TUH)_{10} = \coprod_{[n]\in\Delta} \Delta([1],[n]) \times H_{n0}$$

while $L_{01}$ and $L_{11}$ are given by the following pullbacks:

$$\begin{array}{ccc} L_{11} & \longrightarrow & (TUH)_{10} \times (TUH)_{10} \\ {\scriptstyle g_{11}}\downarrow & & \downarrow{\scriptstyle h_{10}\times h_{10}} \\ H_{11} & \xrightarrow[(d_0,d_1)]{} & H_{10} \times H_{10} \end{array}$$

$$\begin{array}{ccc} L_{01} & \longrightarrow & (TUH)_{00} \times (TUH)_{00} \\ {\scriptstyle g_{01}}\downarrow & & \downarrow{\scriptstyle h_{00}\times h_{00}} \\ H_{00} & \xrightarrow[(\mathrm{id},\mathrm{id})]{} & H_{00} \times H_{00} \end{array}$$

## 7. Weakly globular double categories, Tamsamani weak 2-categories, and bicategories

In this section we prove our main result (Theorem 7.10) that there is a biequivalence between the 2-category of weakly globular double categories with pseudo-functors and the 2-category of bicategories with icons.

We obtain this result first by comparing weakly globular double categories to Tamsamani weak 2-categories. Using the results of Sections 5 and 6 we build a rigidification functor from the latter to weakly globular double categories. In section 7.3 we build a discretization functor in the opposite direction. We show in Theorem 7.6 that these two functors give a biequivalence between the 2-categories of weakly globular double categories and Tamsamani weak 2-categories. Together with the comparison result Theorem 4.7 from Tamsamani weak 2-categories to bicategories, this yields our main result in Theorem 7.10.

7.1. Rigidifying Tamsamani weak 2-categories.
We now use the results of Sections 5 and 6 to build a rigidification functor from Tamsamani weak 2-categories to weakly globular double categories.

7.2. Theorem. *There is a functor*

$$Q\colon \mathsf{Ta}_2 \to \mathsf{WGDbl}_{\mathrm{st},v}$$

*such that, for all* $X \in \mathsf{Ta}_2$*, there is a pseudo-morphism* $\alpha_X\colon N_h QX \to X$*, natural in* $X$*, which is a levelwise categorical equivalence.*

Proof. Let $P$ be the left adjoint to the horizontal nerve $N_h$, $PN_h = id$. Let $Q$ be the composite

$$Q\colon (\mathsf{Ta}_2)_{\mathrm{ps}} \xrightarrow{S} \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \xrightarrow{St} N_h(\mathsf{WGDbl}) \xrightarrow{P} \mathsf{WGDbl}_{\mathrm{ps},v}$$

where $S\colon (\mathsf{Ta}_2)_{\mathrm{ps}} \to \widetilde{\mathrm{Ps}}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ is as in Corollary 5.7 and $St\colon \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \to N_h(\mathsf{WGDbl})$ is as in Corollary 6.9. Now take $X \in (\mathsf{Ta}_2)_{\mathrm{ps}}$. By Corollaries 5.7 and 6.9 there are pseudo-natural transformations $SX \to X$ and $StSX \to SX$ which are levelwise equivalences of categories. Since $StSX \cong N_h PStSX = N_h QX$, we obtain by composition a pseudo-natural transformation $N_h QX \to X$, which is a levelwise equivalence of categories. ■

7.3. Discretization.
In the next proposition we construct a functor in the opposite direction, from weakly globular double categories to Tamsamani weak 2-categories. The idea is to replace the category of objects and vertical morphisms in a weakly globular double category by its equivalent discrete category. This recovers the globularity condition, but at the expense of the strictness of the Segal maps: from being isomorphisms they become equivalences.

7.4. Proposition. *There is a functor*

$$D : \mathsf{WGDbl}_{\mathrm{st},v} \to \mathsf{Ta}_2$$

*such that, for every* $\mathbb{X} \in \mathsf{WGDbl}$ *there is a pseudo-morphism* $\eta_{\mathbb{X}}\colon D\mathbb{X} \to N_h\mathbb{X}$*, natural in* $\mathbb{X}$*, which is a levelwise categorical equivalence.*

Proof. Let $\mathbb{X} \in \mathsf{WGDbl}$. By definition, there is a categorical equivalence $\gamma\colon \mathbb{X}_0 \to \mathbb{X}_0^d$. Write $\gamma'\colon \mathbb{X}_0^d \to \mathbb{X}_0$ for its pseudo-inverse. Then $\gamma\gamma' = \mathrm{id}$, since $\mathbb{X}_0^d$ is discrete. Let

$$\begin{aligned} (D\mathbb{X})_0 &= \mathbb{X}_0^d, \\ (D\mathbb{X})_1 &= \mathbb{X}_1, \\ (D\mathbb{X})_k &= \mathbb{X}_1 \times_{\mathbb{X}_0} \overset{k}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1, \text{ for } k \geq 2. \end{aligned}$$

Let $\partial_i, \sigma_i$ be the face and degeneracy operators of $N_h\mathbb{X}$. Define $d_i\colon (D\mathbb{X})_1 \to (D\mathbb{X})_0$ and $s_0\colon (D\mathbb{X})_0 \to (D\mathbb{X})_1$ by $d_i = \gamma\partial_i$, for $i = 0, 1$ and $s_0 = \sigma_0\gamma'$. All other face and degeneracy operators in $D\mathbb{X}$ are as in $N_h\mathbb{X}$. Notice that, since $\gamma\gamma' = \mathrm{id}$, $D\mathbb{X} \in [\Delta^{\mathrm{op}}, \mathsf{Cat}]$.

By construction, $(D\mathbb{X})_0$ is discrete. To show that $D\mathbb{X} \in \mathsf{Ta}_2$ we need to show that all Segal maps are categorical equivalences. Since $\mathbb{X}$ is weakly globular, by definition we have for $n \geq 2$,

$$\begin{aligned}(D\mathbb{X})_n &= \mathbb{X}_1\times_{\mathbb{X}_0}\overset{n}{\cdots}\times_{\mathbb{X}_0}\mathbb{X}_1\\ &\simeq \mathbb{X}_1\times_{\mathbb{X}_0^d}\overset{n}{\cdots}\times_{\mathbb{X}_0^d}\mathbb{X}_1\\ &= (D\mathbb{X})_1\times_{(D\mathbb{X})_0}\overset{n}{\cdots}\times_{(D\mathbb{X})_0}(D\mathbb{X})_1.\end{aligned}$$

This shows that $D\mathbb{X} \in \mathsf{Ta}_2$.

Let $\eta_0 = \gamma'$, $\eta_k = \mathrm{id}$ for $k > 0$. This defines a pseudo-natural transformation $\eta\colon D\mathbb{X} \to N_h\mathbb{X}$ which is a levelwise categorical equivalence. ∎

7.5. Remark. From the pseudo-inverses $(\mu_{\mathbb{X}})_k$ to $(\eta_{\mathbb{X}})_k$ for each $k$, using Lemma 5.3 we construct the pseudo-inverse $\mu_{\mathbb{X}} : N_h\mathbb{X} \to D\mathbb{X}$ to the pseudo-functor $\eta_{\mathbb{X}} : D\mathbb{X} \to N_h\mathbb{X}$ in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$. Likewise, for each $Y \in \mathsf{Ta}_2$ the pseudo-functor $N_hQY \to Y$ of Theorem 7.2 has a pseudo-inverse $Y \to N_hQY$.

Notice that the functors $Q : \mathsf{Ta}_2 \to \mathsf{WGDbl}_{\mathrm{st},v}$ and $D : \mathsf{WGDbl}_{\mathrm{st},v} \to \mathsf{Ta}_2$ of Theorem 7.2 and Proposition 7.4 extend to functors $Q : (\mathsf{Ta}_2)_{\mathrm{ps}} \to (\mathsf{WGDbl})_{\mathrm{ps},v}$ and $D : (\mathsf{WGDbl})_{\mathrm{ps}} \to (\mathsf{Ta}_2)_{\mathrm{ps},v}$ (we shall denote them with the same letters).

7.6. Theorem. *There is a biequivalence of 2-categories:*

$$(\mathsf{WGDbl})_{\mathrm{ps},v} \simeq (\mathsf{Ta}_2)_{\mathrm{ps}} \ .$$

Proof. Since the horizontal nerve functor $N_h : (\mathsf{WGDbl})_{\mathrm{ps}} \to \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ is fully faithful, there is an isomorphism

$$(\mathsf{WGDbl})_{\mathrm{ps},v}(\mathbb{X}, \mathbb{Y}) \cong \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}](\mathsf{N_h}\mathbb{X}, \mathsf{N_h}\mathbb{Y}) \ . \tag{13}$$

We claim that there is an equivalence of categories

$$F\colon \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}](\mathsf{N_h}\mathbb{X}, \mathsf{N_h}\mathbb{Y}) \simeq (\mathsf{Ta}_2)_{\mathrm{ps}}(\mathsf{D}\mathbb{X}, \mathsf{D}\mathbb{Y}) \ : \mathsf{G} \tag{14}$$

This is constructed as follows. Let $\eta_{\mathbb{X}}\colon D\mathbb{X} \to N_h\mathbb{X}$ and $\mu_{\mathbb{X}}\colon N_h\mathbb{X} \to D\mathbb{X}$ be as in Remark 7.5.

Define

$$Ff = \mu_{\mathbb{Y}} f \eta_{\mathbb{X}}, \qquad\qquad Gg = \eta_{\mathbb{Y}} g \mu_{\mathbb{X}} \ .$$

Then $FGg = \mu_{\mathbb{Y}}(\eta_{\mathbb{Y}} g \mu_{\mathbb{X}})\eta_{\mathbb{X}} \cong g$ and $GFf = \eta_{\mathbb{Y}}(\mu_{\mathbb{Y}} f \eta_{\mathbb{X}})\mu_{\mathbb{X}} \cong f$, showing that (14) is an equivalence of categories as claimed.

From (13) and (14) we deduce

$$(\mathsf{WGDbl})_{\mathrm{ps},v}(\mathbb{X}, \mathbb{Y}) \simeq (\mathsf{Ta}_2)_{\mathrm{ps}}(\mathsf{D}\mathbb{X}, \mathsf{D}\mathbb{Y}) \ ,$$

that is, the functor $D$ is locally an equivalence of categories.

On the other hand, $D$ is also biessentially surjective on objects. In fact by Theorem 7.2 and by Proposition 7.4, for every $X \in (\mathsf{Ta}_2)_{\mathrm{ps}}$ there is a composite morphism $DQX \to N_hQX \to X$ in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ which is levelwise a categorical equivalence, hence an equivalence in $(\mathsf{Ta}_2)_{\mathrm{ps}}$. In conclusion, $D$ is a biequivalence. ∎

7.7. Remark. The biequivalence of the previous theorem is not an adjoint equivalence. The functor $D$ cannot be a right adjoint, since it doesn't preserve general limits, as it is $\pi_0$ at level 0. On the other hand, $Q$ clearly does not preserve products, so it cannot be a right adjoint either.

7.8. The main result.
Theorems 4.7 and 7.6 imply our main result, which is a biequivalence between the 2-categories of weakly globular double categories with pseudo-functors and vertical transformations, and the 2-category of bicategories with homomorphisms and icons. We give a more descriptive name to each of the functors realizing this biequivalence in the following definition:

7.9. Definition. The *fundamental bicategory functor* $\mathbf{Bic}\colon \mathsf{WGDbl}_{\text{ps}} \to \mathsf{Bicat}_{\text{icon}}$ is the composite $GD$. The *associated double category functor* $\mathbf{Dbl}\colon \mathsf{Bicat}_{\text{icon}} \to \mathsf{WGDbl}_{\text{ps}}$ is the composite $QN$. This last functor will also be called the *double category of marked paths functor*, for reasons that will be come clear in Section 8.2 below.

7.10. Theorem. *There is a biequivalence of* 2*-categories*

$$\mathbf{Bic} : \mathsf{WGDbl}_{\text{ps},v} \simeq \mathsf{Bicat}_{\text{icon}} \ : \mathbf{Dbl}.$$

Proof. This follows immediately from theorems 4.7 and 7.6. ∎

7.11. Remark. Note that we have $\mathbf{Dbl} \circ \mathbf{Bic} \simeq \mathrm{Id}_{\mathsf{WGDbl}}$ and $\mathbf{Bic} \circ \mathbf{Dbl} \simeq \mathrm{Id}_{\mathsf{Bicat}}$, but these functors are not adjoint, since $D$ and $Q$ are not adjoint as shown in Remark 7.7, but $G$ and $N$ form a biadjoint biequivalence as proved in [20].

## 8. Bicategories and Double Categories

In the previous section we established our main result, namely the existence of a biequivalence of 2-categories,

$$\mathbf{Bic} : \mathsf{WGDbl}_{\text{ps},v} \simeq \mathsf{Bicat}_{\text{icon}} \ : \mathbf{Dbl}.$$

Both 2-functors, **Bic** and **Dbl**, were obtained as compositions of other functors. Whenever we want to translate bicategorical concepts into concepts of weakly globular double categories, or vice versa, we will need to have an explicit description of the correspondence. In other words, we will need an explicit description of $\mathbf{Bic}\mathbb{X}$ for a weakly globular double category $\mathbb{X}$ and of $\mathbf{Dbl}(\mathcal{B})$ for a bicategory $\mathcal{B}$. Studying the explicit description of $\mathbf{Bic}(\mathbb{X})$ will also give us a clearer understanding of the role of the induced Segal maps condition characterizing weakly globular double categories. Conversely, the explicit description of $\mathbf{Dbl}(\mathcal{B})$ will give us a better understanding of how rigidification of a bicategory is different from the strictification of a bicategory into a strict 2-category.

8.1. The fundamental bicategory.
Let $\mathbb{X}$ be a weakly globular double category. The *objects* of $\mathbf{Bic}\mathbb{X}$ are obtained as the connected components $\pi_0\mathbb{X}_0$ of the vertical arrow category $\mathbb{X}_0$. When $A$ is an object of $\mathbb{X}$, i.e., an element of $\mathbb{X}_{00}$, we write $\bar{A}$ for the corresponding object in $\mathbf{Bic}\mathbb{X}$. Note that $\bar{A} = \bar{B}$ if and only if there is a (unique) vertical arrow $v\colon A \to B$ in $\mathbb{X}$ (since the vertical arrow category $\mathbb{X}_0$ is posetal and groupoidal).

For any two objects $\bar{A}$ and $\bar{B}$ in $\mathbb{X}$, the *set of arrows*, $\mathbf{Bic}\mathbb{X}(\bar{A}, \bar{B})$ is obtained as a disjoint union of horizontal hom-sets in $\mathbb{X}$,

$$\mathbf{Bic}\mathbb{X}(\bar{A}, \bar{B}) = \coprod_{\substack{\bar{A}' = \bar{A} \\ \bar{B}' = \bar{B}}} \mathbb{X}_h(A', B').$$

Note that we do not put an equivalence relation on the horizontal arrows of $\mathbb{X}$ to obtain the arrows of the fundamental bicategory; we will therefore use the same symbol to denote a horizontal arrow in $\mathbb{X}$ and the corresponding arrow in $\mathbf{Bic}(\mathbb{X})$.

For any two arrows $\bar{A} \underset{g}{\overset{f}{\rightrightarrows}} \bar{B}$ in $\mathbf{Bic}\mathbb{X}$ represented by horizontal arrows $A_1 \xrightarrow{f} B_1$ and $A_2 \xrightarrow{g} B_2$ in $\mathbb{X}$, the *2-cells* from $f$ to $g$ correspond to double cells of the form

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & B_1 \\ {\scriptstyle v}\downarrow & \alpha & \downarrow{\scriptstyle w} \\ A_2 & \xrightarrow[g]{} & B_2 \end{array} \quad .$$

Since $v$ and $w$ are unique, we will denote the corresponding 2-cell in $\mathbf{Bic}\mathbb{X}$ by $\alpha\colon f \Rightarrow g$.

Let $f\colon A_1 \to B_1$ and $g\colon B_2 \to C_2$ be horizontal arrows such that there is an invertible vertical arrow $v\colon B_2 \to B_1$. Then the induced Segal maps condition gives rise to a diagram

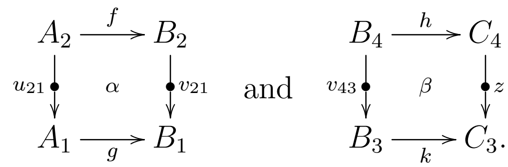
$$\begin{array}{ccccc} A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C_3 \\ \Big| & & \downarrow{\scriptstyle y} & \varphi_{g_3,g} & \downarrow{\scriptstyle z} \\ {\scriptstyle x}\Big| & \varphi_{f_3,f} & B_2 & \xrightarrow[g]{} & C_2 \\ \Big| & & \downarrow{\scriptstyle v} & & \\ A_1 & \xrightarrow[f]{} & B_1 & & \end{array}$$

as in Remark 6.8. Then composition of $f\colon \bar{A}_1 \to \bar{B}_1$ and $g\colon \bar{B}_2 \to \bar{C}_2$ (where $\bar{B}_1 = \bar{B}_2$) in $\mathbf{Bic}\mathbb{X}$ is defined to be the horizontal composite $g_3 \circ f_3\colon \bar{A}_3 \to \bar{C}_3$.

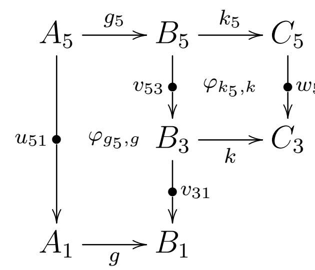
The horizontal composition of 2-cells is defined as follows: Let

$$\bar{A} \underset{g}{\overset{f}{\rightrightarrows}} \bar{B} \underset{k}{\overset{h}{\rightrightarrows}} \bar{C} \qquad (\Downarrow\alpha,\ \Downarrow\beta)$$

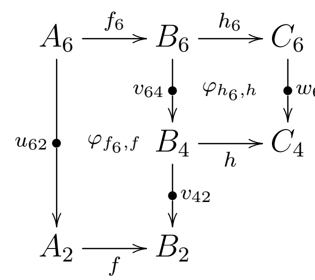
be a diagram of arrows and cells in the fundamental bicategory, represented by double cells in $\mathbb{X}$,



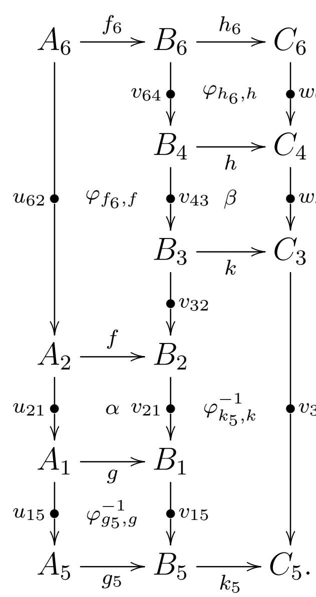
Let the composite of $g$ and $k$ in $\mathbf{Bic}\mathbb{X}$ be the arrow $k_5 \circ g_5$ with a corresponding diagram (as in Remark 6.8),



and let the composite of $f$ and $h$ be the arrow $h_6 \circ f_6$ as in the diagram



Then the composition of $\alpha$ and $\beta$ is represented by the following pasting of double cells:



(Here, $u_{ij} = u_{ji}^{-1}$ and $u_{jk} \cdot u_{ij} = u_{ik}$, and analogous for $v$ and $w$, since the vertical category is groupoidal posetal. Furthermore, the same holds for the cells, because they are components of a vertical transformation.)

The units for the composition are obtained from the functor $\mu_0 \colon \mathbb{X}_0^d \to \mathbb{X}_0$ which is part of the equivalence of categories, $\mathbb{X}_0^d \simeq \mathbb{X}_0$. For an object $\bar{A}$ in **Bic**$\mathbb{X}$, $1_{\bar{A}}$ is the horizontal arrow $\mathrm{Id}_{\mu_0(\bar{A})}$.

There are associativity and unit isomorphisms for this composition that satisfy the usual coherence conditions by the results in [33] and [20].

8.2. The double category of marked paths.

Let $\mathcal{B}$ be a bicategory. Before we begin the construction of $\mathbf{Dbl}(\mathcal{B})$, we first choose a composite $\varphi_{f_1,\dots,f_n}$ for each finite path $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ of (composable) arrows in $\mathcal{B}$. If the path is empty, we take $\varphi_{A_0} = 1_{A_0}$. For each path of such paths,

$$(\xrightarrow{f_{i_1}} \cdots \xrightarrow{f_{i_{n_1}}})(\xrightarrow{f_{i_{n_1+1}}} \cdots \xrightarrow{f_{i_{n_2}}}) \quad \cdots \quad (\xrightarrow{f_{i_{n_{m-1}+1}}} \cdots \xrightarrow{f_{i_{n_m}}})$$

the associativity and unit cells give rise to unique invertible comparison 2-cells, which we denote by

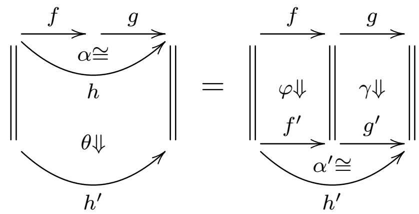
$$\Phi_{f_{i_1}\cdots f_{i_{n_1}}, f_{i_{n_1+1}}\cdots f_{i_{n_2}},\dots,f_{i_{n_{m-1}+1}}\cdots f_{i_{n_m}}} \colon \varphi_{\varphi_{f_{i_1},\dots,f_{i_{n_1}}},\varphi_{f_{i_{n_1+1}},\dots,f_{i_{n_2}}},\dots,\varphi_{f_{i_{n_{m-1}+1}},\dots,f_{i_{n_m}}}} \Rightarrow \varphi_{f_{i_1},\dots,f_{i_{n_m}}}.$$

(The uniqueness of these cells follows from the associativity and unit coherence axioms.) With these chosen composites and cells, we will walk through the constructions corresponding to the functors in the composition $\mathbf{Dbl} = QN = PSt\,SN$. The 2-nerve $N\mathcal{B} \colon \Delta^{\mathrm{op}} \to \mathsf{Cat}$ has the following components:

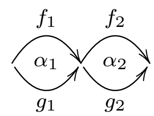
- $(N\mathcal{B})_0$ is the discrete category with objects $\mathcal{B}_0$.
- $(N\mathcal{B})_1$ has objects $\mathcal{B}_1$, i.e., the arrows of $\mathcal{B}$, and arrows the 2-cells of $\mathcal{B}$.
- $(N\mathcal{B})_2$ has objects diagrams of the form

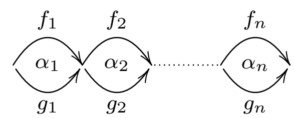
  $$\xrightarrow{f}\xrightarrow{g}, \quad \alpha \colon g f \overset{\cong}{\Rightarrow} h$$

  in $\mathcal{B}$, and arrows cylinders between such diagrams, i.e., an arrow from $(f,g,h,\alpha)$ to $(f',g',h',\alpha')$ is a triple $(\varphi,\gamma,\theta)$ of 2-cells, $\varphi\colon f \Rightarrow f'$, $\gamma\colon g \Rightarrow g'$, and $\theta\colon h \Rightarrow h'$, such that $\theta \cdot \alpha = \alpha' \cdot (\gamma \circ \varphi)$,




The pseudo-functor $SN\mathcal{B}\Delta^{\text{op}} \to \mathsf{Cat}$ has then

- $(SN\mathcal{B})_0$ is the discrete category on the objects of $\mathcal{B}$;
- $(SN\mathcal{B})_1$ has as objects the arrows of $\mathcal{B}$ and as arrows the 2-cells of $\mathcal{B}$;
- $(SN\mathcal{B})_2$ has as objects paths of length 2 in $\mathcal{B}$, $\xrightarrow{f_1}\xrightarrow{f_2}$ and as arrows horizontal paths of 2-cells of length 2 in $\mathcal{B}$, .

- $(SN\mathcal{B})_n$ has as objects paths of length $n$ in $\mathcal{B}$, $\xrightarrow{f_1}\xrightarrow{f_2}\cdots\xrightarrow{f_n}$ and as arrows horizontal paths of 2-cells of length $n$ in $\mathcal{B}$, .


By the construction described in Remark 6.10 and taking the pseudo-inverse of the horizontal nerve, we obtain the following description of the double category $\mathbf{Dbl}(\mathcal{B})$.

The *objects* of $\mathbf{Dbl}(\mathcal{B})$ are given as pairs of an arrow $\psi\colon [0] \to [n]$ in $\Delta$ with a path, $A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$, of length $n$ in $\mathcal{B}$, for all $n$. Since the arrow $\psi$ is determined by its image $i_0 = \psi(0) \in [n]$, we will denote this object in $\mathbf{Dbl}(\mathcal{B})$ by

$$(\ A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\ ; i_0)$$

and think of $A_{i_0}$ as a marked object along the path. So we will also use the notation

$$A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n\,.$$

There is a unique *vertical arrow* from $A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n$ to $B_0 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{j_0}} [B_{j_0}] \xrightarrow{g_{j_0+1}} \cdots \xrightarrow{g_m} A_m$ if and only if $A_{i_0} = B_{j_0}$. In diagrams we will include this vertical arrow in the following way

$$\begin{array}{ccccccccc} A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i_0}} & [A_{i_0}] & \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n \\ & \| & \\ B_0 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{j_0}} & [B_{j_0}] & \xrightarrow{g_{j_0+1}} \cdots \xrightarrow{g_m} A_n \end{array}$$

*Horizontal arrows* in $\mathbf{Dbl}(\mathcal{B})$ are given as pairs of an arrow $\psi\colon [1] \to [n]$ in $\Delta$ with a path of length $n$ in $\mathcal{B}$, for all $n$. Analogous to what we did for objects we denote horizontal arrows by

$$(\,A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\,; i_0, i_1) \quad \text{with} \quad i_0 \leq i_1,$$

or by

$$A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n\,.$$

The domain of $(\,A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\,; i_0, i_1)$ is $(\,A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\,; i_0)$ and the codomain is $(\,A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\,; i_1)$. For a horizontal identity arrow,

$$(\,A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\,; i_0, i_0)$$

we will also use the notation

$$A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \quad \xrightarrow{f_{i_0}} [[A_{i_0}]] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n$$

or

$$A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \quad \xrightarrow{f_{i_0}} [A_{i_0}] = [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n$$

when this makes it easier to fit such an arrow into a diagram representing a double cell as shown below.

A *double cell* consists of two horizontal arrows

$$(\,A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n\,; i_0, i_1) \text{ and } (\,B_0 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_m} B_m\,; j_0, j_1)$$

(for the vertical domain and codomain respectively), such that $A_{i_0} = B_{j_0}$ and $A_{i_1} = B_{j_1}$ (such that there are unique vertical arrows between the domains of these arrows and between the codomains of these arrows), together with a 2-cell in $\mathcal{B}$ between the chosen composites,

$$\varphi_{f_{i_0+1},\ldots,f_{i_1}} \overset{\Downarrow\alpha}{\Longrightarrow} \varphi_{g_{j_0+1},\ldots,g_{j_1}}\,.$$

We combine all this information together in the following diagram

$$\begin{array}{ccccccc}
A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} & [A_{i_0}] & \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} & [A_{i_1}] & \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n \\
 & \| & \varphi_{f_{i_0+1},\ldots,f_{i_1}} & \| & \\
 & \| & \alpha & \| & \\
 & \| & \varphi_{g_{j_0+1},\ldots,g_{j_1}} & \| & \\
B_0 \xrightarrow[g_1]{} \cdots \xrightarrow[g_{j_0}]{} & [B_{j_0}] & \xrightarrow[g_{j_0+1}]{} \cdots \xrightarrow[g_{j_1}]{} & [B_{j_1}] & \xrightarrow[g_{j_1+1}]{} \cdots \xrightarrow[g_m]{} B_m
\end{array}$$

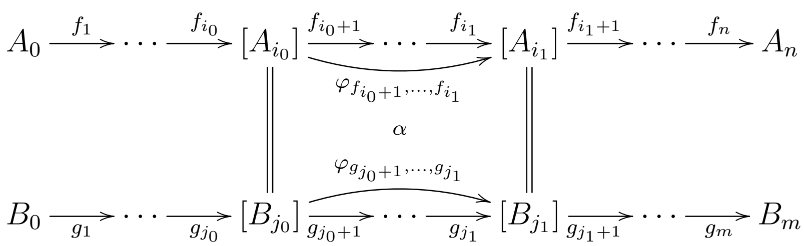
So this represents a double cell in $\mathbf{Dbl}(\mathcal{B})$.

Two horizontal arrows,

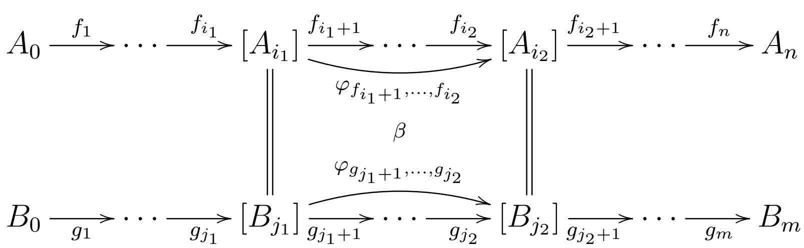
$$(A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n; i_0, i_1) \text{ and } (B_0 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} A_n; j_0, j_1),$$

are composable if and only if the two paths are the same, i.e., $m = n$, $A_i = B_i$ and $f_i = g_i$ for all $i = 1, \ldots, n$, and furthermore, $i_1 = j_0$. In that case, the *horizontal composition* of these arrows is given by $(A_0 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n; i_0, j_1)$.

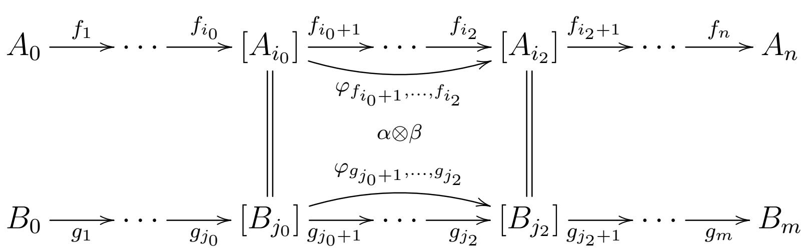
The *horizontal composition of double cells*



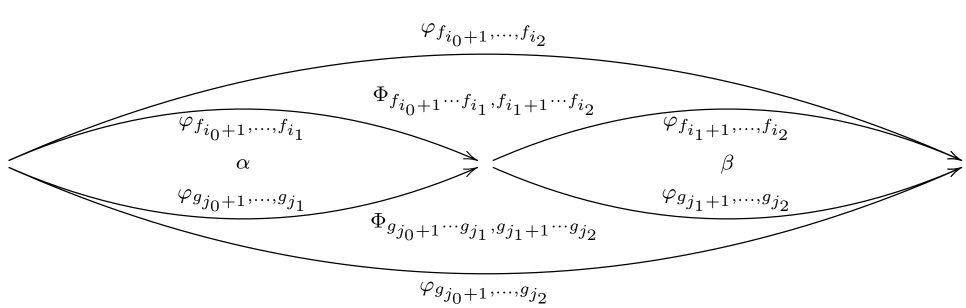

and




is defined to be




where $\alpha \otimes \beta$ is the 2-cell in $\mathcal{B}$ given by the following pasting diagram



8.3. Remarks.

1. Note that both the category of horizontal arrows and the category of vertical arrows of $\mathbf{Dbl}(\mathcal{B})$ are posetal.

2. We call $\mathbf{Dbl}(\mathcal{B})$ the *double category of marked paths in* $\mathcal{B}$.

3. For a 2-category $\mathcal{C}$ there is a double category $H\mathcal{C}$ with the arrows of $\mathcal{C}$ in the horizontal direction and only identity arrows in the vertical direction, and the double cells correspond to the 2-cells in $\mathcal{C}$. The double category $\mathbf{Dbl}(\mathcal{C})$ is not isomorphic to this double category, $H\mathcal{C}$, but it is 2-equivalent to it. And the same statement applies to a category $\mathbf{C}$: $H\mathbf{C} \ncong \mathbf{Dbl}(\mathbf{C})$, but $H\mathbf{C} \simeq_2 \mathbf{Dbl}(\mathbf{C})$.

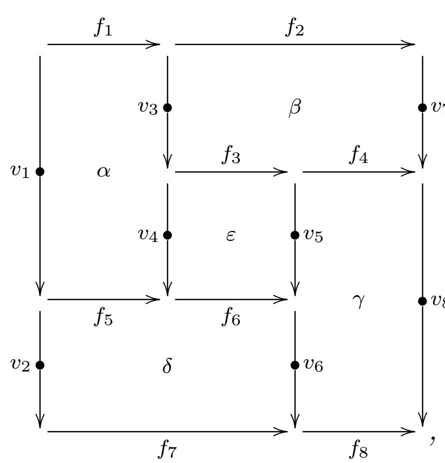
## 9. Pasting diagrams in weakly globular double categories

It was observed by Dawson and Paré [9] that not every rectangular tiling diagram of double cells is composable. However, when it is composable, their general associativity result states that any two ways of composing it by successive binary compositions will give the same result.

Since the tilings in our weakly globular double categories are related to pasting diagrams in bicategories via the functors **Bic** and **Dbl** and all pasting diagrams in a bicategory are composable, we expect that all rectangular tilings of double cells in a weakly globular double category are composable.

Whereas Dawson and Paré proved in [9] that every rectangular tiling in a double category can be composed under certain factorization conditions we will show in an upcoming paper [27] that the existence of certain vertically invertible double cells is sufficient to prove this result. Although the complete proof has several technical parts, the main idea can be illustrated with a single pinwheel tiling. So we want to include this here as an illustration of how one uses the special properties of a weakly globular double category.

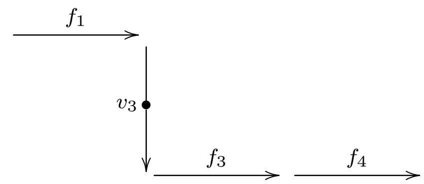
9.1. Proposition. *Every pinwheel diagram,*



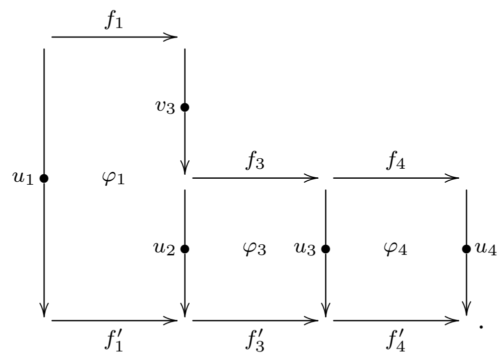

(15)

*of double cells in a weakly globular double category is composable.*

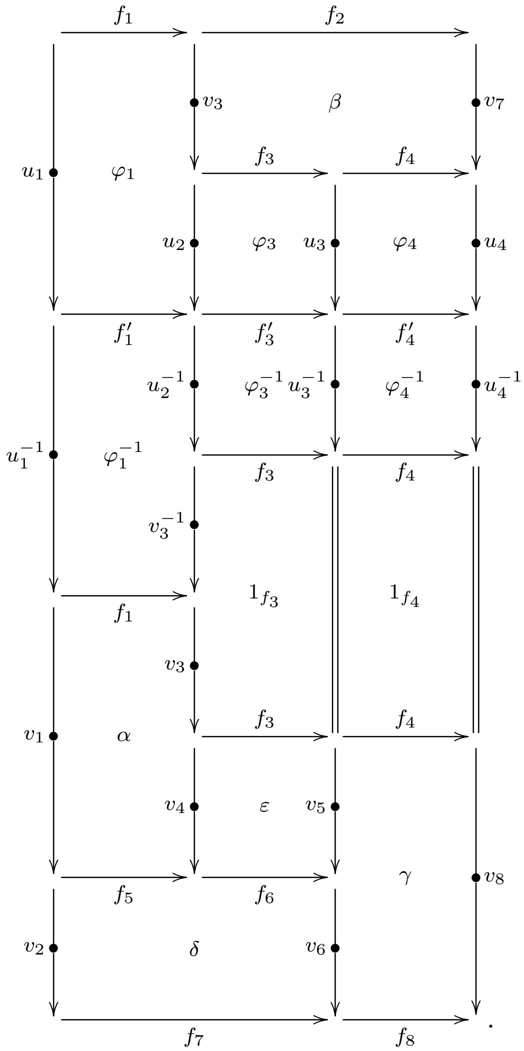
PROOF. We begin by applying the induced Segal maps condition as in Remark 6.8 with $n = 3$ to the configuration




in the diagram (15). This gives us vertically invertible double cells as in the following diagram,

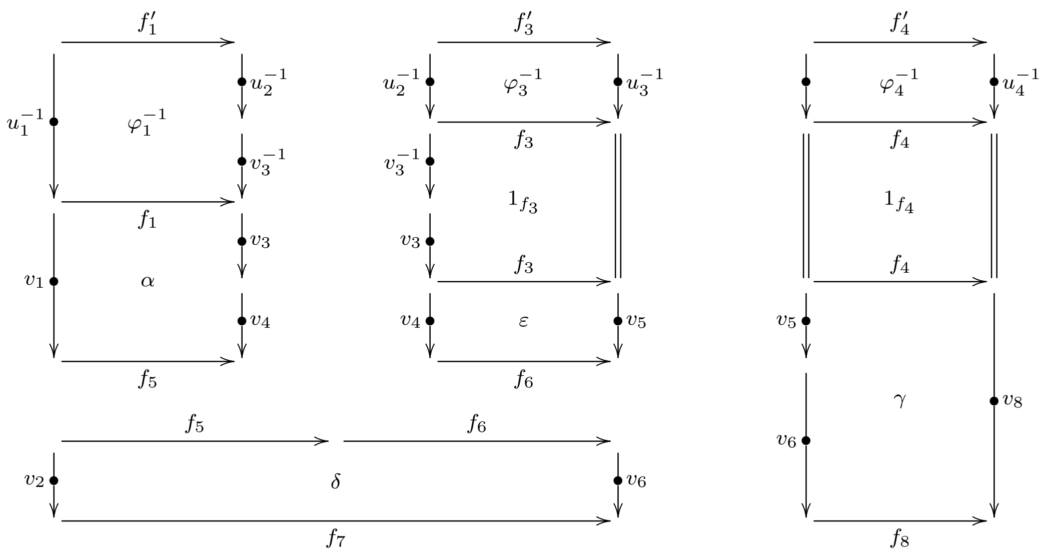



These cells and their inverses can be inserted in the pinwheel diagram so that its pasting, if it exists, is equal to the pasting of the following diagram:



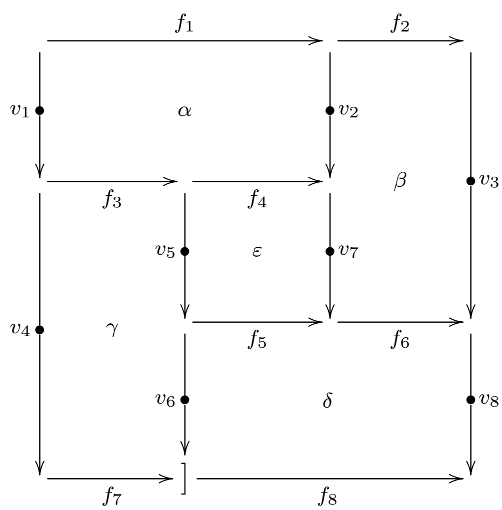

The effect of inserting these cells is the creation of a new horizontal line - a vertical factorization of the whole tiling that we did not have before. The tiling above this line is clearly composable: first compose $\varphi_3$ and $\varphi_4$ horizontally, and then compose the result vertically with $\beta$ and finally compose $\varphi_1$ horizontally with this composite, $((\varphi_4 \circ \varphi_3) \cdot \beta) \circ \varphi_1$.

The pasting diagram below the horizontal line is also composable as indicated in the following diagram




So the pinwheel diagram (15) is indeed composable. ■

Note that the opposite pinwheel,




can be shown to be composable by the mirror image of this factorization.

## 10. Homotopical application

In this section we give a homotopical application of weakly globular double categories. We introduce a subcategory of the latter, whose objects we call groupoidal weakly globular double categories.

We show that this category provides an algebraic model of 2-types. This generalizes a result of [4], where weakly globular double groupoids are proved to model 2-types.

10.1. 2-equivalences in weakly globular double categories.
We are going to introduce a notion of equivalence in $(\mathsf{WGDbl})_{\rm ps}$ modeled over the comparison with Tamsamani weak 2-categories. We need the following preliminary lemma. Let $\pi_0 : \mathsf{Cat} \to \mathsf{Set}$ associate to a category the set of isomorphism classes of its objects.

10.2. Lemma.

*i) There is a functor*

$$\Pi_0 : (\mathsf{WGDbl})_{\rm ps} \to \mathsf{Cat}$$

*such that, for all* $\mathbb{X} \in (\mathsf{WGDbl})_{\rm ps}$ *and* $n \geq 0$, $(N\Pi_0\mathbb{X})_n = (\pi_0^* N_h \mathbb{X})_n$ *where* $N : \mathsf{Cat} \to [\Delta^{\rm op}, \mathsf{Set}]$ *is the nerve functor.*

*ii) If* $\mathbb{X} \in (\mathsf{WGDbl})_{\rm ps}$ *and* $a, b \in \mathbb{X}_0^d$, *let* $\mathbb{X}_{(a,b)}$ *be the full subcategory of* $\mathbb{X}_1$, *whose objects* $z$ *are such that* $\gamma\partial_0(z) = a$, $\gamma\partial_1(z) = b$, *where* $\partial_i : \mathbb{X}_1 \to \mathbb{X}_0$ *are the face operators and* $\gamma : \mathbb{X}_0 \to \mathbb{X}_0^d$. *Then* $\mathbb{X}_1 \cong \coprod\limits_{a,b\in\mathbb{X}_0^d} \mathbb{X}_{(a,b)}$.

*iii) A morphism* $F : \mathbb{X} \to \mathbb{Y}$ *in* $(\mathsf{WGDbl})_{\rm ps}$ *induces functors* $F_{(a,b)} : \mathbb{X}_{(a,b)} \to \mathbb{Y}_{(Fa,Fb)}$ *for all* $a, b \in \mathbb{X}_0^d$.

Proof.

i) By definition, since $\mathbb{X}$ is weakly globular, there is a categorical equivalence

$$\mathbb{X}_1 \times_{\mathbb{X}_0} \overset{n}{\cdots} \times_{\mathbb{X}_0} \mathbb{X}_1 \simeq \mathbb{X}_1 \times_{\mathbb{X}_0^d} \overset{n}{\cdots} \times_{\mathbb{X}_0^d} \mathbb{X}_1$$

for all $n \geq 2$. Since $\pi_0$ sends categorical equivalences to isomorphisms and preserves fiber products over discrete objects, this implies

$$\pi_0(\mathbb{X}_1 \times_{\mathbb{X}_0^d} \overset{n}{\cdots} \times_{\mathbb{X}_0^d} \mathbb{X}_1) \cong \pi_0\mathbb{X}_1 \times_{\pi_0\mathbb{X}_0^d} \overset{n}{\cdots} \times_{\pi_0\mathbb{X}_0^d} \pi_0\mathbb{X}_1.$$

This shows that $\pi_0 N_h \mathbb{X}$ is the nerve of a category, which we denote by $\Pi_0\mathbb{X}$. The functor $\pi_0$ induces a functor $\pi_0^* : \mathrm{Ps}[\Delta^{\rm op}, \mathsf{Cat}] \to [\Delta^{\rm op}, \mathsf{Set}]$. In particular, if $F$ is a morphism in $(\mathsf{WGDbl})_{\rm ps}$, $\pi_0^* F$ is a morphism between nerves of categories. This defines $\Pi_0$ on morphisms.

ii) This follows immediately by considering the functor $\gamma(\partial_0, \partial_1) : \mathbb{X}_1 \to \mathbb{X}_0^d \times \mathbb{X}_0^d$, since $\mathbb{X}_0^d$ is discrete.

iii) Since $F$ is a pseudo-natural transformation, there is a pseudo-commutative diagram

$$\begin{array}{ccc} \mathbb{X}_1 & \xrightarrow{(\partial_0,\partial_1)} & \mathbb{X}_0 \times \mathbb{X}_0 \\ \downarrow & & \downarrow \\ \mathbb{Y}_1 & \xrightarrow[(\partial'_0,\partial'_1)]{} & \mathbb{Y}_0 \times \mathbb{Y}_0 \end{array}$$

and therefore, since $\mathbb{X}_0^d$ and $\mathbb{Y}_0^d$ are discrete, a commutative diagram

$$\begin{array}{ccc} \mathbb{X}_1 & \xrightarrow{\gamma_{\mathbb{X}}(\partial_0,\partial_1)} & \mathbb{X}_0^d \times \mathbb{X}_0^d \\ \downarrow & & \downarrow \\ \mathbb{Y}_1 & \xrightarrow[\gamma_{\mathbb{Y}}(\partial'_0,\partial'_1)]{} & \mathbb{Y}_0^d \times \mathbb{Y}_0^d \end{array}$$

This determines the functor $F_{(a,b)}\colon \mathbb{X}_{(a,b)} \to \mathbb{Y}_{(Fa,Fb)}$ for all $a, b \in \mathbb{X}_0^d$.

■

10.3. DEFINITION. We say that a morphism $F : \mathbb{X} \to \mathbb{Y}$ in $(\mathsf{WGDbl})_{\mathrm{ps}}$ is a 2-equivalence if

i) For all $a, b \in \mathbb{X}_0^d$, $F_{(a,b)} : \mathbb{X}_{(a,b)} \to \mathbb{Y}_{(Fa,Fb)}$ is an equivalence of categories.

ii) $\Pi_0 F$ is an equivalence of categories, where $\Pi_0$ is as in Lemma 10.2

10.4. PROPOSITION. *The functors $Q$ and $D$ preserve 2-equivalences.*

PROOF. The fact that $D$ preserves 2-equivalences is immediate from the definitions. Let $f : X \to Y$ be a 2-equivalence in $(\mathsf{Ta}_2)_{\mathrm{ps}}$. By Theorem 7.2, there is a pseudo-commutative diagram in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$

$$\begin{array}{ccc} N_h Q X & \xrightarrow{\alpha_X} & X \\ {\scriptstyle N_hQf}\downarrow & \sim & \downarrow{\scriptstyle f} \\ N_h Q Y & \xrightarrow[\alpha_Y]{} & Y \end{array} \tag{16}$$

in which $\alpha_X$ and $\alpha_Y$ are levelwise categorical equivalences. Applying the functor $\pi_0^* : \mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}] \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$ we obtain a commutative diagram in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$

$$\begin{array}{ccc} \pi_0^* N_h Q X & \xrightarrow{\pi_0^*\alpha_X} & \pi_0^* X \\ {\scriptstyle \pi_0^*N_hQf}\downarrow & & \downarrow{\scriptstyle \pi_0^* f} \\ \pi_0^* N_h Q Y & \xrightarrow[\pi_0^*\alpha_Y]{} & \pi_0^* Y. \end{array}$$

Recalling that $\pi_0^* N_h Q X = N\Pi_0 Q X$, $\pi_0^* X = N\Pi_0 X$ and similarly for the other terms, and applying the functor $P : [\Delta^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Cat}$ which is left adjoint to the nerve, we obtain the commutative diagram in $\mathsf{Cat}$

$$\begin{array}{ccc} \Pi_0 Q X & \xrightarrow{P\pi_0^*\alpha_X} & \Pi_0 X \\ {\scriptstyle \Pi_0Qf}\downarrow & & \downarrow{\scriptstyle \Pi_0 f} \\ \Pi_0 Q Y & \xrightarrow[P\pi_0^*\alpha_Y]{} & \Pi_0 Y. \end{array} \tag{17}$$

Since $\alpha_X$ and $\alpha_Y$ are levelwise categorical equivalences, $\pi_0^*\alpha_X$, and $\pi_0^*\alpha_Y$ are isomorphisms, hence $P\pi_0^*\alpha_X$ and $P\pi_0^*\alpha_Y$ are isomorphisms. Since $f$ is a 2-equivalence in $(\mathsf{Ta}_2)_{\mathrm{ps}}$, by definition $\Pi_0 F$ is an equivalence of categories. Hence the commutativity of (17) implies that $\Pi_0 Qf$ is an equivalence of categories. Also, for each $a,b \in (N_h QX)^d_{0*} \cong X_{0*}$, by (16) we obtain a pseudo-commutative diagram in $\mathsf{Cat}$

$$\begin{array}{ccc} (QX)_{(a,b)} & \xrightarrow{(\alpha_X)_{(a,b)}} & X_{(a,b)} \\ {\scriptstyle (Qf)_{(a,b)}}\big\downarrow & \sim & \big\downarrow{\scriptstyle f_{(a,b)}} \\ (QY)_{(fa,fb)} & \xrightarrow[(\alpha_Y)_{(fa,fb)}]{} & Y_{(fa,fb)}. \end{array} \tag{18}$$

But $(\alpha_X)_{(a,b)}$ and $(\alpha_Y)_{(fa,fb)}$ are categorical equivalences because $(\alpha_X)_1$ and $(\alpha_Y)_1$ are categorical equivalences. Also $f_{(a,b)}$ is a categorical equivalence because $f$ is a 2-equivalence in $(\mathsf{Ta}_2)_{\mathrm{ps}}$. Hence by (18), $(Qf)_{(a,b)}$ is a categorical equivalence. In conclusion, $Qf$ is a 2-equivalence in $(\mathsf{WGDbl})_{\mathrm{ps}}$.

∎

10.5. Tamsamani weak 2-groupoids.
By imposing suitable invertibility conditions to Tamsamani weak 2-categories one obtains the notion of Tamsamani weak 2-groupoid, which was shown in [33] to give an algebraic model of 2-types of topological spaces. We now recall this background.

10.6. Definition. The category $\mathsf{GTa}_2$ of *Tamsamani weak 2-groupoids* is the full subcategory of $\mathsf{Ta}_2$ whose objects $X$ are such that, for all $a,b \in X_0$, $X_{(a,b)}$ and $\Pi_0 X$ are groupoids.

10.7. Theorem. *[33] There are functors*

$$B : \mathsf{GTa}_2 \rightleftarrows \text{2-}\mathsf{Types} : F$$

*inducing an equivalence of categories*

$$B : \mathsf{GTa}_2/{\sim^2} \rightleftarrows \mathcal{H}o(\text{2-}\mathsf{Types}) : F$$

*In particular, for each $X \in \mathsf{GTa}_2$, there is a 2-equivalence, $\gamma_X : X \to FBX$, natural in $X$ and, for each 2-type $Y$, there is a weak homotopy equivalence $\Gamma_Y : BFY \to Y$, natural in $Y$.*

10.8. Remark. Given $X \in \mathsf{Ta}_2$, $BX$ is homotopy equivalent to the nerve of the Grothendieck construction on $X \in [\Delta^{\mathrm{op}}, \mathsf{Cat}]$, see [7], [32]. The latter can also be used to define the classifying space for objects of $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$, so that in fact we have a functor $B : (\mathsf{GTa}_2)_{\mathrm{ps}} \to \text{2-}\mathsf{Types}$. In particular if $f \in (\mathsf{GTa}_2)_{\mathrm{ps}}$ is such that $Bf_k$ is a weak equivalence for all $k$, then $Bf$ is a weak equivalence (see [7] or [32]).

10.9. Groupoidal weakly globular double categories.
We now introduce the subcategory of groupoidal weakly globular double categories, which we are going to show form an algebraic model of 2-types.

10.10. Definition. The category $\mathsf{GWGDbl}$ of groupoidal weakly globular double categories is the full subcategory of $\mathsf{WGDbl}$ whose objects $\mathbb{X}$ are such that

i) For all $a, b \in \mathbb{X}_0^d$, $\mathbb{X}_{(a,b)}$ is a groupoid.

ii) $\Pi_0\mathbb{X}$ is a groupoid, where $\Pi_0 : \mathsf{WGDbl} \to \mathsf{Cat}$ is as in Lemma 10.2.

A morphism in $\mathsf{GWGDbl}$ is a 2-equivalence if and only if it is a 2-equivalence in $\mathsf{WGDbl}$. The category $(\mathsf{GWGDbl})_{\mathrm{ps}}$ is the full subcategory of $(\mathsf{WGDbl})_{\mathrm{ps}}$ whose objects are in $\mathsf{GWGDbl}$.

10.11. Remark. It follows immediately from the definitions that the functors

$$D : (\mathsf{WGDbl})_{\mathrm{ps}} \leftrightarrows (\mathsf{Ta}_2)_{\mathrm{ps}} : Q$$

restrict to functors

$$D : (\mathsf{GWGDbl})_{\mathrm{ps}} \leftrightarrows (\mathsf{GTa}_2)_{\mathrm{ps}} : Q$$

The notion of groupoidal weakly globular double category is more general than the one of weakly globular double groupoid introduced in [4]. In particular, objects of $\mathsf{GWGDbl}$ are not necessarily double groupoids, as the horizontal categories $\mathbb{X}_{*0}$ and $\mathbb{X}_{*1}$ are not required to be groupoids if $\mathbb{X} \in \mathsf{GWGDbl}$. This notion is similar to the one of Tamsamani weak 2-groupoids, where inverses of horizontal morphisms exist only in a weak sense. Indeed, the next proposition establishes a comparison with Tamsamani's model.

10.12. Modelling 2-types.
We will not prove that groupoidal weakly globular double categories do indeed model homotopy 2-types, and find that they provide an algebraic Postnikov decomposition for their classifying space.

10.13. Theorem. *The functors*

$$BD : (\mathsf{GWGDbl})_{\mathrm{ps}} \leftrightarrows \mathsf{2\text{-}Types} : QF$$

*induce an equivalence of categories*

$$(\mathsf{GWGDbl})_{\mathrm{ps}}/\!\sim^2 \;\cong \mathcal{H}o(\mathsf{2\text{-}Types})\ .$$

Proof. Given $\mathbb{X} \in \mathsf{GWGDbl}$, there is a commutative diagram in $\mathrm{Ps}[\Delta^{\mathrm{op}}, \mathsf{Cat}]$

$$\begin{array}{ccc} N_h Q D\mathbb{X} & \xrightarrow{N_h Q \gamma_{D\mathbb{X}}} & N_h Q F B D\mathbb{X} \\ {\scriptstyle \alpha_{D\mathbb{X}}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{FBDX}} \\ D\mathbb{X} & \longrightarrow & FBD\mathbb{X} \end{array}$$

in which the vertical maps are levelwise equivalences and the horizontal maps are 2-equivalences.

On the other hand, by Remark 7.5, there is a map $\mu_{\mathbb{X}} : N_h\mathbb{X} \to D\mathbb{X}$ which is a levelwise equivalence. Since levelwise equivalences are 2-equivalences, we obtain a zig-zag of 2-equivalences between $N_h\mathbb{X}$ and $N_hQFBD\mathbb{X}$; hence $\mathbb{X} \cong QFBD\mathbb{X}$ in $(\mathsf{GWGDbl})_{\text{ps}}/\!\sim^2$.

Conversely, let $Y \in$ 2-type. By Theorem 7.2 and Proposition 7.4, we have a map in $\text{Ps}[\Delta^{\text{op}}, \mathsf{Gpd}]$

$$DQFY \to FY$$

which is a levelwise equivalence of categories. It follows (see Remark 10.8) that there is a weak homotopy equivalence $BDQFY \to BFY$. On the other hand, by Theorem 10.7 there is a weak equivalence $BFY \simeq Y$. Hence we have a zig-zag of weak homotopy equivalences between $BDQFY$ and $Y$, so that $BDQFY \cong Y$ in $\mathcal{H}o(\mathsf{2\text{-}Types})$. ∎

10.14. Remark. For every object $\mathbb{X}$ of $\mathsf{GWGDbl}$ it is possible, as in the case of the weakly globular double groupoids of [4], to give an algebraic description of the homotopy groups and of the Postnikov decomposition of the classifying space $BD\mathbb{X}$. This is done as follows. Let $B\Pi_0\mathbb{X}$ be the classifying space of the groupoid $\Pi_0\mathbb{X}$; for each $x \in \mathbb{X}_0^d$ let $\text{id}_x \in \mathbb{X}_{10}$ be $\text{id}_x = \sigma_0\gamma'(x)$, where $\gamma' : \mathbb{X}_0^d \to \mathbb{X}_0$ is the inverse of $\gamma : \mathbb{X}_0 \to \mathbb{X}_0^d$ and $\sigma_0 : \mathbb{X}_{00} \to \mathbb{X}_{10}$ is the degeneracy operator. Then it follows from [33] that

$$\pi_i(BD\mathbb{X}, x) = \pi_i(B\Pi_0\mathbb{X}, x) \quad i = 0, 1$$
$$\pi_2(BD\mathbb{X}, x) = \text{Hom}_{\mathbb{X}_1}(\text{id}_x, \text{id}_x).$$

Further, there is a morphism $\mathbb{X} \to c\Pi_0\mathbb{X}$, where $c\Pi_0\mathbb{X}$ is the double category which, as a category object in $\mathsf{Cat}$ in the vertical direction, is discrete with object of objects $\Pi_0\mathbb{X}$. From above, we see that the morphism $\mathbb{X} \to c\Pi_0\mathbb{X}$ induces isomorphisms of homotopy groups $\pi_i(BD\mathbb{X}) \cong \pi_i(Bc\Pi_0\mathbb{X}) \quad i = 0, 1$. Hence this gives algebraically the Postnikov decomposition of $BD\mathbb{X}$.

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