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GALOIS THEORIES OF COMMUTATIVE SEMIGROUPS VIA SEMILATTICES

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ABSTRACT. The classes of stably-vertical, normal, separable, inseparable, purely inseparable and covering morphisms, defined in categorical Galois theory, are characterized for the reflection of the variety of commutative semigroups into its subvariety of semilattices. It is also shown that there is an inseparable-separable factorization, but there is no monotone-light factorization.

1. Introduction

The present work is done at three levels, namely category theory, universal algebra and semigroup theory. The classes of morphisms we will characterize are defined generally at the level of category theory, in particular in categorical Galois theory. Some results are then stated at the level of universal algebra, concerning reflections into idempotent subvarieties of universal algebras (that is, in which every one-element subset is a subalgebra). At the level of semigroup theory, we analyse the reflection of the variety of commutative semigroups into its subvariety of semilattices, making use of the results stated for universal algebra. Some of the results included in this paper were previously obtained by the first-named author and are included in her PhD thesis [10].

A semi-left-exact reflection (in the sense of [3]) can be seen as an admissible Galois structure (in the sense of categorical Galois theory), in which the classes of admissible morphisms are the classes of all the morphisms in both category and subcategory. In the special case of the variety of semigroups, semi-left-exactness was called attainability in [8], where it was proved that the reflection of semigroups into semilattices is the unique attainable reflection of the variety of semigroups into a subvariety. Furthermore, this reflection satisfies the stronger property of having stable units, as shown in [6].

First, we state some results for admissible (= semi-left-exact) reflections of varieties of universal algebras. Second, Proposition 5.3 is stated, proving that the I-images of

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the projections of any pullback diagram are jointly monic, in the (sub)reflection of the variety of commutative semigroups into its subvariety of semilattices. Then, in the latter subreflection, with the help of the results for varieties and crucial Proposition 5.3, we will be able to characterize the classes of stably-vertical, covering, normal, separable, purely inseparable, and inseparable (homo)morphisms defined in categorical Galois theory. The first table at the end summarizes the characterizations of these classes. The second table addresses the existence of inseparable-separable (see [5]) and monotone-light factorizations (see [2]). In both tables the results used in each case are specified.

2. Preliminaries

Admissible reflection into idempotent subvarieties

In this section we define semi-left-exact reflection, in the sense of [3], and recall a characterization of a semi-left-exact reflection of a variety of universal algebras into an idempotent subvariety (that is, a variety in which every one-element subset is a subalgebra), which follows from a more general result in [12]. A semi-left-exact reflection is also called an admissible reflection in categorical Galois theory (see [1, §5.5]).

2.1. Definition. *A reflection $H \vdash I : \mathcal{C} \to \mathcal{M}$ into a full subcategory $\mathcal{M}$, with unit $\eta : 1_{\mathcal{C}} \to HI$, is said to be admissible if every pullback diagram*

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "v"'] & M \arrow[d, "g"] \\ B \arrow[r, "\eta_B"] & I(B) \end{tikzcd} ,$$

with $M \in \mathcal{M}$ has $u \in \mathrm{E}_I$, where E_I is the class of morphisms f in $\mathcal{C}$ such that $I(f)$ is an isomorphism.

Let us recall some data of a reflection of a variety of universal algebras into one of its subvarieties.

1. Let $\mathcal{C}$ be a variety of universal algebras. It is well known that there exists an adjunction $U \vdash F : \mathbf{Set} \to \mathcal{C}$ between the variety $\mathcal{C}$ and the category of sets, with unit $\lambda : 1_{\mathbf{Set}} \to UF$ and counit $\varepsilon : FU \to 1_{\mathcal{C}}$, such that:

   (a) F is a functor which assigns to a set S the free algebra $F(S)$ in $\mathcal{C}$.

   (b) U is the underlying functor which assigns to any $C \in \mathcal{C}$ its underlying set $U(C) \in \mathbf{Set}$, satisfying the following properties:

      i. U reflects isomorphisms, since an isomorphism in a variety of universal algebras is just a bijective homomorphism;

      ii. U preserves finite limits, since it has a left adjoint F.

(c) The unit $\lambda_S : S \to UF(S)$ is the inclusion map.

(d) The counit $\varepsilon_C : FU(C) \to C$ assigns to the letter $x \in FU(C)$ the element $x \in C$, since $U(\varepsilon_C)\lambda_{U(C)} = 1_{U(C)}$. On the other hand, $\varepsilon_C(\theta_{FU(C)}(x, y, ..., z)) = \theta_C(x, y, ..., z)$, for every n-ary operation θ in the variety $\mathcal{C}$, since ε_C is a (surjective) homomorphism in $\mathcal{C}$.

2. Let $H \vdash I : \mathcal{C} \to \mathcal{M}$ be a reflection of a variety $\mathcal{C}$ of universal algebras into its subvariety $\mathcal{M}$, with unit $\eta : 1_{\mathcal{C}} \to HI$.

   (a) Note that $\eta_A : A \to HI(A) = A/\sim_A$ is the canonical projection of A onto the quotient of A by the congruence $\sim_A$ associated by the reflection to A, hence a surjective homomorphism, for every A in $\mathcal{C}$.

   (b) It is easy to check that the following conditions are equivalent:

      i. the subvariety $\mathcal{M}$ is idempotent, in the sense that every one-element set $\{x\} \subseteq M$ is a subalgebra in any $M \in \mathcal{M}$;

      ii. every map $U_{T,M} : \mathcal{C}(T, M) \to \mathbf{Set}(\{*\}, U(M))$ is a surjection, for any object $M \in \mathcal{M}$, with T a terminal object in $\mathcal{C}$.

According to the following Lemma 2.3, stated in [12, §2], a reflection $H \vdash I : \mathcal{C} \to \mathcal{M}$ into an idempotent subvariety is admissible if and only if every congruence class $[x]_{\sim_C}$ of the decomposition of any $C(\in \mathcal{C})$ associated to the reflection is itself I-indecomposable.

2.2. Definition. *Consider any morphism $\mu : T \to HI(C)$ from a terminal object T into $HI(C)$, for some $C \in \mathcal{C}$. The connected component associated to the morphism μ is the pullback C_μ in the following pullback square.*

$$\begin{tikzcd} C_\mu \arrow[r] \arrow[d] & T \arrow[d, "\mu"] \\ C \arrow[r, "\eta_C"] & HI(C) \end{tikzcd} .$$

2.3. Lemma. *Let $H \vdash I : \mathcal{C} \to \mathcal{M}$ be a reflection into an idempotent subvariety of universal algebras. The following conditions are equivalent:*

1. *$H \vdash I$ is admissible;*

2. *$HI(C_\mu) \cong T$, for every connected component C_μ, where T is the one-element algebra.*

Factorization system derived from an admissible reflection

It is well known that there is a factorization system $(\mathrm{E}_I, \mathrm{M}_I)$ associated to the reflection. The class of vertical morphisms E_I is the class of homomorphisms f in $\mathcal{C}$ whose I-images $I(f)$ are isomorphisms. The class of trivial coverings M_I is the class of homomorphisms $f : A \to B$ such that the commutative diagram associated to the identity

$HI(f) \circ \eta_A = \eta_B \circ f$ is a pullback diagram in $\mathcal{C}$ (see [2]). The following Proposition 2.4 states, in other words, that $f \in \mathrm{M}_I$ if, for every congruence class $[a]_{\sim_A} \in A/\sim_A$, its restriction $f_{|_{[a]_{\sim_A}}} : [a]_{\sim_A} \to [f(a)]_{\sim_B}$ is a bijection.

2.4. Proposition. *A homomorphism $f : A \to B$ in $\mathcal{C}$ belongs to M_I if and only if the following two conditions hold:*

1. *for each $b \in B$ and each $a \in A$ such that $b \sim_B f(a)$, there exists $a^* \in A$ for which $f(a^*) = b$ and $a^* \sim_A a$;*
2. *for all $a, a^* \in A$, if $f(a) = f(a^*)$ and $a \sim_A a^*$ then $a = a^*$.*

Proof. The proof follows trivially from the fact that, since the reflection is admissible (hence simple in the sense of [2]), in the pullback diagram (1)

$$\begin{tikzcd}
A \arrow[dr, "e_f"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\
& B \times_{HI(B)} HI(A) \arrow[r] \arrow[d, "m_f"] & HI(A) \arrow[d, "HI(f)"] \\
& B \arrow[r, "\eta_B"] & HI(B)\ ,
\end{tikzcd} \tag{1}$$

$f \in \mathrm{M}_I$ if and only if the uniquely determined homomorphism $e_f = \langle f, \eta_A \rangle$ is an isomorphism. ■

Stably-vertical, Covering morphisms

The class of stably-vertical morphisms E'_I is the class of homomorphisms $f : A \to B$ in $\mathcal{C}$, whose pullback $h^*(f)$ along any homomorphism $h : C \to B$ is a vertical homomorphism. The class M^*_I of covering morphisms (coverings, for short) is the class of homomorphisms $f : A \to B$ in $\mathcal{C}$, such that the pullback $p^*(f)$ along some effective descent morphism[1] $p : E \to B$ is a trivial covering. Note that a variety of universal algebras $\mathcal{C}$ is an exact category, hence an effective descent morphism $p : E \to B$ in $\mathcal{C}$ is just a regular epimorphism (see [2, §4.7]). On the other hand, a regular epimorphism is a surjective homomorphism in universal algebra. Then, an effective descent morphism is just a surjective homomorphism in a variety of universal algebras.

The following Lemma 2.5 states in particular that, for a reflection $H \vdash I : \mathcal{C} \to \mathcal{M}$ of a variety of universal algebras $\mathcal{C}$ into one of its subvarieties $\mathcal{M}$, a covering morphism $f : A \to B$ in $\mathcal{C}$ is just a homomorphism such that its pullback $\varepsilon^*_B(f)$ along $\varepsilon_B : FU(B) \to B$ is a trivial covering, where ε_B is a counit morphism of the adjunction $U \vdash F : \mathbf{Set} \to \mathcal{C}$.

[1]A morphism $p : E \to B$ in $\mathcal{C}$ is an effective descent morphism when the functor "pullback along p" $p^* : \mathcal{C}/B \to \mathcal{C}/E$ is monadic.

2.5. Lemma. *Let $\langle F, U, \lambda, \varepsilon\rangle : \mathcal{S} \to \mathcal{C}$ be an adjunction with unit λ and counit ε.*

If $U(p) : U(E) \to U(B)$ is a split epimorphism in $\mathcal{S}$, then there exists $f : FU(B) \to E$ in $\mathcal{C}$ such that $\varepsilon_B = p \circ f$.

Proof. Since $U(p)$ is a split epimorphism in $\mathcal{S}$, there exists a morphism $h : U(B) \to U(E)$ such that

$$U(p) \circ h = 1_{U(B)}. \tag{2}$$

Since $\lambda_{U(B)}$ is universal from $U(B)$ to U, there exists a unique $f : FU(B) \to E$ such that the following diagram commutes.

$$\begin{tikzcd} U(B) \arrow[r, "\lambda_{U(B)}"] \arrow[dr, "h"'] & UFU(B) \arrow[d, "U(f)"] \\ & U(E) \end{tikzcd} \tag{3}$$

Therefore we have,

$$\begin{aligned} U(p \circ f) \circ \lambda_{U(B)} &= U(p) \circ U(f) \circ \lambda_{U(B)}, \\ &= U(p) \circ h, \text{ by (3)} \\ &= 1_{U(B)}, \text{ by (2)} \\ &= U(\varepsilon_B) \circ \lambda_{U(B)}, \text{ because } \lambda \text{ and } \varepsilon \text{ are respectively the unit and} \\ &\qquad \text{the counit of the adjunction } \langle F, U, \lambda, \varepsilon\rangle. \end{aligned}$$

Therefore, $\varepsilon_B = p \circ f$, since $\lambda_{U(B)}$ is universal from $U(B)$ to U. ■

Hence, since the class M_I is pullback stable, $f : A \to B$ is a covering morphism if and only if the pullback $\varepsilon_B^*(f)$ of f along the surjective homomorphism $\varepsilon_B : FU(B) \to B$ is a trivial covering.

Separable, Purely Inseparable, Normal morphisms

2.6. Definition. *Consider the following pullback diagram*

$$\begin{tikzcd} A \arrow[dr, "\delta_f"] \arrow[drr, "1_A"] \arrow[ddr, "1_A"'] & & \\ & A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "f"] \\ & A \arrow[r, "f"] & B \end{tikzcd}, \tag{4}$$

where (u, v) is the kernel-pair of the homomorphism $f : A \to B$, and $\delta_f = \langle 1_A, 1_A\rangle$ is the uniquely determined homomorphism.

- *f is called a separable homomorphism ($f \in Sep$) if δ_f is a trivial covering.*

- *f is called a purely inseparable homomorphism ($f \in Pin$) if δ_f is vertical.*
- *f is called a normal homomorphism ($f \in Normal$) if u is a trivial covering.*

Inseparable-separable factorization system

For the factorization system $(\mathrm{E}_I, \mathrm{M}_I)$ we can form a derived *weak factorization system* (see [5]) as follows:

$$\begin{tikzcd}
& {} \arrow[dr, "m_{\delta_f}"] & & & & {} \arrow[dr, "\bar{m}_f"] & \\
A \arrow[ur, "e_{\delta_f}"] \arrow[rr, "\delta_f"'] & & A \times_B A \arrow[rr, shift left, "u"] \arrow[rr, shift right, "v"'] & & A \arrow[ur, "\bar{e}_f"] \arrow[rr, "f"'] & & B\,;
\end{tikzcd}$$

here, $\bar{e}_f$ is just the coequalizer of $(u \circ m_{\delta_f}, v \circ m_{\delta_f})$, where (u, v) is the kernel-pair of f, and $m_{\delta_f} \circ e_{\delta_f}$ is the $(\mathrm{E}_I, \mathrm{M}_I)$-factorization of the morphism δ_f of diagram (4).

2.7. Definition. *A morphism $f : A \to B$ is called inseparable if, in its decomposition given in the diagram just above, $\bar{m}_f$ is an isomorphism. So, we define the class of inseparable morphisms $Ins := \{f \in \mathcal{C} \mid \bar{m}_f \ iso\}$.*

There is a factorization system (Ins, Sep) if and only if the class Ins of inseparable homomorphisms is closed under composition (see [5, §3,§4]). The following two inclusions $Pin \cap \mathrm{E} \subseteq Ins \subseteq \mathrm{E}_I \cap \mathrm{E}$, where E is the class of surjective homomorphisms, are satisfied by the class Ins in an admissible reflection into a subvariety of universal algebras (see [11], or the original source [5, §4.2]).

3. Admissible reflections of universal algebras into subvarieties

Next Propositions 3.3 and 3.6, concerning admissible reflections into an idempotent subvariety, that is, a variety in which every one-element subset is a subalgebra, help to characterize the classes of stably-vertical and covering homomorphisms, respectively.

Stably-vertical morphisms

In order to characterize the class of stably-vertical morphisms for an admissible reflection $H \vdash I : \mathcal{C} \to \mathcal{M}$ into an idempotent subvariety, in which the I-images of the projections of certain pullbacks are jointly monic, we define the class F of "almost surjective" homomorphisms.

3.1. Definition. *A homomorphism f is said to be almost surjective if it belongs to the class $\mathrm{F} = \{f : A \to B \text{ in } \mathcal{C} \mid \forall_{b \in B}\ f(A) \cap \langle b \rangle_B \neq \emptyset\}$, where $\langle b \rangle_B$ denotes the subalgebra of B generated by b. In words, a homomorphism is almost surjective if its homomorphic image intersects all the monogenic subalgebras in the codomain.*

The following Proposition 3.2 states a necessary condition for a homomorphism to be stably-vertical in any reflection of a variety of universal algebras into a subvariety. Next

Proposition 3.3 characterizes the class of stably-vertical morphisms for reflections into idempotent subvarieties, such that the I-images of the projections of any pullback of a homomorphism belonging to $\mathrm{E}_I \cap \mathrm{F}$ are jointly monic.

3.2. Proposition. *The class of stably-vertical morphisms,* E'_I*, satisfies the inclusion* $\mathrm{E}'_I \subseteq \mathrm{E}_I \cap \mathrm{F}$*, for any reflection* $H \vdash I : \mathcal{C} \to \mathcal{M}$ *of a variety of universal algebras* $\mathcal{C}$ *into a subvariety* $\mathcal{M}$*.*

Proof. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "e"] \\ \langle d\rangle_D \arrow[r, "\subseteq"] & D \end{tikzcd} , \tag{5}$$

where $e \in \mathrm{E}'_I$. Suppose that $\langle d\rangle_D \cap e(C) = \emptyset$, for some $d \in D$. Then, $P = \emptyset = I(P)$. Since $\langle d\rangle_D \neq \emptyset$, $I(\langle d\rangle_D) \neq \emptyset$ and $I(\pi_1) : I(P) \to I(\langle d\rangle_D)$ is not an isomorphism, which contradicts the assumption of e belonging to E'_I. ∎

3.3. Proposition. *Consider a reflection* $H \vdash I : \mathcal{C} \to \mathcal{M}$ *of a variety* $\mathcal{C}$ *of universal algebras into an idempotent subvariety* $\mathcal{M}$*. The following two conditions are equivalent:*

(a) $I(\pi_1)$ *and* $I(\pi_2)$ *are jointly monic for all the pullback diagrams*

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r] & C \end{tikzcd} ,$$

in $\mathcal{C}$*, such that* $g \in \mathrm{E}_I \cap \mathrm{F}$;

(b) $\mathrm{E}'_I = \mathrm{E}_I \cap \mathrm{F}$.

Proof. (a)⇒(b): Consider the pullback diagram

$$\begin{tikzcd} I(A \times_C B) \arrow[dr, "w"] \arrow[drr, "I(\pi_2)"] \arrow[ddr, "I(\pi_1)"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & I(B) \arrow[d, "I(g)"] \\ & I(A) \arrow[r, "I(f)"] & I(C) \end{tikzcd} , \tag{6}$$

where f is an arbitrary homomorphism, $g \in \mathrm{E}_I \cap \mathrm{F}$, $P = I(A) \times_{I(C)} I(B)$ and w is the unique homomorphism making the diagram commute. Since $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, w is a monomorphism, which is an injective homomorphism in a variety of universal algebras. Since $g \in \mathrm{E}_I$, $I(g)$ is an isomorphism. Hence, p_1 is also an isomorphism. Therefore, $I(\pi_1) = p_1 \circ w$ is an injective homomorphism.

On the other hand, let $[l]_{\sim_A}$ be an arbitrary class of $A/{\sim_A} = HI(A)$. Since every element of any object M in $\mathcal{M}$ is a subalgebra and $\eta_{\langle l \rangle_A} : \langle l \rangle_A \to HI(\langle l \rangle_A)$ is a homomorphism in a variety of universal algebras, $HI(\langle l \rangle_A) = \eta_{\langle l \rangle_A}(\langle l \rangle_A) = \langle \eta_{\langle l \rangle_A}(l) \rangle_{HI(A)} = T$, the one-element algebra, which means that the elements of $\langle l \rangle_A$ are in the same congruence class in A. Therefore, $\langle l \rangle_A \subseteq [l]_{\sim_A}$. Since $f : A \to C$ is a homomorphism in a variety of universal algebras, $f(\langle l \rangle_A) = \langle f(l) \rangle_C$. Since $g \in \mathrm{F}$, $\langle f(l) \rangle_C \cap g(B) \neq \emptyset$, which implies $\langle l \rangle_A \cap \pi_1(A \times_C B) \neq \emptyset$. Hence, $[l]_{\sim_A} \cap \pi_1(A \times_C B) \neq \emptyset$. Therefore, $I(\pi_1)$ is also surjective and so $g \in \mathrm{E}'_I$. Thus, $\mathrm{E}'_I \supseteq \mathrm{E}_I \cap \mathrm{F}$. By Proposition 3.2, $\mathrm{E}'_I \subseteq \mathrm{E}_I \cap \mathrm{F}$.

(b)⇒(a): Consider the pullback diagram (6), where $g \in \mathrm{E}_I \cap \mathrm{F}$. Since by hypothesis $\mathrm{E}'_I = \mathrm{E}_I \cap \mathrm{F}$, $I(\pi_1)$ is an isomorphism, and therefore $I(\pi_1)$, $I(\pi_2)$ are jointly monic. ∎

Covering morphisms

We will show that the class of coverings is just the class of trivial coverings in admissible reflections into idempotent subvarieties, provided that the I-images of the projections of any pullback of a covering $f : A \to B$ along the counit $\varepsilon_B : FU(B) \to B$ (see the paragraph above Lemma 2.5) are jointly monic, and that a certain property holds for the reflection (see Definition 3.4, here it is named ***Property*** F).

Consider the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$, of the variety of bands (idempotent semigroups) into its subvariety of semilattices. It is well known that every band is a semilattice of rectangular bands. This means that, in this reflection, two elements a and b in a band B are in the same congruence class, $a \sim_B b$, if and only if $a = aba$ and $b = bab$. The canonical homomorphism h, from the kernel pair of the unit morphism $\eta_{FU(B)}$ to the kernel pair of the unit morphism η_B (see Definition 3.4), is surjective, since given (a, b) in $B \times_{HI(B)} B$, that is, $a \sim_B b$, there exists (aba, bab) in $FU(B) \times_{HIFU(B)} FU(B)$ such that $h(aba, bab) = (a, b)$ (note that words in any rectangular band in $FU(B)$ are just finite sequences of the same letters).

On the other hand, in the reflection of the variety of commutative semigroups into its subvariety of semilattices, the canonical homomorphism h is not always surjective (see Remark 5.5), but it is almost surjective ($h \in \mathrm{F}$), as we will see in Proposition 5.4. Which is enough, together with Proposition 3.6 and Proposition 5.3, to conclude that coverings and trivial coverings coincide in this reflection.

3.4. Definition. *A reflection $H \vdash I : \mathcal{C} \to \mathcal{M}$, of a variety of universal algebras into a subvariety, is said to satisfy* ***Property*** F *if the canonical homomorphism $h = \langle \varepsilon_B \circ \pi_1, \varepsilon_B \circ \pi_2 \rangle : FU(B) \times_{HIFU(B)} FU(B) \to B \times_{HI(B)} B$, from the kernel pair of the unit morphism $\eta_{FU(B)}$ to the kernel pair of the unit morphism η_B, is almost surjective, that is, $h \in \mathrm{F}$, for all $B \in \mathcal{C}$. Where π_1, π_2 are the projections of the kernel pair of $\eta_{FU(B)}$, and $\varepsilon_B : FU(B) \to B$ is a counit morphism of the adjunction $U \vdash F : \mathbf{Set} \to \mathcal{C}$, such that U is the forgetful functor into sets.*

3.5. Remark. This property means that if $a \sim_B b$, with $a, b \in B$, then there exist w_1 and w_2 in $FU(B)$ such that $w_1 \sim_{FU(B)} w_2$, and $h(w_1, w_2)$ belongs to the subalgebra $\langle (a,b)\rangle_{B\times_{HI(B)}B}$ of $B \times_{HI(B)} B$ generated by (a, b). Note that this implies that if $a \sim_B b$, then there exist w_1 and w_2 in $FU(B)$, such that $w_1 \sim_{FU(B)} w_2$, and $\varepsilon_B(w_1)$ belongs to the subalgebra $\langle a\rangle_B$ of B generated by a, and $\varepsilon_B(w_2)$ belongs to the subalgebra $\langle b\rangle_B$ of B generated by b.

3.6. Proposition. *Let $H \vdash I : \mathcal{C} \to \mathcal{M}$ be an admissible reflection of a variety of universal algebras $\mathcal{C}$ into an idempotent subvariety $\mathcal{M}$, which satisfies **Property** F. Then, the following two conditions are equivalent:*

(a) *$I(\pi_1)$ and $I(\pi_2)$ are jointly monic for all pullback diagrams*

$$\begin{tikzcd} FU(S)\times_S L \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & L \arrow[d, "f"] \\ FU(S) \arrow[r, "\varepsilon_S"] & S \end{tikzcd}, \tag{7}$$

*where $f \in \mathrm{M}_I^*$ and ε_S is a counit morphism of the adjunction $U \vdash F :$ **Set** $\to \mathcal{C}$, such that U is the forgetful functor into sets;*

(b) $\mathrm{M}_I^* = \mathrm{M}_I$.

Proof. Suppose that $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, that is, the canonical homomorphism $\alpha = \langle I(\pi_1), I(\pi_2)\rangle : I(FU(S) \times_S L) \to IFU(S) \times_{I(S)} I(L)$ is an injective homomorphism. First we will prove that the pullback (7) is preserved by I. It remains to show that α is also surjective.

Note that α is surjective if and only if for every $t \in L$ and $w \in FU(S)$, such that $f(t) \sim_S \varepsilon_S(w)$, there exists $(w^*, t^*) \in FU(S) \times_S L$, with $w \sim_{FU(S)} w^*$ and $t \sim_L t^*$.

Since the reflection satisfies ***Property*** F and $f(t) \sim_S \varepsilon_S(w)$, there exists $(w_1, w_2) \in FU(S) \times_{HI(FU(S))} FU(S)$, such that $\varepsilon_S(w_1) \in \langle \varepsilon_S(w)\rangle_S$, $\varepsilon_S(w_2) \in \langle f(t)\rangle_S$, by Remark 3.5.

The existence of $t^* \in L$, with $t \sim_L t^*$ follows from:

On one hand, there exists $t' \in \langle t\rangle_L$, such that $\varepsilon_S(w_2) = f(t')$, that is, $(w_2, t') \in FU(S) \times_S L$, since $\langle f(t)\rangle_S = f(\langle t\rangle_L)$. On the other hand, since $f \in \mathrm{M}_I^*$, $\pi_1 \in \mathrm{M}_I$, and so the restriction of π_1 to any congruence class is a bijection. Then, since $w_1 \sim_{FU(S)} w_2$, there exists $t^* \in L$, such that $(w_1, t^*) \in FU(S) \times_S L$, that is, $\varepsilon_S(w_1) = f(t^*)$, and $(w_2, t') \sim_{FU(S)\times_S L} (w_1, t^*)$. Hence, $t' \sim_L t^*$. Since $t' \in \langle t\rangle_L$ and $\mathcal{M}$ is an idempotent subvariety, $t \sim_L t'$. Therefore, $t \sim_L t^*$.

The existence of $w^* \in FU(S)$, with $w \sim_{FU(S)} w^*$ follows from:

On one hand, there exists $w^* \in \langle w\rangle_{FU(S)}$, such that $\varepsilon_S(w^*) = \varepsilon_S(w_1)$, since $\varepsilon_S(w_1) \in \langle \varepsilon_S(w)\rangle_S = \varepsilon_S(\langle w\rangle_{FU(S)})$. On the other hand, since $w^* \in \langle w\rangle_{FU(S)}$ and $\mathcal{M}$ is an idempotent subvariety, $w^* \sim_{FU(S)} w$.

The fact that $(w^*, t^*) \in FU(S) \times_S L$ follows from: $\varepsilon_S(w^*) = \varepsilon_S(w_1) = f(t^*)$.

Therefore, there exists $(w^*, t^*) \in FU(S) \times_S L$, such that $w \sim_{FU(S)} w^*$ and $t \sim_L t^*$. Thus, α is also surjective.

Now, we have to prove that $f \in \mathrm{M}_I$.

Consider the commutative diagram

$$\begin{tikzcd}
FU(S) \times_S L \arrow[r, "\eta_{FU(S)\times_S L}"] \arrow[d, "\pi_1"] & HI(FU(S) \times_S L) \arrow[r, "HI(\pi_2)"] \arrow[d, "HI(\pi_1)"] & HI(L) \arrow[d, "HI(f)"] \\
FU(S) \arrow[r, "\eta_{FU(S)}"'] & HIFU(S) \arrow[r, "HI(\varepsilon_S)"'] & HI(S)
\end{tikzcd} \quad , \tag{8}$$

where both squares are pullbacks. The right square is a pullback, since the pullback diagram (7) is preserved, and the left square is a pullback, since $\pi_1 \in \mathrm{M}_I$, according to the paragraph above Proposition 2.4.

As $\eta_S \circ \varepsilon_S = HI(\varepsilon_S) \circ \eta_{FU(S)}$ and $\eta_L \circ \pi_2 = HI(\pi_2) \circ \eta_{FU(S)\times_S L}$, the outside square of the following commutative diagram (9) is the same as in the former (8), and therefore a pullback.

$$\begin{tikzcd}
FU(S) \times_S L \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & L \arrow[r, "\eta_L"] \arrow[d, "f"] & HI(L) \arrow[d, "HI(f)"] \\
FU(S) \arrow[r, "\varepsilon_S"'] & S \arrow[r, "\eta_S"'] & HI(S)
\end{tikzcd} \tag{9}$$

According to Lemma 4.6 in [2], since the outside square in diagram (9) is a pullback, and ε_S is an effective descent morphism in $\mathcal{C}$, and the left square is a pullback, the right square is a pullback, too. Hence, $f \in \mathrm{M}_I$, by the paragraph above Proposition 2.4.

Conversely, let $\mathrm{M}_I^* = \mathrm{M}_I$. Since the reflection is admissible, it is known that I preserves the pullback (7) of ε_S and f because the latter is a trivial covering (see [2, 3.6]). Hence, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic. ∎

Separable, Purely Inseparable and Normal morphisms

The following Proposition 3.7 characterizes the classes of separable, purely inseparable and normal morphisms, in admissible reflections of varieties such that the I-images of the projections of every kernel pair are jointly monic.

3.7. Proposition. *Let $H \vdash I : \mathcal{C} \to \mathcal{M}$ be an admissible reflection of a variety of universal algebras $\mathcal{C}$ into a subvariety $\mathcal{M}$. Let Δ denote the identity relation, $Ker(f)$ the congruence associated to the kernel pair of $f : A \to B$, $\sim_A$ the congruence on A induced by the reflection, and $\circ$ the composition of congruences.*

If, for the kernel pair (u, v) of f, $I(u)$ and $I(v)$ are jointly monic, then:

(a) f *is separable if and only if* $Ker(f) \cap \sim_A = \Delta$*;*

(b) f *is purely inseparable if and only if* $Ker(f) \subseteq \sim_A$*;*

(c) f *is normal if and only if*

$\sim_A \circ Ker(f) \subseteq Ker(f) \circ \sim_A$ *and* $Ker(f) \cap \sim_A = \Delta$.

PROOF. Consider the pullback diagram (4) in Definition 2.6 and note that $I(u)$ and $I(v)$ are jointly monic when, for all pairs $(a, b), (c, d)$ in $A \times_B A$, $a \sim_A c$ and $b \sim_A d$ if and only if $(a, b) \sim_{A \times_B A} (c, d)$.

(a) A homomorphism $f : A \to B$ is separable if, for all $a \in A$, the restriction $\delta_f|_{[a]_{\sim_A}} : [a]_{\sim_A} \to [\delta_f(a)]_{\sim_{A\times_B A}}$ is a bijection. On one hand, this restriction is injective for all $a \in A$, since $v \circ \delta_f = 1_A$. On the other hand, this restriction is surjective if and only if, for all $a, a' \in A$, such that $f(a) = f(a')$ and $b \sim_A a$ and $b \sim_A a'$, for some $b \in A$, there exists $c \in A$ such that $(c, c) = (a, a')$. That is, a homomorphism $f : A \to B$ is separable if and only if, for all $a, a' \in A$, such that $f(a) = f(a')$ and $a \sim_A a'$, then $a = a'$. Which is to say that a homomorphism $f : A \to B$ is separable if and only if $Ker(f) \cap \sim_A = \Delta$.

(b) A homomorphism $f : A \to B$ is purely inseparable if and only if $I(\delta_f)$ is an isomorphism. On one hand, $I(\delta_f)$ is always injective, since $v \circ \delta_f = 1_A$. On the other hand, $I(\delta_f)$ is a surjection if and only if, for all $a, a' \in A$, such that $f(a) = f(a')$, there exists $b \in A$ such that $b \sim_A a$ and $b \sim_A a'$. That is, a homomorphism $f : A \to B$ is purely inseparable when, for all $a, a' \in A$, if $f(a) = f(a')$ then $a \sim_A a'$. Which is to say that a homomorphism $f : A \to B$ is separable if and only if $Ker(f) \subseteq \sim_A$.

(c) A homomorphism $f : A \to B$ is normal if and only if the next two conditions hold:

   (1) The restrictions of u to the congruence classes of $\sim_{A\times_B A}$ are surjective. That is, for all $a, b \in A$, such that $a \sim_A c$ and $f(c) = f(b)$ for some $c \in A$, there exists $d \in A$, such that $f(a) = f(d)$ and $d \sim_A b$, or equivalently, $\sim_A \circ Ker(f) \subseteq Ker(f) \circ \sim_A$.

   (2) The restrictions of u to the congruence classes of $\sim_{A\times_B A}$ are injective. That is, for all $a, b \in A$ such that $a \sim_A b$ and $f(a) = f(b)$ then, $a = b$, or equivalently, $Ker(f) \cap \sim_A = \Delta$.

■

4. Inseparable-Separable factorization

In Proposition 4.1 it will be proved that there is an inseparable-separable factorization system, for admissible reflections into subvarieties, provided the class of stably-vertical morphisms is the class of almost surjective vertical homomorphisms.

4.1. Proposition. *Let $H \vdash I : \mathcal{C} \to \mathcal{M}$ be an admissible reflection of a variety of universal algebras $\mathcal{C}$ into a subvariety $\mathcal{M}$. If $\mathrm{E}'_I = \mathrm{E}_I \cap \mathrm{F}$ then there exists a factorization system (Ins, Sep), with $Ins = \mathrm{E}_I \cap \mathrm{E}$.*

Proof. First we will show that $\mathrm{E}'_I \subseteq Pin$. Consider the pullback diagram (4) and suppose $f \in \mathrm{E}'_I$. Then $v \in \mathrm{E}_I$. Since $I(v)$ is an isomorphism and $I(v) \circ I(\delta_f) = 1_{I(A)}$, $I(\delta_f)$ is an isomorphism. Hence, $\delta_f \in \mathrm{E}_I$, thus $f \in Pin$. Therefore, by the paragraph below Definition 2.7,

$$\mathrm{E}'_I \cap \mathrm{E} \subseteq Ins \subseteq \mathrm{E}_I \cap \mathrm{E}.$$

Now we have to prove that the class Ins is closed under composition. Since $\mathrm{E} \subseteq \mathrm{F}$, $\mathrm{E}_I \cap \mathrm{E} \subseteq \mathrm{E}_I \cap \mathrm{F} = \mathrm{E}'_I$. Hence, $\mathrm{E}_I \cap \mathrm{E} \subseteq \mathrm{E}_I \cap \mathrm{F} \cap \mathrm{E} = \mathrm{E}'_I \cap \mathrm{E}$. On the other hand, since $\mathrm{E}'_I \subseteq \mathrm{E}_I$, $\mathrm{E}'_I \cap \mathrm{E} \subseteq \mathrm{E}_I \cap \mathrm{E}$. Hence, $\mathrm{E}'_I \cap \mathrm{E} = \mathrm{E}_I \cap \mathrm{E}$. Therefore, $Ins = \mathrm{E}_I \cap \mathrm{E}$. Since E_I and E are closed under composition, so it is $\mathrm{E}_I \cap \mathrm{E}$. Therefore, (Ins, Sep) is a factorization system. ∎

4.2. Remark. Under the assumptions of Proposition 4.1, the (Ins, Sep)-factorization of $f : A \to B$ given in the paragraph above Definition 2.7, could be obtained alternatively by applying the surjection-injection factorization to the homomorphism e_f in diagram (1). Hence, $f = \bar{m}_f \circ \bar{e}_f = (m_f \circ m) \circ \bar{e}_f$, with $e_f = m \circ \bar{e}_f$. This means that inseparable-separable and concordant-dissonant factorizations coincide (see Proposition 2 in [13, §3], which is a generalization of the results given in [2, §3.9] or [3, Proposition 5.5]; see also [5, §2.11] in the context of pointed endofunctors, where previous references are provided).

5. The reflection of commutative semigroups into semilattices

Consider the admissible reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$ of commutative semigroups into semilattices. It is well known that a commutative semigroup C is semilattice indecomposable if and only if it is archimedean, in the sense of [9]. That is, a commutative semigroup C is archimedean if for every ordered pair (a, b) of elements in C there exist $n \in \mathbf{N}$ (positive integers) and $x \in C$, with $a^n = bx$. Hence, every commutative semigroup is a semilattice of archimedean subsemigroups.

(I) If C is a finitely generated commutative semigroup, any of its archimedean components A has some power $A^k = \{a \in C \mid a = a_1...a_k,\text{ with } a_1,,a_k \in A\}$ with cancellation (see Proposition 9.6 in [4, §IV]).

(II) Since the subvariety of semilattices is idempotent, each archimedean component of the semilattice decomposition of a commutative semigroup S is a subalgebra of S. Therefore, every power of each archimedean component is a subsemigroup of S.

(III) Any archimedean component of the semilattice decomposition of a commutative semigroup is semilattice indecomposable (see [8]). That is, let S be a commutative semigroup and consider an archimedean class $[x]_{\sim_S}$ in S : then, since $I([x]_{\sim S}) = \mathbf{1}$, for every $y, z \in [x]_{\sim_S}$, there exist $m, n \in \mathbf{N}$ and there exist $c, d \in [x]_{\sim_S}$, such that $z^m = yc$ and $y^n = zd$.

The following Proposition 5.1 will be used to prove Proposition 5.2, which in turn will be used to prove the more general Proposition 5.3.

5.1. Proposition. *Consider the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$, and the following pullback diagram,*

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "e"] \\ A \arrow[r, "f"] & C \end{tikzcd} , \tag{10}$$

in **CommSgr**.

Then, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, provided C is a commutative semigroup with cancellation.

Proof. Let $(a_1, b_1), (a_2, b_2) \in A \times_C B$ be such that $a_1 \sim_A a_2$ and $b_1 \sim_B b_2$, that is, there exist $m, n, p, q \in \mathbf{N}$ and $c, d \in A$ and $u, v \in B$, such that $a_1^m = a_2 c$, $a_2^n = a_1 d$, $b_1^p = b_2 u$ and $b_2^q = b_1 v$.

We need $(x, y), (z, t) \in A \times_C B$ and $r, s \in \mathbf{N}$ such that $(a_1, b_1)^r = (a_2, b_2)(x, y)$ and $(a_2, b_2)^s = (a_1, b_1)(z, t)$.

We take $(x, y) = (a_2^{p-1} c^p, b_2^{m-1} u^m)$, $(z, t) = (a_1^{q-1} d^q, b_1^{n-1} v^n)$ and $r = mp, s = nq$.

Clearly we have $(a_1, b_1)^r = (a_1^r, b_1^r) = (a_2, b_2)(x, y)$ and $(a_2, b_2)^s = (a_2^s, b_2^s) = (a_1, b_1)(z, t)$.

We have to prove (1) $f(x) = e(y)$, and (2) $f(z) = e(t)$.

The equality (1) follows from $f(a_2)f(x) = f(a_2^p c^p) = f(a_1^{mp}) = e(b_1^{mp}) = e(b_2^m u^m) = e(b_2)e(y)$, by cancellation.

The equality (2) follows from $f(a_1)f(z) = f(a_1^q d^q) = f(a_2^{nq}) = e(b_2^{nq}) = e(b_1^n v^n) = e(b_1)e(t)$, by cancellation. ∎

5.2. Proposition. *Consider the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$, and the pullback diagram (10). Then, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, provided C is a finitely generated commutative semigroup.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in A \times_C B$ be such that $a_1 \sim_A a_2$ and $b_1 \sim_B b_2$. Let $k \in \mathbf{N}$ be such that $[f(a_1)]^k_{\sim_C} = [e(b_1)]^k_{\sim_C}$ has cancellation, by (I).

Consider the following pullback

$$\begin{tikzcd} X \times_H Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "e_{|Y}"] \\ X \arrow[r, "f_{|X}"] & H \end{tikzcd} , \tag{11}$$

where

- $H = [f(a_1)]^k_{\sim_C} = [e(b_1)]^k_{\sim_C}$, $X = [a_1]^k_{\sim_A} = [a_2]^k_{\sim_A}$, $Y = [b_1]^k_{\sim_B} = [b_2]^k_{\sim_B}$,
- $f_{|X} : X \to H$ and $e_{|Y} : Y \to H$ are restrictions, respectively, of $f : A \to C$ and $e : B \to C$.

Let us prove that:

(1) $(a_1^k, b_1^k), (a_2^k, b_2^k) \in X \times_H Y$;

(2) $a_1^k \sim_X a_2^k$ and $b_1^k \sim_Y b_2^k$.

(1) follows from:
$f(a_1^k) = e(b_1^k)$ and $f(a_2^k) = e(b_2^k)$, since $f(a_1) = e(b_1)$ and $f(a_2) = e(b_2)$. Therefore $f_{|X}(a_1^k) = e_{|Y}(b_1^k)$ and $f_{|X}(a_2^k) = e_{|Y}(b_2^k)$, that is, $(a_1^k, b_1^k), (a_2^k, b_2^k) \in X \times_H Y$.

(2) follows from (III) above:
by (III), there exist $m, n, p, q \in \mathbf{N}$ and $c, d \in [a_1]_{\sim_A}$ and $u, v \in [b_1]_{\sim_B}$, such that $a_1^m = a_2 c$, $a_2^n = a_1 d$, $b_1^p = b_2 u$ and $b_2^q = b_1 v$; therefore, $(a_1^k)^m = a_2^k c^k$, $(a_2^k)^n = a_1^k d^k$, $(b_1^k)^p = b_2^k u^k$ and $(b_2^k)^q = b_1^k v^k$, with $c^k, d^k \in X$ and $u^k, v^k \in Y$.

Hence, by Proposition 5.1, $(a_1^k, b_1^k) \sim_{X \times_H Y} (a_2^k, b_2^k)$, since H is cancellative. Thus, (a_1^k, b_1^k) $\sim_{A \times_C B}$ (a_2^k, b_2^k), since $X \times_H Y$ is a subsemigroup of $A \times_C B$. Therefore, $(a_1, b_1) \sim_{A \times_C B}$ (a_2, b_2), since $(a_1^k, b_1^k) \sim_{A \times_C B} (a_1, b_1)$ and $(a_2^k, b_2^k) \sim_{A \times_C B} (a_2, b_2)$. ■

5.3. Proposition. *Consider the reflection $H \vdash I$:* **CommSgr** $\to$ **SLat**, *of commutative semigroups into semilattices. The I-images of the projections of any pullback diagram in* **CommSgr**, *are jointly monic.*

Proof. We have to show that, for any commutative semigroup C in the pullback diagram (10), the homomorphisms $I(\pi_1)$ and $I(\pi_2)$ are jointly monic.

Let $(a_1, b_1), (a_2, b_2) \in A \times_C B$ be such that $a_1 \sim_A a_2$ and $b_1 \sim_B b_2$. That is, there exist m, n, p, q in $\mathbf{N}$, c and d in A, and u and v in B, with $a_1^m = a_2 c, a_2^n = a_1 d, b_1^p = b_2 u$ and $b_2^q = b_1 v$. Let H be the subsemigroup in C generated by $f(a_1), f(a_2), f(c), f(d), e(u)$ and $e(v)$.

Recall that $f(a_1) = e(b_1)$ and $f(a_2) = e(b_2)$, and consider the pullback diagram

$$\begin{tikzcd} X \times_H Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "e_{|Y}"] \\ X \arrow[r, "f_{|X}"] & H \end{tikzcd},$$

in which $X = f^{-1}(H)$, $Y = e^{-1}(H)$, and $f_{|X}$ and $e_{|Y}$ are the obvious restrictions.
Note that:

(i) $a_1, a_2, c, d \in X$ and $b_1, b_2, u, v \in Y$;

(ii) $(a_1, b_1), (a_2, b_2) \in X \times_H Y$;

(iii) $a_1 \sim_X a_2$ and $b_1 \sim_Y b_2$, since $c, d \in X$ and $u, v \in Y$.

Hence, by Proposition 5.2, $(a_1, b_1) \sim_{X\times_H Y} (a_2, b_2)$. Therefore, $(a_1, b_1) \sim_{A\times_C B} (a_2, b_2)$, since $X \times_H Y$ is a subsemigroup of $A \times_C B$. ∎

5.4. Proposition. *The reflection $H \vdash I$: **CommSgr** $\to$ **SLat**, of commutative semigroups into semilattices, satisfies **Property** F.*

Proof. Remark that, in the following, referring to a word in a free semigroup, x^k stands for the concatenation of a positive integer number k of letters x.

Let B be a commutative semigroup and $a, b \in B$ such that $a \sim_B b$. Recall that $a \sim_B b$ if and only if there exist $n, m \in \mathbf{N}$ and $c, d \in B$, such that $a^n = bc$ and $b^m = ad$. Then, $a^{nm} = b^m c^m = adc^m$ and $b^{mn} = a^n d^n = bcd^n$.

Hence, $a^{n^2m} = a^n d^n c^{mn} = bcd^n c^{mn} = bc^{mn+1}d^n$.

Thus, $a^{n^2m} = bc^{nm+1}d^n$ and $b^{nm} = bcd^n$.

Finally, $a^{n^2m} = bc^{nm+1}d^n$ and $b^{n^2m} = b^n c^n d^{n^2}$.

Since the words in $FU(B)$ are in the same archimedean component if and only if they have the same letters, the words $w_1 = bc^{nm+1}d^n$ and $w_2 = b^n c^n d^{n^2}$, are in the same congruence class in $FU(B)$, that is, $(bc^{nm+1}d^n, b^n c^n d^{n^2}) \in FU(B) \times_{HIFU(B)} FU(B)$.

Hence, applying the canonical homomorphism $h = \langle \varepsilon_B \circ \pi_1, \varepsilon_B \circ \pi_2 \rangle$ of Definition 3.4, we have $h(bc^{nm+1}d^n, b^n c^n d^{n^2}) = (a, b)^{n^2m} \in \langle (a, b) \rangle_{B\times_{HI(B)}B}$. ∎

5.5. Remark. Consider the reflection $H \vdash I$: **CommSgr** $\to$ **SLat**, of the variety of commutative semigroups into its subvariety of semilattices, and the canonical homomorphism $h : FU(B) \times_{HIFU(B)} FU(B) \to B \times_{HI(B)} B$ of Definition 3.4, where $B = F(a, b; a^2 = (ab)^2)$ is the quotient of the free commutative semigroup on the set $\{a, b\}$, by the congruence generated by the identity $a^2 = (ab)^2$.

The archimedean components of B are $\{a^k, a^m b^n\}$, with k, m, n arbitrary positive integers, since $a \sim_B ab$, and $\{b^l\}$, with l an arbitrary positive integer, since a and b are not in the same archimedean component of B. While the archimedean components in $FU(B)$ are its subsemigroups whose elements have the same letters. Hence, $a \sim_B ab$, but the words $w_1 = a$ and $w_2 = ab$ are not in the same archimedean component in $FU(B)$, and they are the unique two words such that $\varepsilon_B(w_1) = a$ and $\varepsilon_B(w_2) = ab$. Therefore, the canonical homomorphism h of Definition 3.4 is not surjective.

The following first table concludes the characterizations of the classes of morphisms considered above, for the reflection of commutative semigroups into semilattices. The second table concludes that there exists an inseparable-separable factorization system, but does not exist a monotone-light factorization system, for that reflection.

Classes of homomorphisms $f : A \to B$ associated to I : CommSgr $\to$ SLat

E'_I	M^*_I	*Normal*	*Pin*	*Sep*	*Ins*
$E_I \cap F$	$M^*_I = M_I$	$\sim_A \circ Ker(f) \subseteq Ker(f) \circ \sim_A$ $Ker(f) \cap \sim_A = \Delta$	$Ker(f) \subseteq \sim_A$	$Ker(f) \cap \sim_A = \Delta$	$E_I \cap E$
Pr. 3.3 *and* *Pr.* 5.3	*Pr.* 3.6 *and* *Pr.* 5.3 *and* *Pr.* 5.4	*Pr.* 3.7 *and* *Pr.* 5.3	*Pr.* 3.7 *and* *Pr.* 5.3	*Pr.* 3.7 *and* *Pr.* 5.3	*Pr.* 4.1

Factorizations associated to I : CommSgr $\to$ SLat

(E_I, M_I)	(Ins, Sep)	(E'_I, M^*_I)
YES	*YES*	*NO*
Admissible reflection	*Pr.* 4.1	$M^*_I = M_I, E'_I = E_I \cap F \neq E_I$ $(i : (\mathbf{N}, +) \subseteq (\mathbf{R}, +), \quad i \in E_I$ *but* $i \notin F)$

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