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MATHEMATICAL PROBLEMS OF THERMO-ELECTRO-MAGNETO-ELASTICITY

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Abstract The monograph is dedicated to the investigation of basic, mixed and crack type three-dimensional boundary value problems (BVP) of the thermo-electro-magneto-elasticity theory. The fundamental matrices of the corresponding differential operators are constructed explicitly and their properties near the origin and at infinity are established. By the potential method the corresponding three-dimensional basic, mixed and crack type BVPs are reduced to the equivalent system of boundary pseudo-differential equations. The solvability of the resulting boundary pseudodifferential equations are analyzed in the Sobolev-Slobodetski (W^s_p), Bessel potential (H^s_p), and Besov ($B^s_{p,t}$) spaces and the corresponding uniqueness and existence theorems for the original boundary value problems are proved. The smoothness properties and singularities of thermo-mechanical and electro-magnetic fields are investigated near the crack edges and the curves where the boundary conditions change their types. It is shown that the smoothness and stress singularity exponents essentially depend on the material parameters and an efficient method for their computation is described.

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Contents

1 Introduction

Modern industrial and technological processes apply widely, on the one hand, composite materials with complex microstructure and, on the other hand, complex composed structures consisting of materials having essentially different physical properties (for example, piezoelectric, piezomagnetic, hemitropic materials, two- and multi-component mixtures, nanomaterials, bio-materials, and solid structures constructed by composition of these materials, such as, e.g., Smart Materials and other meta-materials). Therefore the investigation and analysis of mathematical models describing the mechanical, thermal, electric, magnetic and other physical properties of such materials have a crucial importance for both fundamental research and practical applications. In particular, the investigation of correctness of corresponding mathematical models (namely, existence, uniqueness, smoothness, asymptotic properties and stability of solutions) and construction of appropriate adequate numerical algorithms have a crucial role for fundamental research.

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. For example, piezoelectric materials (electro-elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal-electric coupling) have been applied in thermal imaging devices; and piezomagnetic materials (elastic-magnetic coupling) are pursued for health monitoring of civil structures (see [Mo1], [Qi1], [Er1], [Pa1], [To1], [To2], [Vo1], [La1]-[La9] and the references therein).

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. In the reference [VTS] it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [Br1], [BV], [HDN1], [HDN2], [AH], [Li2], [Nan], [Ben], [LD1], [LD2], [LD3], [WYD1], [He1], [Silva et al].

The mathematical model of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

For these equations the uniqueness theorems of solutions for some mixed initial-boundary value problems are well studied. In particular, in the reference [Li2] the uniqueness theorem is proved without making restrictions on the positive definiteness on the elastic moduli. However, to the best of our knowledge, the existence of solutions of the boundary value, transmission and crack type problems for homogeneous and composed bodies are not studied in the scientific literature systematically.

As it is well known, solutions to mixed and crack type boundary value problems and

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corresponding mechanical, electrical, magnetic and thermal characteristics usually have singularities at the so called *exceptional curves*: the crack edges and the curves where the boundary conditions change their type. Along with the existence and uniqueness questions our main goal is a detailed theoretical investigation of regularity properties of the thermo-mechanical and electro-magnetic fields near the exceptional curves and qualitative description of their singularities. In particular, the most important question is description of the dependence of the stress singularity exponents on the material parameters.

With the help of the potential method we reduce the three-dimensional basic, mixed and crack type boundary value problems of the thermo-electro-magneto-elasticity to the equivalent 6×6 system of pseudo-differential equations which live on proper parts of the boundary of the elastic body under consideration.

We analyze the solvability of the resulting boundary pseudodifferential equations in the Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s), and Besov ($B_{p,t}^s$) spaces and prove the corresponding uniqueness and existence theorems for the original problems.

We show that the principal homogeneous symbol matrices of the corresponding pseudodifferential operators yield information on the existence and regularity of the solution fields and establish global C^α-regularity results with some $\alpha \in (\,0,\, \frac{1}{2}\,)$. The exponent α is determined with the help of the eigenvalues λ_j, $j = \overline{1,6}$, of special 6×6 matrices which are explicitly constructed by means of the principal homogeneous symbol matrices of the corresponding pseudodifferential operators. These eigenvalues depend on the material parameters, in general, and actually they define the singularity exponents for the first order derivatives of solutions. We give an efficient method for computation of the stress singularity exponents.

Essential difficulties arise in the study of exterior BVPs of statics for unbounded domains. The case is that one has to consider the problem in a class of vector functions which are bounded at infinity. This complicates the proof of uniqueness and existence theorems since Green's formulas do not hold for such vector functions and analysis of null spaces of the corresponding integral operators needs special consideration. We have found efficient and natural asymptotic conditions at infinity which ensure the uniqueness of solutions in the space of bounded vector functions. Moreover, for the interior Neumann-type boundary-value problem, the complete system of linearly independent solutions of the corresponding homogeneous adjoint integral equation is constructed in polynomials and the necessary and sufficient conditions of solvability of the problem are written explicitly.

2 Basic equations and formulation of boundary value problems

2.1 Field equations

Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, $E = (E_1, E_2, E_3)^\top$ and $H = (H_1, H_2, H_3)^\top$ are electric and magnetic fields respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials and

$$E = -\operatorname{grad}\varphi\,, \qquad H = -\operatorname{grad}\psi\,, \tag{2.1}$$

ϑ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

We employ also the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation. In what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators.

Constitutive relations:

$$\sigma_{rj} = \sigma_{jr} = c_{rjkl}\,\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \qquad r, j = 1, 2, 3, \tag{2.2}$$

$$D_j = e_{jkl}\,\varepsilon_{kl} + \varkappa_{jl}\,E_l + a_{jl}\,H_l + p_j\,\vartheta, \qquad j = 1, 2, 3, \tag{2.3}$$

$$B_j = q_{jkl}\,\varepsilon_{kl} + a_{jl}\,E_l + \mu_{jl}H_l + m_j\,\vartheta, \qquad j = 1, 2, 3, \tag{2.4}$$

$$\mathcal{S} = \lambda_{kl}\,\varepsilon_{kl} + p_k\,E_k + m_k\,H_k + \gamma\,\vartheta. \tag{2.5}$$

Fourier Law:

$$q_j = -\eta_{jl}\,\partial_l\vartheta, \quad j = 1, 2, 3. \tag{2.6}$$

Equations of motion:

$$\partial_j\sigma_{rj} + X_r = \varrho\ \partial_t^2 u_r, \qquad r = 1, 2, 3. \tag{2.7}$$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free):

$$\partial_j D_j = \varrho_e\,, \qquad \partial_j B_j = 0\,. \tag{2.8}$$

Linearized equation of the entropy balance:

$$T_0\,\partial_t\mathcal{S} - Q = -\partial_j q_j. \tag{2.9}$$

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho\, c\, T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity.

The constants involved in these equations satisfy the symmetry conditions:

$$
\begin{aligned}
&c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \\
&\varkappa_{kj} = \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk}, \quad r, j, k, l = 1, 2, 3.
\end{aligned} \tag{2.10}
$$

From physical considerations it follows that (see, e.g., [No1], [Li1]):

$$
\begin{gathered}
c_{rjkl}\, \xi_{rj}\, \xi_{kl} \geq c_0\, \xi_{kl}\, \xi_{kl}, \quad \varkappa_{kj}\, \xi_k\, \xi_j \geq c_1\, |\xi|^2, \quad \mu_{kj}\, \xi_k\, \xi_j \geq c_2\, |\xi|^2, \quad \eta_{kj}\, \xi_k\, \xi_j \geq c_3\, |\xi|^2, \\
\text{for all} \quad \xi_{kj} = \xi_{jk} \in \mathbb{R} \quad \text{and for all} \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,
\end{gathered} \tag{2.11}
$$

where c_0, c_1, c_2, and c_3 are positive constants.

It easy to see that due to the symmetry conditions (2.11)

$$
\begin{gathered}
c_{rjkl}\, \xi_{rj}\, \overline{\xi_{kl}} \geq c_0\, \xi_{kl}\, \overline{\xi_{kl}}, \quad \varkappa_{kj}\, \xi_k\, \overline{\xi_j} \geq c_1\, |\xi|^2, \quad \mu_{kj}\, \xi_k\, \overline{\xi_j} \geq c_2\, |\xi|^2, \quad \eta_{kj}\, \xi_k\, \overline{\xi_j} \geq c_3\, |\xi|^2, \\
\text{for all} \quad \xi_{kj} = \xi_{jk} \in \mathbb{C} \quad \text{and for all} \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3.
\end{gathered} \tag{2.12}
$$

More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure that for arbitrary $\zeta', \zeta'' \in \mathbb{C}^3$ and $\theta \in \mathbb{C}$ there is a positive constant δ_0 depending on the material constants such that (cf. [No1])

$$
\begin{aligned}
\varkappa_{kj}\, \zeta'_k\, \overline{\zeta'_j} + a_{kj}\, (\zeta'_k\, \overline{\zeta''_j} + \overline{\zeta'_k}\, \zeta''_j) + \mu_{kj}\, \zeta''_k\, \overline{\zeta''_j} \pm 2\, \Re\, [\, \overline{\theta}\, (p_j\, \zeta'_j + m_j\, \zeta''_j)\,] + \gamma\, |\theta|^2& \\
\geq \delta_0\, (|\zeta'|^2 + |\zeta''|^2 + |\theta|^2).&
\end{aligned} \tag{2.13}
$$

This condition is equivalent to positive definiteness of the matrix

$$
\Xi := \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} & [p_j]_{3\times1} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} & [m_j]_{3\times1} \\ [p_j]_{1\times3} & [m_j]_{1\times3} & \gamma \end{bmatrix}_{7\times7}. \tag{2.14}
$$

In particular, it follows that the matrix

$$
\Lambda := \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} \end{bmatrix}_{6\times6} \tag{2.15}
$$

is positive definite, i.e.,

$$
\varkappa_{kj}\, \zeta'_k\, \overline{\zeta'_j} + a_{kj}\, (\zeta'_k\, \overline{\zeta''_j} + \overline{\zeta'_k}\, \zeta''_j) + \mu_{kj}\, \zeta''_k\, \overline{\zeta''_j} \geq \kappa\, (|\zeta'|^2 + |\zeta''|^2) \tag{2.16}
$$

with some positive constant κ depending on the material parameters involved in (2.15). A sufficient condition for the quadratic form in the left hand side of (2.13) to be positive definite then reads as

$$\nu^2 < \frac{\kappa\,\gamma}{6} \qquad \text{with} \qquad \nu = \max\{|p_1|, |p_2|, |p_3|, |m_1|, |m_2|, |m_3|\}. \tag{2.17}$$

With the help of the symmetry conditions (2.11) we can rewrite the constitutive relations (2.2)-(2.5) as follows

$$\sigma_{rj} = c_{rjkl}\,\partial_l u_k + e_{lrj}\,\partial_l\varphi + q_{lrj}\,\partial_l\psi - \lambda_{rj}\,\vartheta, \quad r,j = 1,2,3, \tag{2.18}$$

$$D_j = e_{jkl}\,\partial_l u_k - \varkappa_{jl}\,\partial_l\varphi - a_{jl}\,\partial_l\psi + p_j\,\vartheta, \quad j = 1,2,3, \tag{2.19}$$

$$B_j = q_{jkl}\,\partial_l u_k - a_{jl}\,\partial_l\varphi - \mu_{jl}\,\partial_l\psi + m_j\,\vartheta, \quad j = 1,2,3, \tag{2.20}$$

$$\mathcal{S} = \lambda_{kl}\,\partial_l u_k - p_l\,\partial_l\varphi - m_l\,\partial_l\psi + \gamma\,\vartheta. \tag{2.21}$$

In the theory of thermo-electro-magneto-elasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{rj}\,n_j = c_{rjkl}\,n_j\,\partial_l u_k + e_{lrj}\,n_j\,\partial_l\varphi + q_{lrj}\,n_j\,\partial_l\psi - \lambda_{rj}\,n_j\,\vartheta\,, \quad r = 1,2,3, \tag{2.22}$$

while the normal components of the electric displacement vector, magnetic induction vector and heat flux vector read as

$$D_j\,n_j = e_{jkl}\,n_j\,\partial_l u_k - \varkappa_{jl}\,n_j\,\partial_l\varphi - a_{jl}\,n_j\,\partial_l\psi + p_j\,n_j\,\vartheta, \tag{2.23}$$

$$B_j\,n_j = q_{jkl}\,n_j\,\partial_l u_k - a_{jl}\,n_j\,\partial_l\varphi - \mu_{jl}\,n_j\,\partial_l\psi + m_j\,n_j\,\vartheta, \tag{2.24}$$

$$q_j\,n_j = -\eta_{jl}\,n_j\,\partial_l\vartheta. \tag{2.25}$$

For convenience we introduce the following matrix differential operator

$$\mathcal{T}(\partial, n) = \big[\, \mathcal{T}_{pq}(\partial, n)\,\big]_{6\times 6}$$

$$:= \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times 3} & [e_{lrj}\,n_j\,\partial_l]_{3\times 1} & [q_{lrj}\,n_j\,\partial_l]_{3\times 1} & [-\lambda_{rj}\,n_j]_{3\times 1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -p_j\,n_j \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times 3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -m_j\,n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l \end{bmatrix}_{6\times 6}, \tag{2.26}$$

i.e.,

$$\begin{aligned} &\mathcal{T}_{rk} = c_{rjkl}\,n_j\,\partial_l\,, && \mathcal{T}_{r4} = e_{lrj}\,n_j\,\partial_l\,, && \mathcal{T}_{r5} = q_{lrj}\,n_j\,\partial_l\,, && \mathcal{T}_{r6} = -\lambda_{rj}\,n_j\,, \\ &\mathcal{T}_{4k} = -e_{jkl}\,n_j\,\partial_l\,, && \mathcal{T}_{44} = \varkappa_{jl}\,n_j\,\partial_l\,, && \mathcal{T}_{45} = a_{jl}\,n_j\,\partial_l\,, && \mathcal{T}_{46} = -p_j\,n_j\,, \\ &\mathcal{T}_{5k} = -q_{jkl}\,n_j\,\partial_l\,, && \mathcal{T}_{54} = a_{jl}\,n_j\,\partial_l\,, && \mathcal{T}_{55} = \mu_{jl}\,n_j\,\partial_l\,, && \mathcal{T}_{56} = -m_j\,n_j\,, \\ &\mathcal{T}_{6k} = 0\,, && \mathcal{T}_{64} = 0\,, && \mathcal{T}_{65} = 0\,, && \mathcal{T}_{66} = \eta_{jl}\,n_j\,\partial_l\,, \\ &r,k = 1,2,3. \end{aligned} \tag{2.27}$$

Denote by $\mathcal{T}^{(0)}(\partial,n)$ the main part of the operator $\mathcal{T}(\partial,n)$,

$$\mathcal{T}^{(0)}(\partial,n) = \big[\ \mathcal{T}^{(0)}_{pq}(\partial,n)\ \big]_{6\times 6}$$

$$:= \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times3} & [e_{lrj}\,n_j\,\partial_l]_{3\times1} & [q_{lrj}\,n_j\,\partial_l]_{3\times1} & [0]_{3\times1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & 0 \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & 0 \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l \end{bmatrix}_{6\times6}. \tag{2.28}$$

Evidently, for a six vector $U := (u,\varphi,\psi,\vartheta)^\top$ we have

$$\mathcal{T}(\partial,n)\,U = (\,\sigma_{1j}\,n_j,\ \sigma_{2j}\,n_j,\ \sigma_{3j}\,n_j,\ -D_j\,n_j,\ -B_j\,n_j,\ -q_j\,n_j)^\top. \tag{2.29}$$

The components of the vector $\mathcal{T}\,U$ given by (2.29) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

As we see all the thermo-mechanical and electro-magnetic characteristics can be determined by the six functions: three displacement components u_j, $j=1,2,3$, temperature distribution ϑ, and the electric and magnetic potentials φ and ψ. Therefore, all the above field relations and the corresponding boundary value problems we reformulate in terms of these six functions.

First of all from the equations (2.2)–(2.9) we derive the basic linear system of dynamics of the theory of thermo-electro-magneto-elasticity:

$$\begin{aligned} & c_{rjkl}\,\partial_j\,\partial_l\,u_k(x,t) + e_{lrj}\,\partial_j\partial_l\varphi(x,t) + q_{lrj}\,\partial_j\partial_l\psi(x,t) - \lambda_{rj}\,\partial_j\vartheta(x,t) \\ &\qquad\qquad -\varrho\,\partial_t^2\,u_r(x,t) = -X_r(x,t),\quad r=1,2,3, \\ & -e_{jkl}\,\partial_j\partial_l u_k(x,t) + \varkappa_{jl}\,\partial_j\,\partial_l\varphi(x,t) + a_{jl}\,\partial_j\,\partial_l\psi(x,t) - p_j\,\partial_j\vartheta(x,t) = -\varrho_e(x,t), \\ & -q_{jkl}\,\partial_j\partial_l u_k(x,t) + a_{jl}\,\partial_j\,\partial_l\varphi(x,t) + \mu_{jl}\,\partial_j\,\partial_l\psi(x,t) - m_j\,\partial_j\vartheta(x,t) = 0, \\ & -T_0\,\lambda_{kl}\,\partial_t\partial_l u_k(x,t) + T_0\,p_l\,\partial_t\partial_l\varphi(x,t) + T_0\,m_l\,\partial_t\partial_l\psi(x,t) + \eta_{jl}\,\partial_j\,\partial_l\vartheta(x,t) \\ &\qquad\qquad -T_0\,\gamma\,\partial_t\vartheta(x,t) = -Q(x,t). \end{aligned} \tag{2.30}$$

If all the functions involved in these equations are harmonic time dependent, that is they can be represented as the product of a function of the spatial variables (x_1,x_2,x_3) and the multiplier $\exp\{\tau\,t\}$, where $\tau=\sigma+i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the theory of thermo-electro-magneto-elasticity. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau=i\omega$ is a pure imaginary number, with the so called frequency parameter $\omega\in\mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau=0$ we get the *equations of statics*.

In particular, the corresponding pseudo-oscillation equations read as

$$
\begin{aligned}
& c_{rjkl}\,\partial_j\,\partial_l\,u_k(x)-\varrho\,\tau^2\,u_r(x)+e_{lrj}\,\partial_j\partial_l\varphi(x)+q_{lrj}\,\partial_j\partial_l\psi(x)\\
&\qquad\qquad\qquad -\lambda_{rj}\,\partial_j\vartheta(x)=-X_r(x),\quad r=1,2,3,\\
& -e_{jkl}\,\partial_j\partial_l u_k(x)+\varkappa_{jl}\,\partial_j\,\partial_l\varphi(x)+a_{jl}\,\partial_j\,\partial_l\psi(x)-p_j\,\partial_j\vartheta(x)=-\varrho_e(x),\\
& -q_{jkl}\,\partial_j\partial_l u_k(x)+a_{jl}\,\partial_j\,\partial_l\varphi(x)+\mu_{jl}\,\partial_j\,\partial_l\psi(x)-m_j\,\partial_j\vartheta(x)=0,\\
& -\tau\,T_0\,\lambda_{kl}\,\partial_l u_k(x)+\tau\,T_0\,p_l\,\partial_l\varphi(x)+\tau\,T_0\,m_l\,\partial_l\psi(x)+\eta_{jl}\,\partial_j\,\partial_l\vartheta(x)\\
&\qquad\qquad\qquad -\tau\,T_0\,\gamma\,\vartheta(x)=-Q(x).
\end{aligned}
\tag{2.31}
$$

In matrix form these equations can be written as

$$
A(\partial,\tau)\,U(x)=\Phi(x)\,,
\tag{2.32}
$$

where

$$
\begin{aligned}
& U=(u_1,u_2,u_3,u_4,u_5,u_6)^\top:=(u,\varphi,\psi,\vartheta)^\top,\\
& \Phi=(\Phi_1,\cdots,\Phi_6)^\top:=(-X_1,-X_2,-X_3,-\varrho_e,0,-Q)^\top,
\end{aligned}
$$

and $A(\partial,\tau)$ is the matrix differential operator generated by the equations (2.31),

$$
\begin{aligned}
& A(\partial,\tau)=[A_{pq}(\partial,\tau)]_{6\times 6}\\
& :=\begin{bmatrix}
[c_{rjkl}\,\partial_j\,\partial_l-\varrho\,\tau^2\,\delta_{rk}]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\,\partial_j]_{3\times 1}\\
[-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l & -p_j\,\partial_j\\
[-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l & -m_j\,\partial_j\\
[-\tau\,T_0\,\lambda_{kl}\,\partial_l]_{1\times 3} & \tau\,T_0\,p_l\,\partial_l & \tau\,T_0\,m_l\,\partial_l & \eta_{jl}\,\partial_j\,\partial_l-\tau\,T_0\,\gamma
\end{bmatrix}_{6\times 6}
\end{aligned}
\tag{2.33}
$$

i.e.,

$$
\begin{aligned}
& A_{rk}(\partial,\tau)=c_{rjkl}\,\partial_j\,\partial_l-\varrho\,\tau^2\,\delta_{rk}, && A_{r4}(\partial,\tau)=e_{lrj}\,\partial_j\partial_l\,,\\
& A_{r5}(\partial,\tau)=q_{lrj}\,\partial_j\partial_l, && A_{r6}(\partial,\tau)=-\lambda_{rj}\,\partial_j\,,\\
& A_{4k}(\partial,\tau)=-e_{jkl}\,\partial_j\partial_l, && A_{44}(\partial,\tau)=\varkappa_{jl}\,\partial_j\,\partial_l\,,\\
& A_{45}(\partial,\tau)=a_{jl}\,\partial_j\,\partial_l, && A_{46}(\partial,\tau)=-p_j\,\partial_j\,,\\
& A_{5k}(\partial,\tau)=-q_{jkl}\,\partial_j\partial_l, && A_{54}(\partial,\tau)=a_{jl}\,\partial_j\,\partial_l\,,\\
& A_{55}(\partial,\tau)=\mu_{jl}\,\partial_j\,\partial_l, && A_{56}(\partial,\tau)=-m_j\,\partial_j\,,\\
& A_{6k}(\partial,\tau)=-\tau\,T_0\,\lambda_{kl}\,\partial_l, && A_{64}(\partial,\tau)=\tau\,T_0\,p_l\,\partial_l\,,\\
& A_{65}(\partial,\tau)=\tau\,T_0\,m_l\,\partial_l, && A_{66}(\partial,\tau)=\eta_{jl}\,\partial_j\,\partial_l-\tau\,T_0\,\gamma\,,\\
& r,k=1,2,3.
\end{aligned}
\tag{2.34}
$$

As we have mentioned, we obtain the equations and operators of statics if $\tau = 0$. Denote by $A^{(0)}(\partial) = [A^{(0)}_{kj}(\partial)]_{6\times 6}$ the main homogeneous part of the operator $A(\partial, \tau)$,

$$A^{(0)}(\partial) := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l & 0 \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{6\times 6}. \tag{2.35}$$

Clearly, the symbol matrix of the operator $A^{(0)}(\partial)$ is the principal homogeneous symbol matrix of the operator $A(\partial, \tau)$ and reads as

$$A^{(0)}(-i\,\xi) := \begin{bmatrix} [-c_{rjkl}\,\xi_j\,\xi_l]_{3\times 3} & [-e_{lrj}\,\xi_j\xi_l]_{3\times 1} & [-q_{lrj}\,\xi_j\xi_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}\,\xi_j\xi_l]_{1\times 3} & -\varkappa_{jl}\,\xi_j\xi_l & -a_{jl}\,\xi_j\xi_l & 0 \\ [q_{jkl}\,\xi_j\xi_l]_{1\times 3} & -a_{jl}\,\xi_j\xi_l & -\mu_{jl}\,\xi_j\xi_l & 0 \\ [0]_{1\times 3} & 0 & 0 & -\eta_{jl}\,\xi_j\,\xi_l \end{bmatrix}_{6\times 6}. \tag{2.36}$$

From the symmetry conditions (2.10), inequalities (2.11) and positive definiteness of the matrix (2.14) it follows that there is a positive constant C depending only on the material parameters, such that

$$\begin{gathered} \Re\Big(- A^{(0)}(-i\,\xi)\zeta\,\cdot\,\zeta\Big) = \Re\Big(- \sum_{k,j=1}^{6} A^{(0)}_{kj}(-i\,\xi)\,\zeta_j\,\overline{\zeta_k}\Big) \geq C\,|\xi|^2\,|\zeta|^2 \\ \text{for all} \quad \xi \in \mathbb{R}^3 \quad \text{and} \quad \zeta \in \mathbb{C}^6 . \end{gathered} \tag{2.37}$$

Therefore, $A(\partial, \tau)$ is a nonselfadjoint strongly elliptic differential operator. Here and in what follows the over bar denotes complex conjugation and the central dot denotes the scalar product in the complex space $\mathbb{C}^N$, i.e., $a \cdot b \equiv (a, b) := \sum_{j=1}^{N} a_j\,\overline{b_j}$ for $a, b \in \mathbb{C}^N$.
By $A^*(\partial, \tau) := [\overline{A(-\partial, \tau)}]^\top = A^\top(-\partial, \overline{\tau})$ we denote the operator formally adjoint to $A(\partial, \tau)$. Below, in Green's formulas there appears also the boundary operator $\mathcal{P}(\partial, n, \tau)$ associated with the adjoint differential operator $A^*(\partial, \tau)$,

$$\mathcal{P}(\partial, n, \tau) = \big[\, \mathcal{P}_{pq}(\partial, n, \tau)\, \big]_{6\times 6}$$

$$= \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times 3} & [-e_{lrj}\,n_j\,\partial_l]_{3\times 1} & [-q_{lrj}\,n_j\,\partial_l]_{3\times 1} & [\overline{\tau}\,T_0\,\lambda_{rj}\,n_j]_{3\times 1} \\ [e_{jkl}\,n_j\,\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -\overline{\tau}\,T_0\,p_j\,n_j \\ [q_{jkl}\,n_j\,\partial_l]_{1\times 3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -\overline{\tau}\,T_0\,m_j\,n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l \end{bmatrix}_{6\times 6} \tag{2.38}$$

i.e.,

$$\begin{aligned} &\mathcal{P}_{rk} = \mathcal{T}_{rk}\,, && \mathcal{P}_{r4} = -\mathcal{T}_{r4}\,, && \mathcal{P}_{r5} = -\mathcal{T}_{r5}\,, && \mathcal{P}_{r6} = -\overline{\tau}\,T_0\,\mathcal{T}_{r6}\,, \\ &\mathcal{P}_{4k} = -\mathcal{T}_{4k}\,, && \mathcal{P}_{44} = \mathcal{T}_{44}\,, && \mathcal{P}_{45} = \mathcal{T}_{45}\,, && \mathcal{P}_{46} = \overline{\tau}\,T_0\,\mathcal{T}_{46}\,, \\ &\mathcal{P}_{5k} = -\mathcal{T}_{5k}\,, && \mathcal{P}_{54} = \mathcal{T}_{54}\,, && \mathcal{P}_{55} = \mathcal{T}_{55}\,, && \mathcal{P}_{56} = \overline{\tau}\,T_0\,\mathcal{T}_{56}\,, \\ &\mathcal{P}_{6k} = 0\,, && \mathcal{P}_{64} = 0\,, && \mathcal{P}_{65} = 0\,, && \mathcal{P}_{66} = \mathcal{T}_{66}\,, \quad r, k = 1, 2, 3, \end{aligned}$$

where $\mathcal{T}_{pq}(\partial,n)$ are defined by (2.27). Note that the boundary matrix operator

$$\mathcal{P}^{(0)}(\partial_y,n(y)) := \mathcal{P}(\partial_y,n(y),0) \tag{2.39}$$

represents the main part of the operator $\mathcal{P}(\partial,n,\tau)$.

2.2 Green's formulas

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a smooth boundary $S = \partial\Omega^+$. Throughout the paper we assume that the origin of the co-ordinate system belongs to Ω^+. Assume that the domain $\overline{\Omega^+}$ is filled with an anisotropic homogeneous material with the above described thermo-electro-magneto-elastic properties.

By L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [Tr1], [LiMa1]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we essentially employ also the spaces:

$$\begin{aligned}
&\widetilde{H}_p^s(\mathcal{M}) := \{f \,:\, f \in H_p^s(\mathcal{M}_0),\ \operatorname{supp} f \subset \overline{\mathcal{M}}\},\\
&\widetilde{B}_{p,q}^s(\mathcal{M}) := \{f \,:\, f \in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp} f \subset \overline{\mathcal{M}}\},\\
&H_p^s(\mathcal{M}) := \{r_{\mathcal{M}} f \,:\, f \in H_p^s(\mathcal{M}_0)\},\\
&B_{p,q}^s(\mathcal{M}) := \{r_{\mathcal{M}} f \,:\, f \in B_{p,q}^s(\mathcal{M}_0)\,\},
\end{aligned}$$

where $\mathcal{M}_0$ is a closed manifold without boundary and $\mathcal{M}$ is an open submanifold of $\mathcal{M}_0$ with nonempty boundary $\partial\mathcal{M} \neq \varnothing$; $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$. Below, sometimes we use also the abbreviations $H_2^s = H^s$ and $W_2^s = W^s$.

Let us also make the following agreement: if we are given only an open manifold $\mathcal{M}$ with nonempty boundary $\partial\mathcal{M}$, then we again use the notation $\widetilde{B}_{p,q}^s(\mathcal{M})$ to denote the space of functions whose extension by zero onto an appropriately chosen closed "enveloping" manifold $\mathcal{M}_0$ without boundary preserves the space, i.e., the extended by zero functions belong to the space $B_{p,q}^s(\mathcal{M}_0)$ and have supports in $\overline{\mathcal{M}}$.

For arbitrary vector-functions

$$U = (u_1,u_2,u_3,\varphi,\psi,\vartheta)^\top \in \big[C^2(\overline{\Omega^+})\big]^6 \quad \text{and} \quad U' = (u_1',u_2',u_3',\varphi',\psi',\vartheta')^\top \in \big[C^2(\overline{\Omega^+})\big]^6$$

we can derive the following Green's identities with the help of the Gauss integration by parts formula:

$$\int\limits_{\Omega^+}\Big[A(\partial,\tau)\,U\cdot U'+\mathcal{E}(U,\overline{U'})\Big]dx=\int\limits_{\partial\Omega^+}\{\mathcal{T}(\partial,n)U\}^+\cdot\{U'\}^+dS\,, \tag{2.40}$$

$$\int\limits_{\Omega^+}\Big[U\cdot A^*(\partial,\tau)\,U'+\mathcal{E}(U,\overline{U'})\Big]dx=\int\limits_{\partial\Omega^+}\{U\}^+\cdot\{\mathcal{P}(\partial,n,\tau)U'\}^+dS\,, \tag{2.41}$$

$$\int\limits_{\Omega^+}\Big[A(\partial,\tau)\,U\cdot U'-U\cdot A^*(\partial,\tau)\,U'\Big]dx=\int\limits_{\partial\Omega^+}\Big[\{\mathcal{T}(\partial,n)U\}^+\cdot\{U'\}^+ -\{U\}^+\cdot\{\mathcal{P}(\partial,n,\tau)\,U'\}^+\Big]dS, \tag{2.42}$$

where the symbol $\{\,\cdot\,\}^+$ denotes the one sided limit (the trace operator) on $\partial\Omega^+$ from Ω^+, the operators $A(\partial,\tau)$, $\mathcal{T}(\partial,n)$ and $\mathcal{P}(\partial,n,\tau)$ are determined by (2.33), (2.26) and (2.38) respectively, $A^*(\partial,\tau)$ is the operator adjoint to $A(\partial,\tau)$, $n(x)=(n_1(x),n_2(x),n_3(x))$ is the outward unit normal vector with respect to Ω^+ at the point $x\in\partial\Omega^+$ and

$$\begin{aligned}\mathcal{E}(U,\overline{U'})&=c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u_r'}+\varrho\,\tau^2\,u_r\,\overline{u_r'}+e_{lrj}\,(\partial_l\varphi\,\overline{\partial_j u_r'}-\partial_j u_r\,\overline{\partial_l\varphi'})\\&\quad+q_{lrj}\,(\partial_l\psi\,\overline{\partial_j u_r'}-\partial_j u_r\,\overline{\partial_l\psi'})+\varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi'}+a_{jl}\,(\partial_l\varphi\,\overline{\partial_j\psi'}+\partial_j\psi\,\overline{\partial_l\varphi'})\\&\quad+\mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi'}+\lambda_{kj}\,(\tau\,T_0\,\overline{\vartheta'}\,\partial_j u_k-\vartheta\,\overline{\partial_j u_k'})-p_l\,(\tau\,T_0\,\overline{\vartheta'}\,\partial_l\varphi+\vartheta\,\overline{\partial_l\varphi'})\\&\quad-m_l\,(\tau\,T_0\,\overline{\vartheta'}\,\partial_l\psi+\vartheta\,\overline{\partial_l\psi'})+\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta'}+\tau\,T_0\,\gamma\,\vartheta\,\overline{\vartheta'}\,.\end{aligned} \tag{2.43}$$

Remark that the above Green's formulas by standard limiting procedure can be generalized to Lipschitz domains and to vector-functions $U\in[W^1_p(\Omega^+)]^6$ and $U'\in[W^1_{p'}(\Omega^+)]^6$ with

$$A(\partial,\tau)U\in\big[L_p(\Omega^+)\big]^6,\quad A^*(\partial,\tau)U'\in\big[L_{p'}(\Omega^+)\big]^6,\quad 1<p<\infty,\quad \frac{1}{p}+\frac{1}{p'}=1.$$

With the help of these Green's formulas we can correctly determine a *generalized trace vector* $\{\mathcal{T}(\partial,n)U\}^+\in[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ for a function $U\in[W^1_p(\Omega^+)]^6$ with $A(\partial,\tau)U\in[L_p(\Omega^+)]^6$ by the following relation

$$\big\langle\,\{\mathcal{T}(\partial,n)U\}^+\,,\,\{U'\}^+\big\rangle_{\partial\Omega^+}:=\int\limits_{\Omega^+}[\,A(\partial,\tau)\,U\cdot U'+\mathcal{E}(U,\overline{U'})\,]\,dx, \tag{2.44}$$

where $U'\in[W^1_{p'}(\Omega^+)]^6$ is an arbitrary vector-function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the function spaces $[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ and $[B^{1/p}_{p',p'}(\partial\Omega^+)]^6$ which extends the usual L_2 scalar product

$$\langle\,f\,,\,g\,\rangle_{\partial\Omega^+}=\int\limits_{\partial\Omega^+}\sum_{j=1}^{6}f_j\,\overline{g_j}\,dS\quad\text{for}\quad f,g\in[L_2(\partial\Omega^+)]^6. \tag{2.45}$$

Evidently we have the following estimate

$$\|\{\mathcal{T}(\partial,n)U\}^{+}\|_{[B_{p,p}^{-1/p}(\partial\Omega^{+})]^{6}} \leq c^{*}\left\{\|A(\partial,\tau)\,U\|_{[L_{p}(\Omega^{+})]^{6}} + \|U\|_{[W_{p}^{1}(\Omega^{+})]^{6}}\right\}, \tag{2.46}$$

where c^* does not depend on U; in general c^* depends on the material parameters and on the geometrical characteristics of the domain.

Let us introduce a sesquilinear form on $[H_2^1(\Omega^+)]^6 \times [H_2^1(\Omega^+)]^6$

$$\mathcal{B}(U,V) := \int\limits_{\Omega^{+}} \mathcal{E}(U,\overline{V})\,dx. \tag{2.47}$$

With the help of the relations (2.11) and (2.43), positive definiteness of the matrix (2.15) and the well known Korn's inequality we easily establish

$$\Re\,\mathcal{B}(U,U) \geq c_1\,||U||^2_{[H_2^1(\Omega^+)]^6} - c_2\,||U||^2_{[H_2^0(\Omega^+)]^6} \tag{2.48}$$

with some positive constants c_1 and c_2 depending on the material parameters (cf. [Fi1], [Ne1]).

2.3 Formulation of boundary value problems

As above let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a smooth simply connected boundary $S = \partial\Omega^+$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\overline{\Omega^\pm}$ are filled with an anisotropic homogeneous material with the above described thermo-electro-magneto-elastic properties. The symbols $\{\,\cdot\,\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$, while $n = (n_1, n_2, n_3)$ stands for the outward unit normal vector on S with respect to Ω^+. Further, let S_D and S_N denote two disjoint sub-manifolds of S such that $S = \overline{S}_D \cup \overline{S}_N$. Put $\partial S_D = \partial S_N =: \ell_m$. In what follows, for simplicity we assume that S, S_D, S_N, ℓ_m are C^∞-smooth if not otherwise stated.

2.3.1 Basic boundary value problems

Here we formulate the basic interior and exterior boundary value problems of the thermo-electro-magneto-elasticity theory. The operators $A(\partial,\tau)$ and $\mathcal{T}(\partial,n)$ involved in the formulations below are determined by (2.33) and (2.26) respectively.

Dirichlet problem $(D)^\pm$: *Find a solution vector* $U = (u,\varphi,\psi,\vartheta)^\top \in [W_p^1(\Omega^+)]^6$ (*respectively* $U = (u,\varphi,\psi,\vartheta)^\top \in [W^1_{p,loc}(\Omega^-)]^6$) *to the system of pseudo-oscillation equations*

$$A(\partial,\tau)\,U = \Phi \quad \text{in} \quad \Omega^\pm \tag{2.49}$$

satisfying the Dirichlet type boundary condition

$$\{\,U\}^{\pm} = g \qquad on \qquad S. \tag{2.50}$$

Neumann problem $(N)^{\pm}$: *Find a solution vector* $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_p(\Omega^+)]^6$ *(respectively* $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_{p,loc}(\Omega^-)]^6$*) to the system of pseudo-oscillation equations* (2.49) *satisfying the Neumann type boundary condition*

$$\big\{\mathcal{T}(\partial,n)U\big\}^{\pm}=G \qquad on \qquad S. \tag{2.51}$$

Mixed problem $(M)^{\pm}$: *Find a solution vector* $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_p(\Omega^+)]^6$ *(respectively* $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_{p,loc}(\Omega^-)]^6$*) to the system of pseudo-oscillation equations* (2.49) *satisfying the mixed Dirichlet-Neumann type boundary conditions*

$$\big\{U\big\}^{\pm}=g^{(D)} \qquad on \qquad S_D, \tag{2.52}$$

$$\big\{\mathcal{T}(\partial,n)U\big\}^{\pm}=G^{(N)} \qquad on \qquad S_N. \tag{2.53}$$

Note that the most general mixed type BVP can be formulated as follows (cf., e.g., [Li1]): *Find a solution* $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_p(\Omega^+)]^6$ *(respectively* $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_{p,loc}(\Omega^-)]^6$*) to the system of pseudo-oscillation equations* (2.49) *satisfying the mixed type boundary conditions*

$$\begin{aligned}
&\big\{u_k\big\}^{\pm}=g^{(D)}_k \quad on \quad S'_k, &&\big\{[\mathcal{T}(\partial,n)U]_k\big\}^{\pm}=G^{(N)}_k \quad on \quad S''_k, \quad k=1,2,3,\\
&\big\{\varphi\big\}^{\pm}=g^{(D)}_4 \quad on \quad S'_4, &&\big\{[\mathcal{T}(\partial,n)U]_4\big\}^{\pm}=G^{(N)}_4 \quad on \quad S''_4,\\
&\big\{\psi\big\}^{\pm}=g^{(D)}_5 \quad on \quad S'_5, &&\big\{[\mathcal{T}(\partial,n)U]_5\big\}^{\pm}=G^{(N)}_5 \quad on \quad S''_5,\\
&\big\{\vartheta\big\}^{\pm}=g^{(D)}_6 \quad on \quad S'_6, &&\big\{[\mathcal{T}(\partial,n)U]_6\big\}^{\pm}=G^{(N)}_6 \quad on \quad S''_6,
\end{aligned} \tag{2.54}$$

where $S'_j\cap S''_j=\varnothing$ *and* $\overline{S'_j}\cup\overline{S''_j}=S$, $j=\overline{1,6}$.

The differential equation (2.49) is understood in the distributional sense, in general. We remark that if $U\in[H^1_p(\Omega)]^6$ solves the homogeneous differential equation in a domain $\Omega\subset\mathbb{R}^3$ then actually we have the inclusion $U\in[C^\infty(\Omega)]^6$ due to the strong ellipticity of the differential operator $A(\partial,\tau)$. In fact, in this case U is a complex valued analytic vector function of the spatial real variables (x_1,x_2,x_3) in the domain Ω.

The Dirichlet type conditions (2.50) and (2.52) are understood in the usual trace sense, while the Neumann type conditions (2.51) and (2.53) involving boundary limiting values of the components of the vector $\mathcal{T}U$ are understood in the above described functional sense (see (2.44)).

We require that the data involved in the above setting possess the natural smoothness properties associated with the trace theorems, more precisely, we assume that

$$\begin{aligned}
&\Phi\in[L_p(\Omega^+)]^6, \quad \Phi\in[L_{p,comp}(\Omega^-)]^6, \quad g\in[B^{1-\frac1p}_{p,p}(S)]^6,\\
&G\in[B^{-\frac1p}_{p,p}(S)]^6, \quad g^{(D)}\in[B^{1-\frac1p}_{p,p}(S_D)]^6, \quad G^{(N)}\in[B^{-\frac1p}_{p,p}(S_N)]^6.
\end{aligned} \tag{2.55}$$

Thus, in the case of the exterior problems the right hand side vector in the differential equation (2.49) is assumed to be compactly supported. In addition, in this case we have to require some decay conditions for the components of solution vectors and its derivatives at infinity. Namely, for $\tau = \sigma + i\,\omega$ with $\sigma > 0$ solutions of the pseudo-oscillation problems should satisfy the following conditions at infinity

$$\begin{aligned}
&u_k(x) = \mathcal{O}(|x|^{-2}), \quad \partial_j u_k(x) = \mathcal{O}(|x|^{-3}), \quad \partial_j\partial_l u_k(x) = \mathcal{O}(|x|^{-4}),\\
&\varphi(x) = \mathcal{O}(|x|^{-1}), \quad \partial_j \varphi(x) = \mathcal{O}(|x|^{-2}), \quad \partial_j\partial_l \varphi(x) = \mathcal{O}(|x|^{-3}),\\
&\psi(x) = \mathcal{O}(|x|^{-1}), \quad \partial_j \psi(x) = \mathcal{O}(|x|^{-2}), \quad \partial_j\partial_l \psi(x) = \mathcal{O}(|x|^{-3}),\\
&\vartheta(x) = \mathcal{O}(|x|^{-2}), \quad \partial_j \vartheta(x) = \mathcal{O}(|x|^{-3}), \quad \partial_j\partial_l \vartheta(x) = \mathcal{O}(|x|^{-4}), \quad k,j,l = 1,2,3,
\end{aligned} \tag{2.56}$$

As we shall see below the fundamental matrix of the operator $A(\partial, \tau)$ possesses these decay properties at infinity (see Section 3).

For BVPs of statics, i.e., when $\tau = 0$, the conditions at infinity will be specified later.

2.3.2 Crack type boundary value problems

Let an elastic solid occupying the domain Ω^+ (respectively Ω^-) contain an interior crack. We identify the crack surface as a two-dimensional, two-sided smooth manifold $\Sigma \subset \Omega^\pm$ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a submanifold of a closed surface Σ_0 surrounding a domain $\overline{\Omega}_0$ which is a proper subdomain of Ω^+ (respectively Ω^-). We choose the direction of the unit normal vector on the fictional surface Σ_0 such that it is outward with respect to the domain Ω_0. This agreement defines uniquely the direction of the normal vector on the crack surface Σ.

As usual, we assume that the crack faces are mechanically traction free, i.e., the traces of the components of the mechanical stress vector $\{\sigma_{lj}\, n_l\}^\pm$, $j = 1,2,3$, equal to zero on Σ.

Depending on the physical properties of the crack gap, one can consider different conditions on the crack faces for the electric, magnetic and thermal fields. In particular,

1. if the crack gap is a dielectric medium, then the traces of the normal component of the electric displacement vector $\{D_l\, n_l\}^\pm$ should be zero on Σ;

2. if the crack gap is a conductor, then the electric potential function and the normal component of the electric displacement vector should satisfy the electrically permeable boundary conditions on the crack surface Σ, i.e., $\{\varphi\}^+ = \{\varphi\}^-$ and $\{D_l\, n_l\}^+ = \{D_l\, n_l\}^-$ on Σ;

3. if the crack gap is not magnetically permeable, then the traces of the normal component of the magnetic induction vector $\{B_l\, n_l\}^\pm$ should be zero on Σ;

4. if the crack gap is magnetically permeable, then the magnetic potential function and the normal component of the magnetic induction vector should be continuous across the crack surface Σ, i.e. $\{\psi\}^+ = \{\psi\}^-$ and $\{B_l\, n_l\}^+ = \{B_l\, n_l\}^-$ on Σ;

5. if the crack gap is thermally insulated, then the traces of the normal heat flux function $\{q_l\, n_l\}^\pm$ should be zero on the crack surface Σ;

6. if the crack gap is not thermally insulated, then the temperature and the normal heat flux functions should be continuous on the crack surface Σ, i.e., $\{\vartheta\}^+ = \{\vartheta\}^-$ and $\{q_l\, n_l\}^+ = \{q_l\, n_l\}^-$ on Σ;

The applicability and effect of the crack-free electrical boundary conditions in piezoelectric fracture are investigated in many papers and by treating flaws in a medium as notches with a finite width, the results from different electrical boundary condition assumptions on the crack faces are compared. It is found that the electrically impermeable boundary is a reasonable one for engineering problems. Unless the flaw interior is filled with conductive media, the permeable crack assumption may not be directly applied to the fracture of piezoelectric materials in engineering applications (see, e.g. [WM1] and the references therein).

As model cases we shall consider the following two type of conditions on the crack surface Σ :

(CN) Neumann type crack conditions - the crack gap is thermally insulated and electrically impermeable:

$$\big\{\,[\mathcal{T}(\partial,n)U]_j\big\}^{\pm} \equiv \big\{\,\sigma_{lj}\, n_l\big\}^{\pm} = G_j^{(\pm)}, \quad j=1,2,3, \tag{2.57}$$

$$\big\{\,[\mathcal{T}(\partial,n)U]_4\big\}^{\pm} \equiv \big\{\,-D_l\, n_l\big\}^{\pm} = G_4^{(\pm)}, \tag{2.58}$$

$$\big\{\,[\mathcal{T}(\partial,n)U]_5\big\}^{\pm} \equiv \big\{\,-B_l\, n_l\big\}^{\pm} = G_5^{(\pm)}, \tag{2.59}$$

$$\big\{\,[\mathcal{T}(\partial,n)U]_6\big\}^{\pm} \equiv \big\{\,-q_l\, n_l\big\}^{\pm} = G_6^{(\pm)}, \tag{2.60}$$

(CT) Transmission type crack conditions - the crack gap is thermally and electrically conductive:

$$\big\{\,[\mathcal{T}(\partial,n)U]_j\big\}^{\pm} \equiv \big\{\,\sigma_{lj}\, n_l\big\}^{\pm} = G_j^{(\pm)}, \quad j=1,2,3, \tag{2.61}$$

$$\{U_4\}^+ - \{U_4\}^- \equiv \{\varphi\}^+ - \{\varphi\}^- = \widetilde{g}_4 \tag{2.62}$$

$$\big\{\,[\mathcal{T}(\partial,n)U]_4\big\}^{+} - \big\{\,[\mathcal{T}(\partial,n)U]_4\big\}^{-} \equiv \big\{D_l\, n_l\big\}^{-} - \big\{D_l\, n_l\big\}^{+} = \widetilde{G}_4, \tag{2.63}$$

$$\{U_5\}^+ - \{U_5\}^- \equiv \{\psi\}^+ - \{\psi\}^- = \widetilde{g}_5, \tag{2.64}$$

$$\big\{\,[\mathcal{T}(\partial,n)U]_5\big\}^{+} - \big\{\,[\mathcal{T}(\partial,n)U]_5\big\}^{-} \equiv \big\{B_l\, n_l\big\}^{-} - \big\{B_l\, n_l\big\}^{+} = \widetilde{G}_5, \tag{2.65}$$

$$\{U_6\}^+ - \{U_6\}^- \equiv \{\vartheta\}^+ - \{\vartheta\}^- = \widetilde{g}_6, \tag{2.66}$$

$$\big\{\,[\mathcal{T}(\partial,n)U]_6\big\}^{+} - \big\{\,[\mathcal{T}(\partial,n)U]_6\big\}^{-} \equiv \big\{q_l\, n_l\big\}^{-} - \big\{q_l\, n_l\big\}^{+} = \widetilde{G}_6, \tag{2.67}$$

where

$$G_k^{(\pm)} \in B_{p,p}^{-\frac{1}{p}}(\Sigma), \;\; G_k^{(+)} - G_k^{(-)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \;\; \widetilde{g}_j \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma), \;\; \widetilde{G}_j \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \tag{2.68}$$
$$k=\overline{1,6}, \quad j=4,5,6.$$

If the domain under consideration contains an interior crack Σ, then we look for solutions of the equation (2.49) in the domain $\Omega^{\pm}_{\Sigma} := \Omega^{\pm} \setminus \overline{\Sigma}$ belonging to the spaces $\big[W_2^1(\Omega^{\pm}_{\Sigma})\big]^6$ or

$\big[W^1_{2,\,loc}(\Omega^-_\Sigma)\big]^6$, and in the formulation of the basic boundary value problems described in Subsection 2.3.1 we have to add either the crack conditions (CN) or the crack conditions (CT). In the case of exterior problems we have to require again the decay conditions (2.56) at infinity.

These problems we refer as the crack type problems $(D)^{\pm}$-(CN), $(N)^{\pm}$-(CN), $(M)^{\pm}$-(CN), $(D)^{\pm}$-(CT), $(N)^{\pm}$-(CT), and $(M)^{\pm}$-(CT), respectively.

2.4 Uniqueness theorems

In this subsection we study uniqueness of solutions of the interior and exterior BVPs for the pseudo-oscillation equations and the interior BVPs of statics. The exterior BVPs of statics will be treated later in subsection 3.4.

2.4.1 Uniqueness theorems for pseudo-oscillation problems

We start with the following uniqueness result for $p = 2$.

Theorem 2.1 *Let S be Lipschitz and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$.*

(i) *The basic boundary value problems $(D)^+$ and $(M)^+$ have at most one solution in the space $\big[W^1_2(\Omega^+)\big]^6$.*

(ii) *Solutions to the Neumann type boundary value problem $(N)^+$ in the space $\big[W^1_2(\Omega^+)\big]^6$ are defined modulo a vector of type $U^{(\mathcal{N})} = (0,0,0,b_1,b_2,0)^\top$, where b_1 and b_2 are arbitrary constants.*

(iii) *The crack type boundary value problems $(D)^+$-(CN), $(M)^+$-(CN), $(D)^+$-(CT), and $(M)^+$-(CT) have at most one solution in the space $\big[W^1_2(\Omega^+_\Sigma)\big]^6$.*

(iv) *Solutions to the crack type boundary value problems $(N)^+$-(CN) and $(N)^+$-(CT) in the space $\big[W^1_2(\Omega^+_\Sigma)\big]^6$ are defined modulo a vector of type $U^{(\mathcal{N})} = (0,0,0,b_1,b_2,0)^\top$, where b_1 and b_2 are arbitrary constants.*

Proof. Due to the linearity of the boundary value problems in question it suffices to prove that the corresponding homogeneous problems have only the trivial solution.

First we demonstrate the proof for the problem $(M)^+$. Let $U = (u,\varphi,\psi,\vartheta)^\top \in \big[W^1_2(\Omega^+)\big]^6$ be a solution to the problem $(M)^+$ with $\Phi = 0$ in Ω^+, $g^{(D)} = 0$ on S_D and $G^{(N)} = 0$ on S_N (see (2.52)-(2.53)). For arbitrary $U' = (u',\varphi',\psi',\vartheta')^\top \in \big[W^1_2(\Omega^+)\big]^6$ from Green's formula (2.44) then we have

$$\int\limits_{\Omega^+} \mathcal{E}(U,\overline{U'})\,dx = \big\langle \{\mathcal{T}U\},\,\{U'\}^+\big\rangle_{\partial\Omega^+}, \tag{2.69}$$

where $\mathcal{E}(U,\overline{U'})$ is given by (2.43).

If in (2.69) we substitute successively the vectors $(u_1,u_2,u_3,0,0,0)^\top$, $(0,0,0,\varphi,0,0)^\top$, $(0,0,0,0,\psi,0)^\top$, and $(0,0,0,0,0,[\overline{\tau}\,T_0]^{-1}\vartheta)^\top$ for the vector U' and take into consideration the homogeneous boundary conditions, we get

$$\int\limits_{\Omega^+} \{c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u_r} + \varrho\,\tau^2\,u_r\,\overline{u_r} + e_{lrj}\,\partial_l\varphi\,\overline{\partial_j u_r} + q_{lrj}\,\partial_l\psi\,\overline{\partial_j u_r} - \lambda_{kj}\,\vartheta\,\overline{\partial_j u_k}\}dx = 0, \tag{2.70}$$

$$\int\limits_{\Omega^+} \{-e_{lrj}\,\partial_j u_r\,\overline{\partial_l\varphi} + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,\partial_j\psi\,\overline{\partial_l\varphi} - p_l\,\vartheta\,\overline{\partial_l\varphi}\}\,dx = 0, \tag{2.71}$$

$$\int\limits_{\Omega^+} \{-q_{lrj}\,\partial_j u_r\,\overline{\partial_l\psi} + a_{jl}\,\partial_l\varphi\,\overline{\partial_j\psi} + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi} - m_l\,\vartheta\,\overline{\partial_l\psi}\}\,dx = 0, \tag{2.72}$$

$$\int\limits_{\Omega^+} \{\lambda_{kj}\,\overline{\vartheta}\,\partial_j u_k - p_l\,\overline{\vartheta}\,\partial_l\varphi - m_l\,\overline{\vartheta}\,\partial_l\psi + [\,\tau\,T_0\,]^{-1}\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta} + \gamma\,\vartheta\,\overline{\vartheta}\}\,dx = 0. \tag{2.73}$$

Add to equation (2.70) the complex conjugate of equations (2.71)-(2.73) and take into account the symmetry properties (2.10) to obtain

$$\begin{aligned}\int\limits_{\Omega^+} \Big\{ & c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u_r} + \varrho\,\tau^2\,|u_r|^2 + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi} \\ & - p_l\,(\vartheta\,\overline{\partial_l\varphi} + \overline{\vartheta}\,\partial_l\varphi) - m_l\,(\vartheta\,\overline{\partial_l\psi} + \overline{\vartheta}\,\partial_l\psi) + \frac{\tau}{|\tau|^2\,T_0}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta} + \gamma\,|\vartheta|^2\Big\}\,dx = 0.\end{aligned} \tag{2.74}$$

Due to the relations (2.12) and (2.13) we have

$$c_{ijlk}\,\partial_i u_j\,\overline{\partial_l u_k} \geq 0, \qquad \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta} \geq 0,$$
$$\Im\,[\varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi}] = 0.$$

Therefore, separating the imaginary part of (2.74) leads to the equation

$$\omega \int\limits_{\Omega^+} \Big\{2\,\varrho\,\sigma\,|u_r|^2 + \frac{1}{|\tau|^2\,T_0}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta}\Big\}\,dx = 0. \tag{2.75}$$

Whence, $u = 0$ and $\vartheta = const$ in Ω^+ follow if $\omega \neq 0$ (since $\sigma > 0$). From (2.74) we then have

$$\begin{aligned}\int\limits_{\Omega^+} \{ & \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi} \\ & - p_l\,(\vartheta\,\overline{\partial_l\varphi} + \overline{\vartheta}\,\partial_l\varphi) - m_l\,(\vartheta\,\overline{\partial_l\psi} + \overline{\vartheta}\,\partial_l\psi) + \gamma\,|\vartheta|^2\}\,dx = 0.\end{aligned} \tag{2.76}$$

Whence, with the help of inequality (2.13) we get $\partial_l\varphi = 0$, $\partial_l\psi = 0$, $l = 1, 2, 3$, and $\vartheta = 0$ in Ω^+. Thus, if $\omega \neq 0$, finally we have

$$u = 0, \quad \varphi = b_1 = const, \quad \psi = b_2 = const, \quad \vartheta = 0 \quad \text{in } \Omega^+. \tag{2.77}$$

If $\omega = 0$, then from (2.74) with the help of (2.11) and (2.13) we easily derive equalities (2.77). Now, in view of the homogeneous Dirichlet condition on S_D we conclude $U = (u, \varphi, \psi, \vartheta)^\top = 0$ in Ω^+.

For the homogeneous BVPs $(D)^+$ and $(N)^+$ we again arrive at the relations (2.75) and (2.76), whence (2.77) follows immediately. Therefore we conclude that the homogeneous Dirichlet BVP $(D)^+$ possesses only the trivial solution. Furthermore, it can easily be shown that a vector $U^{(\mathcal{N})} = (0,0,0,b_1,b_2,0)^\top$, where b_1 and b_2 are arbitrary constants, solves the homogeneous Neumann BVP $(N)^+$. This proves the items (i) and (ii).

To prove the remaining items of the theorem we have to add together two Green's formulas of type (2.69) for the domains $\Omega \setminus \overline{\Omega}_0$ and Ω_0, where the auxiliary domain $\Omega_0 \subset \Omega^+$ is introduced in the beginning of Subsection 2.3.2. We recall that the crack surface Σ is a proper part of the boundary $\Sigma_0 = \partial\Omega_0 \subset \Omega^+$ and any solution to the homogeneous differential equation $A(\partial,\tau)\,U = 0$ of the class $\big[W^1_2(\Omega^+_\Sigma)\big]^6$ and its derivatives are continuous across the surface $\Sigma_0 \setminus \overline{\Sigma}$. If U is a solution to one of the homogeneous crack type BVPs listed in items (iii) and (iv), by the same approach as above, we arrive at the relation

$$\int\limits_{\Omega^+_\Sigma} \Big\{ c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u_r} + \varrho\,\tau^2\,|u_r|^2 + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi}$$
$$-p_l\,(\vartheta\,\overline{\partial_l\varphi} + \overline{\vartheta}\,\partial_l\varphi) - m_l\,(\vartheta\,\overline{\partial_l\psi} + \overline{\vartheta}\,\partial_l\psi) + \frac{\tau}{|\tau|^2\,T_0}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta} + \gamma\,|\vartheta|^2 \Big\}\,dx = 0. \quad (2.78)$$

The surface integrals vanish due to the homogeneous boundary and crack type conditions and the above mentioned continuity of solutions and its derivatives across the auxiliary surface $\Sigma_0 \setminus \overline{\Sigma}$. Therefore, the proof of items (iii) and (iv) can be verbatim performed. $\square$

For the exterior BVPs of pseudo-oscillations we have the following uniqueness results.

Theorem 2.2 *Let S be Lipschitz and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. The exterior basic boundary value problems $(D)^-$, $(N)^-$ and $(M)^-$, and the crack type boundary value problems $(D)^-$-(CN), $(N)^-$-(CN), $(M)^-$-(CN), $(D)^-$-(CT), $(N)^-$-(CT) and $(M)^-$-(CT) have at most one solution in the space $\big[W^1_{2,\,loc}(\Omega^-)\big]^6$ and $\big[W^1_{2,\,loc}(\Omega^-_\Sigma)\big]^6$, respectively, satisfying the decay conditions* (2.56) *at infinity.*

Proof. From (2.40) and (2.42) by standard limiting procedure we can obtain Green's formulas in Ω^- for arbitrary vectors $U \in \big[C^2(\overline{\Omega^-})\big]^6$ and $U' \in \big[C^2(\overline{\Omega^-})\big]^6$ satisfying the decay conditions at infinity (2.56)

$$\int\limits_{\Omega^-} \big[A(\partial,\tau)\,U \cdot U' + \mathcal{E}(U,\overline{U'})\big]\,dx = -\int\limits_S \{\mathcal{T}U\}^- \cdot \{U'\}^- dS\,, \quad (2.79)$$

$$\int\limits_{\Omega^-} \big[A(\partial,\tau)\,U \cdot U' - U \cdot A^*(\partial,\tau)\,U'\big]\,dx = -\int\limits_S \big[\{\mathcal{T}U\}^- \cdot \{U'\}^- - \{U\}^- \cdot \{\mathcal{P}\,U'\}^-\big]\,dS, \quad (2.80)$$

where the symbol $\{\,\cdot\,\}^-$ denotes the one sided limit (the trace operator) on $S = \partial\Omega^-$ from Ω^- and n is the outward normal to S with respect to Ω^+.

As in the case of bounded domains, these formulas can be extended to vectors U and U' from the space $\big[W^1_{2,loc}(\Omega^-)\big]^6$ satisfying the decay conditions at infinity (2.56) and

$$A(\partial,\tau)\,U,\ A^*(\partial,\tau)\,U' \in \big[L_{2,comp}(\Omega^-)\big]^6.$$

In these generalized Green's formulas the surface integrals in the right hand side expressions in (2.79) and (2.80) are understood in the appropriate duality sense.

Note that since the operator $A(\partial,\tau)$ is strongly elliptic and $A(\partial,\tau)\,U$ has a compact support, then actually U is an analytic vector function of real variables (x_1, x_2, x_3) in a vicinity of infinity and the conditions (2.56) can be understood in the usual classical sense. Therefore, the integrals over Ω^- in formulas (2.79) and (2.80) are convergent and well defined.

With the help of formula (2.79) and the decay conditions (2.56) by the word for word arguments applied in the proof of Theorem 2.1 we can show that the homogeneous basic and crack type exterior BVPs possess only the trivial solution. □

2.4.2 Uniqueness theorems for interior static problems

The setting of the BVPs of statics coincides with the above formulated pseudo-oscillation BVPs with $\tau = 0$. Note that the differential equation for the temperature function and the corresponding boundary conditions are then decoupled and we obtain a separated BVPs for ϑ, since

$$[A(\partial,0)U]_6 = \eta_{jl}\partial_j\partial_l\vartheta \quad \text{and} \quad \big\{\mathcal{T}(\partial,n)U\big\}_6 = \big\{\,\eta_{jl}n_j\partial_l\vartheta\big\}.$$

Note that in static problems, without loss of generality, we can assume that all unknowns and given data are real functions, since the coefficients of the differential operators in Ω^+ and the boundary operators on $\partial\Omega^+$ are real quantities. For static BVPs we have the following uniqueness results.

Theorem 2.3 *Let S be a Lipschitz surface.*
(i) *The homogeneous boundary value problems of statics $(D)^+$ and $(M)^+$ have only the trivial solution in the space $\big[W^1_2(\Omega^+)\big]^6$.*
(ii) *The crack type boundary value problems of statics $(D)^+$-(CN), $(M)^+$-(CN), $(D)^+$-(CT), and $(M)^+$-(CT) have at most one solution in the space $\big[W^1_2(\Omega^+_\Sigma)\big]^6$.*

Proof. Let $U = (u,\varphi,\psi,\vartheta)^\top$ be a solution to the homogeneous BVP $(M)^+$. Then ϑ solves the following decoupled mixed BVP

$$\eta_{jl}\,\partial_j\partial_l\vartheta = 0 \quad \text{in} \ \ \Omega^+, \tag{2.81}$$

$$\{\vartheta\}^+ = 0 \quad \text{on} \ \ S_D, \tag{2.82}$$

$$\{\eta_{jl}\ n_j\ \partial_l\vartheta\}^+ = 0 \quad \text{on} \ \ S_N. \tag{2.83}$$

By Green's formula

$$\int\limits_{\Omega^+} \eta_{jl}\,\partial_l\vartheta\,\partial_j\vartheta\,dx = \langle\{\eta_{jl}\ n_j\,\partial_l\vartheta\}^+\,,\{\vartheta\}^+\rangle_S \tag{2.84}$$

and with the help of the homogeneous boundary conditions we derive $\vartheta = const$ in Ω^+, since the right hand side duality expression in (2.84) vanishes and the matrix $[\eta_{jl}]_{3\times3}$ is positive definite. Consequently, $\vartheta = 0$ in Ω^+ due to the homogeneous Dirichlet condition (2.82). Therefore, the five dimensional vector $V = (u, \varphi, \psi)^\top$, constructed by the first five components of the solution vector U, solves the following homogeneous mixed BVP (see the formulation of BVP $(M)^+$, formulas (2.33) and (2.26) and take into account that $\vartheta = 0$)

$$\begin{aligned} &\widetilde{A}^{(0)}(\partial)V = 0 \quad \text{in} \;\; \Omega^+, \\ &\{V\}^+ = 0 \quad \text{on} \;\; S_D, \\ &\{T(\partial, n)V\}^+ = 0 \quad \text{on} \;\; S_N, \end{aligned} \tag{2.85}$$

where $\widetilde{A}^{(0)}(\partial)$ is the 5×5 differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects

$$\widetilde{A}^{(0)}(\partial) = \Big[\widetilde{A}^{(0)}_{pq}(\partial)\Big]_{5\times5} := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times3} & [e_{lrj}\,\partial_j\partial_l]_{3\times1} & [q_{lrj}\,\partial_j\partial_l]_{3\times1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{5\times5}, \tag{2.86}$$

and $T(\partial, n)$ is the corresponding 5×5 generalized stress operator (cf. (2.26), (2.28) and (2.38))

$$T(\partial, n) = \big[\, T_{pq}(\partial, n)\, \big]_{5\times5} = \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times3} & [e_{lrj}\,n_j\,\partial_l]_{3\times1} & [q_{lrj}\,n_j\,\partial_l]_{3\times1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l \end{bmatrix}_{5\times5}. \tag{2.87}$$

In this case, Green's identity for arbitrary vectors $V = (u, \varphi, \psi)^\top, V' = (u', \varphi', \psi')^\top \in \big[W^1_2(\Omega^+)\big]^5$ reads as

$$\int\limits_{\Omega^+} [\, \widetilde{A}^{(0)}(\partial)\, V\cdot V' + \widetilde{\mathcal{E}}(V, V')\,]\, dx = \big\langle \{TV\}^+,\, \{V'\}^+ \big\rangle_{\partial\Omega^+}, \tag{2.88}$$

where

$$\begin{aligned} \widetilde{\mathcal{E}}(V, V') = {} & c_{rjkl}\,\partial_l u_k\,\partial_j u'_r + e_{lrj}\,(\partial_l\varphi\,\partial_j u'_r - \partial_j u_r\,\partial_l\varphi') + q_{lrj}\,(\partial_l\psi\,\partial_j u'_r - \partial_j u_r\,\partial_l\psi') \\ & + \varkappa_{jl}\,\partial_l\varphi\,\partial_j\varphi' + a_{jl}\,(\partial_l\varphi\,\partial_j\psi' + \partial_j\psi\,\partial_l\varphi') + \mu_{jl}\,\partial_l\psi\,\partial_j\psi'\,. \end{aligned} \tag{2.89}$$

Write the above Green's formula for a solution V of the problem (2.85) and $V' = V$ to obtain

$$\int\limits_{\Omega^+} \widetilde{\mathcal{E}}(V, V)\, dx = 0, \tag{2.90}$$

where

$$\widetilde{\mathcal{E}}(V,V) := c_{rjkl}\,\partial_l u_k\,\partial_j u_r + \varkappa_{jl}\,\partial_l\varphi\,\partial_j\varphi + 2\,a_{jl}\,\partial_l\varphi\,\partial_j\psi + \mu_{jl}\,\partial_l\psi\,\partial_j\psi\,. \tag{2.91}$$

Due to the inequalities (2.11) and positive definiteness of the matrix (2.15) we conclude that $\partial_j\varphi = 0$ and $\partial_j\psi = 0$ in Ω^+ for $j = 1,2,3$, and

$$c_{rjkl}\,\partial_l u_k\,\partial_j u_r = 0 \quad \text{in} \;\; \Omega^+. \tag{2.92}$$

As it is well known (see, e.g., [KGBB]), the general solution to the equation (2.92) is a rigid displacement vector which reads as

$$\chi(x) = a \times x + b, \tag{2.93}$$

where $a = (a_1,a_2,a_3)^\top$ and $b = (b_1,b_2,b_3)^\top$ are arbitrary real constant vectors and the symbol " $\times$ " denotes the cross product.

Therefore, from (2.91) it follows that

$$u(x) = \chi(x), \quad \varphi = b_4, \;\; \psi = b_5, \tag{2.94}$$

where $\chi(x)$ is an arbitrary rigid displacement vector and b_4 and b_5 are arbitrary real constants. Now, the homogeneous Dirichlet condition in (2.85) implies $u = 0$, $\varphi = 0$, and $\psi = 0$ in Ω^+, which proves the uniqueness theorem for the homogenous problem $(M)^+$.

It is clear that the proof for the problem $(D)^+$ is word for word.

The uniqueness results for the homogeneous problems $(D)^+$-(CN), $(M)^+$-(CN), $(D)^+$-(CT), and $(M)^+$-(CT) follow from the identities

$$\int\limits_{\Omega^+_\Sigma} \eta_{jl}\,\partial_l\vartheta\,\partial_j\vartheta\,dx = 0, \qquad \int\limits_{\Omega^+_\Sigma} \widetilde{\mathcal{E}}(V,V)\,dx = 0,$$

which can be obtained with the help of the same arguments as in the proof of Theorem 2.1; here $U = (u,\varphi,\psi,\vartheta)^\top$ is a solution vector to one of the above listed homogeneous crack type static problems, $V = (u,\varphi,\psi)^\top$, and $\widetilde{\mathcal{E}}(V,V)$ is defined by (2.91). Therefore the proof can be verbatim performed. $\square$

Further, we analyse the homogenous Neumann type boundary value problem $(N)^+$. Let a vector $U = (u,\varphi,\psi,\vartheta)^\top$ solve the homogenous problem $(N)^+$. In this case the temperature function ϑ solves the following decoupled problem

$$\eta_{jl}\,\partial_j\partial_l\vartheta = 0 \quad \text{in} \;\; \Omega^+, \tag{2.95}$$

$$\{\eta_{jl}\;n_j\,\partial_l\vartheta\}^+ = 0 \quad \text{on} \;\; S = \partial\Omega^+. \tag{2.96}$$

Whence, by (2.84), we get $\vartheta = b_6 = const$ in Ω^+. Therefore, the vector $V = (u,\varphi,\psi)^\top$ solves then the nonhomogeneous BVP (see the formulation of BVP $(N)^+$, formulas (2.33) and (2.26) and take into account that $\vartheta = b_6 = const$ in Ω^+)

$$\widetilde{A}^{(0)}(\partial)V = 0 \quad \text{in} \;\; \Omega^+, \tag{2.97}$$

$$\{T(\partial,n)V\}^+ = b_6\,G^* \quad \text{on} \;\; S, \tag{2.98}$$

where $\widetilde{A}^{(0)}(\partial)$ and $T(\partial,n)$ are defined by (2.86) and (2.87), and G^* is a special type given five dimensional vector function

$$G^* = (\lambda_{1j}n_j,\, \lambda_{2j}n_j,\, \lambda_{3j}n_j,\, p_j\, n_j,\, m_j\, n_j)^\top. \tag{2.99}$$

Due to Green's formula (2.88) we easily derive that a solution to the BVP (2.97)-(2.98) is defined modulo the summand

$$\widetilde{V} = (\chi(x),\, b_4,\, b_5)^\top, \tag{2.100}$$

where $\chi(x)$ is an arbitrary rigid displacement vector and b_4 and b_5 are arbitrary real constants. This follows from the fact that the vector (2.100) is a general solution of the equation $\mathcal{E}(\widetilde{V},\widetilde{V}) = 0$ in Ω^+ and $T(\partial,n)\widetilde{V} = 0$ everywhere for arbitrary unit vector n. Therefore, an arbitrary solution to the homogeneous Neumann type BVP (2.97)-(2.98) is represented as

$$V = \widetilde{V} + b_6\, V^*, \tag{2.101}$$

where $\widetilde{V}$ is given by (2.100) and $V^* = (u^*,\varphi^*,\psi^*)^\top$ is a particular solution to the BVP

$$\widetilde{A}^{(0)}(\partial)V^* = 0 \quad \text{in } \ \Omega^+, \tag{2.102}$$

$$\{T(\partial,n)V^*\}^+ = G^* \quad \text{on } \ S, \tag{2.103}$$

with G^* defined by (2.99).

Now, we show that the vector V^* can be constructed explicitly in terms of linear functions for arbitrary domain Ω^+. Namely, let

$$V^* = (u^*,\varphi^*,\psi^*)^\top, \quad u_k^* = \widetilde{b}^*_{kq}x_q, \ k = 1,2,3, \quad \varphi^* = \widetilde{c}^*_q x_q, \quad \psi^* = \widetilde{d}^*_q x_q, \tag{2.104}$$

where $\widetilde{b}^*_{kq} = \widetilde{b}^*_{qk}$, $\widetilde{c}^*_q$ and $\widetilde{d}^*_q$, $k,q = 1,2,3$, are unknown real coefficients. Evidently, the vector V^* solves the differential equation (2.102) and in view of (2.87) the boundary condition (2.103) leads to the equations

$$\begin{aligned}
&c_{rjkl}\, n_j\, \widetilde{b}^*_{kl} + e_{lrj}\, n_j\, \widetilde{c}^*_l + q_{lrj}\, n_j\, \widetilde{d}^*_l = \lambda_{rj}\, n_j, \quad r = 1,2,3,\\
&-e_{jkl}\, n_j\, \widetilde{b}^*_{kl} + \varkappa_{jl}\, n_j\, \widetilde{c}^*_l + a_{jl}\, n_j\, \widetilde{d}^*_l = p_j\, n_j,\\
&-q_{jkl}\, n_j\, \widetilde{b}^*_{kl} + a_{jl}\, n_j\, \widetilde{c}^*_l + \mu_{jl}\, n_j\, \widetilde{d}^*_l = m_j\, n_j.
\end{aligned} \tag{2.105}$$

Further, we equate the expressions which stand at the components n_j of the normal vector to obtain 12 linear equations with 12 unknown coefficients

$$\begin{aligned}
&c_{rjkl}\, \widetilde{b}^*_{kl} + e_{lrj}\, \widetilde{c}^*_l + q_{lrj}\, \widetilde{d}^*_l = \lambda_{rj}, \quad r,j = 1,2,3,\\
&-e_{jkl}\, \widetilde{b}^*_{kl} + \varkappa_{jl}\, \widetilde{c}^*_l + a_{jl}\, \widetilde{d}^*_l = p_j, \quad j = 1,2,3,\\
&-q_{jkl}\, \widetilde{b}^*_{kl} + a_{jl}\, \widetilde{c}^*_l + \mu_{jl}\, \widetilde{d}^*_l = m_j, \quad j = 1,2,3.
\end{aligned} \tag{2.106}$$

Due to the first inequality in (2.11) and positive definiteness of the matrix (2.15), and since $\widetilde{b}^*_{kq} = \widetilde{b}^*_{qk}$, it follows that the homogeneous version of the system (2.106) possesses only the trivial solution, i.e., the determinant of the system is different from zero. Therefore, the nonhomogeneous system (2.106) is uniquely solvable and we can define the twelve unknown coefficients $\widetilde{b}^*_{kq} = \widetilde{b}^*_{qk}$, $\widetilde{c}^*_q$ and $\widetilde{d}^*_q$, $k, q = 1, 2, 3$. It is evident that then the boundary conditions (2.105) are satisfied and, consequently, the vector V^* solves the BVP (2.102)-(2.103) for arbitrary domain Ω^+.

Thus, we have constructed the general solution of the homogeneous Neumann problem $(N)^+$ of statics explicitly $U = (V, b_6)^\top = (\widetilde{V}, 0)^\top + b_6(V^*, 1)^\top$, where V is defined by (2.101), and $\widetilde{V}$ and V^* are given by (2.100) and (2.104).

It is easy to check that the same vector is a general solution to the homogeneous crack type problems $(N)^+$-(CN) and $(N)^+$-(CT) for arbitrary domain Ω^+_Σ with arbitrary crack surface Σ.

Thus, we have the following uniqueness theorem.

Theorem 2.4 *A general solution to the homogeneous Neumann type boundary value problem of statics $(N)^+$ and to the homogeneous crack type boundary value problems of statics $(N)^+$-(CN) and $(N)^+$-(CT) in the space $\big[W^1_2(\Omega^+)\big]^6$ and $\big[W^1_2(\Omega^+_\Sigma)\big]^6$, respectively, reads as*

$$U = (\widetilde{V}, 0)^\top + b_6\,(V^*, 1)^\top,$$

where $\widetilde{V} = (a \times x + b,\, b_4,\, b_5)^\top$ with $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ and V^* is given by (2.104) with coefficients $\widetilde{b}^*_{kq} = \widetilde{b}^*_{qk}$, $\widetilde{c}^*_q$, $\widetilde{d}^*_q$, $k, q = 1, 2, 3$, defined by the uniquely solvable system (2.106), and where a_1, a_2, a_3, and $b_1, \cdots, b_6$ are arbitrary real constants.

Uniqueness theorems for exterior BVPs of statics will be considered later since it needs a quite different approach based on the properties of the corresponding fundamental matrix of the operator $A(\partial, 0)$.

2.5 Auxiliary boundary value problems for $A^*(\partial, \tau)$

In our analysis we need also uniqueness theorems for some auxiliary BVPs for the operator $A^*(\partial, \tau)$ adjoint to $A(\partial, \tau)$. In particular, in the study of properties of boundary operators generated by the layer potentials we will use the uniqueness theorems for the following homogeneous Dirichlet and Neumann type BVPs.

Dirichlet problem $(D^*_0)^\pm$: *Find a solution vector $U = (u_1, \cdots, u_6)^\top \in [W^1_2(\Omega^+)]^6$ (respectively $U = (u_1, \cdots, u_6)^\top \in [W^1_{2,loc}(\Omega^-)]^6$) to the equation*

$$A^*(\partial, \tau)\, U = 0 \quad \text{in} \quad \Omega^\pm \tag{2.107}$$

satisfying the Dirichlet type boundary condition

$$\big\{ U \big\}^\pm = 0 \qquad on \qquad S. \tag{2.108}$$

Neumann problem $(N_0^*)^{\pm}$: *Find a solution vector* $U=(u_1,\cdots,u_6)^\top\in[W_2^1(\Omega^+)]^6$ *(respectively* $U=(u_1,\cdots,u_6)^\top\in[W^1_{2,\,loc}(\Omega^-)]^6$*) to equation* (2.79) *satisfying the Neumann type boundary condition*

$$\big\{\,\mathcal{P}(\partial,n,\overline{\tau})U\big\}^{\pm}=0\qquad on\qquad S, \tag{2.109}$$

where the operator $\mathcal{P}(\partial,n,\tau)$ *is defined by* (2.38).
In the case of exterior BVPs we assume that solutions satisfy the decay conditions (2.56) at infinity.

We have the following uniqueness results for these auxiliary problems.

Theorem 2.5 *Let* $\tau=\sigma+i\omega$ *with* $\sigma>0$ *and* $\omega\in\mathbb{R}$.

(i) *The homogeneous boundary value problems* $(D_0^*)^{\pm}$ *and* $(N_0^*)^{-}$ *have only the trivial solutions.*

(ii) *A general solution to the homogeneous Neumann type boundary value problem* $(N_0^*)^{+}$ *reads as* $U=b_1\,U^{(1)}+b_2\,U^{(1)}$*, where* b_1 *and* b_2 *are arbitrary constants,* $U^{(1)}=(0,0,0,1,0,0)^\top$ *and* $U^{(2)}=(0,0,0,0,1,0)^\top$.

Proof. The proof is quite similar to the proof of Theorem 2.1 and follows from Green's formula (2.41). $\square$

3 Fundamental matrices

3.1 Fundamental matrix of the operator $A^{(0)}(\partial)$

We start with construction of a fundamental matrix of the operator $A^{(0)}(\partial)$ given by (2.35). Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^3)$) which for regular summable functions f and g read as follows

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)\, e^{i\,x\cdot\xi} dx, \qquad \mathcal{F}^{-1}_{\xi\to x}[g\,] = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} g(\xi)\, e^{-i\,x\cdot\xi} d\xi, \tag{3.1}$$

where $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$. Note that for arbitrary multi–index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in \mathcal{S}'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\,\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}^{\,-1}[\xi^\alpha\, g\,] = (i\,\partial)^\alpha \mathcal{F}^{\,-1}[g\,], \tag{3.2}$$

where $\xi^\alpha = \xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$ and $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

Denote by $\Gamma^{(0)}(x) = [\Gamma^{(0)}_{kj}(x)]_{6\times 6}$ the matrix of fundamental solutions of the operator $A^{(0)}(\partial)$

$$A^{(0)}(\partial)\,\Gamma^{(0)}(x) = \delta(x)\, I_6. \tag{3.3}$$

Here $\delta(\,\cdot\,)$ is the Dirac's delta distribution and I_k stands for the unit $k\times k$ matrix. By standard arguments we can show that (cf., e.g., [BCNS2])

$$\begin{aligned}\Gamma^{(0)}(x) &= \mathcal{F}^{-1}_{\xi\to x}[\{A^{(0)}(-i\,\xi)\}^{-1}] = \frac{1}{8\pi^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}\{A^{(0)}(-i\,\xi)\}^{-1}\, e^{-i\,x\cdot\xi}\, d\xi \\ &= \frac{1}{8\pi^2\,|x|}\int_0^{2\pi}\{A^{(0)}(-i\,E(\widetilde{x})\,\eta)\}^{-1}\, d\phi\,, \quad \eta = (\cos\phi, \sin\phi, 0)^\top, \ \ \widetilde{x} = \frac{x}{|x|},\end{aligned} \tag{3.4}$$

where $E(\widetilde{x})$ is an orthogonal matrix with properties $E^\top(\widetilde{x})\, x^\top = (0, 0, |x|)^\top$ and $\det E(\widetilde{x}) = 1$,

$$\{A^{(0)}(-i\,\xi)\}^{-1} = \frac{1}{\det A^{(0)}(-i\,\xi)}\, A^{(0)*}(-i\,\xi)$$

is the inverse to the symbol matrix $A^{(0)}(-i\,\xi)$ given by (2.36) and $A^{(0)*}(-i\,\xi) = [A^{(0)*}_{kj}(-i\,\xi)]_{6\times 6}$ is the corresponding matrix of cofactors.

Note that the entries of the matrix $\Gamma^{(0)}(x)$ are homogeneous even functions of order -1 and

$$\Gamma^{(0)}(x) = \begin{bmatrix} \widetilde{\Gamma}^{(0)}(x) & [0]_{5\times 1} \\ [0]_{1\times 5} & \Gamma^{(0)}_{66}(x) \end{bmatrix}_{6\times 6}, \quad \Gamma^{(0)}(-x) = \Gamma^{(0)}(x), \tag{3.5}$$

where $\widetilde{\Gamma}^{(0)}(x) = [\Gamma^{(0)}_{kj}(x)]_{5\times 5}$ is a fundamental matrix of the operator $\widetilde{A}^{(0)}(\partial)$ defined by (2.86) and $\Gamma^{(0)}_{66}(x)$ is a fundamental solution of the operator $A^{(0)}_{66}(\partial) = \eta_{jl}\partial_j\partial_l$ which reads as

$$\Gamma^{(0)}_{66}(x) = -\frac{\alpha_0}{4\,\pi\,(D\,x\cdot x)^{1/2}} = -\frac{\alpha_0}{4\,\pi\,[d_{kj}\,x_k\,x_j]^{1/2}}\,, \quad \alpha_0 = (\det D)^{1/2}, \tag{3.6}$$

where $D = D^\top = [d_{kj}]_{3\times 3}$ is the inverse to the positive definite matrix $[\eta_{kj}]_{3\times 3}$.

With the help of the Cauchy integral theorem for analytic functions, we can represent the matrix $\Gamma^{(0)}(x)$ in the form

$$\begin{aligned}\Gamma^{(0)}(x) &= \mathcal{F}^{-1}_{\xi\to x}\big[\,\{A^{(0)}(-i\,\xi)\}^{-1}\,\big] = \mathcal{F}^{-1}_{\xi'\to x'}\big[\,\mathcal{F}^{-1}_{\xi_3\to x_3}\,\{A^{(0)}(-i\,\xi)\}^{-1}\,\big]\\ &= \mathcal{F}^{-1}_{\xi'\to x'}\big[\,\Psi(\xi',x_3)\,\big]\,,\end{aligned} \tag{3.7}$$

where $\xi' = (\xi_1,\xi_2)$, $x' = (x_1', x_2')$ and

$$\begin{aligned}\Psi(\xi',x_3) &= \frac{1}{2\pi}\int_{\mathbb{R}^1}\{A^{(0)}(-i\,\xi)\}^{-1}\,e^{-i\,x_3\,\xi_3}\,d\xi_3\\ &= \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{\ell^+}\{A^{(0)}(-i\,\xi)\}^{-1}\,e^{-i\,x_3\,\xi_3}\,d\xi_3 & \text{for} \quad x_3\le 0,\\[2mm] \dfrac{1}{2\pi}\displaystyle\int_{\ell^-}\{A^{(0)}(-i\,\xi)\}^{-1}\,e^{-i\,x_3\,\xi_3}\,d\xi_3 & \text{for} \quad x_3\ge 0.\end{cases}\end{aligned} \tag{3.8}$$

Here ℓ^+ (respectively ℓ^-) is a closed simple curve of positive counterclockwise orientation (respectively negative clockwise orientation) in the upper (respectively lower) complex half-plane $\Re\,\xi_3 > 0$ (respectively $\Re\,\xi_3 < 0$) enveloping all the roots with respect to ξ_3 of the equation $\det A^{(0)}(-i\,\xi) = 0$ with positive (respectively negative) imaginary parts. Clearly, (3.8) does not depend on the shape of ℓ^+ (respectively ℓ^-). It can easily be shown that the entries of the matrix (3.8) with $x_3 = 0$ are even, homogeneous functions in ξ' of order -1. Moreover, from (3.8) and the inequality (2.37) it follows that there is a positive constant c depending on the material parameters, such that

$$\Re\,[\,-\Psi(\xi',0)\,\zeta\,\cdot\,\zeta\,] \ge c\,|\xi'|^{-1}\,|\zeta|^2 \quad \text{for all} \ \ \xi'\in\mathbb{R}^2\setminus\{0\} \quad \text{and for all} \ \ \zeta\in\mathbb{C}^6. \tag{3.9}$$

3.2 Fundamental matrix of the operator $A(\partial,\tau)$

Now let us construct a fundamental matrix $\Gamma(x,\tau) = [\Gamma_{kj}(x,\tau)]_{6\times 6}$ of the operator $A(\partial,\tau)$ given by (2.33),

$$A(\partial,\tau)\,\Gamma(x,\tau) = \delta(x)\,I_6.$$

Applying the Fourier transform we get

$$A(-i\,\xi,\tau)\,\mathcal{F}_{x\to\xi}[\Gamma(x,\tau)] = I_6, \tag{3.10}$$

where

$$A(-i\,\xi,\tau) = [A_{pq}(-i\,\xi,\tau)]_{6\times 6}$$

$$= -\begin{bmatrix} [c_{rjkl}\,\xi_j\,\xi_l + \varrho\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\,\xi_j\xi_l]_{3\times 1} & [q_{lrj}\,\xi_j\xi_l]_{3\times 1} & [-i\,\lambda_{rj}\xi_j]_{3\times 1} \\ [-e_{jkl}\,\xi_j\xi_l]_{1\times 3} & \varkappa_{jl}\,\xi_j\xi_l & a_{jl}\,\xi_j\xi_l & -i\,p_j\xi_j \\ [-q_{jkl}\,\xi_j\xi_l]_{1\times 3} & a_{jl}\,\xi_j\xi_l & \mu_{jl}\,\xi_j\xi_l & -i\,m_j\xi_j \\ [-i\,\tau\,T_0\,\lambda_{kl}\,\xi_l]_{1\times 3} & i\,\tau\,T_0\,p_l\,\xi_l & i\,\tau\,T_0\,m_l\,\xi_l & \eta_{jl}\,\xi_j\,\xi_l + \tau\,T_0\,\gamma \end{bmatrix}_{6\times 6} \tag{3.11}$$

First we establish some properties of the matrix $A(-i\,\xi,\tau)$ needed in the further analysis and prove some technical lemmata.

Lemma 3.1 *Let $\tau = \sigma + i\,\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then*

$$\det A(-i\,\xi,\tau) \neq 0 \quad \textit{for all} \quad \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{3.12}$$

Proof. It suffices to show that for all $\xi \in \mathbb{R}^3 \setminus \{0\}$ the following homogeneous system of linear algebraic equations for unknowns $\zeta_1, \cdots, \zeta_6$

$$\begin{aligned} &c_{rjkl}\,\xi_j\,\xi_l\,\zeta_k + \varrho\tau^2\,\zeta_r + e_{lrj}\,\xi_j\xi_l\,\zeta_4 + q_{lrj}\,\xi_j\xi_l\,\zeta_5 - i\,\lambda_{rj}\xi_j\,\zeta_6 = 0, \quad r = 1,2,3, \\ &-e_{jkl}\,\xi_j\xi_l\,\zeta_k + \varkappa_{jl}\,\xi_j\xi_l\,\zeta_4 + a_{jl}\,\xi_j\xi_l\,\zeta_5 - i\,p_j\xi_j\,\zeta_6 = 0, \\ &-q_{jkl}\,\xi_j\xi_l\,\zeta_k + a_{jl}\,\xi_j\xi_l\,\zeta_4 + \mu_{jl}\,\xi_j\xi_l\,\zeta_5 - i\,m_j\xi_j\,\zeta_6 = 0, \\ &-i\,\tau\,T_0\,\lambda_{kl}\,\xi_l\,\zeta_k + i\,\tau\,T_0\,p_l\,\xi_l\,\zeta_4 + i\,\tau\,T_0\,m_l\,\xi_l\,\zeta_5 + \eta_{jl}\,\xi_j\,\xi_l\,\zeta_6 + \tau\,T_0\,\gamma\,\zeta_6 = 0, \end{aligned} \tag{3.13}$$

have only the trivial solution.

Multiply the first three equations by $\overline{\zeta}_r$, $r = 1,2,3$, the complex conjugate of the fourth equation by ζ_4, the complex conjugate of the fifth equation by ζ_5, the complex conjugate of the sixth equation by $[\overline{\tau}\,T_0]^{-1}\,\zeta_6$, and add together to obtain

$$\begin{aligned} &c_{rjkl}\,(\xi_l\,\zeta_k\,)\,(\xi_j\,\overline{\zeta}_r) + \varrho\tau^2\,\zeta_r\,\overline{\zeta}_r + \varkappa_{jl}\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_4) + \mu_{jl}\,(\xi_j\,\zeta_5)\,(\xi_l\,\overline{\zeta}_5) \\ &+a_{jl}\,[\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_5) + (\xi_j\,\overline{\zeta}_4)\,(\xi_l\,\zeta_5)] + i\,p_l(\xi_l\,\zeta_4\,\overline{\zeta_6} - \xi_l\,\overline{\zeta_4}\,\zeta_6) + i\,m_l(\xi_l\,\zeta_5\,\overline{\zeta_6} - \xi_l\,\overline{\zeta_5}\,\zeta_6) \\ &+\frac{\tau}{|\tau|^2\,T_0}\,\eta_{jl}\,(\xi_j\,\zeta_6)\,(\xi_l\,\overline{\zeta}_6) + \gamma\,\zeta_6\,\overline{\zeta}_6 = 0, \end{aligned} \tag{3.14}$$

In view of (2.12) and (2.13) with $\zeta_k' = \xi_k\zeta_4$, $\zeta_k'' = \xi_k\zeta_5$, $k = 1, 2, 3$, and $\theta = i\,\zeta_6$, we have

$$c_{rjkl}\,(\xi_l\,\zeta_k\,)\,(\xi_j\,\overline{\zeta}_r) = \frac{1}{4}\,c_{rjkl}\,(\xi_l\,\zeta_k + \xi_k\,\zeta_l)\,(\xi_j\,\overline{\zeta}_r + \xi_r\,\overline{\zeta}_j) \geq \frac{c_0}{4}\,\sum_{k,j=1}^{3}|\xi_l\,\zeta_k + \xi_k\,\zeta_l|^2\,,$$

$$\begin{aligned}
&\varkappa_{jl}\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_4) + \mu_{jl}\,(\xi_j\,\zeta_5)\,(\xi_l\,\overline{\zeta}_5) + a_{jl}[(\xi_j\,\zeta_4)(\xi_l\,\overline{\zeta}_5) + (\xi_j\,\overline{\zeta}_4)(\xi_l\,\zeta_5)]\\
&\qquad + i\,p_l(\xi_l\,\overline{\zeta_6}\,\zeta_4 - \xi_l\,\zeta_6\,\overline{\zeta_4}) + i\,m_l(\xi_l\,\overline{\zeta_6}\,\zeta_5 - \xi_l\,\zeta_6\,\overline{\zeta_5}) + \gamma\,\zeta_6\,\overline{\zeta}_6\\
&\qquad = \varkappa_{jl}\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_4) + \mu_{jl}\,(\xi_j\,\zeta_5)\,(\xi_l\,\overline{\zeta}_5) + a_{jl}\,[\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_5) + (\xi_j\,\overline{\zeta}_4)\,(\xi_l\,\zeta_5)]\\
&\qquad - p_l\,[\,\xi_l\,\overline{(i\,\zeta_6)}\,\zeta_4 + \xi_l\,(i\,\zeta_6)\,\overline{\zeta_4}\,] - m_l\,[\,\xi_l\,\overline{(i\,\zeta_6)}\,\zeta_5 + \xi_l\,(i\,\zeta_6)\,\overline{\zeta_5}\,] + \gamma\,(i\,\zeta_6)\,\overline{(i\,\zeta_6)}\\
&\qquad \geq \delta_0\,(\,|\xi|^2|\zeta_4|^2 + |\xi|^2|\zeta_5|^2 + |\zeta_6|^2\,)\,,
\end{aligned}$$

$$\eta_{jl}\,(\xi_j\,\zeta_6)\,(\xi_l\,\overline{\zeta}_6) \geq c_3\,|\xi|^2|\zeta_6|^2.$$

Therefore, separating the imaginary part of (3.14) we get

$$2\varrho\,\sigma\,\omega\,\zeta_r\,\overline{\zeta_r} + \frac{\omega}{|\tau|^2\,T_0}\,|\zeta_6|^2 = 0,$$

whence $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_6 = 0$ follow if $\omega \neq 0$. Then from (3.14)

$$\varkappa_{jl}\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_4) + \mu_{jl}\,(\xi_j\,\zeta_5)\,(\xi_l\,\overline{\zeta}_5) + a_{jl}\,[\,(\xi_j\,\zeta_4)\,(\xi_l\,\overline{\zeta}_5) + (\xi_j\,\overline{\zeta}_4)\,(\xi_l\,\zeta_5)] = 0$$

and by positive definiteness of the matrix (2.15) we conclude $\zeta_4 = \zeta_5 = 0$ since $\xi \in \mathbb{R}^3 \setminus \{0\}$. Consequently, the system (3.13) possesses only the trivial solution.

Now, if $\omega = 0$, then $\tau = \sigma > 0$ and form (3.14) we again get $\zeta_j = 0$, $j = 1, \cdots, 6$, due to the above inequalities. □

It can be shown that the determinant $\det A(-i\,\xi, \tau)$ is representable as

$$\det A(-i\,\xi, \tau) = P_{12}(\xi) + P_{10}(\xi, \tau) + P_8(\xi, \tau) + P_6(\xi, \tau) + P_4(\xi, \tau), \tag{3.15}$$

where P_k are homogeneous polynomials in ξ of order k. In particular,

$$P_{12}(\xi) = \det A^{(0)}(-i\,\xi), \tag{3.16}$$

where $A^{(0)}(-i\,\xi)$ is given by (2.36). In view of (2.37) we have $P_{12}(\xi) \neq 0$ for $\xi \in \mathbb{R}^3 \setminus \{0\}$ and consequently there is a positive constant c_{12}^* depending only on the material parameters, such that

$$|P_{12}(\xi)| \geq c_{12}^*\,|\xi|^{12} \quad \text{for all} \quad \xi \in \mathbb{R}^3. \tag{3.17}$$

In particular, we can take

$$c_{12}^* = \min_{|\xi|=1}|\det A^{(0)}(-i\,\xi)| > 0 \tag{3.18}$$

Further, the polynomial $P_4(\xi, \tau)$ reads as

$$P_4(\xi, \tau) = T_0\varrho^3\tau^7\det B(\xi), \tag{3.19}$$

where

$$B(\xi) = \begin{bmatrix} \varkappa_{jl}\,\xi_j\xi_l & a_{jl}\,\xi_j\xi_l & p_j\,\xi_j \\ a_{jl}\,\xi_j\xi_l & \mu_{jl}\,\xi_j\xi_l & m_j\,\xi_j \\ p_j\,\xi_j & m_j\,\xi_j & \gamma \end{bmatrix}. \tag{3.20}$$

Due to positive definiteness of the matrix (2.14) the matrix $B(\xi)$ is positive definite for all $\xi \in \mathbb{R}^3 \setminus \{0\}$. Therefore, there is a positive constant c_4^* depending only on the material parameters, such that

$$|P_4(\xi)| \geq c_4^*\,|\tau|^7\,|\xi|^4 \quad \text{for all} \quad \xi \in \mathbb{R}^3. \tag{3.21}$$

In particular, we can take

$$c_4^* = T_0\varrho^3 \min_{|\xi|=1} |\det B(\xi)| > 0. \tag{3.22}$$

Lemma 3.2 *Let $\tau = \sigma + i\,\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. There hold the following asymptotic relations*

$$\det A(-i\,\xi, \tau) = |\xi|^{12}[\,\widetilde{a}(\widetilde{\xi}) + \mathcal{O}(|\xi|^{-2})\,] \quad as \quad |\xi| \to \infty, \tag{3.23}$$

$$\det A(-i\,\xi, \tau) = |\xi|^{4}[\,\widetilde{b}(\widetilde{\xi}) + \mathcal{O}(|\xi|^{2})\,] \quad as \quad |\xi| \to 0, \tag{3.24}$$

where $\widetilde{\xi} = \xi/|\xi|$ and

$$c_{12}^* \leq |\widetilde{a}(\widetilde{\xi})| \leq c_{12}^{**}, \qquad |\tau|^7\,c_4^* \leq |\widetilde{b}(\widetilde{\xi})| \leq |\tau|^7\,c_4^{**},$$

with c_{12}^* and c_4^* are given by (3.18) *and* (3.22) *respectively and*

$$c_{12}^{**} = \max_{|\xi|=1} |\det A^{(0)}(-i\,\xi)|, \qquad c_4^{**} = T_0\varrho^3 \max_{|\xi|=1} |\det B(\xi)| > 0. \tag{3.25}$$

Proof. It immediately follows from inequalities (3.17) and (3.21). □

Now, with the help of the above results we can investigate behaviour of the inverse matrix $A^{-1}(-i\,\xi, \tau)$ at infinity and near the origin. By (3.11) and the representation formula

$$A^{-1}(-i\,\xi, \tau) = \frac{1}{\det A(-i\,\xi, \tau)}\,A^{(c)}(-i\,\xi, \tau)\,,$$

where $A^{(c)}(-i\,\xi, \tau) = [A^{(c)}_{kj}(-i\,\xi, \tau)]_{6\times 6}$ is the corresponding matrix of cofactors, we derive the following asymptotic relations for sufficiently large $|\xi|$, i.e., as $|\xi| \to \infty$,

$$A^{(c)}(-i\,\xi, \tau) = \begin{bmatrix} [\mathcal{O}(|\xi|^{10})]_{5\times 5} & [\mathcal{O}(|\xi|^{9})]_{5\times 1} \\ [\mathcal{O}(|\xi|^{9})]_{1\times 5} & \mathcal{O}(|\xi|^{10}) \end{bmatrix}_{6\times 6}, \tag{3.26}$$

$$A^{-1}(-i\,\xi, \tau) = \begin{bmatrix} [\mathcal{O}(|\xi|^{-2})]_{5\times 5} & [\mathcal{O}(|\xi|^{-3})]_{5\times 1} \\ [\mathcal{O}(|\xi|^{-3})]_{1\times 5} & \mathcal{O}(|\xi|^{-2}) \end{bmatrix}_{6\times 6}. \tag{3.27}$$

For sufficiently small $|\xi|$, i.e., as $|\xi|\to 0$, we have

$$A^{(c)}(-i\,\xi,\tau)=\begin{bmatrix} \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^5)\\ \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^5)\\ \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^6) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^5)\\ \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^3)\\ \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^4) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^3)\\ \mathcal{O}(|\xi|^5) & \mathcal{O}(|\xi|^5) & \mathcal{O}(|\xi|^5) & \mathcal{O}(|\xi|^3) & \mathcal{O}(|\xi|^3) & \mathcal{O}(|\xi|^4) \end{bmatrix}_{6\times 6}, \tag{3.28}$$

$$A^{-1}(-i\,\xi,\tau)=\begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|)\\ \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|)\\ \mathcal{O}(|\xi|^2) & \mathcal{O}(|\xi|^2) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|)\\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-1})\\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-1})\\ \mathcal{O}(|\xi|) & \mathcal{O}(|\xi|) & \mathcal{O}(|\xi|) & \mathcal{O}(|\xi|^{-1}) & \mathcal{O}(|\xi|^{-1}) & \mathcal{O}(1) \end{bmatrix}_{6\times 6}. \tag{3.29}$$

Since the entries of the matrix $A^{-1}(-i\,\xi,\tau)$ are rational functions in ξ it follows that the first dominant terms in asymptotic expansions at infinity and at the origin are homogenous functions of order mentioned in the relations (3.27) and (3.29). Furthermore, in (3.27) the entries $A^{-1}_{6j}(-i\,\xi,\tau)$ and $A^{-1}_{j6}(-i\,\xi,\tau)$, $j=\overline{1,5}$, with the asymptotic $\mathcal{O}(|\xi|^{-3})$ have dominant terms of type $|\xi|^{-3}\chi(\xi)$, where $\chi(\xi)$ is an odd homogeneous function of order 0. Therefore

$$\int_{|\xi|=1}\chi(\xi)\,dS=0,$$

and consequently the generalized inverse Fourier transform of the function $|\xi|^{-3}\chi(\widetilde{\xi})$, considered in the Principal Value sense, is a homogeneous function of order 0 (see, e.g., [MP], [Esk1]).

Let h be an infinitely differentiable function with compact support,

$$h\in C^\infty(\mathbb{R}^3),\quad h(\xi)=\begin{cases}1 & \text{for}\quad |\xi|<1,\\ 0 & \text{for}\quad |\xi|>2.\end{cases}$$

Then we can represent the fundamental matrix $\Gamma(x,\tau)$ in the form

$$\Gamma(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\,\xi,\tau)]=\Gamma^{(1)}(x,\tau)+\Gamma^{(2)}(x,\tau), \tag{3.30}$$

where

$$\Gamma^{(1)}(x,\tau):=\mathcal{F}^{-1}_{\xi\to x}[h(\xi)\,A^{-1}(-i\,\xi,\tau)], \tag{3.31}$$

$$\Gamma^{(2)}(x,\tau):=\mathcal{F}^{-1}_{\xi\to x}[(1-h(\xi))\,A^{-1}(-i\,\xi,\tau)]. \tag{3.32}$$

Applying properties (3.2) of the generalized Fourier transform we derive that the entries of the matrix $\Gamma^{(2)}(x,\tau)$ decay at infinity faster than any power of $|x|^{-1}$, while at the origin (as $|x|\to 0$) the singularity is defined by the asymptotic behaviour (3.27) and with the help of Fourier transform of homogeneous functions we get (see, e.g., [Esk1])

$$\Gamma^{(2)}(x,\tau)=\begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [\mathcal{O}(1)]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}. \tag{3.33}$$

Here the dominant parts of the entries of block matrices are homogeneous functions of the corresponding order.

On the other hand, we easily establish that the entries of the matrix $\Gamma^{(1)}(x,\tau)$ are infinitely differentiable functions in $\mathbb{R}^3$ and due to formula (3.29) they have the following asymptotic behaviour at infinity (as $|x|\to\infty$)

$$\Gamma^{(1)}(x,\tau)=\begin{bmatrix}
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2}) \\
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2}) \\
\mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-3})
\end{bmatrix}_{6\times 6}. \tag{3.34}$$

As above, here the dominant parts of the entries of the matrix $\Gamma^{(1)}(x,\tau)$ are homogeneous functions of the corresponding order. Therefore the above obtained asymptotic formulas (3.33) and (3.34) can be differentiated any times with respect to the variables x_j, $j=1,2,3$, to obtain similar asymptotic formulas for $\partial^\alpha\Gamma^{(2)}(x,\tau)$ and $\partial^\alpha\Gamma^{(1)}(x,\tau)$ with arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$.

Finally, we arrive at the following asymptotic behaviour for the fundamental matrix

$$\begin{aligned}
&\Gamma(x,\tau)=\mathcal{O}\big(\Gamma^{(2)}(x,\tau)\big) \quad \text{as} \;\; |x|\to 0,\\
&\Gamma(x,\tau)=\mathcal{O}\big(\Gamma^{(1)}(x,\tau)\big) \quad \text{as} \;\; |x|\to \infty,\\
&\partial^\alpha\Gamma(x,\tau)=\mathcal{O}\big(\partial^\alpha\Gamma^{(2)}(x,\tau)\big) \quad \text{as} \;\; |x|\to 0,\\
&\partial^\alpha\Gamma(x,\tau)=\mathcal{O}\big(\partial^\alpha\Gamma^{(1)}(x,\tau)\big) \quad \text{as} \;\; |x|\to \infty,
\end{aligned} \tag{3.35}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is an arbitrary multi-index.

Note that the fundamental matrices $\Gamma^{(0)}(x)$ and $\Gamma(x,\tau)$ have essentially different properties at infinity. To describe the relationship between them in a vicinity of the origin we prove the following assertion.

Lemma 3.3 *For an arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and for sufficiently small $|x|$ (i.e. as $|x|\to 0$) there hold the estimates*

$$\begin{gathered}
\Gamma_{kj}(x,\tau)-\Gamma^{(0)}_{kj}(x)=\mathcal{O}(1), \quad \partial^\alpha\Gamma_{kj}(x,\tau)-\partial^\alpha\Gamma^{(0)}_{kj}(x)=\mathcal{O}(|x|^{-|\alpha|}),\\
k,j=\overline{1,6}, \quad |\alpha|=|\alpha_1|+|\alpha_2|+|\alpha_3|.
\end{gathered} \tag{3.36}$$

Proof. One can easily verify the identity

$$\frac{1}{P+Q}=\frac{1}{P}+\sum_{k=1}^{N}\frac{(-1)^k\,Q^k}{P^{k+1}}+\frac{(-1)^{N+1}\,Q^{N+1}}{P^{N+1}\,(P+Q)}\,. \tag{3.37}$$

Take into consideration (3.15) and (3.16) and apply (3.37) with $P=P_{12}(\xi)$ and $Q=P_{10}(\xi,\tau)+P_8(\xi,\tau)+P_6(\xi,\tau)+P_4(\xi,\tau)$ to obtain

$$\begin{aligned} A^{-1}(-i\,\xi,\tau) &= \frac{1}{\det A(-i\,\xi,\tau)}\,A^{(c)}(-i\,\xi,\tau)\\ &= \frac{1}{P_{12}(\xi)}\,A^{(c)}(-i\,\xi,\tau)+\mathcal{O}(|\xi|^{-4}) \quad \text{as} \quad |\xi|\to\infty. \end{aligned} \tag{3.38}$$

It can be easily checked that for sufficiently large $|\xi|$

$$A^{(c)}(-i\,\xi,\tau)-A^{(c,0)}(-i\,\xi)=[\mathcal{O}(|\xi|^9)]_{6\times 6}\,,$$

where $A^{(c,0)}(-i\,\xi)$ is the matrix of cofactors of $A^{(0)}(-i\,\xi)$ and the dominant parts of the entries of the right hand side matrix are homogeneous polynomials of order 9. Therefore, in view of (3.16) we get

$$A^{-1}(-i\,\xi,\tau)=\{A^{(0)}(-i\,\xi)\}^{-1}+|\xi|^{-3}[\,\chi_{kj}(\xi)\,]_{6\times 6}+[\mathcal{O}(|\xi|^{-4})]_{6\times 6} \quad \text{as} \quad |\xi|\to\infty,$$

where $\chi_{kj}(\xi)$ are odd homogeneous functions of order 0. Whence the relations (3.36) follow due to the above mentioned properties of the generalized Fourier transform of homogeneous functions. □

Remark 3.4 *Note that the matrix $\Gamma^*(x,\tau):=[\Gamma(-x,\tau)]^\top$ represents a fundamental solution to the formally adjoint differential operator $A^*(\partial,\tau)\equiv[A(-\partial,\tau)]^\top$,*

$$A^*(\partial,\tau)[\Gamma(-x,\tau)]^\top=I_6\,\delta(x). \tag{3.39}$$

3.3 Fundamental matrix of the operator $A(\partial,0)$

If $\tau=0$, than it is evident that $\det A(-i\,\xi,0)=\det A^{(0)}(-i\,\xi)$ due to (3.11) and (2.36), and by the same approach as above we get the following expression for the fundamental matrix of the operator of statics $A(\partial,0)$

$$\Gamma(x,0)=\mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\,\xi,0)] \tag{3.40}$$

where

$$A^{-1}(-i\,\xi,0)=\frac{1}{\det A(-i\,\xi,0)}\,A^{(c)}(-i\,\xi,0)\,. \tag{3.41}$$

Note that $\det A(-i\,\xi,0)$ is a homogeneous polynomial of order 12. Moreover, it is easy to see that the cofactors $A^{(c)}_{kj}(-i\,\xi,0)$ are homogeneous polynomials in ξ as well, namely,

$$\begin{aligned} &\operatorname{ord}A^{(c)}_{kj}(-i\,\xi,0)=10,\ k,j=\overline{1,5}, && \operatorname{ord}A^{(c)}_{66}(-i\,\xi,0)=10,\\ &\operatorname{ord}A^{(c)}_{j6}(-i\,\xi,0)=9, && A^{(c)}_{6j}(-i\,\xi,0)=0,\ \ j=\overline{1,5}. \end{aligned}$$

Therefore, the functions

$$K_j(\xi) := \frac{A^{(c)}_{j6}(-i\,\xi,0)}{\det A(-i\,\xi,0)}\,,\quad j=\overline{1,5}, \tag{3.42}$$

are odd homogeneous rational functions of order -3 in ξ and, consequently,

$$\int_{|\xi|=1} K_j(\xi)\,dS = 0\,,\quad j=\overline{1,5}. \tag{3.43}$$

Then it follows that the inverse Fourier transform of the function $K_j(\xi)$, considered in the Principal Value sense, is a homogeneous function of order 0 (see, e.g., [MP], [Esk1]) and

$$\int_{|x|=1} \mathcal{F}^{-1}_{\xi\to x}[K_j(\xi)]\,dS_x = 0\,,\quad j=\overline{1,5}, \tag{3.44}$$

i.e.,

$$\int_{|x|=1} \Gamma_{j6}(x,0)\,dS = 0\,,\quad j=\overline{1,5}, \tag{3.45}$$

Therefore, the entries of the fundamental matrix $\Gamma(x,0)$ are homogeneous functions in x and

$$\Gamma(x,0) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times5} & [\mathcal{O}(1)]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times6}. \tag{3.46}$$

Moreover, since $A^{(c)}_{kj}(-i\,\xi,0) = A^{(c,0)}_{kj}(-i\,\xi)$ for $k,j=\overline{1,5}$ and $A^{(c)}_{66}(-i\,\xi,0) = A^{(c,0)}_{66}(-i\,\xi)$, we conclude that (see (3.4))

$$\Gamma_{kj}(x,0) = \Gamma^{(0)}_{kj}(x),\ \ k,j=\overline{1,5},\quad \Gamma_{66}(x,0) = \Gamma^{(0)}_{66}(x).$$

As we see from formulas (3.46), (3.34) and (3.35) the entries of the fundamental matrices $\Gamma(x,0)$ and $\Gamma(x,\tau)$ with $\Re\,\tau=\sigma>0$ have essentially different properties at infinity.

3.4 Integral representation formulae of solutions

For simplicity, in this subsection we assume (if not otherwise stated) that

$$\begin{aligned} &S=\partial\Omega^{\pm}\in C^{m,\kappa}\quad\text{with integer }\ m\geq 1\ \text{ and }\ 0<\kappa\leq 1;\\ &\tau=\sigma+i\,\omega\quad\text{with }\ \sigma>0,\ \omega\in\mathbb{R}. \end{aligned} \tag{3.47}$$

Let us introduce the generalized single and double layer potentials, and the Newton type volume potential

$$V_S(g)(x) = V(g)(x) = \int_S \Gamma(x-y,\tau)\,g(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S, \tag{3.48}$$

$$W_S(g)(x) = W(g)(x) = \int_S [\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(x-y,\tau)]^\top\,g(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S, \tag{3.49}$$

$$N_{\Omega^\pm}(h)(x) = \int_{\Omega^\pm} \Gamma(x-y,\tau)\,h(y)\,dy,\quad x\in\mathbb{R}^3, \tag{3.50}$$

where $\mathcal{P}(\partial, n, \tau)$ is the boundary differential operator defined by (2.38), $\Gamma(\cdot\,,\tau)$ is the fundamental matrix of the operator $A(\partial,\tau)$, $g=(g_1,\cdots,g_6)^\top$ is a density vector-function defined on S, while a density vector-function $h=(h_1,\cdots,h_6)^\top$ is defined on $\Omega^\pm$. We assume that in the case of unbounded domain Ω^- the support of the vector function h is a compact domain. Due to the equality

$$\sum_{j=1}^{6} A_{kj}(\partial_x,\tau)\big([\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(x-y,\tau)]^\top\big)_{jp}$$

$$=\sum_{j,\,q=1}^{6} A_{kj}(\partial_x,\tau)\mathcal{P}_{pq}(\partial_y,n(y),\overline{\tau})\Gamma_{jq}(x-y,\tau)$$

$$=\sum_{j,\,q=1}^{6} \mathcal{P}_{pq}(\partial_y,n(y),\overline{\tau})A_{kj}(\partial_x,\tau)\Gamma_{jq}(x-y,\tau)=0,\quad x\neq y,\quad k,p=\overline{1,6},$$

it can easily be checked that the potentials defined by (3.48) and (3.49) are C^∞–smooth in $\mathbb{R}^3\setminus S$ and solve the homogeneous equation $A(\partial,\tau)U(x)=0$ in $\mathbb{R}^3\setminus S$ for an arbitrary L_p-summable vector function g. The volume potential solves the nonhomogeneous equation

$$A(\partial,\tau)N_{\Omega^\pm}(h)=h \quad \text{in} \quad \Omega^\pm \quad \text{for} \quad h\in[C^{0,\kappa}(\Omega^\pm)]^6.$$

This formula holds true almost everywhere in $\Omega^\pm$ also for $h\in[L_p(\Omega^\pm)]^6$, provided that in the case of unbounded domain Ω^- the support of the vector function h is a compact domain.

With the help of Green's formulas (2.42) and (2.80) by standard arguments we can prove the following assertions (cf., e.g., [KGBB], [NDS1], Ch. I, Lemma 2.1; Ch. II, Lemma 8.2).

Theorem 3.5 *Let $S=\partial\Omega^+$ be $C^{1,\kappa}$-smooth with $0<\kappa\leq 1$ and let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^6$. Then there holds the integral representation formula*

$$W(\{U\}^+)(x)-V(\{\mathcal{T}U\}^+)(x)+N_{\Omega^+}(A(\partial,\tau)U)(x)=\begin{cases} U(x) & \text{for} \quad x\in\Omega^+,\\ 0 & \text{for} \quad x\in\Omega^-.\end{cases} \tag{3.51}$$

This formula can be extended to Lipschitz domains and to vector functions satisfying the conditions $U\in[W_p^1(\Omega^+)]^6$ and $A(\partial,\tau)U\in[L_p(\Omega^+)]^6$ with $1<p<\infty$.

Proof. For the smooth case it easily follows from Green's formula (2.42) with the domain of integration $\Omega^+\setminus B(x,\varepsilon)$, where $x\in\Omega^+$ is treated as a fixed parameter and $B(x,\varepsilon)\subset\Omega^+$ is a ball centered at x and radius ε. One needs to take the j-th column of the fundamental matrix $\Gamma^*(y-x,\tau)=[\Gamma(x-y,\tau)]^\top$ for $V(y)$, calculate the surface integrals over $\Sigma(x,\varepsilon)=\partial B(x,\varepsilon)$ and pass to the limit as $\varepsilon\to 0$.
The second part of the theorem can be shown by standard limiting procedure. □

Similar representation formula holds in the exterior domain Ω^-.

Theorem 3.6 *Let $S = \partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0 < \kappa \le 1$ and let U be a regular vector of the class $[C^2(\overline{\Omega^-})]^6$ satisfying the decay conditions at infinity* (2.56)*. Then there holds the integral representation formula*

$$-W(\{U\}^-)(x) + V(\{\mathcal{T}U\}^-)(x) + N_{\Omega^-}(A(\partial,\tau)U)(x) = \begin{cases} 0 & \text{for} \quad x \in \Omega^+, \\ U(x) & \text{for} \quad x \in \Omega^-, \end{cases} . \tag{3.52}$$

This formula can be extended to Lipschitz domains and to vector functions satisfying the conditions (2.56)*, $U \in [W^1_{p,loc}(\Omega^-)]^6$ and $A(\partial,\tau)U \in [L_{p,loc}(\Omega^-)]^6$ with $1 < p < \infty$.*

Proof. The proof immediately follows from Theorem 3.5. Indeed, one needs to write the integral representation formula (3.51) for bounded domain $\Omega^- \cap B(0,R)$, send then R to $+\infty$ and take into consideration that the surface integrals over $\Sigma(0,R)$ tend to zero due to the conditions (2.56) and the decay properties of the fundamental matrix at infinity.
The second part of the theorem again can be shown by standard limiting procedure. □

Corollary 3.7 *Let $S = \partial\Omega^\pm$ be $C^{1,\kappa}$-smooth with $0 < \kappa \le 1$ and $U \in [C^2(\overline{\Omega^\pm})]^6$ be a solution to the homogeneous equations $A(\partial,\tau)U = 0$ in Ω^+ and Ω^- satisfying the conditions* (2.56)*. Then the following representation formula holds*

$$U(x) = W([U]_S)(x) - V([\mathcal{T}U]_S)(x), \quad x \in \Omega^+ \cup \Omega^-, \tag{3.53}$$

where $[U]_S = \{U\}^+ - \{U\}^-$ and $[\mathcal{T}U]_S = \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^-$ on S.

This formula can be extended to Lipschitz domains and to solution vector functions $U \in [W^1_{p,loc}(\Omega^-)]^6$ with $1 < p < \infty$ satisfying the conditions (2.56)*.*

Proof. It immediately follows from Theorems 3.5 and 3.6. □

It is evident that representation formulas similar to (3.51), (3.52) and (3.53) hold also for domains $\Omega^\pm_\Sigma$ with interior cracks. For example, for a solution vector $U \in [W^1_p(\Omega^+_\Sigma)]^6$ to the homogeneous equations $A(\partial,\tau)U = 0$ in Ω^+_Σ the following representation formula holds true

$$\begin{aligned} &W_S(\{U\}^+_S)(x) - V_S(\{\mathcal{T}U\}^+_S)(x) + N_{\Omega^+_\Sigma}(A(\partial,\tau)U)(x) \\ &+W_\Sigma([U]_\Sigma)(x) - V_\Sigma([\mathcal{T}U]_\Sigma)(x) = \begin{cases} U(x) & \text{for} \quad x \in \Omega^+_\Sigma, \\ 0 & \text{for} \quad x \in \Omega^-. \end{cases} \end{aligned} \tag{3.54}$$

Note that if $U \in [H^1_p(\Omega^+_\Sigma)]^6$ and $A(\partial,\tau)U \in [L_p(\Omega^+_\Sigma)]^6$ or $U \in [H^1_{p,loc}(\Omega^-_\Sigma)]^6$ and $A(\partial,\tau)U \in [L_{p,loc}(\Omega^-_\Sigma)]^6$, then $U \in [H^2_p(\Omega)]^6$ for arbitrary $\overline{\Omega} \subset \Omega^\pm_\Sigma$ due to the interior regularity results and by trace theorem we have

$$\begin{aligned} &\{U\}^\pm \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \quad \{\mathcal{T}U\}^\pm \in [B^{-\frac{1}{p}}_{p,p}(S)]^6, \\ &[U]_\Sigma \in r_{\Sigma}[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6, \quad [\mathcal{T}U]_\Sigma \in r_{\Sigma}[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \end{aligned} \tag{3.55}$$

Therefore, if necessary, the surface integrals over the exterior boundary manifold S or over the interior crack surface Σ, containing the traces of the generalized stress vector, one can understand as dualities between the pairs of the adjoint spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^6$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^6$, or $[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ and $[B^{\frac{1}{p}}_{p',p'}(\Sigma)]^6$, respectively.

3.5 Uniqueness results for exterior BVPs of statics

Here we study the uniqueness of solutions to exterior BVPs of statics of thermo-electro-magneto-elasticity, which corresponds to the case $\tau = 0$. Throughout this subsection we assume that S and Σ are Lipschitz if not otherwise stated. First we analyze the temperature field.

3.5.1 Asymptotic behaviour of the temperature field at infinity

As we have mentioned above, in Subsection 2.4.2, in the case of static problems the differential equation (see (2.33) and (2.35))

$$A_{66}(\partial, 0)\,\vartheta \equiv A^{(0)}_{66}(\partial)\,\vartheta \equiv \eta_{jl}\partial_j\partial_l\vartheta = \Phi_6 \tag{3.56}$$

and the corresponding boundary and crack type conditions for temperature field are separated. Here the right hand side function Φ_6 has a compact support. Therefore, one can easily prove the corresponding uniqueness theorems for the homogenous BVPs for the temperature function $\vartheta \in W^1_{2,loc}(\Omega^-)$ or $\vartheta \in W^1_{2,loc}(\Omega^-_\Sigma)$ satisfying the decay condition $\vartheta = o(1)$ at infinity. This decay condition automatically implies that

$$\partial^\alpha\vartheta(x) = \mathcal{O}(|x|^{-|\alpha|-1}) \quad \text{as} \quad |x| \to \infty \tag{3.57}$$

for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

For such solutions to the differential equation (3.56) we have the following integral representation formula (for the domain Ω^-_Σ say)

$$\begin{aligned}\vartheta(x) \;=\; & \int_S \Gamma^{(0)}_{66}(x-y)\,\{\partial_{n(y)}\vartheta(y)\}^-\,dS_y - \int_S \partial_{n(y)}\Gamma^{(0)}_{66}(x-y)\,\{\vartheta(y)\}^-\,dS_y \\ & - \int_\Sigma \Gamma^{(0)}_{66}(x-y)\,[\partial_{n(y)}\vartheta(y)]_\Sigma\,dS_y + \int_\Sigma \partial_{n(y)}\Gamma^{(0)}_{66}(x-y)\,[\vartheta(y)]_\Sigma\,dS_y \\ & + \int_{\Omega^-_\Sigma} \Gamma^{(0)}_{66}(x-y)\,\Phi_6(y)\,dy, \qquad x \in \Omega^-_\Sigma,\end{aligned} \tag{3.58}$$

where $\Gamma^{(0)}_{66}(x)$ is the fundamental solution of the operator $A_{66}(\partial, 0) \equiv A^{(0)}_{66}(\partial)$ defined by (3.6), $\partial_{n(y)} = \mathcal{T}_{66}(\partial_y, n(y)) = \eta_{kj}n_j(y)\partial_k$ denotes the co-normal derivative,

$$[\vartheta(y)]_\Sigma = \{\vartheta(y)\}^+ - \{\vartheta(y)\}^-, \;\; [\partial_{n(y)}\vartheta(y)]_\Sigma = \{\partial_{n(y)}\vartheta(y)\}^+ - \{\partial_{n(y)}\vartheta(y)\}^- \quad \text{on } \Sigma\,.$$

If Ω^- does not contain an interior crack Σ, then in (3.58) the surface integrals over Σ vanish. Applying (3.58) we derive the following asymptotic relation

$$\vartheta(x) = \frac{\theta_0}{(Dx\cdot x)^{1/2}} + \mathcal{O}(|x|^{-2}) \quad \text{as} \quad |x| \to \infty, \tag{3.59}$$

where $D = [d_{kj}]_{3\times3}$ is defined in (3.6), θ_0 is a real constant, which is calculated explicitly

$$\begin{aligned}\theta_0 &= \lim_{|x|\to\infty} (D\,x\cdot x)^{1/2}\,\vartheta(x) \\ &= -\frac{\alpha_0}{4\,\pi}\Big[\int_S \{\partial_{n(y)}\vartheta(y)\}^-\,dS_y - \int_\Sigma [\partial_{n(y)}\vartheta(y)]_\Sigma\,dS_y + \int_{\Omega^-_\Sigma} \Phi(y)\,dy\Big]\end{aligned} \tag{3.60}$$

with α_0 as in (3.6). Note that (3.59) can be differentiated any times with respect to x_j, $j=1,2,3$. In particular,

$$\partial_j \vartheta(x) = -\frac{\theta_0\, d_{jl}\, x_l}{(D\,x\cdot\, x)^{3/2}} + \mathcal{O}(|x|^{-3}) \quad \text{as} \quad |x|\to\infty, \quad j=1,2,3. \tag{3.61}$$

3.5.2 General uniqueness results

First, let us consider the exterior Dirichlet problem of statics of thermo-electro-magneto-elasticity:

$$A(\partial,0)U = \Phi \quad \text{in} \quad \Omega^-, \tag{3.62}$$

$$\{U\}^- = g \quad \text{on} \quad S=\partial\Omega^-, \tag{3.63}$$

where $U=(u,\varphi,\psi,\vartheta)^\top \in [W^1_{2,\,loc}(\Omega^-)]^6$ is a sought for vector and

$$\Phi = (\Phi_1,\cdots,\Phi_6)^\top \in [L_{2,\,comp}(\Omega^-)]^6, \qquad g=(g_1,\cdots,g_6)^\top \in [H_2^{\frac{1}{2}}(S)]^6.$$

Our goal is to establish asymptotic conditions at infinity which guarantee the uniqueness for the BVP (3.62)-(3.63).

For the temperature function ϑ we have the separated exterior Dirichlet problem

$$A_{66}(\partial,0)\vartheta = \eta_{kj}\partial_k\partial_j\vartheta = \Phi_6 \quad \text{in} \quad \Omega^-, \tag{3.64}$$

$$\{\vartheta\}^- = g_6 \quad \text{on} \quad S=\partial\Omega^-. \tag{3.65}$$

We assume that

$$\vartheta(x) = \mathcal{O}(|x|^{-1}) \quad \text{as} \quad |x|\to\infty. \tag{3.66}$$

Then the BVP (3.64)-(3.66) is uniquely solvable for arbitrary Φ_6 and g_6, and there holds the asymptotic relation (3.59) with

$$\theta_0 = -\frac{\alpha_0}{4\,\pi}\Big[\int_S \{\partial_{n(y)}\vartheta(y)\}^-\,dS_y - \int_{\Omega_0}\Phi(y)\,dy\Big] \tag{3.67}$$

which follows from the representation

$$\begin{aligned}\vartheta(x) \;\;=\;\; & \int_S \Gamma^{(0)}_{66}(x-y)\,\{\partial_{n(y)}\vartheta(y)\}^-\,dS_y - \int_S \partial_{n(y)}\Gamma^{(0)}_{66}(x-y)\,\{\vartheta(y)\}^-\,dS_y \\ & + \int_{\Omega_0}\Gamma^{(0)}_{66}(x-y)\,\Phi_6(y)\,dy, \quad x\in\Omega^-, \end{aligned}\tag{3.68}$$

where $\Omega_0 = \operatorname{supp}\Phi_6 \subset \Omega^-$ is compact.

Since Φ_6 has a compact support we see that outside of $\operatorname{supp}\Phi_6$ the temperature function ϑ is C^∞-smooth.

Thus, assuming that the temperature function is known we can substitute it in the first five equations in (3.62). Then from (3.62)-(3.63) we obtain the following BVP for the unknown vector function $\widetilde{U} = (u, \psi, \varphi)^\top \in [W^1_{2,loc}(\Omega^-)]^5$ (see (2.31), (2.33) and (2.35))

$$\widetilde{A}(\partial, 0)\widetilde{U} = \widetilde{\Psi} + \widetilde{\Phi} \quad \text{in} \quad \Omega^-, \tag{3.69}$$

$$\{\widetilde{U}\}^- = \widetilde{g} \quad \text{on} \quad S = \partial\Omega^-, \tag{3.70}$$

where $\widetilde{\Phi} = (\Phi_1, \cdots, \Phi_5)^\top \in [L_{2,comp}(\Omega^-)]^5$, $\widetilde{g} = (g_1, \cdots, g_5)^\top \in [H_2^{\frac{1}{2}}(S)]^5$, the differential operator $\widetilde{A}(\partial, 0) = \widetilde{A}^{(0)}(\partial) = [\widetilde{A}^{(0)}_{kj}(\partial)]_{5\times 5}$ is defined by (2.86) and

$$\widetilde{\Psi} = (\lambda_{1j}\,\partial_j\vartheta,\, \lambda_{2j}\,\partial_j\vartheta,\, \lambda_{3j}\,\partial_j\vartheta,\, p_j\,\partial_j\vartheta,\, m_j\,\partial_j\vartheta)^\top \in [L_2(\Omega^-)]^5. \tag{3.71}$$

Note that $\widetilde{\Psi}$ has not a compact support and due to formulas (3.61)

$$\widetilde{\Psi}(x) = \theta_0\,\widetilde{P}(x) + \widetilde{Q}(x), \tag{3.72}$$

where $\widetilde{Q} \in [L_2(\Omega^-)]^5 \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi_6)]^5$ and

$$\widetilde{Q}(x) = \mathcal{O}(|x|^{-3}) \quad \text{as} \quad |x| \to \infty, \tag{3.73}$$

while $\widetilde{P}(x)$ is an odd, C^∞-smooth homogeneous vector function of order -2,

$$\widetilde{P}(x) = -\frac{1}{(D\,x,\, x)^{3/2}}\,(\lambda_{1j}\,d_{jl}\,x_l,\, \lambda_{2j}\,d_{jl}\,x_l,\, \lambda_{3j}\,d_{jl}\,x_l,\, p_j\,d_{jl}\,x_l,\, m_j\,d_{jl}\,x_l)^\top. \tag{3.74}$$

Therefore, it is easy to see that in a vicinity of infinity, more precisely, outside of $\operatorname{supp}\Phi$ the solution vector $\widetilde{U}$ of equation (3.69) is C^∞-smooth but we can not assume that $\widetilde{U}$ decays at infinity, in general.

Now, we establish asymptotic properties of $\widetilde{U}(x)$ as $|x| \to \infty$. To this end, let us consider the equation

$$\widetilde{A}^{(0)}(\partial)\widetilde{U} = \theta_0\,\widetilde{P} \quad \text{in} \quad \mathbb{R}^3 \setminus \{0\}, \tag{3.75}$$

where θ_0 is given by (3.67). In view of (3.74) and in accordance with Lemma A.2, equation (3.75) possesses a unique solution $\widetilde{W}^{(0)} \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^5$ in the space of zero order homogeneous vector functions satisfying the condition

$$\int_{|x|=1} \widetilde{W}^{(0)}(x)\,dS = 0. \tag{3.76}$$

This solution reads as (cf. (A.17))

$$\widetilde{W}^{(0)}(x) = \theta_0\,\widetilde{U}^{(0)}(x) \quad \text{with} \quad \widetilde{U}^{(0)}(x) := \mathcal{F}^{-1}_{\xi\to x}\Big(\text{v.p.}\,[\widetilde{A}^{(0)}(-i\,\xi)]^{-1}\,\mathcal{F}\widetilde{P}(\xi)\Big). \tag{3.77}$$

Equation (3.69) can be rewritten as

$$\widetilde{A}^{(0)}(\partial)\widetilde{U} = \theta_0\,\widetilde{P} + \widetilde{Q} + \widetilde{\Phi} \quad \text{in} \quad \Omega^-, \tag{3.78}$$

and by Lemmas A.1-A.3 and Corollary A.4 we conclude that a solution of (3.78), which is bounded at infinity, has the form

$$\widetilde{U}(x) = C + \theta_0\, \widetilde{U}^{(0)}(x) + \widetilde{U}^{*}(x) \quad x \in \Omega^{-}, \tag{3.79}$$

where $C = (C_1, \cdots, C_5)^\top$ is an arbitrary constant, $\widetilde{U}^{(0)}$ is given by (3.77) and satisfies the condition (3.76), $\widetilde{U}^{*} \in [W^1_{2,\,loc}(\Omega^-)]^5 \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi)]^5$ and

$$\partial^\alpha \widetilde{U}^{*}(x) = \mathcal{O}(|x|^{-1-|\alpha|} \ln |x|) \quad \text{as} \quad |x| \to \infty \tag{3.80}$$

for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Along with the boundedness at infinity, if we require that the mean value of a solution vector $\widetilde{U}$ over the sphere $\Sigma(O,R)$ tends to zero as $R \to \infty$, i.e.,

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma(O,R)} \widetilde{U}(x)\, d\Sigma(O,R) = 0\,, \tag{3.81}$$

then the constant summand C in the formula (3.79) vanishes and we arrive at the following assertion.

Lemma 3.8 *Let S be Lipschitz and $\widetilde{U} \in [W^1_{2,\,loc}(\Omega^-)]^5$ be a solution of equation* (3.78)*, i.e., equation* (3.69)*, which is bounded at infinity and satisfies the condition* (3.81)*. Then*

$$\widetilde{U}(x) = \theta_0\, \widetilde{U}^{(0)}(x) + \widetilde{U}^{*}(x), \quad x \in \Omega^{-}, \tag{3.82}$$

where $\widetilde{U}^{(0)}$ is given by (3.77) *and $\widetilde{U}^{*}$ is as in* (3.79)*.*

Now, let us return to the exterior Dirichlet BVP (3.62)-(3.63) and analyse the uniqueness question.

Theorem 3.9 *Let S be Lipschitz. The exterior Dirichlet boundary value problem* (3.62)-(3.63) *has at most one solution $U = (u, \varphi, \psi, \vartheta)^\top$ in the space $[W^1_{2,\,loc}(\Omega^-)]^6$, provided*

$$\vartheta(x) = \mathcal{O}(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \tag{3.83}$$

and $\widetilde{U} = (u, \varphi, \psi)^\top$ is bounded at infinity and satisfies the condition (3.81)*.*

Proof. Let $U^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})^\top$ be two solutions of the problem under consideration with properties indicated in the theorem. It is evident that the difference

$$V = (u', \varphi', \psi', \vartheta')^\top = U^{(1)} - U^{(2)}$$

solves then the corresponding homogeneous problem.

Therefore, for the temperature function ϑ' we get the homogeneous Dirichlet problem of type (3.64)-(3.65) and since ϑ' satisfies the decay condition (3.83), it is identical zero in Ω^-.

Consequently, the vector $\widetilde{V} = (u', \varphi', \psi')^\top$ is a solution of the homogeneous exterior Dirichlet problem

$$A^{(0)}(\partial)\widetilde{V} = 0 \quad \text{in} \quad \Omega^-, \tag{3.84}$$

$$\{\widetilde{V}\}^- = 0 \quad \text{on} \quad S = \partial\Omega^-. \tag{3.85}$$

Moreover, the vector $\widetilde{V}$ satisfies the condition (3.81) with $\widetilde{V}$ for $\widetilde{U}$ since both vectors $\widetilde{U}^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)})^\top$ and $\widetilde{U}^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top$ satisfy the same condition.

In accordance with Lemma 3.8 then $\widetilde{V}$ is representable in the form (3.82),

$$\widetilde{V}(x) = \theta'_0\, \widetilde{U}^{(0)}(x) + \widetilde{V}^*(x), \quad x \in \Omega^-,$$

where $\widetilde{U}^{(0)}$ is given by (3.77), $\partial^\alpha \widetilde{V}^*(x) = \mathcal{O}(|x|^{-1-|\alpha|} \ln|x|)$ as $|x| \to \infty$ for arbitrary multi-index α and

$$\theta'_0 = \lim_{|x|\to\infty} (D\, x \cdot x)^{1/2}\, \vartheta'(x) = 0$$

since $\vartheta' = 0$ in Ω^- (cf. (3.60)). Therefore,

$$\partial^\alpha \widetilde{V} = \mathcal{O}(|x|^{-1-|\alpha|} \ln|x|) \quad \text{as} \quad |x| \to \infty. \tag{3.86}$$

For vectors satisfying the decay conditions (3.86) we can easily derive the following Green's formula (cf. (2.88))

$$\int\limits_{\Omega^-} [\, \widetilde{A}^{(0)}(\partial)\, \widetilde{V} \cdot \widetilde{V} + \widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V})\,]\, dx = -\big\langle \{T\widetilde{V}\}^-, \{\widetilde{V}\}^- \big\rangle_{\partial\Omega^-}, \tag{3.87}$$

where $T(\partial, n)$ is given by (2.87) and

$$\widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V}) = c_{rjkl}\, \partial_l u'_k\, \partial_j u'_r + \varkappa_{jl}\, \partial_l \varphi'\, \partial_j \varphi' + a_{jl}\, (\partial_l \varphi'\, \partial_j \psi' + \partial_j \psi'\, \partial_l \varphi') + \mu_{jl}\, \partial_l \psi'\, \partial_j \psi'. \tag{3.88}$$

From (3.84)-(3.85) and (3.87)-(3.88) along with the inequalities (2.11) we get

$$u'(x) = a \times x + b, \quad \varphi'(x) = b_4, \quad \psi'(x) = b_5,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are arbitrary constant vectors, and b_4 and b_5 are arbitrary constants. Now, in view of (3.86) we arrive at the equalities $u'(x) = 0$, $\varphi'(x) = 0$ and $\psi'(x) = 0$ for $x \in \Omega^-$. Consequently, $U^{(1)} = U^{(2)}$ in Ω^-. $\square$

The proof of the following theorem is word for word.

Theorem 3.10 *th3.10th2.3 Let S be Lipschitz. The exterior Neumann and mixed boundary value problems of statics of thermo-electro-magneto-elasticity have at most one solution $U = (u, \varphi, \psi, \vartheta)^\top$ in the space $[W^1_{2,\,loc}(\Omega^-)]^6$, provided*

$$\vartheta(x) = \mathcal{O}(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \tag{3.89}$$

and $\widetilde{U} = (u, \varphi, \psi)^\top$ is bounded at infinity and satisfies the condition (3.81).

4 Properties of generalized potentials

4.1 Mapping properties

Here we establish the mapping and regularity properties of the single and double layer potentials and the boundary pseudodifferential operators generated by them in the Hölder ($C^{k,\kappa}$), Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces. They can be established by standard methods (see, e.g., [KGBB], [MP], [Se1], [Esk1], [DNS1], [DNS2], [NCS1], [NDS1], [Mc1], [NBC1] and [Du1]). We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter τ (τ' and τ'' say) have the same smoothness properties and possess the same jump relations, since the entries of the difference of the fundamental matrices $\Gamma(x,\tau') - \Gamma(x,\tau'')$ are bounded and their derivatives of order m have a singularity of type $\mathcal{O}(|x|^{-m})$ in a vicinity of the origin. This implies that the boundary integral operators generated by single layer potentials (respectively, by double layer potentials) constructed with the help of the kernels $\Gamma(x,\tau')$ and $\Gamma(x,\tau'')$ differ by a compact perturbations. Therefore, using the technique and word for word arguments given in [KGBB], [Mc1], [DNS1], [BCNS1] and [Du1] we can prove the following theorems concerning the above introduced generalized potentials.

Theorem 4.1 *Let S, m, and κ be as in* (3.47)*, $0<\kappa'<\kappa$, and let $k\leq m-1$ be integer. Then the operators*

$$\begin{aligned} V\ &:\ [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6, \\ W\ &:\ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(\overline{\Omega^\pm})]^6, \end{aligned} \tag{4.1}$$

are continuous.

For any $g\in[C^{0,\kappa'}(S)]^6$, $h\in[C^{1,\kappa'}(S)]^6$, and any $x\in S$

$$[\,V(g)(x)\,]^\pm = V(g)(x) = \mathcal{H}\,g(x), \tag{4.2}$$

$$[\,\mathcal{T}(\partial_x,n(x))\,V(g)(x)\,]^\pm = [\,\mp 2^{-1}I_6 + \mathcal{K}\,]\,g(x), \tag{4.3}$$

$$[\,W(g)(x)\,]^\pm = [\,\pm 2^{-1}I_6 + \mathcal{N}\,]\,g(x), \tag{4.4}$$

$$[\,\mathcal{T}(\partial_x,n(x))\,W(h)(x)\,]^+ = [\,\mathcal{T}(\partial_x,n(x))\,W(h)(x)\,]^- = \mathcal{L}\,h(x), \quad m\geq 2, \tag{4.5}$$

where

$$\mathcal{H}\,g(x) \equiv \mathcal{H}_S\,g(x) := \int_S \Gamma(x-y,\tau)\,g(y)\,dS_y\,, \tag{4.6}$$

$$\mathcal{K}\,g(x) \equiv \mathcal{K}_S\,g(x) := \int_S [\,\mathcal{T}(\partial_x,n(x))\,\Gamma(x-y,\tau)\,]\;g(y)\,dS_y\,, \tag{4.7}$$

$$\mathcal{N}\,g(x) \equiv \mathcal{N}_S\,g(x) := \int_S [\,\mathcal{P}(\partial_y,n(y),\overline{\tau})\,\Gamma^\top(x-y,\tau)\,]^\top\,g(y)\,dS_y\,, \tag{4.8}$$

$$\mathcal{L}\,h(x) \equiv \mathcal{L}_S\,h(x) := \lim_{\Omega^\pm\ni z\to x\in S} \mathcal{T}(\partial_z,n(x))\int_S [\mathcal{P}(\partial_y,n(y),\overline{\tau})\Gamma^\top(z-y,\tau)]^\top h(y)\,dS_y\,. \tag{4.9}$$

Proof. The proof of the relations (4.2)-(4.4) can be performed by standard arguments (see, e.g., [KGBB], Ch. 5). We demonstrate here only a simplified proof of the relation (4.5), the so called *Liapunov-Tauber type theorem.* Let $h \in [C^{1,\kappa'}(S)]^6$, $S \in C^{2,\kappa}$, and consider the double layer potential $U := W(h) \in [C^{1,\kappa'}(\overline{\Omega^{\pm}})]^6$. Then by Corollary 3.7 and the jump relations (4.4) we have

$$U(x) = W([U]_S)(x) - V([\mathcal{T}U]_S)(x), \quad x \in \Omega^{\pm},$$

i.e.,

$$W(h)(x) = W(h)(x) - V([\mathcal{T}W(h)]_S)(x), \quad x \in \Omega^{\pm},$$

since $[U]_S = \{W(h)\}^+ - \{W(h)\}^- = h$ on S due to (4.4). Therefore $V([\mathcal{T}W(h)]_S) = 0$ in $\Omega^{\pm}$ and in view of (4.3) we conclude

$$\{\mathcal{T}V([\mathcal{T}W(h)]_S)\}^- - \{\mathcal{T}V([\mathcal{T}W(h)]_S)\}^+ = [\mathcal{T}W(h)]_S = \{\mathcal{T}W(h)\}^+ - \{\mathcal{T}W(h)\}^- = 0$$

on S, which completes the proof. $\square$

With the help of the explicit form of the fundamental matrix $\Gamma(x-y,\tau)$ it can easily be shown that the operators $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator. For a C^∞-smooth surfaces S all these operators can be treated as pseudodifferential operators on S (cf., [Ag1], [HW], [DNS2]). In contrast to the classical elasticity case, neither $\mathcal{H}$ and $\mathcal{L}$ are self-adjoint and nor $\mathcal{K}$ and $\mathcal{N}$ are mutually adjoint operators. For the adjoint operators $\mathcal{H}^*$, $\mathcal{K}^*$ and $\mathcal{N}^*$ we have

$$\mathcal{H}^*\, g(x) \equiv \mathcal{H}^*_\tau\, g(x) := \int_S \Gamma^*(x-y,\tau)\, g(y)\, dS_y\,, \tag{4.10}$$

$$\mathcal{K}^*\, g(x) \equiv \mathcal{K}^*_\tau\, g(x) := \int_S [\,\mathcal{T}(\partial_y, n(y))\, [\Gamma^*(x-y,\tau)]^\top\,]^\top\, g(y)\, dS_y\,, \tag{4.11}$$

$$\mathcal{N}^*\, g(x) \equiv \mathcal{N}^*_\tau\, g(x) := \int_S \mathcal{P}(\partial_x, n(x), \overline{\tau})\, \Gamma^*(x-y,\tau)\, g(y)\, dS_y\,, \tag{4.12}$$

where $\Gamma^*(x-y,\tau) = [\Gamma(y-x,\tau)]^\top$ is a fundamental matrix of the operator $A^*(\partial,\tau)$ (see Remark 3.4). Note that by these relations the adjoint operators are defined without complex conjugation, which means that (cf. (2.45))

$$\langle\, \mathcal{H}\, g\,,\, \overline{h}\,\rangle_S = \langle\, g\,,\, \overline{\mathcal{H}^*\, h}\,\rangle_S\,, \quad \langle\, \mathcal{K}\, g\,,\, \overline{h}\,\rangle_S = \langle\, g\,,\, \overline{\mathcal{K}^*\, h}\,\rangle_S\,, \quad \langle\, \mathcal{N}\, g\,,\, \overline{h}\,\rangle_S = \langle\, g\,,\, \overline{\mathcal{N}^*\, h}\,\rangle_S\,. \tag{4.13}$$

It is easy to see that the adjoint boundary operators are generated by the single layer and double layer potentials constructed with the help of the fundamental matrix $\Gamma^*(x-y,\tau)$. In particular, let

$$V^*_S(g)(x) = V^*(g)(x) := \int_S \Gamma^*(x-y,\tau)\, g(y)\, dS_y, \tag{4.14}$$

$$W^*_S(g)(x) = W^*(g)(x) = \int_S [\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y,\tau)]^\top]^\top\, g(y)\, dS_y. \tag{4.15}$$

Then for any solution U of the equation $A^*(\partial,\tau)U=0$ we have the representation formula

$$U(x)=W^*\big(\{U\}^+\big)(x)-V^*\big(\{\mathcal{P}U\}^+\big)(x)\,,\quad x\in\Omega^+, \tag{4.16}$$

which can be obtained by Green's identity (2.42). The right hand side expression in (4.16) vanishes for $x\in\Omega^-$. Clearly the layer potential operators V^* and W^* have the same mapping properties as the operators V and W, namely

$$\begin{aligned} V^* \;&:\; [C^{k,\kappa'}(S)]^6\to[C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6,\\ W^* \;&:\; [C^{k,\kappa'}(S)]^6\to[C^{k,\kappa'}(\overline{\Omega^\pm})]^6, \end{aligned} \tag{4.17}$$

where S, m, κ, κ', and $k\le m-1$ are as in Theorem 4.1. Moreover, for $g\in[C^{0,\kappa'}(S)]^6$ and $h\in[C^{1,\kappa'}(S)]^6$ the following jump relations hold on S

$$[\,V^*(g)(x)\,]^\pm=V^*(g)(x)=\mathcal{H}^*\,g(x), \tag{4.18}$$

$$[\,\mathcal{P}(\partial_x,n(x),\overline{\tau})\,V^*(g)(x)\,]^\pm=[\,\mp2^{-1}I_6+\mathcal{N}^*\,]\,g(x), \tag{4.19}$$

$$[\,W^*(g)(x)\,]^\pm=[\,\pm2^{-1}I_6+\mathcal{K}^*\,]\,g(x), \tag{4.20}$$

$$[\,\mathcal{P}(\partial_x,n(x),\overline{\tau})\,W^*(h)(x)\,]^+=[\,\mathcal{P}(\partial_x,n(x),\overline{\tau})\,W^*(h)(x)\,]^-=\mathcal{L}^*\,h(x),\quad m\ge2 \tag{4.21}$$

Theorem 4.2 *Let S be a Lipschitz surface. The operators V, W, V^*, and W^* can be extended to the continuous mappings*

$$\begin{aligned} V,\ V^* \;&:\; [H_2^{-\frac12}(S)]^6\to[H_2^1(\Omega^+)]^6\quad\Big[\,[H_2^{-\frac12}(S)]^6\to[H^1_{2,\,loc}(\Omega^-)]^6\,\Big],\\ W,\ W^* \;&:\; [H_2^{\frac12}(S)]^6\to[H_2^1(\Omega^+)]^6\quad\Big[\,[H_2^{\frac12}(S)]^6\to[H^1_{2,\,loc}(\Omega^-)]^6\,\Big]. \end{aligned}$$

The jump relations (4.2)-(4.5) *and* (4.18)-(4.21) *on S remain valid for the extended operators in the corresponding function spaces.*

Proof. It is word for word of the proofs of the similar theorems in [Mc1]. □

Theorem 4.3 *Let S, m, κ, κ' and k be as in Theorem* 4.1. *Then the operators*

$$\mathcal{H},\ \mathcal{H}^* \;:\; [C^{k,\kappa'}(S)]^6\to[C^{k+1,\kappa'}(S)]^6\,,\quad m\ge1, \tag{4.22}$$

$$:\; [H_2^{-\frac12}(S)]^6\to[H_2^{\frac12}(S)]^6\,,\quad m\ge1, \tag{4.23}$$

$$\mathcal{K},\ \mathcal{N}^* \;:\; [C^{k,\kappa'}(S)]^6\to[C^{k,\kappa'}(S)]^6\,,\quad m\ge1, \tag{4.24}$$

$$:\; [H_2^{-\frac12}(S)]^6\to[H_2^{-\frac12}(S)]^6\,,\quad m\ge1, \tag{4.25}$$

$$\mathcal{N},\ \mathcal{K}^* \;:\; [C^{k,\kappa'}(S)]^6\to[C^{k,\kappa'}(S)]^6\,,\quad m\ge1, \tag{4.26}$$

$$:\; [H_2^{\frac12}(S)]^6\to[H_2^{\frac12}(S)]^6\,,\quad m\ge1, \tag{4.27}$$

$$\mathcal{L},\ \mathcal{L}^* \;:\; [C^{k,\kappa'}(S)]^6\to[C^{k-1,\kappa'}(S)]^6\,,\quad m\ge2,\ k\ge1, \tag{4.28}$$

$$:\; [H_2^{\frac12}(S)]^6\to[H_2^{-\frac12}(S)]^6\quad m\ge2, \tag{4.29}$$

are continuous. The operators (4.23), (4.25), (4.27), *and* (4.29) *are bounded if S is a Lipschitz surface and the following equalities hold true in appropriate function spaces:*

$$\begin{aligned} &\mathcal{N}\mathcal{H} = \mathcal{H}\mathcal{K}, && \mathcal{L}\mathcal{N} = \mathcal{K}\mathcal{L}, \\ &\mathcal{H}\mathcal{L} = -4^{-1} I_6 + \mathcal{N}^2, && \mathcal{L}\mathcal{H} = -4^{-1} I_6 + \mathcal{K}^2. \end{aligned} \tag{4.30}$$

Proof. It is word for word of the proofs of the similar theorems in [KGBB], [DNS2], [Co1] and [Mc1]. □

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [Ag1], [Esk1], [Se1], [DNS2], [Du1], and the references therein).

Theorem 4.4 *Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$, $\mathcal{L}$, V^*, W^*, $\mathcal{H}^*$, $\mathcal{K}^*$, $\mathcal{N}^*$, and $\mathcal{L}^*$ be as in Theorems* 4.1, 4.2 *and* 4.3 *and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The layer potential operators* (4.1), (4.17) *and the boundary integral* (*pseudodifferential*) *operators* (4.22)-(4.29) *can be extended to the following continuous operators*

$$\begin{aligned}
V,\ V^* \ &: \ [B^s_{p,p}(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[[B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \Big], \\
&: \ [B^s_{p,q}(S)]^6 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B_{p,q,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \Big], \\
W,\ W^* \ &: \ [B^s_{p,p}(S)]^6 \to [H_p^{s+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[[B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)]^6 \Big], \\
&: \ [B^s_{p,q}(S)]^6 \to [B_{p,q}^{s+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B_{p,q,loc}^{s+\frac{1}{p}}(\Omega^-)]^6 \Big], \\
\mathcal{H},\ \mathcal{H}^* \ &: \ [H^s_p(S)]^6 \to [H^{s+1}_p(S)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6 \Big], \\
\mathcal{K},\ \mathcal{N}^* \ &: \ [H^s_p(S)]^6 \to [H^s_p(S)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B^s_{p,q}(S)]^6 \Big], \\
\mathcal{N},\ \mathcal{K}^* \ &: \ [H^s_p(S)]^6 \to [H^s_p(S)]^6 \quad \Big[[B^s_{p,q}(S)]^6 \to [B^s_{p,q}(S)]^6 \Big], \\
\mathcal{L},\ \mathcal{L}^* \ &: \ [H^{s+1}_p(S)]^6 \to [H^s_p(S)]^6 \quad \Big[[B^{s+1}_{p,q}(S)]^6 \to [B^s_{p,q}(S)]^6 \Big].
\end{aligned}$$

The jump relations (4.2)-(4.5) *remain valid for arbitrary $g \in [B^s_{p,q}(S)]^6$ with $s \in \mathbb{R}$ if the limiting values* (*traces*) *on S are understood in the sense described in* [Se1].

Proof. It is word for word of the proofs of the similar theorems in [DNS2] and [Du1]. □

Remark 4.5 *Let either $\Phi \in [L_p(\Omega^+)]^6$ or $\Phi \in [L_{p,comp}(\Omega^-)]^6$, $p > 1$. Then the Newtonian volume potential $N_{\Omega^\pm}(\Phi)$ possesses the following properties* (*see* (3.50)) :

$$\begin{aligned} &N_{\Omega^+}(\Phi) \in [W^2_p(\Omega^+)]^6, \quad N_{\Omega^-}(\Phi) \in [W^2_{p,loc}(\Omega^-)]^6, \\ &A(\partial,\tau) N_{\Omega^\pm}(\Phi) = \Phi \quad \text{almost everywhere in} \quad \Omega^\pm. \end{aligned}$$

4.2 Coercivity and strong ellipticity properties of the operator $\mathcal{H}$

Here we establish that the boundary integral operator $\mathcal{H}$, defined by (4.6), satisfies Gårding type inequality. By $\mathcal{H}^{(0)}$, $\mathcal{K}^{(0)}$, $\mathcal{N}^{(0)}$ and $\mathcal{L}^{(0)}$ we denote the boundary operators generated by the single and double layer potentials constructed with the help of the fundamental matrix $\Gamma^{(0)}(\cdot)$ associated with the operator $A^{(0)}(\partial)$. Note that $\Gamma^{(0)}(\cdot)$ is the principal singular part of the fundamental matrix $\Gamma(\cdot,\tau)$ (see Subsection 3.1). So we have

$$\mathcal{H}^{(0)}h = \{V^{(0)}(h)\}^{+} = \{V^{(0)}(h)\}^{-}\,, \tag{4.31}$$

$$[\,\mp 2^{-1}I_6 + \mathcal{K}^{(0)}\,]\,g = [\,\mathcal{T}^{(0)}(\partial_x, n(x))\,V^{(0)}(g)\,]^{\pm}, \tag{4.32}$$

$$[\,\pm 2^{-1}I_6 + \mathcal{N}^{(0)}\,]\,h = [\,W^{(0)}(h)\,]^{\pm}, \tag{4.33}$$

$$\mathcal{L}^{(0)}g = \{\mathcal{T}^{(0)}(\partial,n)W^{(0)}(g)\}^{+} = \{\mathcal{T}^{(0)}(\partial,n)W^{(0)}(g)\}^{-}\,, \tag{4.34}$$

where

$$V^{(0)}(h)(x) = \int_S \Gamma^{(0)}(x-y)\,h(y)\,dS_y, \tag{4.35}$$

$$W^{(0)}(g)(x) = \int_S [\mathcal{P}^{(0)}(\partial_y, n(y))[\Gamma^{(0)}(x-y)]^{\top}]^{\top}\,g(y)\,dS_y. \tag{4.36}$$

Here the boundary differential operators $\mathcal{T}^{(0)}(\partial,n)$ and $\mathcal{P}^{(0)}(\partial_y,n(y)) := \mathcal{P}(\partial_y,n(y),0)$ are defined by (2.28) and (2.38) respectively. Clearly, for a Lipschitz surface S the operators

$$\mathcal{H}-\mathcal{H}^{(0)}\ :\ [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6\,, \tag{4.37}$$

$$\mathcal{K}-\mathcal{K}^{(0)}\ :\ [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,, \tag{4.38}$$

$$\mathcal{N}-\mathcal{N}^{(0)}\ :\ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6\,, \tag{4.39}$$

$$\mathcal{L}-\mathcal{L}^{(0)}\ :\ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,, \tag{4.40}$$

are compact for arbitrary τ due to Lemma 3.3. Moreover, if S, m, κ, κ' and k are as in Theorem 4.1, then the operators

$$\mathcal{H}-\mathcal{H}^{(0)}\ \ :\ \ [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6\,,\quad m\geq 1, \tag{4.41}$$

$$\mathcal{K}-\mathcal{K}^{(0)}\ \ :\ \ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6\,,\quad m\geq 1, \tag{4.42}$$

$$\mathcal{N}-\mathcal{N}^{(0)}\ \ :\ \ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6\,,\quad m\geq 1, \tag{4.43}$$

$$\mathcal{L}-\mathcal{L}^{(0)}\ \ :\ \ [C^{k,\kappa'}(S)]^6 \to [C^{k-1,\kappa'}(S)]^6\,,\quad m\geq 2,\ k\geq 1, \tag{4.44}$$

are compact for arbitrary τ due to Lemma 3.3.

Remark 4.6 *Note that Theorems* 4.1-4.4 *hold true for the potentials* $V^{(0)}$ *and* $W^{(0)}$*, and for the boundary operators generated by them.*

Theorem 4.7 *Let $\partial\Omega^+ = S$ be a Lipschitz surface. Then there is a positive constant c such that for all $h \in [H_2^{-\frac{1}{2}}(S)]^6$ there holds the inequality*

$$\Re\,\langle -\mathcal{H}^{(0)}h\,,\,h\rangle_S \geq c\,\|h\|^2_{[H_2^{-\frac{1}{2}}(S)]^6}\,, \tag{4.45}$$

where $\langle\cdot\,,\,\cdot\rangle_S$ denotes the duality between the spaces $[H_2^{\frac{1}{2}}(S)]^6$ and $[H_2^{-\frac{1}{2}}(S)]^6$.

Proof. Note that the single layer potential $V^{(0)}(h)$ with $h \in [H_2^{-\frac{1}{2}}(S)]^6$ belongs to the space $[W^1_{2,loc}(\mathbb{R}^3)]^6$, solves the homogeneous equation $A^{(0)}(\partial)V^{(0)}(h) = 0$ in $\Omega^\pm$ and possesses the following asymptotic property at infinity: $\partial^\alpha V^{(0)}(h)(x) = \mathcal{O}(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Therefore in view of formulas (2.84) and (2.88) we easily derive the following Green's identities

$$\int\limits_{\Omega^+} \mathcal{E}^{(0)}(U,\overline{U})\,dx = \big\langle\{U\}^+,\,\{\mathcal{T}^{(0)}U\}^+\big\rangle_{\partial\Omega^+}\,, \tag{4.46}$$

$$\int\limits_{\Omega^-} \mathcal{E}^{(0)}(U,\overline{U})\,dx = -\big\langle\{U\}^-,\,\{\mathcal{T}^{(0)}U\}^-\big\rangle_{\partial\Omega^-}\,, \tag{4.47}$$

with $U = (u,\varphi,\psi,\vartheta)^\top = V^{(0)}(h)$ and

$$\begin{aligned}\mathcal{E}^{(0)}(U,\overline{U}) &= c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u_r} + e_{lrj}\,(\partial_l\varphi\,\overline{\partial_j u_r} - \partial_j u_r\,\overline{\partial_l\varphi})\\ &+q_{lrj}\,(\partial_l\psi\,\overline{\partial_j u_r} - \partial_j u_r\,\overline{\partial_l\psi}) + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\varphi\,\overline{\partial_j\psi} + \partial_j\psi\,\overline{\partial_l\varphi})\\ &+\mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi} + \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta}\,.\end{aligned} \tag{4.48}$$

Applying the properties of the single layer potential treated in Theorem 4.1, from (4.46) and (4.47) we get

$$\int\limits_{\Omega^+\cup\Omega^-} \mathcal{E}^{(0)}(U,\overline{U})\,dx = \langle -\mathcal{H}^{(0)}h\,,\,h\rangle_S. \tag{4.49}$$

With the help of inequalities (2.11) and (2.16) we derive from (4.49)

$$\begin{aligned}\Re\,\langle -\mathcal{H}^{(0)}h\,,\,h\rangle_S &= \Re \int\limits_{\Omega^+\cup\Omega^-} \mathcal{E}^{(0)}(U,\overline{U})\,dx\\ &\geq c_1 \int\limits_{\Omega^+\cup\Omega^-} \{\varepsilon_{kj}\varepsilon_{kj} + |\nabla\varphi|^2 + |\nabla\psi|^2 + |\nabla\vartheta|^2\}\,dx\,,\end{aligned} \tag{4.50}$$

where c_1 is a positive constant independent of h. Now, using the Korn's inequality for $\mathbb{R}^3$ (see [KO1]) we have

$$\Re\,\langle -\mathcal{H}^{(0)}h\,,\,h\rangle_S \geq c_2\Big\{\sum_{k,j=1}^{3}\|\partial_j u_k\|^2_{L_2(\mathbb{R}^3)} + \|\nabla\varphi\|^2_{L_2(\mathbb{R}^3)} + \|\nabla\psi\|^2_{L_2(\mathbb{R}^3)} + \|\nabla\vartheta\|^2_{L_2(\mathbb{R}^3)}\Big\}. \tag{4.51}$$

Due to the properties of the single layer potential $V^{(0)}(h)$ it follows that

$$V^{(0)}(h) \in BL(\mathbb{R}^3) := \Big\{ U \in [W^1_{2,loc}(\mathbb{R}^3)]^6 : (1+|x|^2)^{-1/2} U_k \in L_2(\mathbb{R}^3), \nabla U_k \in [L_2(\mathbb{R}^3)]^3 \Big\}, \tag{4.52}$$

where $BL(\mathbb{R}^3)$ denotes the Beppo-Levy type space (for details see [DaLi1], Ch.XI). It is well known that the norm in this space defined by

$$\|U\|^2_{BL(\mathbb{R}^3)} := \|(1+|x|^2)^{-1/2}\, U\|^2_{[L_2(\mathbb{R}^3)]^6} + \sum_{k=1}^{6}\sum_{j=1}^{3} \|\partial_j U_k\|^2_{L_2(\mathbb{R}^3)} \tag{4.53}$$

is equivalent to the seminorm

$$\|U\|^2_{*BL(\mathbb{R}^3)} := \sum_{k=1}^{6}\sum_{j=1}^{3} \|\partial_j U_k\|^2_{L_2(\mathbb{R}^3)}\,. \tag{4.54}$$

Therefore, from (4.51) it follows that

$$\Re\,\langle -\mathcal{H}^{(0)} h\,,\, h\rangle_S \geq c_3\, \|V^{(0)}(h)\|^2_{BL(\mathbb{R}^3)}\,. \tag{4.55}$$

Since $A^{(0)}(\partial)V^{(0)}(h) = 0$ in $\Omega^+ \cup \Omega^-$ and $V^{(0)}(h) \in [W^1_{2,loc}(\mathbb{R}^3)]^6$, the boundary functionals $[\mathcal{T}^{(0)}V^{(0)}(h)]^{\pm} \in [H_2^{-\frac{1}{2}}(S)]^6$ are defined correctly and the norms $\|[\mathcal{T}^{(0)}V^{(0)}(h)]^{\pm}\|_{[H_2^{-\frac{1}{2}}(S)]^6}$ can be controlled by the norm $\|V^{(0)}(h)\|_{BL(\mathbb{R}^3)}$ (see (2.46)). Consequently, there is a positive constant c_5 such that

$$\|[\mathcal{T}^{(0)}V^{(0)}(h)]^{\pm}\|_{[H_2^{-\frac{1}{2}}(S)]^6} \leq c_5\, \|V^{(0)}(h)\|_{BL(\mathbb{R}^3)}. \tag{4.56}$$

Whence the inequality

$$\|h\|_{[H_2^{-\frac{1}{2}}(S)]^6} = \|[\mathcal{T}^{(0)}V^{(0)}(h)]^{-} - [\mathcal{T}^{(0)}V^{(0)}(h)]^{+}\|_{[H_2^{-\frac{1}{2}}(S)]^6} \leq c_6\, \|V^{(0)}(h)\|_{BL(\mathbb{R}^3)} \tag{4.57}$$

follows immediately which along with (4.55) completes the proof. □

Corollary 4.8 *Let $\partial\Omega^+ = S$ be a Lipschitz surface. Then the operator*

$$\mathcal{H}^{(0)} : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6$$

is invertible.

Proof. It follows from Theorem 4.7 and the Lax-Milgram theorem. □

Corollary 4.9 *Let $\partial\Omega^+ = S$ be a Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then there is a positive constant c_1 such that for all $h \in [H_2^{-\frac{1}{2}}(S)]^6$ there holds the inequality*

$$\Re \langle(-\mathcal{H} + \mathcal{C})h\,,\, h\rangle_S \geq c_1 \, \|h\|^2_{[H_2^{-\frac{1}{2}}(S)]^6}\,, \tag{4.58}$$

where $\mathcal{C} \,:\, [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6$ is a compact operator. The operator

$$\mathcal{H} \;:\; [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6 \tag{4.59}$$

is invertible.

Proof. The first part of the corollary follows from Theorem 4.7 and from the fact that the operator (4.37) is compact. In turn, (4.58) implies that the index of the operator (4.59) is zero. On the other hand, from the uniqueness Theorem 2.1 for the Dirichlet BVP, we conclude that the null space of the operator (4.59) is trivial and consequently (4.59) is invertible. $\square$

Corollary 4.10 *Let $\partial\Omega^\pm = S$ be a Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Further, let either $U \in [H_2^1(\Omega^+)]^6$ or $U \in [H^1_{2,\,loc}(\Omega^-)]^6$ be a solution to the homogeneous equation $A(\partial, \tau)U = 0$ in $\Omega^\pm$, satisfying the decay conditions* (2.56) *in the case of exterior domain Ω^-. Then U is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}[U]^\pm\big)(x), \quad x \in \Omega^\pm, \tag{4.60}$$

where $[U]^\pm$ are the interior and exterior traces of U on S from $\Omega^\pm$ respectively.

Proof. It follows from Corollary 4.9 and the uniqueness Theorems 2.1 and 2.2. $\square$

Remark 4.11 *If S is a sufficiently smooth surface (C^∞ regular surface say), then for arbitrary $\tau \in \mathbb{C}$, the operator $\mathcal{H}$ is a pseudodifferential operator of order -1 with the principal homogeneous symbol matrix given by the following relation (see Subsection 3.1 an the Appendix C)*

$$\begin{aligned}
\mathfrak{S}\big(\mathcal{H}; x, \xi_1, \xi_2\big) &= \mathfrak{S}\big(\mathcal{H}^{(0)}; x, \xi_1, \xi_2\big) = M(x, \xi_1, \xi_2) = [\, M_{kj}(x, \xi_1, \xi_2)\,]_{6\times 6} \\
&:= \begin{bmatrix} [M_{kj}(x, \xi_1, \xi_2)]_{5\times 5} & [\,0\,]_{5\times 1} \\ [\,0\,]_{1\times 5} & M_{66}(x, \xi_1, \xi_2) \end{bmatrix}_{6\times 6} \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} [\,A^{(0)}(\,B_n\,\xi\,)\,]^{-1}\, d\xi_3 = -\frac{1}{2\pi}\int_{\ell^\pm} [\,A^{(0)}(\,B_n\,\xi\,)\,]^{-1}\, d\xi_3,
\end{aligned} \tag{4.61}$$

$$B_n = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \quad \text{for } x \in \partial\Omega,\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}, \tag{4.62}$$

where $B_n(x)$ *is an orthogonal matrix with* $\det B_n(x) = 1$*,* $n(x) = \big(n_1(x), n_2(x), n_3(x)\big)$ *is the outward unit normal vectors to* S*, while* $l(x) = \big(l_1(x), l_2(x), l_3(x)\big)$ *and* $m(x) = \big(m_1(x), m_2(x), m_3(x)\big)$ *are orthogonal unit vectors in the tangential plane associated with some local chart at the point* $x \in S$*; here* ℓ^+ *(respectively* ℓ^-*) is a closed contours in the upper (respectively lower) complex half-plane* $\Re\,\xi_3 > 0$ *(respectively* $\Re\,\xi_3 < 0$*), orientated counterclockwise (respectively clockwise) and enveloping all the roots with positive (respectively negative) imaginary parts of the equation* $\det A^{(0)}(\,B_n\,\xi\,) = 0$ *with respect to* ξ_3*;* ξ_1 *and* ξ_2 *are to be considered as real parameters.*

From the representation (4.61) *it follows that the entries of the principal homogeneous symbol matrix* $\mathfrak{S}\big(\mathcal{H}; x, \xi_1, \xi_2\big)$ *are odd, real valued and homogeneous of order* -1 *functions in* $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$*,*

$$\begin{aligned} &\Im\,\mathfrak{S}_{kj}\big(\mathcal{H}; x, \xi_1, \xi_2\big) = 0, \quad k, j = \overline{1,6}, \\ &\mathfrak{S}_{kj}\big(\mathcal{H}; x, -\xi_1, -\xi_2\big) = \mathfrak{S}_{kj}\big(\mathcal{H}; x, \xi_1, \xi_2\big), \\ &\mathfrak{S}_{kj}\big(\mathcal{H}; x, t\xi_1, t\xi_2\big) = t^{-1}\mathfrak{S}_{kj}\big(\mathcal{H}; x, \xi_1, \xi_2\big) \quad \text{for all } \ t > 0\,. \end{aligned} \tag{4.63}$$

In accordance with (3.5) *we have*

$$\mathfrak{S}_{5j}\big(\mathcal{H}; x, \xi_1, \xi_2\big) = \mathfrak{S}_{j5}\big(\mathcal{H}; x, \xi_1, \xi_2\big) = 0, \quad j = \overline{1,5}, \tag{4.64}$$

Moreover, with the help of the relations (2.37) *and* (4.61) *one can easily show that the matrix* $-\mathfrak{S}\big(\mathcal{H}; x, \xi_1, \xi_2\big)$ *is strongly elliptic, i.e., there is a positive constant* C *such that for all* $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ *and for all* $\zeta \in \mathbb{C}^6$

$$\Re\,\{-\mathfrak{S}\big(\mathcal{H}; x, \xi_1, \xi_2\big)\,\zeta\cdot\zeta\} \geq C\,|\xi|^{-1}\,|\zeta|^2\,. \tag{4.65}$$

4.3 Steklov-Poincaré type operators

Now we introduce the so called Steklov-Poincaré type operators $\mathcal{A}^{\pm}$ which map Dirichlet data to the corresponding Neumann data,

$$\mathcal{A}^{\pm}[U]^{\pm} = [\mathcal{T}U]^{\pm} \quad \text{on } \ S. \tag{4.66}$$

From (4.60) and (4.3) it is clear that

$$\mathcal{A}^{\pm} := \big(\mp 2^{-1}\,I_6 + \mathcal{K}\big)\mathcal{H}^{-1} \tag{4.67}$$

and by Theorem 4.3 and Corollary 4.9

$$\mathcal{A}^{\pm} \ : \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,. \tag{4.68}$$

Lemma 4.12 *Let* $\partial\Omega^+ = S$ *be a Lipschitz surface and* $\tau = \sigma + i\omega$ *with* $\sigma > 0$ *and* $\omega \in \mathbb{R}$*. Then there is a positive constant* C_1 *such that for all* $h \in [H_2^{\frac{1}{2}}(S)]^6$ *there holds the inequality*

$$\Re\,\langle(\pm\,\mathcal{A}^{\pm} + \mathcal{C}_0)h\,,\; h\rangle_S \geq C_1\,\|h\|^2_{[H_2^{\frac{1}{2}}(S)]^6}\,, \tag{4.69}$$

where $\mathcal{C}_0 \,:\, [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6$ *is a compact operator. The operator*

$$\mathcal{A}^- \,:\, [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,, \tag{4.70}$$

is continuously invertible, while

$$\mathcal{A}^+ \,:\, [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \tag{4.71}$$

is a Fredholm operator of index zero and with the null space of dimension two.

Proof. Mapping properties (4.70) and (4.71) follow from Theorem 4.3 and Corollary 4.9. With the help of Green's identities for the vector function $U = V\big(\mathcal{H}^{-1}h\big)$ with $h \in [H_2^{\frac{1}{2}}(S)]^6$ we get (see (2.40), (2.43), (2.79))

$$\big\langle\, \mathcal{A}^+ h\,,\, h\big\rangle_{\partial\Omega^+} = \int\limits_{\Omega^+} \mathcal{E}(U,\overline{U})\,dx\,, \tag{4.72}$$

$$-\big\langle\, \mathcal{A}^- h\,,\, h\big\rangle_{\partial\Omega^-} = \int\limits_{\Omega^-} \mathcal{E}(U,\overline{U})\,dx\,. \tag{4.73}$$

Applying the same reasoning as in the proof of Corollaries 4.8 and 4.9 we arrive at the inequalities (4.69) which in turn imply that the operators (4.70) and (4.71) are Fredholm and have zero index.

The null space of the operator (4.70) is trivial. Indeed, the homogeneous equation $\mathcal{A}^- h = 0$ corresponds to the exterior homogeneous Neumann type problem for the vector function $U = V\big(\mathcal{H}^{-1}h\big)$. Therefore by Theorem 2.2 we get $U = V\big(\mathcal{H}^{-1}h\big) = 0$ in Ω^-. Hence $h = 0$ follows. Thus the operator (4.70) is invertible.

Now we show that the null space $\ker\mathcal{A}^+$ of the operator (4.71) is two dimensional. Set

$$\Psi = b_1\,\Psi^{(1)} + b_2\,\Psi^{(2)}, \tag{4.74}$$

where b_1 and b_2 are arbitrary constants and

$$\Psi^{(1)} = (0,0,0,1,0,0)^\top, \quad \Psi^{(2)} = (0,0,0,0,1,0)^\top. \tag{4.75}$$

Consider the vector $U^{(\mathcal{N})} := V\big(\mathcal{H}^{-1}\Psi\big)$ in Ω^+. Since $[U^{(\mathcal{N})}]^+ = [V\big(\mathcal{H}^{-1}\Psi\big)]^+ = \Psi$ on $\partial\Omega^+$ and the interior Dirichlet problem possesses a unique solution we conclude that

$$U^{(\mathcal{N})} = V\big(\mathcal{H}^{-1}\Psi\big) = (0,0,0,b_1,b_2,0)^\top \ \text{ in } \ \Omega^+.$$

Therefore, $[\mathcal{T}(\partial,n)U^{(\mathcal{N})}]^+ \equiv \mathcal{A}^+\Psi = 0$ on S; hence it follows that $\dim\ker\mathcal{A}^+ \geq 2$, since $\Psi^{(1)}$ and $\Psi^{(2)}$ are linearly independent.

On the other hand, if $\mathcal{A}^+\psi = 0$ on S, then $[\mathcal{T}(\partial,n)V\big(\mathcal{H}^{-1}\psi\big)]^+ = 0$ on S, and by Theorem 2.1 we have $V\big(\mathcal{H}^{-1}\psi\big) = (0,0,0,b_1',b_2',0)^\top$ in Ω^+, where b_1' and b_2' are arbitrary constants. Consequently,

$$[V\big(\mathcal{H}^{-1}\psi\big)]^+ = \psi = (0,0,0,b_1',b_2',0)^\top = b_1'\,\Psi^{(1)} + b_2'\,\Psi^{(2)} \quad \text{on } \ \partial\Omega^+ \tag{4.76}$$

and $\dim\ker\mathcal{A}^+ \leq 2$. Therefore $\dim\ker\mathcal{A}^+ = 2$. Moreover, from (4.76) it follows that the null space $\ker\mathcal{A}^+$ is the linear span of the vectors (4.75). $\square$

Remark 4.13 *If S is a sufficiently smooth surface (C^∞ regular surface say), then for arbitrary $\tau \in \mathbb{C}$, the operators $\pm\mathcal{A}^{\pm}$ are strongly elliptic pseudodifferential operators of order 1, i.e., there is a positive constant C such that for all $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and $x \in S$*

$$\Re\{\mathfrak{S}\big(\pm\mathcal{A}^{\pm}; x, \xi_1, \xi_2)\,\zeta\cdot\zeta\} \geq C\,|\xi|\,|\zeta|^2\,, \tag{4.77}$$

which follow from equalities (4.72) and (4.73). Here $\mathfrak{S}\big(\pm\mathcal{A}^{\pm}; x, \xi_1, \xi_2\big)$ stand for the principal homogeneous symbol matrices of the operators $\pm\mathcal{A}^{\pm}$. With the help of the strong elipticity of the principal homogeneous symbol matrices $\mathfrak{S}\big(\pm\mathcal{A}^{\pm}; x, \xi_1, \xi_2\big)$ and Lemma 4.12, and applying the general theory of pseudodifferential operators on manifolds without boundary we infer that the operator

$$\mathcal{A}^- \;:\; [B^{s+1}_{p,q}(S)]^6 \to [B^{s}_{p,q}(S)]^6, \qquad s\in\mathbb{R},\;\; p>1,\;\; q\geq 1, \tag{4.78}$$

is invertible, while the operator

$$\mathcal{A}^+ \;:\; [B^{s+1}_{p,q}(S)]^6 \to [B^{s}_{p,q}(S)]^6, \qquad s\in\mathbb{R},\;\; p>1,\;\; q\geq 1, \tag{4.79}$$

is Fredholm of zero index and the corresponding two-dimensional null space $\ker\mathcal{A}^+$ *is a linear span of the vectors (4.75) as it is shown in the proof of Lemma 4.12.*

5 Investigation of basic BVPs of pseudo-oscillations

Throughout this section we assume that $\Re\tau = \sigma > 0$ and investigate the Dirichlet $(D)^{\pm}$, Neumann $(N)^{\pm}$ and mixed boundary value problems for the pseudo-oscillation equation (2.49). Note that with the help of the Newtonian volume potential $N_{\Omega^{\pm}}(\Phi)$ (see (3.50) and Remark 4.5) we can reduce the nonhomogeneous equation (2.49) to the homogeneous one. Therefore without loss of generality in what follows we consider the homogeneous differential equation (2.49) with $\Phi = 0$.

5.1 The interior Dirichlet BVPs: a regular case

We assume that

$$S = \partial\Omega^{\pm} \in C^{m,\kappa} \quad \text{with integer } \ m \geq 2 \ \text{ and } \ 0 < \kappa \leq 1, \tag{5.1}$$

$$g \in [C^{k,\kappa'}(S)]^6, \quad 0 < \kappa' < \kappa, \quad 1 \leq k \leq m-1, \tag{5.2}$$

and look for a solution to the interior Dirichlet problem (see Subsection 2.3.1, (2.49), (2.50)) in the form of double layer potential

$$U(x) = W(h)(x), \quad x \in \Omega^{+}, \tag{5.3}$$

where $h \in [C^{k,\kappa'}(S)]^6$ is an unknown density vector. By Theorem 4.1 and in view of the boundary condition (2.50) we get the following integral equation for the density vector function h:

$$[\,2^{-1}I_6 + \mathcal{N}\,]\,h = g \quad \text{on} \quad S. \tag{5.4}$$

Our goal is to prove that this integral equation is unconditionally solvable for an arbitrary right hand side vector function. To this end we prove the following assertion.

Theorem 5.1 *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the singular integral operator*

$$2^{-1}I_6 + \mathcal{N} \ : \ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \quad 0 < \kappa' < \kappa, \tag{5.5}$$

is continuously invertible.

Proof. The mapping property (5.5) follows from Theorem 4.3. With the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, by standard arguments (see, e.g., [NCS1], [Na1], [JN1]) we can show that $2^{-1}I_6 + \mathcal{N}$ is a singular integral operator with elliptic principal homogeneous symbol matrix $\mathfrak{S}\big(2^{-1}I_6 + \mathcal{N}; x, \xi_1, \xi_2\big)$, i.e., $\det \mathfrak{S}\big(2^{-1}I_6 + \mathcal{N}; x, \xi_1, \xi_2\big) \neq 0$ for all $x \in S$ and $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ (see the Appendix C).

Next we show that the index of the operator

$$2^{-1}I_6 + \mathcal{N} \ : \ [L_2(S)]^6 \to [L_2(S)]^6 \tag{5.6}$$

equals to zero. First we establish that the operator (5.6) is injective. Let $h_0 \in [L_2(S)]^6$ be a solution of the homogeneous equation $[2^{-1}I_6 + \mathcal{N}]\,h_0 = 0$ on S. Then by the embedding theorems (see, e.g., [KGBB], Ch.IV) we conclude that $h_0 \in [C^{k,\kappa'}(S)]^6$ and construct the double layer potential $U_0(x) := W(h_0)(x)$. Clearly, U_0 is a regular vector function of the class $[C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ due the Theorem 4.1 and solves the homogeneous interior Dirichlet problem. By the uniqueness Theorems 2.1 then U_0 vanishes in Ω^+ and in accordance with Theorem 4.1

$$[\,\mathcal{T}(\partial_x, n(x))\, W(h_0)(x)\,]^- = [\,\mathcal{T}(\partial_x, n(x))\, W(h_0)(x)\,]^+ = 0, \quad x \in S.$$

So, U_0 solves the exterior homogeneous Neumann type problem in the domain Ω^- and possesses the decay property (2.56) at infinity. Therefore, by the uniqueness Theorem 2.2, it follows that U_0 vanishes in the exterior domain Ω^- as well. But then in view of the jump relation (4.5) we finally conclude $\{W(h_0)(x)\}^+ - \{W(h_0)(x)\}^- = h_0(x) = 0$ on S, whence the injectivity of the operator (5.5) follows.

Further, we show that the null space of the adjoint operator

$$2^{-1} I_6 + \mathcal{N}^* \; : \; [L_2(S)]^6 \to [L_2(S)]^6 \tag{5.7}$$

is trivial as well. Indeed, let $h_0^* \in [L_2(S)]^6$ be a solution of the equation $[2^{-1} I_6 + \mathcal{N}^*]\, h_0^* = 0$ on S. Then, using again the the embedding theorems we conclude that $h_0^* \in [C^{k,\kappa'}(S)]^6$ and construct the single layer potential $U_0^*(x) := V^*(h_0^*)(x)$. Clearly, U_0^* is a regular vector function of the class $[C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ due the Theorem 4.2 and satisfies the decay conditions (2.56) at infinity. It is clear that U_0^* solves the exterior homogeneous Neumann type auxiliary BVP (see Subsection 2.5 and (4.19))

$$\begin{aligned} &A^*(\partial, \tau)\, U_0^* = 0 \quad \text{in} \quad \Omega^-, \\ &\big\{\, \mathcal{P}(\partial, n, \overline{\tau}) U_0^* \big\}^- = [2^{-1} I_6 + \mathcal{N}^*] h_0^* = 0 \qquad \text{on} \qquad S. \end{aligned} \tag{5.8}$$

Therefore by Theorem 2.5 we get $U_0^*(x) = V^*(h_0^*)(x) = 0$, $x \in \Omega^-$. Now by (4.18) we see that $U_0^* = V^*(h_0^*)$ is a solution to the interior homogeneous Dirichlet type auxiliary BVP (see Subsection 2.5)

$$\begin{aligned} &A^*(\partial, \tau)\, U_0^* = 0 \quad \text{in} \quad \Omega^+, \\ &\big\{\, U_0^* \big\}^+ = 0 \qquad \text{on} \qquad S. \end{aligned} \tag{5.9}$$

Hence $U_0^*(x) = V^*(h_0^*) = 0$ in Ω^+ and consequently, in view of jump relations (4.19), we finally get $h_0^* = 0$ on S. Thus the null spaces of the operators (5.6) and (5.7) are trivial and the index of the operator (5.6) equals to zero. Therefore, the operator (5.6) is invertible, which implies that the operator (5.5) is continuously invertible as well. $\square$

From the invertibility of the operator (5.5) the following existence result follows immediately.

Theorem 5.2 *Let S, m, κ, κ' and k be as in Theorem* 5.1. *Then the Dirichlet interior problem* (2.49), (2.50) *with $\Phi = 0$ and $g \in [C^{k,\kappa'}(S)]^6$ is uniquely solvable in the space $[C^{k,\kappa'}(\overline{\Omega^+})]^6$ and the solution is representable in the form of double layer potential* (5.3), *where the density vector h is defined by the singular integral equation* (5.4).

5.2 The exterior Dirichlet BVPs: a regular case

We again assume that the conditions (5.1), (5.2) are fulfilled and look for a solution to the exterior Dirichlet type BVP (see Subsection 2.3.1, (2.49), (2.50)) in the form of the linear combination of the single and double layer potentials

$$U(x) = W(h)(x) + \alpha V(h)(x), \quad x \in \Omega^-, \tag{5.10}$$

where $h \in [C^{k,\kappa'}(S)]^6$ is an unknown density vector and $\alpha > 0$ is a constant. By Theorem 4.1 and in view of the boundary condition (2.50) we get the following integral equation for the density vector function h:

$$[-2^{-1}I_6 + \mathcal{N} + \alpha\,\mathcal{H}]\,h = g \quad \text{on} \quad S. \tag{5.11}$$

Theorem 5.3 *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the singular integral operator*

$$-2^{-1}I_6 + \mathcal{N} + \alpha\,\mathcal{H} \ : \ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \quad 0 < \kappa' < \kappa, \tag{5.12}$$

is continuously invertible.

Proof. With the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, as in the previous case, by standard arguments (see, e.g., [NCS1], [Na1], [JN1]) we can show that $-2^{-1}I_6 + \mathcal{N} + \alpha\,\mathcal{H}$ is a singular integral operator with elliptic principal homogeneous symbol matrix $\mathfrak{S}\big(-2^{-1}I_6+\mathcal{N}; x,\xi_1,\xi_2\big)$, i.e., det $\mathfrak{S}\big(-2^{-1}I_6+\mathcal{N}; x,\xi_1,\xi_2\big) \neq 0$ for all $x \in S$ and $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$ (see the Appendix C). Note that the summand $\alpha\,\mathcal{H}$ in (5.12) is a compact perturbation of the operator $-2^{-1}I_6 + \mathcal{N}$.

Now we show that the index of the operator

$$-2^{-1}I_6 + \mathcal{N} + \alpha\,\mathcal{H} \ : \ [L_2(S)]^6 \to [L_2(S)]^6 \tag{5.13}$$

equals to zero. To this end let us consider the homogeneous equation on S

$$[-2^{-1}I_6 + \mathcal{N} + \alpha\,\mathcal{H}]\,h = 0. \tag{5.14}$$

By the embedding theorems we conclude that if $h_0 \in [L_2(S)]^6$ solves equation (5.14), then $h_0 \in [C^{k,\kappa'}(S)]^6$ and consequently the vector $U_0 = W(h_0) + \alpha V(h_0) \in [C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ is a regular solution of the homogenous equation $A(\partial,\tau)\,U_0 = 0$ in $\Omega^\pm$ satisfying the decay conditions (2.56) at infinity. In view of (5.14) we see that U_0 solves the exterior Dirichlet BVP and by the uniqueness Theorem 2.2 we have $U_0(x) = 0$, $x \in \Omega^-$. Due to the jump relations for the layer potentials (see Theorem 4.1), we then have $\{U_0\}^+ = h_0$ and $\{\mathcal{T}U_0\}^+ = -\alpha\,h_0$ on S, i.e.,

$$\{\mathcal{T}U_0\}^+ + \alpha\,\{U_0\}^+ = 0 \quad \text{on} \ \ S. \tag{5.15}$$

With the help of Green's formula (2.40) we get

$$\int\limits_{\Omega^+} \mathcal{E}(U_0,\overline{U'})\,dx + \alpha \int\limits_{\partial\Omega^+} \{U_0\}^+ \cdot \{U'\}^+\,dS = 0 \tag{5.16}$$

for arbitrary $U' \in \big[W_2^1(\Omega^+)\big]^6$. By the word for word arguments applied in the proof of Theorem 2.1 we conclude that $U_0(x) = 0$, $x \in \Omega^+$. Therefore $h_0 = 0$ on S and the null space of the operator (5.13) is trivial.

Quite similarly we can show that the null space of the adjoint operator

$$-2^{-1}I_6 + \mathcal{N}^* + \alpha\,\mathcal{H}^* \;:\; [L_2(S)]^6 \to [L_2(S)]^6 \tag{5.17}$$

is trivial. Indeed, if $h_0^* \in [L_2(S)]^6$ solves the homogeneous equation

$$[\,-2^{-1}I_6 + \mathcal{N}^* + \alpha\,\mathcal{H}^*]\,h_0^* = 0, \tag{5.18}$$

then $h_0^* \in [C^{k,\kappa'}(S)]^6$ by the embedding theorems and consequently the vector $U_0^* = V^*(h_0^*) \in [C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ is a regular solution of the homogenous equation $A^*(\partial,\tau)\,U_0^* = 0$ in $\Omega^\pm$ satisfying the decay conditions (2.56) at infinity. In view of (5.18) we find that U_0^* satisfies the Robin type BVP on S (see (4.19))

$$\big\{\,\mathcal{P}(\partial,n,\overline{\tau})U_0^*\big\}^+ + \alpha\big\{U_0^*\big\}^+ = [-2^{-1}I_6 + \mathcal{N}^* + \alpha\,\mathcal{H}^*]h_0^* = 0 \quad \text{on } S. \tag{5.19}$$

By Green's formula (2.41) we have

$$\int\limits_{\Omega^+} \mathcal{E}(U,\overline{U_0^*})dx + \int\limits_{\partial\Omega^+} \{U\}^+ \cdot \{U_0^*\}^+ dS = 0 \tag{5.20}$$

for arbitrary $U \in \big[W_2^1(\Omega^+)\big]^6$. By the same arguments as in the proof of the uniqueness Theorem 2.1 we derive that $U_0^*(x) = V^*(h_0^*)(x) = 0$, $x \in \Omega^+$. Since the single layer potential is continuous across the surface S (see (4.18)), we see that U_0^* solves the homogeneous exterior Dirichlet type BVP for the operator $A^*(\partial,\tau)$. Hence, with the help of Green's formulas (2.79)-(2.80), it follows that $U_0^*(x) = V^*(h_0^*)(x) = 0$, $x \in \Omega^-$. Due to the jump relations for the single layer potential (see (4.19)), we then have $h_0^* = 0$ on S, i.e., the null space of the adjoint operator (5.17) is trivial.

Thus, the operator (5.13) is injective and has the zero index. Consequently, it is continuously invertible. Then it follows that the operator (5.12) is continuously invertible as well. $\square$

This theorem leads to the following existence result for the exterior Dirichlet problem.

Theorem 5.4 *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the Dirichlet exterior problem* (2.49), (2.50), (2.56) *with* $\Phi = 0$ *and* $g \in [C^{k,\kappa'}(S)]^6$ *is uniquely solvable in the space of regular vector functions* $[C^{k,\kappa'}(\overline{\Omega^-})]^6$ *and the solution is representable in the form* (5.10), *where the density vector* h *is defined by the uniquely solvable singular integral equation* (5.11).

5.3 Single layer approach for the interior and exterior Dirichlet BVPs: a regular case

From the results of Subsection 4.2 we have the following existence results for the Dirichlet problems. We look for solutions to the interior and exterior Dirichlet BVPs (2.49), (2.50) with $\Phi = 0$ and $g \in [C^{k,\kappa'}(S)]^6$ in the form of single layer potential

$$U(x) = V(h)(x), \quad x \in \Omega^\pm, \tag{5.21}$$

where h is a solution to the following equation

$$\mathcal{H}\, h(x) = g(x), \quad x \in \partial\Omega^{\pm}. \tag{5.22}$$

Due to the results outlined in Remark 4.12 and under the conditions (5.1) and (5.2) we see that the operator

$$\mathcal{H} \;:\; [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6\,, \;\; 1 \le k \le m-1, \quad m \ge 2, \tag{5.23}$$

is a strongly elliptic pseudodifferential operator of order -1 with index zero. Since the null space of the operator (5.23) is trivial we conclude that it is continuously invertible and

$$\mathcal{H}^{-1} \;:\; [C^{k+1,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6\,. \tag{5.24}$$

This leads to the following existence results and representation formulas of solutions.

Theorem 5.5 *Let conditions* (5.1) *and* (5.2) *be fulfilled. Then the Dirichlet interior and exterior problems* (2.49), (2.50), (2.56) *with* $\Phi = 0$ *and* $g \in [C^{k,\kappa'}(S)]^6$ *is uniquely solvable in the space of regular vector functions* $[C^{k,\kappa'}(\overline{\Omega^{\pm}})]^6$ *and the solution is representable in the form* (5.21)*, where the density vector h is defined by the uniquely solvable pseudodifferential equation* (5.22)*.*

In the regular case under consideration, we have the following counterpart of Corollary 4.10.

Corollary 5.6 *Let conditions* (5.1) *be fulfilled and* $U \in [C^{1,\kappa'}(\overline{\Omega^{\pm}})]^6$ *be a solution to the homogeneous equation* $A(\partial,\tau)U = 0$ *in* $\Omega^{\pm}$*, satisfying the decay conditions* (2.56) *in the case of exterior domain* Ω^{-}*. Then U is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}[U]^{\pm}\big)(x), \quad x \in \Omega^{\pm}, \tag{5.25}$$

where $[U]^{\pm}$ *are the interior and exterior limiting values (traces) of U on S from* $\Omega^{\pm}$ *respectively.*

5.4 The interior and exterior Neumann BVPs: a regular case

Here we assume that

$$\partial\Omega^{\pm} = S \in C^{m,\kappa}, \quad m \ge 2, \quad 0 < \kappa \le 1, \tag{5.26}$$
$$G \in [C^{k,\kappa'}(S)]^6, \quad 0 \le k \le m-1, \quad 0 < \kappa' < \kappa, \tag{5.27}$$

and look for a solution of the interior Neumann BVP (2.49), (2.51) with $\Phi = 0$ in the form of single layer potential

$$U(x) = V(h)(x), \quad x \in \Omega^{+}, \tag{5.28}$$

where $h \in [C^{k,\kappa'}(S)]^6$ is an unknown density vector function. By Theorem 4.1 and in view of the boundary condition (2.51) we get the following integral equation for the density vector h:

$$[\,-2^{-1}I_6 + \mathcal{K}\,]\, h = G \quad \text{on} \quad S. \tag{5.29}$$

Theorem 5.7 *Let S and $G=(G_1,\cdots,G_6)^\top$ satisfy the conditions* (5.26) *and* (5.27).

(i) *The operator*

$$-2^{-1}I_6+\mathcal{K}\ :\ [L_2(S)]^6\to[L_2(S)]^6 \tag{5.30}$$

is a singular integral operator of normal type with zero index and has a two-dimensional null space $\Lambda(S):=\ker(-2^{-1}I_6+\mathcal{K})\subset[C^{m-1,\kappa'}(S)]^6$, which represents a linear span of the vector functions

$$h^{(1)}\in\Lambda(S),\qquad h^{(2)}\in\Lambda(S), \tag{5.31}$$

such that

$$V(h^{(1)})=\Psi^{(1)}:=(0,0,0,1,0,0)^\top\quad and\ \ V(h^{2)})=\Psi^{(2)}:=(0,0,0,0,1,0)^\top\quad in\ \ \Omega^+. \tag{5.32}$$

(ii) *The null space of the operator adjoint to* (5.30),

$$-2^{-1}I_6+\mathcal{K}^*\ :\ [L_2(S)]^6\to[L_2(S)]^6 \tag{5.33}$$

is a linear span of the vectors $(0,0,0,1,0,0)^\top$ and $(0,0,0,0,1,0)^\top$.

(iii) *Equation* (5.29) *is solvable if and only if*

$$\int_S G_4(x)\,dS=\int_S G_5(x)\,dS=0. \tag{5.34}$$

(iv) *If the conditions* (5.34) *hold, then solutions to equation* (5.29) *belong to the space $[C^{k,\kappa'}(S)]^6$ and are defined modulo a linear combination of the vector functions $h^{(1)}$ and $h^{(2)}$.*

(v) *If the conditions* (5.34) *hold, then the interior Neumann BVP is solvable and its solution is representable in the form of single layer potential* (5.28)*, where the density vector function h is defined by the singular integral equation* (5.29)*. A solutions to the interior Neumann BVP is defined in Ω^+ modulo a linear combination of the constant vector functions $\Psi^{(1)}$ and $\Psi^{(2)}$ given by* (5.32).

Proof. The mapping property (5.30) follows from Theorem 4.3. With the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, by standard arguments one can show that $-2^{-1}I_6+\mathcal{K}$ is a singular integral operator with elliptic principal homogeneous symbol matrix $\mathfrak{S}\big(-2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2\big)$, i.e., $\det\mathfrak{S}\big(-2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2\big)\neq 0$ for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ (see the Appendix C). Therefore, (5.29) is normally solvable ([MP], [KGBB]).

Further, we prove that the index of the operator (5.30) equals to zero. To this end let us consider the operator

$$-2^{-1}I_6+\mathcal{K}+\alpha\,\mathcal{H}\ :\ [L_2(S)]^6\to[L_2(S)]^6 \tag{5.35}$$

with $\alpha>0$. Clearly, the operator (5.35) is a compact perturbation of the operator (5.30) due to Theorem 4.4 since $\mathcal{H}:[L_2(S)]^6\to[H_2^1(S)]^6$ and $[H_2^1(S)]^6$ is compactly embedded in $[L_2(S)]^6$. One can easily show that the homogeneous equation

$$[-2^{-1}I_6+\mathcal{K}+\alpha\,\mathcal{H}]\,h_0=0\quad\text{on}\ \ S \tag{5.36}$$

has only the trivial solution in $[L_2(S)]^6$. Indeed, by the embedding theorem we have $h_0 \in [C^{1,\kappa'}(S)]^6$ and the vector $U_0 = V(h_0) \in [C^{2,\kappa'}(\overline{\Omega^\pm})]^6$ is a regular solution of the homogeneous equation $A(\partial,\tau)U_0 = 0$ in $\Omega^\pm$ and satisfies the following Robin type condition

$$\{\mathcal{T}U_0\}^+ + \alpha\,\{U_0\}^+ = 0 \quad \text{on} \;\; S.$$

Therefore, by Green's formula (2.40) we derive $U_0(x) = V(h_0)(x) = 0$ in Ω^+, and consequently, $h_0 = 0$, since $U_0 = V(h_0)$ possesses the decay conditions (2.56). Thus $\ker(-2^{-1}I_6 + \mathcal{K} + \alpha\,\mathcal{H}) = \{0\}$.

Now let us consider the adjoint homogeneous equation (see (4.10)-(4.11))

$$[-2^{-1}I_6 + \mathcal{K}^* + \alpha\,\mathcal{H}^*]\,h_0^* = 0 \quad \text{on} \;\; S. \tag{5.37}$$

Again by the embedding theorem we have that $h_0^* \in [C^{1,\kappa'}(S)]^6$ and the vector

$$U_0^* = W^*(h_0^*) + \alpha\,V^*(h_0) \in [C^{2,\kappa'}(\overline{\Omega^+})]^6 \tag{5.38}$$

is a regular solution to the homogeneous equation $A^*(\partial,\tau)U_0^* = 0$ in $\Omega^\pm$, satisfies the decay conditions of type (2.56) at infinity and the homogeneous Dirichlet condition $\{U_0^*\}^- = 0$ on S. Therefore $U_0^* = 0$ in Ω^- by Theorem 2.2. In view of (5.38) and the jump relations for the layer potentials involved in (5.38), then it follows that $\{\mathcal{P}U_0^*\}^+ + \alpha\,\{U_0^*\}^+ = 0$ on S since $\{\mathcal{P}U_0^*\}^+ - \{\mathcal{P}U_0^*\}^+ = -\alpha\,h_0^*$ and $\{U_0^*\}^+ - \{U_0^*\}^+ = h_0^*$. As in the proof of Theorem 5.3 with the help of formula (5.20) we derive that $U_0^* = 0$ in Ω^+ which implies $h_0^* = 0$ on S and consequently $\ker(-2^{-1}I_6 + \mathcal{K}^* + \alpha\,\mathcal{H}^*) = \{0\}$. Thus the index of the operator (5.35) is zero. The same conclusion holds true for the operator (5.30) due to the above mentioned compactness property of the operator $\mathcal{H}^*$. Note that the operator (5.35) is invertible.

Now we study that the null spaces of the operator (5.30) and its adjoint one

$$-2^{-1}I_6 + \mathcal{K}^* \;:\; [L_2(S)]^6 \to [L_2(S)]^6. \tag{5.39}$$

Clearly $\dim\ker(-2^{-1}I_6 + \mathcal{K}) = \dim\ker(-2^{-1}I_6 + \mathcal{K}^*)$.

From the integral representation formula (4.16) it follows that for the vector

$$\Psi = (0,0,0,b_1',b_2',0)^\top = b_1'\,\Psi^{(1)} + b_2'\,\Psi^{(2)}, \tag{5.40}$$

where b_1' and b_2' are arbitrary constants and vector functions $\Psi^{(1)}$ and $\Psi^{(2)}$ are given by (5.32), the following formula

$$\Psi = W^*\big(\Psi\big) \quad \text{in} \;\; \Omega^+ \tag{5.41}$$

holds, since $A^*(\partial,\tau)\Psi = 0$ in $\mathbb{R}^3$ and $\mathcal{P}(\partial,n,\overline{\tau})\Psi = 0$ for arbitrary n and $x \in \mathbb{R}^3$ (see Section 4). From (5.41) we get

$$[-2^{-1}I_6 + \mathcal{K}^*]\,\Psi = 0 \quad \text{on} \;\; S. \tag{5.42}$$

Hence $\Psi \in \ker(-2^{-1}I_6 + \mathcal{K}^*)$ which shows that $\dim\ker(-2^{-1}I_6 + \mathcal{K}^*) \geq 2$. On the other hand, it is clear that if $\Phi \in \ker(-2^{-1}I_6 + \mathcal{K}) \equiv \Lambda(S)$, then $(-2^{-1}I_6 + \mathcal{K})\Phi = 0$ on S which is equivalent to the relation $\{\mathcal{T}V(\Phi)\}^+ = 0$ on S. Therefore $V(\Phi) = (0,0,0,b_1,b_2,0)^\top$ in Ω^+ with arbitrary constants b_1 and b_2 due to Theorem 2.1, i.e., $V(\Phi) = b_1\,\Psi^{(1)} + b_2\,\Psi^{(2)}$, where $\Psi^{(1)}$ and $\Psi^{(2)}$ are given by (5.32). Since the operator (5.23) is invertible there are vector functions $h^{(1)} \in \Lambda(S)$ and $h^{(2)} \in \Lambda(S)$ such that

$$\mathcal{H}\,h^{(j)} = \Psi^{(j)} \quad \text{on} \;\; S, \qquad h^{(j)} \in [C^{m-1,\kappa'}(S)]^6, \quad j=1,2,$$

which in view of the uniqueness theorem for the interior Dirichlet problem lead to the equalities

$$V(h^{(j)}) = \Psi^{(j)} \quad \text{in} \;\; \Omega^+ \quad j=1,2.$$

In turn these formulas imply that

$$V(\Phi) = b_1\,V(h^{(1)}) + b_2\,V(h^{(2)}) \quad \text{in} \;\; \Omega^+, \qquad \Phi = b_1\,\mathcal{H}^{-1}\Psi^{(1)} + b_2\,\mathcal{H}^{-1}\Psi^{(2)} \quad \text{on} \;\; S. \qquad (5.43)$$

Therefore $\dim\ker(-2^{-1}I_6 + \mathcal{K}) \le 2$ since the vector functions $h^{(1)} := \mathcal{H}^{-1}\Psi^{(1)}$ and $h^{(2)} := \mathcal{H}^{-1}\Psi^{(2)}$ are linearly independent. Consequently we finally get

$$\dim\ker(-2^{-1}I_6 + \mathcal{K}) = \dim\ker(-2^{-1}I_6 + \mathcal{K}^*) = 2,$$

and the vector functions $h^{(1)}$ and $h^{(2)}$ represent the basis of the null space $\Lambda(S)$, while the null space $\ker(-2^{-1}I_6 + \mathcal{K}^*)$ represents a linear span of the vector functions $\Psi^{(1)}$ and $\Psi^{(2)}$. From the above arguments the items (i) and (ii) of the theorem follow.

It is clear that the necessary and sufficient conditions for the integral equation (5.29) to be solvable reads then as (5.34) which proves the item (iii).

The item (iv) follows then from the embedding theorems (see, e.g., [KGBB], Ch. IV), while the item (v) is a direct consequence of items (i)-(iv). □

The exterior Neumann BVP (2.49), (2.51) with $\Phi = 0$ and G as in (5.27) can be studied quite similarly. Indeed, if we look for a solution again in the form of single layer potential

$$U(x) = V(h)(x), \quad x \in \Omega^-, \qquad (5.44)$$

we arrive at the following singular integral equation for the sought for density vector function h

$$[\,2^{-1}I_6 + \mathcal{K}\,]\,h = G \quad \text{on} \quad S. \qquad (5.45)$$

Theorem 5.8 *Let S and $G = (G_1,\cdots,G_6)^\top$ satisfy the conditions* (5.26) *and* (5.27).

(i) *The operators*

$$2^{-1}I_6 + \mathcal{K} \;\; : \;\; [L_2(S)]^6 \to [L_2(S)]^6, \qquad (5.46)$$

$$: \;\; [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \;\; 0 \le k \le m-1, \qquad (5.47)$$

are singular integral operators of normal type with zero index and have the trivial null spaces.

(ii) *Operators* (5.46) *are invertible; equation* (5.45) *is uniquely solvable in the space* $[C^{k,\kappa'}(S)]^6$.

(iii) *The exterior Neumann BVP is uniquely solvable and the solution is representable in the form of single layer potential* (5.44), *where the density vector function h is defined by the singular integral equation* (5.45).

Proof. Again, with the help of the strong ellipticity property of the differential operator $A(\partial,\tau)$, by standard arguments one can show that $2^{-1}I_6+\mathcal{K}$ is a singular integral operator of normal type ([MP], [KGBB]) with elliptic principal homogeneous symbol matrix $\mathfrak{S}\big(2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)$, i.e., det $\mathfrak{S}\big(2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2)\neq 0$ for all $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ (see the Appendix C).

Now we show that the operator (3.34) and its adjoint one have trivial null spaces. Let $h_0\in[L_2(S)]^6$ be a solution to the homogeneous equation $[2^{-1}I_6+\mathcal{K}]h_0=0$ on S. Then by embedding theorems we conclude that $h_0\in[C^{m-1,\kappa'}(S)]^6$ and consequently the single layer potential $U_0(x)=V(h_0)(x)$ is a regular vector function of the class $[C^{m,\kappa'}(\overline{\Omega^-})]^6$ which solves the homogeneous exterior Neumann BVP. Therefore, $U_0=V(h_0)=0$ in Ω^- by Theorem 2.2. Due to continuity of the single layer potential we see that $U_0=V(h_0)$ solves then the homogeneous interior Dirichlet BVP in Ω^+ and by Theorem 2.1 $U_0=V(h_0)$ vanishes identically in Ω^+. In view of jump formulas (4.3) we arrive at the equation $[\mathcal{T}V(h_0)]^- - [\mathcal{T}V(h_0)]^+=h_0=0$ on S implying that ker $[2^{-1}I_6+\mathcal{K}]$ is trivial.

Now, let $h_0^*\in[L_2(S)]^6$ be a solution to the adjoint homogeneous equation $[2^{-1}I_6+\mathcal{K}^*]h_0^*=0$ on S. Then by embedding theorems we conclude that $h_0^*\in[C^{m-1,\kappa'}(S)]^6$ and consequently the double layer potential $U_0^*(x)=W^*(h_0^*)(x)$ is a regular vector function of the class $[C^{m-1,\kappa'}(\overline{\Omega^+})]^6$ which solves the homogeneous interior Dirichlet problem (2.107), (2.108) for the adjoint operator $A^*(\partial,\tau)$. Therefore, $U_0^*=W^*(h_0^*)=0$ in Ω^+ by Theorem 2.2. Since $[\mathcal{P}W^*(h_0^*)]^-=[\mathcal{P}W^*(h_0^*)]^+=0$ on S, by Theorem 2.5 we get $U_0^*=W^*(h_0^*)=0$ in Ω^-. Hence, in accordance with the jump relations (4.20) we finally derive $[W^*(h_0^*)]^+-[W^*(h_0^*)]^-=h_0$ on S, implying that ker $[2^{-1}I_6+\mathcal{K}^*]$ is trivial.

From these results the items (i), (ii) and (iii) follow immediately. $\square$

5.5 Double layer approach for the interior and exterior Neumann BVPs: a regular case

Let conditions (5.26) and (5.27) be satisfied with $1\leq k\leq m-1$ and look for a solution of the interior and exterior Neumann BVPs (2.49), (2.51) with $\Phi=0$ in the form of double layer potential

$$U(x)=W(h)(x),\quad x\in\Omega^{\pm},\tag{5.48}$$

where $h\in[C^{k,\kappa'}(S)]^6$ is an unknown density vector function. By Theorem 4.1 and in view of the boundary conditions (2.51) we get the following integral equation for the density vector h:

$$\mathcal{L}\,h=G\quad\text{on}\quad S.\tag{5.49}$$

The mapping properties of the operator $\mathcal{L}$ is described in Theorems 4.3 and 4.4. Due to the equalities

$$\begin{aligned}&\mathcal{H}\,\mathcal{L}=-4^{-1}I_6+\mathcal{N}^{\,2}, &&\mathcal{L}\,\mathcal{H}=-4^{-1}I_6+\mathcal{K}^{\,2},\\ &\mathcal{L}^*\mathcal{H}^*=-4^{-1}I_6+[\mathcal{N}^*]^{\,2}, &&\mathcal{H}^*\mathcal{L}^*=-4^{-1}I_6+[\mathcal{K}^*]^{\,2},\end{aligned}\tag{5.50}$$

we see that

$$\ker\mathcal{L}=\ker\,(-2^{-1}I_6+\mathcal{N}),\qquad \ker\mathcal{L}^*=\ker\,(-2^{-1}I_6+\mathcal{K}^*).\tag{5.51}$$

Now we show that the null spaces of the operators $\mathcal{L}$ and $\mathcal{L}^*$ are the same and coincide with the linear span of the vectors $\Psi^{(1)} = (0,0,0,1,0,0)^\top$ and $\Psi^{(2)} = (0,0,0,0,1,0)^\top$ (see (5.32)).

From the integral representation formula (3.51) and Theorem 4.1 it follows that $\Psi^{(1)}$ and $\Psi^{(2)}$ are linearly independent solutions of the homogeneous equation $[-2^{-1}\, I_6 + \mathcal{N}]h = 0$ on S, since for the vector

$$\Psi = (0,0,0,b_1,b_2,0)^\top = b_1\, \Psi^{(1)} + b_2\, \Psi^{(2)}, \tag{5.52}$$

where b_1 and b_2 are arbitrary constants, we have $A(\partial,\tau)\Psi = 0$ in $\mathbb{R}^3$ and $\mathcal{T}(\partial,n)\Psi = 0$ for arbitrary n and $x \in \mathbb{R}^3$. Consequently, in view of (3.51), the following formula

$$\Psi = W\big(\Psi\big) \quad \text{in } \ \Omega^+ \tag{5.53}$$

holds which implies $[-2^{-1}I_6 + \mathcal{N}]\Psi = 0$ on S. Hence $\Psi \in \ker\big(-2^{-1}I_6 + \mathcal{N}\big)$ which shows that $\dim\ker\big(-2^{-1}I_6 + \mathcal{N}\big) \geq 2$. On the other hand, it is clear that if $\Phi \in \ker\big(-2^{-1}I_6 + \mathcal{N}\big)$, then $\Phi \in [C^{m-1,\kappa'}(S)]^6$ and $(-2^{-1}I_6 + \mathcal{N})\Phi = 0$ on S which is equivalent to the relation $\{W(\Phi)\}^- = 0$ on S. Therefore $W(\Phi) = 0$ in Ω^- due to Theorem 2.1 and $[\mathcal{T}(\partial,n)W(\Phi)]^+ = [\mathcal{T}(\partial,n)W(\Phi)]^- = 0$ by Theorem 4.1. In accordance with Theorem 2.1 the double layer potential $W(\Phi)$, as a solution to the interior homogeneous Neumann BVP in Ω^+, belongs to the linear span of the vectors $\Psi^{(1)}$ and $\Psi^{(2)}$, i.e., $W(\Phi) = c_1\, \Psi^{(1)} + c_2\, \Psi^{(2)}$ in Ω^+ with some constants c_1 and c_2. By the jump relations we derive $\Phi = [W(\Phi)]^+ - [W(\Phi)]^- = [W(\Phi)]^+ = c_1\, \Psi^{(1)} + c_2\, \Psi^{(2)}$ on S. Thus $\ker\mathcal{L}$ represents the linear span of the vectors $\Psi^{(1)}$ and $\Psi^{(2)}$.

By Theorem 5.7 the same holds for the null space of the operator $[-2^{-1}I_6 + \mathcal{K}^*]$. Therefore

$$\begin{aligned} \ker\mathcal{L} &= \ker\big(-2^{-1}I_6 + \mathcal{N}\big) = \ker\mathcal{L}^* = \ker\big(-2^{-1}I_6 + \mathcal{K}^*\big) \\ &= \big\{c_1\, \Psi^{(1)} + c_2\, \Psi^{(2)},\ c_1, c_2 \in \mathbb{C}\big\} \end{aligned} \tag{5.54}$$

with $\Psi^{(1)}$ and $\Psi^{(2)}$ defined by (5.32). Consequently, the index of the operator $\mathcal{L}$ equals to zero.

Due to invertibility of the operator (5.24) we have the representation

$$\mathcal{L} = \mathcal{H}^{-1}\big(-4^{-1}I_6 + \mathcal{N}^{\,2}\big). \tag{5.55}$$

Taking into account that the principal homogeneous symbol matrices of the pseudodifferential operators $\pm 2^{-1}\, I_6 + \mathcal{N}$, $\pm 2^{-1}\, I_6 + \mathcal{K}$ and $\mathcal{H}$ are elliptic, we infer that $\mathcal{L}$ is an elliptic pseudodifferential operator of order $+1$ with the principal homogeneous symbol matrix

$$\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2) = \big[\mathfrak{S}\big(\mathcal{H};x,\xi_1,\xi_2\big)\big]^{-1}\mathfrak{S}(-4^{-1}I_6 + \mathcal{N}^{\,2};x,\xi_1,\xi_2) \tag{5.56}$$

for all $x \in S$ and $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$.

Note that the entries of the matrix $\mathfrak{S}\big(\mathcal{H};x,\xi_1,\xi_2\big)$ are even and homogeneous functions of order -1 (see (4.63)), while the entries of the matrix $\mathfrak{S}\big(\mathcal{N};x,\xi_1,\xi_2\big)$ are odd functions of zero order sine they represent the Fourier transforms of odd singular kernel functions. Therefore, from (5.56) we conclude that $\det\mathfrak{S}\big(\mathcal{L};x,\xi_1,\xi_2\big) \neq 0$ for all $x \in S$ and $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$, and $\mathfrak{S}\big(\mathcal{L};x,\xi_1,\xi_2\big)$ is even and homogeneous of order $+1$ matrix function in $\xi' = (\xi_1,\xi_2)$.

Further, let us show that the symbol matrix $\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)=\mathfrak{S}(\mathcal{L}^{(0)};x,\xi_1,\xi_2)$ is strongly elliptic. To this end we recall that the operator $\mathcal{L}^{(0)}\,:\,[H_2^{\frac{1}{2}}(S)]^6\to[H_2^{-\frac{1}{2}}(S)]^6$ is introduced in Subsection 4.2 and that the operator $\mathcal{L}-\mathcal{L}^{(0)}\,:\,[H_2^{\frac{1}{2}}(S)]^6\to[H_2^{-\frac{1}{2}}(S)]^6$ is compact.

Formulas (4.46) and (4.47) with $U^{(0)}=(u^{(0)},\varphi^{(0)},\psi^{(0)},\vartheta^{(0)})^\top:=W^{(0)}(g)$ for U imply (see Subsection 4.2)

$$\big\langle\,\mathcal{L}^{(0)}\,g\,,\,g\,\big\rangle_S=\int\limits_{\Omega^+\cup\Omega^-}\mathcal{E}(U^{(0)},\overline{U^{(0)}})\,dx \tag{5.57}$$

for arbitrary $g\in[C^{1,\alpha}(S)]^6$. Note that $W^{(0)}(g)\in[H_2^1(\Omega^\pm)]^6$, but $W^{(0)}(g)\notin[H_2^1(\mathbb{R}^3)]^6$ if $g\neq0$. With the help of (4.48) and (5.57), using the Korn inequalities for $\Omega^\pm$ (see [KO1]), the trace theorem and the jump relations for the double layer potential $U^{(0)}=W^{(0)}(g)$, we derive the following Gårding type inequality

$$\Re\,\big\langle\,\mathcal{L}^{(0)}\,g\,,\,g\,\big\rangle_S\geq\int\limits_{\Omega^+\cup\Omega^-}\big\{\,\varepsilon_{kj}^{(0)}\,\overline{\varepsilon_{kj}^{(0)}}+|\nabla\varphi^{(0)}|^2+|\nabla\psi^{(0)}|^2+|\nabla\vartheta^{(0)}|^2\big\}\,dx \tag{5.58}$$

$$\begin{aligned}
&\geq C_1\left(||U||^2_{[H_2^1(\Omega^+)]^6}+||U||^2_{[H_2^1(\Omega^-)]^6}\right)-C_2\,||U^{(0)}||^2_{[L_2(\Omega^+)]^6}\\
&\geq C_3\left(||\{U^{(0)}\}^+||^2_{[H_2^{1/2}(S)]^6}+||\{U^{(0)}\}^-||^2_{[H_2^{1/2}(S)]^6}\right)-C_4\,||U^{(0)}||^2_{[L_2(\Omega^+)]^6}\\
&\geq C_3\,||\{U^{(0)}\}^+-\{U^{(0)}\}^-||^2_{[H_2^{1/2}(S)]^6}-C_4\,||U^{(0)}||^2_{[L_2(\Omega^+)]^6}\\
&\geq C_5\,||g||^2_{[H_2^{1/2}(S)]^6}-C_6\,||g||^2_{[H_2^{-1/2}(S)]^6}
\end{aligned} \tag{5.59}$$

where C_j, $j=\overline{1,6}$, are some positive constants.

Next, we consider unbounded half-spaces $\mathbb{R}^3_+(n):=\{x_1n_1+x_2n_2+x_3n_3<0\}$ and $\mathbb{R}^3_-(n):=\{x_1n_1+x_2n_2+x_3n_3>0\}$ instead of Ω^+ and Ω^- respectively and assume that n is the unit "outward" normal vector to the plane $S_n:=\{x_1n_1+x_2n_2+x_3n_3=0\}$ with respect to $\mathbb{R}^3_+(n)$. Further, let us note that the double layer potential $U^{(0)}=W^{(0)}(g)$ with the unbounded integration surface S_n and the density g being an arbitrary rapidly decreasing vector function of the Schwartz space, decays at infinity as $\mathcal{O}(|x|^{-2})$. Moreover, $\partial^\alpha W^{(0)}(g)(x)=\mathcal{O}(|x|^{-2-|\alpha|})$ as $|x|\to\infty$ for arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ due to the homogeneity property of the fundamental matrix $\Gamma^{(0)}(x)$ given by (3.5) and since $g(\widetilde{x})$ decays at infinity faster than any negative power of $|\widetilde{x}|$, $\widetilde{x}\in S_n$. Therefore, $\partial^\alpha W^{(0)}(g)\in[L_2(\mathbb{R}^3_\pm(n))]^6$ for $|\alpha|\geq0$ and with the help of the Korn inequalities for unbounded domains $\mathbb{R}^3_\pm(n)$ (see [KO1]), one can show a counterpart of formula (5.57) which yields the following relation (cf.

(5.58))

$$
\begin{aligned}
\Re\,\big\langle\,\mathcal{L}^{(0)}\,g\,,\,g\,\big\rangle_{S_n} &= \Re\int\limits_{S_n}\mathcal{L}^{(0)}\,g\cdot g\,dS_n = \Re\int\limits_{\mathbb{R}^3_+(n)\cup\mathbb{R}^3_-(n)}\mathcal{E}(U^{(0)},\overline{U^{(0)}})\,dx \\
&\geq \int\limits_{\mathbb{R}^3_+(n)\cup\mathbb{R}^3_-(n)}\big\{\,\varepsilon^{(0)}_{kj}\,\overline{\varepsilon^{(0)}_{kj}}+|\nabla\varphi^{(0)}|^2+|\nabla\psi^{(0)}|^2+|\nabla\vartheta^{(0)}|^2\big\}\,dx \qquad (5.60)\\
&\geq C_1^*\Big(\,||U^{(0)}||^2_{[H^1_2(\mathbb{R}^3_+(n))]^6}+||U^{(0)}||^2_{[H^1_2(\mathbb{R}^3_-(n))]^6}\Big)\\
&\geq C_2^*\Big(\,||\{U^{(0)}\}^+||^2_{[H^{1/2}_2(S_n)]^6}+||\{U^{(0)}\}^-||^2_{[H^{1/2}_2(S_n)]^6}\Big)\\
&\geq C_2^*\,||\{U^{(0)}\}^+-\{U^{(0)}\}^-||^2_{[H^{1/2}_2(S_n)]^6}\\
&\geq C_3^*\,||g||^2_{[H^{1/2}_2(S_n)]^6}\,, \qquad (5.61)
\end{aligned}
$$

where C_j^*, $j=\overline{1,3}$, are some positive constants. Now, let us take into account that $\mathcal{L}^{(0)}$ is a convolution operator and perform an orthogonal transform $x=B(n)\,x'$ of the half-spaces $\mathbb{R}^3_\pm(n)$ onto the usual standard half-spaces $\mathbb{R}^3_\pm:=\{x'\in\mathbb{R}^3\,:\,\pm x_3'\geq 0\}$ having the boundary $S=\mathbb{R}^2:=\{x'\in\mathbb{R}^3\,:\,x_3'=0\}$. Here $B(n)$ is an orthogonal matrix given by (4.61) where $n=\big(n_1,n_2,n_3\big)$, $l'(x)=\big(l_1',l_2',l_3'\big)$ and $l''=\big(l_1'',l_2'',l_3''\big)$ are mutually orthogonal constant unit vectors. Applying the Parseval equality we then easily deduce that the corresponding homogeneous symbol matrix $\mathfrak{S}\,(\mathcal{L}^{(0)};\xi_1,\xi_2)$ is strongly elliptic, i.e., there is a positive constant c such that

$$
\Re\,[\,\mathfrak{S}\,(\mathcal{L}^{(0)};\xi_1,\xi_2)\,\zeta\cdot\zeta\,]=\Re\,[\,\mathfrak{S}\,(\mathcal{L};\xi_1,\xi_2)\,\zeta\cdot\zeta\,]\geq c\,|\xi'|\,|\zeta|^2 \qquad (5.62)
$$

for arbitrary normal vector n and for all $\zeta\in\mathbb{C}^6$, $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$, and $x\in S_n$ (see the Appendix C). The constant c depends only on the material parameters.

Thus we have proved the following

Lemma 5.9 *Let condition* (5.26) *be satisfied and* $\tau=\sigma+i\omega$ *with* $\sigma>0$ *and* $\omega\in\mathbb{R}$. *Then there is a positive constant* C_1 *such that for all* $g\in[H_2^{\frac{1}{2}}(S)]^6$ *there holds the inequality*

$$
\Re\,\langle(\mathcal{L}+\mathcal{C}_0)g\,,\,g\rangle_S\geq C_1\,\|g\|^2_{[H_2^{\frac{1}{2}}(S)]^6}\,, \qquad (5.63)
$$

where $\mathcal{C}_0\,:\,[H_2^{\frac{1}{2}}(S)]^6\to[H_2^{-\frac{1}{2}}(S)]^6$ *is a compact operator. The operator*

$$
\mathcal{L}\,:\,[H_2^{\frac{1}{2}}(S)]^6\to[H_2^{-\frac{1}{2}}(S)]^6\,, \qquad (5.64)
$$

is a strongly elliptic pseudodifferential operator of index zero and the corresponding two dimensional null space is defined by (5.54).

Equation (5.49) *with* $G \in [C^{k,\kappa'}(S)]^6$ *is solvable if and only if*

$$\int_S G_4(x)\, dS = \int_S G_5(x)\, dS = 0 \tag{5.65}$$

and solution $h \in [C^{k,\kappa'}(S)]^6$ *is defined modulo a vector summand given by* (5.52).

From Lemma 5.9 one can derive the corresponding existence results and representability of solutions to the Neumann BVPs by double layer potentials.

Theorem 5.10 *Let* S *and* $G = (G_1, \cdots, G_6)^\top$ *satisfy the conditions* (5.26) *and* (5.27).

(i) *If conditions* (5.65) *hold, then the interior Neumann BVP is solvable in the space of vector functions* $[C^{k,\kappa'}(\overline{\Omega^+})]^6$ *and its solutions are representable in the form of double layer potential* (5.48), *where the density vector function* h *is defined by the pseudodifferential equation* (5.49) *and* $h \in [C^{k,\kappa'}(S)]^6$ *is defined modulo a vector summand given by* (5.52). *A solutions to the interior Neumann BVP for the domain* Ω^+ *is defined modulo a linear combination of the constant vector functions* $\Psi^{(1)} = (0,0,0,1,0,0)^\top$ *and* $\Psi^{(2)} = (0,0,0,0,1,0)^\top$.

(ii) *If conditions* (5.65) *hold, then the exterior Neumann BVP is solvable in the space of functions* $[C^{k,\kappa'}(\overline{\Omega^-})]^6$ *satisfying the decay conditions* (2.56) *at infinity and its solution is representable in the form of double layer potential* (5.48), *where the density vector function* h *is defined by the pseudodifferential equation* (5.49) *and* $h \in [C^{k,\kappa'}(S)]^6$ *is defined modulo a vector summand given by* (5.52). *A solution to the exterior Neumann BVP for the domain* Ω^- *is uniquely defined since the double layer potentials* $W(\Psi^{(j)})$, $j = 1, 2$, *vanish identically in* Ω^-.

Remark 5.11 *Note that if we seek a solution to the exterior Neumann BVP in the form of linear combination of the single and double layer potentials*

$$U(x) = W(h)(x) + \alpha\, V(h)(x), \quad x \in \Omega^-, \quad \alpha = const > 0, \tag{5.66}$$

we arrive at the equation (see Theorem 4.1, (4.7), (4.9)*)*

$$\mathcal{L}h + \alpha\,[2^{-1}I_6 + \mathcal{K}]h = G \quad on \;\; S. \tag{5.67}$$

It can be shown that the operator

$$\mathcal{L} + \alpha\,[2^{-1}I_6 + \mathcal{K}] \;:\; [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,, \tag{5.68}$$

is invertible. Indeed, since the index of the operator (5.68) *is zero by Lemma 5.9, it suffices to prove that the corresponding null space is trivial. Let* $\mathcal{L}h_0 + \alpha\,[2^{-1}I_6 + \mathcal{K}]h_0 = 0$ *on* S. *Then* $h_0 \in [C^{k,\kappa'}(S)]^6$ *and the regular vector* $U_0 = W(h_0) + \alpha\, V(h_0) \in [C^{k,\kappa'}(\overline{\Omega^-})]^6$ *solves the homogeneous exterior Neumann BVP. In view of the uniqueness Theorem 2.2 then* $U_0 = 0$ *in* Ω^-. *Due to the jump relations we then get that* U_0 *solves the homogeneous Robin type BVP in* Ω^+,

$$\{\mathcal{T}U_0\}^+ + \alpha\,\{U_0\}^+ = 0 \quad on \;\; S. \tag{5.69}$$

As we have shown in the proof of Theorem 5.3 this problem has only the trivial solution, i.e., $U_0 = 0$ *in* Ω^+. *Therefore* $h_0 = \{U_0\}^+ - \{U_0\}^- = 0$ *on* S *and consequently the null space the operator* (5.68) *is trivial.*

Thus (5.68) *is invertible and equation* (5.67) *is uniquely solvable. This proves that a unique solution to the exterior Neumann BVP can be represented in the form* (5.66) *with the density* $h \in [C^{k,\kappa'}(S)]^6$ *defined by the pseudodifferential equation* (5.67).

5.6 The interior and exterior Dirichlet and Neumann BVPs in Bessel potential and Besov spaces

If not otherwise stated, throughout this subsection we assume that

$$S \in C^\infty, \quad p > 1, \ \ q \geq 1, \ \ s \in \mathbb{R}. \tag{5.70}$$

Applying the general theory of pseudodifferential equations on manifolds without boundary (see, e.g., [Esk1], [Sh1], [Grb1]), we can generalize the existence results obtained in the previous subsections to more wide classes of boundary data. In particular, using Theorem 4.4 and the fact that the null spaces of strongly elliptic pseudodifferential operators acting in Bessel potential $H^s_p(S)$ and Besov $B^s_{p,q}(S)$ spaces actually do not depend on the parameters s, p, and q, we arrive at the following existence theorems.

Theorem 5.12 *Let condition* (5.70) *be fulfilled and* $g \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6$. *Then the pseudodifferential operator*

$$2^{-1} I_6 + \mathcal{N} \ : \ [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \to [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \tag{5.71}$$

is continuously invertible and the interior Dirichlet BVP (2.49),(2.50) *with* $\Phi = 0$ *is uniquely solvable in the space* $[H^1_p(\Omega^+)]^6$ *and the solution is representable in the form of double layer potential* $U = W(h)$ *with the density vector function* $h \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6$ *being a unique solution of the equation*

$$[\,2^{-1} I_6 + \mathcal{N}\,]\, h = g \quad \text{on} \quad S. \tag{5.72}$$

Proof. The invertibility of the operator (5.71) immediately follows from the invertibility of the operator (5.6). Hence equation (5.72) is uniquely solvable and it is easy to see that the vector function $U = W(h)$ is a solution to the interior Dirichlet BVP. It remains to show that the homogenous interior Dirichlet BVP possesses only the trivial solution in the space $[H^1_p(\Omega^+)]^6$. Let $U_0 \in [H^1_p(\Omega^+)]^6$ be a solution to the homogenous interior Dirichlet BVP. Due to the general integral representation formula (3.51) we then get

$$U_0 = -V(\{\mathcal{T} U_0\}^+) \ \ \text{in} \ \ \Omega^+\,, \tag{5.73}$$

where $\{\mathcal{T} U_0\}^+ \in [B^{-\frac{1}{p}}_{p,p}(S)]^6$. In view of the homogeneous Dirichlet condition on S

$$\{U_0\}^+ = -\mathcal{H}\,\{\mathcal{T} U_0\}^+ = 0 \ \ \text{on} \ \ S. \tag{5.74}$$

But the operator

$$\mathcal{H} \ : \ [B_{p,p}^{-\frac{1}{p}}(S)]^6 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^6 \tag{5.75}$$

is invertible, since for a particular value of the parameter $p = 2$ it is invertible (see Corollary 4.9 and (4.59)). Therefore (5.74) and (5.73) yields $U_0 = 0$ in Ω^+. $\square$

Quite similarly one can prove the following assertions.

Theorem 5.13 *Let condition* (5.70) *be fulfilled and* $g \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$. *Then the Dirichlet exterior problem* (2.49), (2.50), (2.56) *with* $\Phi = 0$ *is uniquely solvable in the space* $[H^1_{p,\,loc}(\Omega^-)]^6$ *and the solution is representable in the form* $U = W(h) + \alpha V(h)$, *where the density vector function* h *is defined by the uniquely solvable pseudodifferential equation*

$$[\,-2^{-1}I_6 + \mathcal{N} + \alpha\,\mathcal{H}]\,h = g \quad \text{on} \quad S. \tag{5.76}$$

Theorem 5.14 *Let a vector function* $U \in [H^1_p(\Omega^\pm)]^6$ *solve the homogeneous differential equation* $A(\partial,\tau)U = 0$ *in* $\Omega^\pm$. *Then it is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}[U]^+\big)(x), \quad x \in \Omega^\pm, \tag{5.77}$$

where $[U]^\pm \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$ *are the traces of* U *on* S *from* $\Omega^\pm$ *and* $\mathcal{H}^{-1}$ *stands for the operator inverse to* (5.75).

Analogous propositions hold true for the interior and exterior Neumann BVPs. In fact, one can prove the following counterparts of Theorems 5.7 and 5.8.

Theorem 5.15 *Let* (5.70) *be fulfilled and* $G = (G_1, \cdots, G_6)^\top \in [B_{p,p}^{-\frac{1}{p}}(S)]^6$.

(i) *The operator*

$$-2^{-1}I_6 + \mathcal{K} \ : \ [B_{p,p}^{-\frac{1}{p}}(S)]^6 \to [B_{p,p}^{-\frac{1}{p}}(S)]^6 \tag{5.78}$$

is an elliptic pseudodifferential operator with zero index and has a two-dimensional null space $\Lambda(S) := \ker(-2^{-1}I_6 + \mathcal{K}) \subset [C^\infty(S)]^6$, *which represents a linear span of the vector functions*

$$h^{(1)} \in \Lambda(S), \quad h^{(2)} \in \Lambda(S), \tag{5.79}$$

such that

$$V(h^{(1)}) = \Psi^{(1)} := (0,0,0,1,0,0)^\top \quad \textit{and} \quad V(h^{2)}) = \Psi^{(2)} := (0,0,0,0,1,0)^\top \quad \textit{in} \quad \Omega^+. \tag{5.80}$$

(ii) *The null space of the operator adjoint to* (5.78),

$$-2^{-1}I_6 + \mathcal{K}^* \ : \ [B_{p',p'}^{1-\frac{1}{p'}}(S)]^6 \to [B_{p',p'}^{1-\frac{1}{p'}}(S)]^6, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{5.81}$$

is a linear span of the vectors $(0,0,0,1,0,0)^\top$ *and* $(0,0,0,0,1,0)^\top$.

(iii) *The equation*

$$[-2^{-1}I_6 + \mathcal{K}]\,h = G \quad on \quad S, \tag{5.82}$$

is solvable if and only if

$$\int_S G_4(x)\,dS = \int_S G_5(x)\,dS = 0. \tag{5.83}$$

(iv) *If the conditions* (5.83) *hold, then solutions to equation* (5.82) *are defined modulo a linear combination of the vector functions* $h^{(1)}$ *and* $h^{(2)}$.

(v) *If the conditions* (5.83) *hold, then the interior Neumann BVP* (2.49), (2.51) *with* $\Phi = 0$ *is solvable in the space* $[H^1_p(\Omega^+)]^6$ *and its solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* h *is defined by equation* (5.82). *A solutions to the interior Neumann BVP in* Ω^+ *is defined modulo a linear combination of the constant vector functions* $\Psi^{(1)}$ *and* $\Psi^{(2)}$ *given by* (5.80).

Theorem 5.16 *Let* (5.70) *be fulfilled and* $G = (G_1, \cdots, G_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(S)]^6$.

(i) *The operator*

$$2^{-1}I_6 + \mathcal{K} \;:\; [B^{-\frac{1}{p}}_{p,p}(S)]^6 \to [B^{-\frac{1}{p}}_{p,p}(S)]^6 \tag{5.84}$$

is an invertible elliptic pseudodifferential operator.

(ii) *The exterior Neumann BVP* (2.49), (2.51) *with* $\Phi = 0$ *is uniquely solvable in the space of vector functions* $[H^1_p(\Omega^+)]^6$ *satisfying the decay conditions* (2.56) *and the solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* h *is defined by the uniquely solvable pseudodifferential equation* $[\,2^{-1}I_6 + \mathcal{K}\,]\,h = G$ *on* S.

Remark 5.17 *From the results obtained in Subsections 4.2 and 4.3 it follows that Theorems 5.13-5.16 with* $p = 2$ *hold true for Lipschitz domains.*

Remark 5.18 *From the general theory of pseudodifferential equations on* C^∞*-smooth manifolds without boundary it follows that*

(i) the elliptic pseudodifferential operators

$$\mathcal{H} \;:\; [B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6, \tag{5.85}$$

$$2^{-1}I_6 + \mathcal{N} \;:\; [B^{s+1}_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6, \tag{5.86}$$

$$2^{-1}I_6 + \mathcal{K} \;:\; [B^s_{p,q}(S)]^6 \to [B^s_{p,q}(S)]^6, \tag{5.87}$$

are invertible for arbitrary $s \in \mathbb{R}$, $p > 1$, $q \geq 1$, *since* (5.85), (5.86) *and* (5.87) *are invertible for* $s = -\frac{1}{2}$ *and* $p = q = 2$ *due to Corollary 4.9, Theorem 5.12 and Theorem 5.16, respectively;*

(ii) the elliptic pseudodifferential operators

$$-2^{-1}I_6 + \mathcal{N} \;:\; [B^s_{p,q}(S)]^6 \to [B^s_{p,q}(S)]^6, \tag{5.88}$$

$$-2^{-1}I_6 + \mathcal{K} \;:\; [B^s_{p,q}(S)]^6 \to [B^s_{p,q}(S)]^6, \tag{5.89}$$

have zero index for arbitrary $s \in \mathbb{R}$, $p > 1$, $q \geq 1$, *and their two-dimensional null spaces do not depend on* s, p, q.

5.7 Mixed type BVPs

Having in hand the results obtained in the previous subsections, we can investigate the mixed type BVPs (see (2.49), (2.52), (2.53)). In general, solutions to the mixed type BVPs are not C^{α}-Hölder continuous with $\alpha > \frac{1}{2}$ at the exceptional curves ℓ_m where different boundary conditions collide. Therefore we are not allowed to look for solutions in the space of regular vector functions even for C^∞ smooth boundary surfaces and C^∞ smooth boundary data.

Here we study in detail the interior mixed type BVP. The exterior problem can be treated quite similarly. So, we have to find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega^+)]^6$ to the homogeneous system of pseudo-oscillation equation

$$A(\partial, \tau)\, U = 0 \quad \text{in} \quad \Omega^+, \tag{5.90}$$

which satisfies the mixed Dirichlet-Neumann type boundary conditions

$$\big\{ U \big\}^+ = g^{(D)} \qquad \text{on} \qquad S_D, \tag{5.91}$$

$$\big\{ \mathcal{T}(\partial, n) U \big\}^+ = G^{(N)} \qquad \text{on} \qquad S_N. \tag{5.92}$$

Here

$$g^{(D)} \in [B^{1-\frac{1}{p}}_{p,p}(S_D)]^6, \qquad G^{(N)} \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^6. \tag{5.93}$$

For simplicity, throughout this subsection we assume that S and $\partial S_D = \partial S_N$ are C^∞-smooth. Denote by $g^{(e)}$ a fixed extension of the vector-function $g^{(D)}$ from S_D onto S preserving the functional space:

$$g^{(e)} \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \qquad r_{_{S_D}} g^{(e)} = g^{(D)} \quad \text{on} \quad S_D. \tag{5.94}$$

Recall that $r_{_\mathcal{M}}$ denotes the restriction operator onto $\mathcal{M}$.

Clearly, an arbitrary extension g of $g^{(D)}$ onto the whole of S, which preserves the functional space, can be then represented as

$$g = g^{(e)} + \widetilde{g} \quad \text{with} \quad \widetilde{g} \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6. \tag{5.95}$$

In accordance with Theorem 5.14, we can seek a solution in the form

$$U = V\big(\mathcal{H}^{-1}\,(g^{(e)} + \widetilde{g})\big), \tag{5.96}$$

where $\widetilde{g} \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6$ is an unknown vector function and $\mathcal{H}^{-1}$ is a strongly elliptic pseudodifferential operator inverse to the operator (5.75) (see Remark 5.18):

$$\mathcal{H}^{-1} \;:\; [B^{1-\frac{1}{p}}_{p,p}(S)]^6 \to [B^{-\frac{1}{p}}_{p,p}(S)]^6. \tag{5.97}$$

In view of (5.94) and (5.95), it is easy to check that the Dirichlet condition (5.91) on S_D is satisfied automatically. It remains only to satisfy the Neumann condition (5.92) on S_N, which leads to the pseudodifferential equation

$$[-2^{-1} I_6 + \mathcal{K}]\, \mathcal{H}^{-1}\,(g^{(e)} + \widetilde{g}) = G^{(N)} \tag{5.98}$$

on the open subsurface S_N for the unknown vector function $\widetilde{g}$.

We recall that

$$\mathcal{A}^+ = [-2^{-1}I_6 + \mathcal{K}]\,\mathcal{H}^{-1} \tag{5.99}$$

is the Steklov-Poincaré operator introduced and studied in Subsection 4.3 for $p=2$. In view of Remark 4.13 it is clear that

$$\mathcal{A}^+ \;:\; [B_{p,p}^{1-\frac{1}{p}}(S)]^6 \to [B_{p,p}^{-\frac{1}{p}}(S)]^6 \tag{5.100}$$

is a strongly elliptic pseudodifferential operator of order 1 with index zero.

Denote

$$G^{(0)} := G^{(N)} - r_{_{S_N}}\,\mathcal{A}^+\,g^{(e)} \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^6 \tag{5.101}$$

and rewrite equation (5.98) as

$$r_{_{S_N}}\,\mathcal{A}^+\,\widetilde{g} = G^{(0)} \quad \text{on} \quad S_N, \tag{5.102}$$

which is a pseudodifferential equation on the submanifold S_N with boundary ∂S_N. We would like to investigate the solvability of equation (5.102). To this end we proceed as follows.

Denote by $\mathfrak{S}(\mathcal{A}^+;x,\xi_1,\xi_2)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}^+$ in some local coordinate system at the point $x\in\overline{S_N}$ and let $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\})$. Let $\lambda_1(x),\cdots,\lambda_6(x)$ be the eigenvalues of the matrix

$$M_{\mathcal{A}^+}(x) := [\,\mathfrak{S}(\mathcal{A}^+;x,0,+1)\,]^{-1}[\,\mathfrak{S}(\mathcal{A}^+;x,0,-1)\,], \quad x\in\partial\overline{S_N}. \tag{5.103}$$

Introduce the notation

$$\delta_j(x) = \Re\,\big[\,(2\pi\,i)^{-1}\ln\lambda_j(x)\,\big],\;\; j=\overline{1,6}, \tag{5.104}$$

$$a_1 = \inf_{x\in\,\partial S_N,\,1\le j\le 6}\delta_j(x), \qquad a_2 = \sup_{x\in\,\partial S_N,\,1\le j\le 6}\delta_j(x) \tag{5.105}$$

here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty,0]$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system (see the Appendix B). Due to the strong ellipticity of the operator $\mathcal{A}^+$ we have the strong inequalities $-\frac{1}{2}<\delta_j(x)<\frac{1}{2}$ for $x\in\overline{S_N}$, $j=\overline{1,6}$. Therefore

$$-\frac{1}{2} < a_1 \le a_2 < \frac{1}{2}. \tag{5.106}$$

Lemma 5.19 *The operators*

$$r_{_{S_N}}\,\mathcal{A}^+ \;\;:\;\; [\widetilde{H}_p^s(S_N)]^6 \to [H_p^{s-1}(S_N)]^6, \tag{5.107}$$

$$:\;\; [\widetilde{B}_{p,q}^s(S_N)]^6 \to [B_{p,q}^{s-1}(S_N)]^6, \quad q\ge 1 \tag{5.108}$$

are invertible if

$$\frac{1}{p}-\frac{1}{2}+a_2 < s < \frac{1}{p}+\frac{1}{2}+a_1, \tag{5.109}$$

where a_1 and a_2 are given by (5.105).

Proof. The mapping properties (5.107) and (5.108) follow from Theorem 4.4 and Remark 5.18. To prove the invertibility of the operators (5.107) and (5.108), we first consider the particular values of the parameters $s = 1/2$ and $p = q = 2$, which fall into the region defined by the inequalities (5.109), and show that the null space of the operator

$$r_{_{S_N}} \mathcal{A}^+ : [\widetilde{H}_2^{\frac{1}{2}}(S_N)]^6 \to [H_2^{-\frac{1}{2}}(S_N)]^6 \tag{5.110}$$

is trivial, i.e., the equation

$$r_{_{S_N}} \mathcal{A}^+ \widetilde{g} = 0 \quad \text{on} \quad S_N \tag{5.111}$$

admits only the trivial solution in the space $[\widetilde{H}_2^{\frac{1}{2}}(S_N)]^6$. Recall that $\widetilde{H}_2^s(S_N) = \widetilde{B}_{2,2}^s(S_N)$ and $H_2^s(S_N) = B_{2,2}^s(S_N)$ for $s \in \mathbb{R}$.

Let $\widetilde{g} \in [\widetilde{H}^{\frac{1}{2}}(S_N)]^6$ be a solution of the homogeneous equation (5.111). It is clear that the vector $U = V(\mathcal{H}^{-1}\widetilde{g})$ belongs to the space $[H_2^1(\Omega^+)]^6 = [W_2^1(\Omega^+)]^6$ and solves the homogeneous mixed BVP (5.90)-(5.92) with $g^{(D)} = 0$ and $G^{(N)} = 0$. Therefore, $U(x) = V(\mathcal{H}^{-1}\widetilde{g})(x) = 0$ for $x \in \Omega^+$, due to Theorem 2.1 and, consequently, $\{U\}^+ = \widetilde{g} = 0$ on S.

By Lemma 4.12 we can easily conclude that the index of the operator (5.110) equals to zero and thus, it is invertible.

Since the principal homogeneous symbol matrix of the operator $\mathcal{A}^+$ is strongly elliptic, by Theorem B.1 (see the Appendix B) we conclude that the operators (5.107) and (5.108) are Fredholm with trivial null spaces for all values of the parameters satisfying the inequalities (5.109). Thus they are invertible. □

With the help of this lemma we can prove the following main existence result.

Theorem 5.20 *Let the conditions* (5.109) *be fulfilled,* a_2 *and* a_1 *be defined by* (5.105)*, and*

$$\frac{4}{3 - 2\,a_2} < p < \frac{4}{1 - 2\,a_1}\,. \tag{5.112}$$

Then the mixed BVP $(M)^+$ (5.90)-(5.92) *has a unique solution* $U \in [W_p^1(\Omega^+)]^6$ *which is representable in the form of single layer potential* (5.96)*,*

$$U = V\big(\mathcal{H}^{-1}\,(g^{(e)} + \widetilde{g})\big), \tag{5.113}$$

where $g^{(e)} \in [B_{p,p}^{1-1/p}(S)]^6$ *is a fixed extension of the vector function* $g^{(D)} \in [B_{p,p}^{1-1/p}(S_D)]^g$ *from* S_D *onto* S *preserving the functional space and* $\widetilde{g} \in [\widetilde{B}_{p,p}^{1-1/p}(S_N)]^6$ *is defined by the uniquely solvable pseudodifferential equation*

$$r_{_{S_N}} \mathcal{A}\,\widetilde{g} = G^{(0)} \quad \text{on} \quad S_N \tag{5.114}$$

with

$$G^{(0)} := G^{(N)} - r_{_{S_N}} \mathcal{A}\, g^{(e)} \in [B_{p,p}^{-1/p}(S_N)]^6.$$

Proof. First we note that in accordance with Lemma 5.19, equation (5.114) is uniquely solvable for $s = 1 - \frac{1}{p}$ where p satisfies the inequality (5.112), since the inequalities (5.109) are fulfilled. This implies that the mixed BVP $(M)^+$ is solvable in the space $[W_p^1(\Omega^+)]^6$ with p as in (5.112).

Next, we show the uniqueness of solution in the space $[W_p^1(\Omega^+)]^6$ for arbitrary p satisfying (5.112). Note that $p = 2$ belongs to the interval defined by the inequality (5.112) and for $p = 2$ the uniqueness has been proved in Theorem 2.1. Now, let $U \in [W_p^1(\Omega^+)]^6$ be some solution of the homogeneous mixed BVP $(M)^+$. Clearly, then

$$\{U\}^+ \in [\widetilde{B}_{p,p}^{1-1/p}(S_N)]^6. \tag{5.115}$$

By Theorem 5.14, we have the representation

$$U(x) = V\left(\mathcal{H}^{-1}\{U\}^+\right)(x), \quad x \in \Omega^+.$$

Since U satisfies the homogeneous Neumann condition (5.92) on S_N, we arrive at the equation

$$r_{S_N}\,\mathcal{A}^+\,\{U\}^+ = 0 \quad \text{on} \quad S_N,$$

whence $\{U\}^+ = 0$ on S follows due to the inclusion (5.115), Lemma 5.19, and the inequality (5.112) implying the conditions (5.109). Therefore, $U = 0$ in Ω^+. $\square$

Further, we prove almost the best regularity results for solutions to the mixed type BVP $(M)^+$.

Theorem 5.21 *Let the inclusions* (5.109) *hold and let*

$$\frac{4}{3-2a_2} < p < \frac{4}{1-2a_1}\ ,\ 1 < r < \infty,\ 1 \le q \le \infty,\ \frac{1}{r} - \frac{1}{2} + a_2 < s < \frac{1}{r} + \frac{1}{2} + a_1, \tag{5.116}$$

with a_2 and a_1 defined by (5.105).

Further, let $U \in [W_p^1(\Omega^+)]^6$ be a unique solution pair to the mixed BVP $(M)^+$ (5.90)-(5.92). *Then the following hold:*

i) *if*

$$g^{(D)} \in [B_{r,r}^s(S_D)]^6, \qquad G^{(D)} \in [B_{r,r}^{s-1}(S_N)]^6,$$

then $U \in [\,H_r^{s+\frac{1}{r}}(\Omega^+)\,]^6$;

ii) *if*

$$g^{(D)} \in [B_{r,q}^s(S_D)]^6, \qquad G^{(D)} \in [B_{r,q}^{s-1}(S_N)]^6,$$

then

$$U \in [\,B_{r,q}^{s+\frac{1}{r}}(\Omega^+)\,]^6; \tag{5.117}$$

iii) *if $\alpha > 0$ is not integer and*

$$g^{(D)} \in [C^{\alpha}(\overline{S_D})]^6, \qquad G^{(D)} \in [B_{\infty,\infty}^{\alpha-1}(S_N)]^6, \tag{5.118}$$

then

$$U \in \bigcap_{\alpha' < \kappa_m} [\,C^{\alpha'}(\,\overline{\Omega^+}\,)\,]^6,$$

where $\kappa_m = \min\{\alpha,\ a_1 + \frac{1}{2}\} > 0$.

Proof. The proofs of items i) and ii) follow easily from Lemma 5.19, and Theorems 5.20, and B.1.
To prove (iii) we use the following embedding (see, e.g., [Tr1], [Tr2])

$$C^{\alpha}(\mathcal{M}) = B^{\alpha}_{\infty,\infty}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,1}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,q}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{r,q}(\mathcal{M}) \subset C^{\alpha-\varepsilon-k/r}(\mathcal{M}), \tag{5.119}$$

where ε is an arbitrary small positive number, $\mathcal{M} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < r < \infty$, $\alpha - \varepsilon - \frac{k}{r} > 0$, and α and $\alpha - \varepsilon - \frac{k}{r}$ are not integers.

From (5.118) and the embedding (5.119) the condition (5.117) follows with any $s \le \alpha - \varepsilon$.

Bearing in mind (5.116) and taking r sufficiently large and ε sufficiently small, we can put

$$s = \alpha - \varepsilon \quad \text{if} \quad \frac{1}{r} - \frac{1}{2} + a_2 < \alpha - \varepsilon < \frac{1}{r} + \frac{1}{2} + a_1, \tag{5.120}$$

and

$$s \in \left(\frac{1}{r} - \frac{1}{2} + a_2, \frac{1}{r} + \frac{1}{2} + a_1\right) \quad \text{if} \quad \frac{1}{r} + \frac{1}{2} + a_1 < \alpha - \varepsilon. \tag{5.121}$$

By (5.117) for the solution vector we have $U \in [\, B^{s+\frac{1}{r}}_{r,q}(\Omega^+)\,]^6$ with

$$s + \frac{1}{r} = \alpha - \varepsilon + \frac{1}{r}$$

if (5.120) holds, and with

$$s + \frac{1}{r} \in \left(\frac{2}{r} - \frac{1}{2} + a_2, \frac{2}{r} + \frac{1}{2} + a_1\right)$$

if (5.121) holds. In the last case we can take

$$s + \frac{1}{r} = \frac{2}{r} + \frac{1}{2} + a_1 - \varepsilon\,.$$

Therefore, we have either

$$U \in [\, B^{\alpha-\varepsilon+\frac{1}{r}}_{r,q}(\Omega^+)\,]^6,$$

or

$$U \in [\, B^{\frac{1}{2}+\frac{2}{r}+a_1-\varepsilon}_{r,q}(\Omega^+)\,]^5,$$

in accordance with the inequalities (5.120) and (5.121). The last embedding in (5.119) (with $k = 3$) yields then that either

$$U \in [\, C^{\alpha-\varepsilon-\frac{2}{r}}(\overline{\Omega^+})\,]^6,$$

or

$$U \in [\, C^{\frac{1}{2}-\varepsilon+a_1-\frac{1}{r}}(\overline{\Omega^+})\,]^6\,.$$

These relations lead to the inclusions

$$U \in [\, C^{\kappa_m-\varepsilon-\frac{2}{r}}(\overline{\Omega^+})\,]^6, \tag{5.122}$$

where $\kappa_m = \min\{\alpha,\ a_1 + \frac{1}{2}\}$ and $\kappa_m > 0$ due to the inequality (5.106). Since r is sufficiently large and ε is sufficiently small, the inclusions (5.122) accomplish the proof. □

Remark 5.22 *Using exactly the same arguments, it can be shown that the similar uniqueness, existence and regularity results hold also true for the exterior mixed BVP* $(M)^-$. *We note only that the solution* $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega^-)]^6$, *satisfying the decay condition* (2.56), *is representable again in the form of the single layer potential* (5.96), *where* $g^{(e)} \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6$ *is the same as above, and* $\widetilde{g} \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6$ *is the unique solution of the pseudodifferential equation*

$$r_{S_N} \mathcal{A}^- \widetilde{g} = \widetilde{G}^{(0)} \quad \text{on} \quad S_N, \tag{5.123}$$

where

$$\mathcal{A}^- := [2^{-1} I_7 + \mathcal{K}]\, \mathcal{H}^{-1}, \qquad \widetilde{G}^{(0)} := G^{(N)} - r_{S_N} \mathcal{A}^- g^{(e)}.$$

The operator $r_{S_N} \mathcal{A}^-$ *has the same Fredholm properties as* $r_{S_N} \mathcal{A}^+$, *in particular, Lemma 5.19 holds with* $r_{S_N} \mathcal{A}^-$ *for* $r_{S_N} \mathcal{A}^+$.

Remark 5.23 *Lemma* 5.19 *with* $p = q = 2$ *and* $s = \frac{1}{2}$ *and Theorems* 5.20 *with* $p = 2$ *remain valid for Lipschitz domains due to Lemma 4.12 and the uniqueness Theorem 2.2.*

6 Investigation of crack type problems of pseudo-oscillations

In this section, first we investigate in detail two model crack type problems (CN) and (CT) formulated in Subsection 2.3.2. To describe principal qualitative aspects of the crack problems, for simplicity, first we assume that an elastic solid occupies a whole space $\mathbb{R}^3$ and contains an interior crack which coincides with a two-dimensional, two-sided smooth manifold Σ with the crack edge $\ell_c := \partial\Sigma$. Denote $\mathbb{R}^3_\Sigma := \mathbb{R}^3 \setminus \overline{\Sigma}$. As in Subsection 2.3.2, the crack surface Σ is considered as a submanifold of a closed surface Σ_0 surrounding a bounded domain $\overline{\Omega}_0$. We choose the direction of the unit normal vector on the fictional surface Σ_0 such that it is outward with respect to the domain Ω_0. This agreement defines uniquely the direction of the normal vector on the crack surface Σ. We prove unique solvability of the problems (CN) and (CT) by the potential method and analyse regularity properties of solutions.

Afterwards we investigate in detail the crack type BVPs $(D)^+$-(CN) and $(M)^+$-(CN) for a bounded domain Ω^+_Σ with an interior crack Σ. The BVPs $(D)^-$-(CN), $(M)^-$-(CN), $(D)^\pm$-(CT), $(M)^\pm$-(CT), $(N)^\pm$-(CN), and $(N)^\pm$-(CT) can be studied in an almost identical way.

For simplicity, throughout this section we assume that Σ, $\ell_c = \partial\Sigma$, $S = \partial\Omega^\pm$ and $\ell_m = \partial S_D = \partial S_N$ are C^∞-smooth if not otherwise stated.

6.1 Crack type problem (CN)

We have to find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6$ to the equation $A(\partial, \tau)U = 0$ in $\mathbb{R}^3_\Sigma$ possessing the decay properties (2.56) and satisfying the boundary conditions (2.57)-(2.60) on the crack faces. Recall that these boundary conditions correspond to the case when the crack gap is *thermally insulated and electrically impermeable.*

Let us rewrite the boundary conditions (2.57)-(2.60) in the following equivalent form

$$\big\{ [\mathcal{T}(\partial, n)U] \big\}^+ + \big\{ [\mathcal{T}(\partial, n)U] \big\}^- = G^{(+)} + G^{(-)} \quad \text{on} \quad \Sigma, \tag{6.1}$$

$$\big\{ [\mathcal{T}(\partial, n)U] \big\}^+ - \big\{ [\mathcal{T}(\partial, n)U] \big\}^- = G^{(+)} - G^{(-)} \quad \text{on} \quad \Sigma, \tag{6.2}$$

where $G^{(\pm)} = (G^{(\pm)}_1, \cdots, G^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ are given vector functions on Σ satisfying the following compatibility condition

$$G^{(+)} - G^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.3}$$

The imbedding (6.3) means that the extension of a vector function $G^{(+)} - G^{(-)}$ form Σ onto the whole of Σ_0 by zero preserves the functional space. It is easy to see that (6.3) is a necessary condition for the problem (CN) to be solvable in the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^6$.

Due to Corollary 3.7 we look for a solution to the crack type BVP (CN) in the form

$$U = W(g) - V(h) \quad \text{in} \quad \mathbb{R}^3_\Sigma, \tag{6.4}$$

where $W(g) = W_\Sigma(g)$ and $V(h) = V_\Sigma(h)$ are double and single layer potentials defined by (3.49) and (3.48) respectively with Σ for S,

$$g = [U]_\Sigma = \{U\}^+ - \{U\}^- \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6 \tag{6.5}$$

is an unknown vector function on Σ, while

$$h = [\mathcal{T}U]_\Sigma = \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^- = G^{(+)} - G^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6 \tag{6.6}$$

is a given vector function on Σ.

In view of Theorem 4.4 and jump relations (4.2)-(4.5), it is clear that the vector function (6.4) with g and h as in (6.5) and (6.6) belongs to the space $[W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$, satisfies the decay condition (2.56) and the crack condition (6.2) on Σ. The remaining crack condition (6.1) leads then to the pseudodifferential equation for the unknown vector function g

$$r_{_\Sigma}\,\mathcal{L}\,g = F \quad \text{ on } \quad \Sigma, \tag{6.7}$$

where $\mathcal{L}$ is a pseudodifferential operator defined by (4.9) and

$$F = \tfrac{1}{2}\,[G^{(+)} + G^{(-)}] + r_{_\Sigma}\mathcal{K}h \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6 \tag{6.8}$$

with h given by (6.6) and where the operator $\mathcal{K}$ is defined by (4.7). In what follows we show that the pseudodifferential equation (6.7) is uniquely solvable for arbitrary right hand side. To this end we first prove the following proposition.

Lemma 6.1 *Let $s \in \mathbb{R}$, $p > 1$, and $q \geq 1$. The operators*

$$r_{_\Sigma}\,\mathcal{L} \;\; : \;\; [\widetilde{H}^s_p(\Sigma)]^6 \to [H^{s-1}_p(\Sigma)]^6, \tag{6.9}$$

$$: \;\; [\widetilde{B}^s_{p,q}(\Sigma)]^6 \to [B^{s-1}_{p,q}(\Sigma)]^6, \tag{6.10}$$

are invertible if

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}. \tag{6.11}$$

Proof. In Subsection 5.5, we have shown that the principal homogeneous symbol matrix $\mathfrak{S}\big(\mathcal{L}; x, \xi'\big)$ of the operator $\mathcal{L}$ is even and homogeneous of order $+1$ in $\xi' = (\xi_1, \xi_2)$. Moreover, in the same subsection we have established that the symbol matrix $\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)$ is strongly elliptic. So we can apply the theory on strongly elliptic pseudodifferential equations on manifolds with boundary exposed in Appendix B (see Theorem B.1).

Note that since the principal homogeneous symbol matrix $\mathfrak{S}\big(\mathcal{L}; x, \xi_1, \xi_2\big)$ is even in $\xi' = (\xi_1, \xi_2)$, we have

$$M_{\mathcal{L}}(x) := [\,\mathfrak{S}(\mathcal{L}; x, 0, +1)\,]^{-1}[\,\mathfrak{S}(\mathcal{L}; x, 0, -1)\,] = I_6, \quad x \in \overline{\Sigma}. \tag{6.12}$$

Therefore all the eigenvalues $\lambda_1(x), \cdots, \lambda_6(x)$ of the matrix $M_{\mathcal{L}}(x)$ equal to 1 and

$$\delta_j(x) = \Re\,\big[\,(2\pi\,i)^{-1} \ln \lambda_j(x)\,\big] = 0, \;\; j = \overline{1,6}, \tag{6.13}$$

where $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$ (see the Appendix B). Therefore, by Theorem B.1 the operators (6.9) and (6.10) are Fredholm with zero index if the conditions (6.11) hold. It remains to show that for some particular values of the parameters s, p, and q, satisfying the inequalities (6.11), they are invertible. Let us take $s = 1/2$, $p = q = 2$ and recall that $\widetilde{H}^s_2(\Sigma) = \widetilde{B}^s_{2,2}(\Sigma)$ and $H^s_2(\Sigma) = B^s_{2,2}(\Sigma)$ for $s \in \mathbb{R}$. Thus the operators (6.9) and (6.10) coincide for these particular values of the parameters and actually we have to prove that the null space of the operator $r_{_\Sigma}\mathcal{L} : [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6 \to [H^{-\frac{1}{2}}_2(\Sigma)]^6$ is trivial. Indeed, let $g_0 \in [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6$ be a solution of the homogeneous equation $r_{_\Sigma}\mathcal{L}\, g_0 = 0$ on Σ and construct the vector function $U_0 = W(g_0)$. By Theorem 4.2 we see that $U_0 = W(g_0) \in [W^1_{2,\,loc}(\mathbb{R}^3_\Sigma)]^6$ and satisfies the decay conditions (2.56). Moreover, it is also easy to see that U_0 satisfies the homogeneous crack conditions (6.1)-(6.2) due to Theorem 4.2 and the homogeneous equation for g_0 on Σ. By the uniqueness Theorem 2.2 we conclude $U_0 = 0$ in $\mathbb{R}^3_\Sigma$. Consequently, $\{U_0\}^+ - \{U_0\}^- = g_0 = 0$ on Σ which completes the proof. $\square$

Now we can prove the following existence result.

Theorem 6.2 *Let* $G^{(\pm)} = (G^{(\pm)}_1, \cdots, G^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$*, the compatibility conditions* (6.3) *be fulfilled and*

$$\frac{4}{3} < p < 4\,. \tag{6.14}$$

Then the crack type BVP (CN) *has a unique solution* $U \in [W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$ *which is representable in the form*

$$U = W(g) - V(G^{(+)} - G^{(-)}) \quad \text{in} \quad \mathbb{R}^3_\Sigma, \tag{6.15}$$

where $g \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$ *is defined by the uniquely solvable pseudodifferential equation* (6.7)-(6.8).

Proof. First we note that in accordance with Lemma 6.1, the pseudodifferential equation (6.7)-(6.8) is uniquely solvable for $s = 1 - \frac{1}{p}$ with p from the interval (6.14), since the inequalities (6.11) are fulfilled. This implies that the crack type BVP (CN) is solvable in the space $[W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$ with p satisfying the inequalities (6.14).

Next, we show the uniqueness of solution in the space $[W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$ with p satisfying (6.14). Note that $p = 2$ belongs to the interval (6.14) and for $p = 2$ the uniqueness has been proved in Theorem 2.2. Now, let $U \in [W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$ be some solution of the homogeneous crack type BVP (CN). Clearly, then

$$\begin{aligned} &\{U\}^\pm \in [B^{1-1/p}_{p,p}(\Sigma)]^6, && \{\mathcal{T}U\}^\pm \in [B^{-1/p}_{p,p}(\Sigma)]^6,\\ &\{U\}^+ - \{U\}^- \in [\widetilde{B}^{1-1/p}_{p,p}(\Sigma)]^6, && \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^- \in [\widetilde{B}^{-1/p}_{p,p}(\Sigma)]^6, \end{aligned} \tag{6.16}$$

since actually $U \in [W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6 \cap [C^\infty(\mathbb{R}^3_\Sigma)]^6$ due to the interior regularity results.

In accordance with Corollary 3.7 for the solution vector U of the homogeneous crack type BVP (CN) we have then the representation

$$U = W(g) \quad \text{in} \quad \mathbb{R}^3_\Sigma, \tag{6.17}$$

with $g = [U]_\Sigma = \{U\}^+ - \{U\}^- \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$. Since U satisfies the homogeneous crack type conditions on Σ, we arrive at the equation $r_{\Sigma}\,\{\mathcal{T}U\}^\pm = r_{\Sigma}\,\mathcal{L}\,g = 0$ on Σ, whence $g = 0$ on Σ follows due to Lemma 6.1 in view of the inequality (6.14). Therefore, $U = 0$ in $\mathbb{R}^3_\Sigma$. $\square$

As in the case of mixed type BVP $(M)^+$, we can prove almost the best regularity results for solutions to the crack type BVP (CN).

Theorem 6.3 *Let the inclusions* $G^{(\pm)} = (G_1^{(\pm)}, \cdots, G_6^{(\pm)})^\top \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6$ *and the compatibility conditions* (6.3) *hold and let*

$$\frac{4}{3} < p < 4\,, \quad 1 < r < \infty, \quad 1 \leq q \leq \infty, \quad \frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}\,. \tag{6.18}$$

Further, let $U \in [W_p^1(\Omega)]^6$ *be a unique solution to the crack type BVP* (CN). *Then the following hold:*

i) *if*

$$G^{(\pm)} \in [B_{r,r}^{s-1}(\Sigma)]^6, \quad G^{(+)} - G^{(-)} \in [\widetilde{B}_{r,r}^{s-1}(\Sigma)]^6\,,$$

then $U \in [\,H^{s+\frac{1}{r}}_{r,\,loc}(\mathbb{R}^3_\Sigma)\,]^6$;

ii) *if*

$$G^{(\pm)} \in [B_{r,q}^{s-1}(\Sigma)]^6, \quad G^{(+)} - G^{(-)} \in [\widetilde{B}_{r,q}^{s-1}(\Sigma)]^6\,,$$

then

$$U \in [\,B^{s+\frac{1}{r}}_{r,q,\,loc}(\mathbb{R}^3_\Sigma)\,]^6; \tag{6.19}$$

iii) *if* $\alpha > 0$ *is not integer and*

$$G^{(\pm)} \in [B_{\infty,\infty}^{\alpha-1}(\Sigma)]^6, \quad G^{(+)} - G^{(-)} \in [\widetilde{B}_{\infty,\infty}^{\alpha-1}(\Sigma)]^6, \tag{6.20}$$

then

$$U \in \bigcap_{\alpha' < \kappa_c} [\,C^{\alpha'}(\,\overline{\Omega}\,)\,]^6,$$

with $\kappa_c = \min\{\alpha,\ \frac{1}{2}\} > 0$; *here* Ω *is either* Ω_0 *or* $\mathbb{R}^3 \backslash \overline{\Omega_0}$, *where* Ω_0 *is a domain with boundary* $\Sigma_0 = \partial\Omega_0$ *whose proper part is the crack surface* Σ.

Proof. It is word for word of the proof of Theorem 5.21 provided $a_1 = a_2 = 0$. $\square$

Remark 6.4 *If we compare the regularity results exposed in Theorems 5.21 and 6.3 for solutions of mixed* $(M)^\pm$ *and crack type* (CN) *BVPs near the exceptional curves, i.e., near the curve* ℓ_m *where the Dirichlet and Neumann conditions collide and near the crack edge* ℓ_c, *we see that the Hölder smoothness exponent for solution vectors at the curve* ℓ_c *is greater than the Hölder smoothness exponent at the curve* ℓ_m, *in general. In particular, if boundary data are sufficiently smooth,* $\alpha > 1/2$ *say, due to Theorem 5.21 (iii) solutions to mixed BVPs belong then to the class* $\bigcap\limits_{\alpha' < \kappa_m} C^{\alpha'}$ *at the curve* ℓ_m *where the positive upper bound* $\kappa_m = a_1 + \frac{1}{2}$ *depends on the material parameters essentially and may take an arbitrary value* *from the interval* $(0,\,\frac{1}{2}]$*, since in general* a_1 *may take an arbitrary value from the interval* $(-\frac{1}{2},\,0]$ *depending on the material parameters.*

In the case of crack type BVPs with $\alpha > 1/2$*, due to Theorem 6.3 (iii) solutions belong to the class* $\bigcap\limits_{\alpha'<1/2} C^{\alpha'}$ *at the crack edge* ℓ_c *and as we see the upper bound* $\kappa_c = 1/2$ *does not depend on the material parameters. Thus* $\kappa_m \le \kappa_c$ *which proves that, in general, solutions to the crack type BVPs possess higher regularity near the crack edge* ℓ_c *than solutions to the mixed type BVPs at the exceptional curve* ℓ_m *(cf. [BCN3], [BC1], [BCD1]).*

6.2 Crack type problem (CT)

In this case we have to find a solution vector $U = (u,\varphi,\psi,\vartheta)^\top \in [W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$ to the equation $A(\partial,\tau)U = 0$ in $\mathbb{R}^3_\Sigma$ possessing the decay properties (2.56) and satisfying the boundary conditions (2.61)-(2.67) on the crack faces. Recall that these boundary conditions correspond to the case when the crack gap is *thermally and electrically conductive.*

As in the previous case, we first reformulate the crack conditions (2.61)-(2.67) on Σ equivalently

$$\big\{[\mathcal{T}(\partial,n)U]_k\big\}^+ - \big\{[\mathcal{T}(\partial,n)U]_k\big\}^- = \widetilde{G}_k := G_k^{(+)} - G_k^{(-)}, \quad k=1,2,3, \tag{6.21}$$

$$\big\{[\mathcal{T}(\partial,n)U]_j\big\}^+ - \big\{[\mathcal{T}(\partial,n)U]_j\big\}^- = \widetilde{G}_j, \quad j=4,5,6, \tag{6.22}$$

$$\{U_j\}^+ - \{U_j\}^- = \widetilde{g}_j, \quad j=4,5,6, \tag{6.23}$$

$$\big\{[\mathcal{T}(\partial,n)U]_k\big\}^+ + \big\{[\mathcal{T}(\partial,n)U]_k\big\}^- = G_k^{(+)} + G_k^{(-)}, \quad k=1,2,3, \tag{6.24}$$

where

$$G_k^{(\pm)} \in B_{p,p}^{-\frac{1}{p}}(\Sigma),\ \widetilde{G}_k := G_k^{(+)} - G_k^{(-)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma),\ \widetilde{g}_j \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma),\ \widetilde{G}_j \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \tag{6.25}$$
$$k=1,2,3,\ \ j=4,5,6.$$

Again, due to Corollary 3.7 we look for a solution to the crack type BVP (CT) in the form

$$U = W(g) - V(h) \quad \text{in} \quad \mathbb{R}^3_\Sigma, \tag{6.26}$$

where $W(g) = W_\Sigma(g)$ and $V(h) = V_\Sigma(h)$ are double and single layer potentials defined by (3.49) and (3.48) respectively with Σ for S,

$$g = (g_1, g_2, \cdots, g_6)^\top = \{U\}^+ - \{U\}^- \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6, \tag{6.27}$$

$$h = (h_1, h_2, \cdots, h_6)^\top = \{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^- \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)]^6\,. \tag{6.28}$$

It is easy to see that the vector function h is defined explicitly from the boundary conditions (6.21)-(6.22),

$$h_j = \widetilde{G}_j \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)\,, \quad j = \overline{1,6} \tag{6.29}$$

while the components g_4, g_5, and g_5 of the vector function g are explicitly defined from the conditions (6.23)

$$g_j = \widetilde{g}_j \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)\,, \quad j = 4,5,6. \tag{6.30}$$

So as we see, if (6.29) and (6.30) hold, then the vector function U defined by (6.26) automatically satisfies all conditions of the crack type BVP (CT) except the three boundary conditions in (6.24). Keeping in mind that in the representation formula (6.26) the only unknowns remain the functions g_1, g_2, and g_3, the first three components of the vector g, and taking into account the boundary conditions (6.24), we arrive at the following pseudodifferential equations:

$$r_{\Sigma} \sum_{j=1}^{3} \mathcal{L}_{kj}\, g_j = F_k \quad \text{ on } \quad \Sigma, \quad k = 1,2,3, \tag{6.31}$$

where $\mathcal{L} = [\mathcal{L}_{kj}]_{6\times 6}$ is a pseudodifferential operator defined by (4.9) and

$$F_k = \tfrac{1}{2}\,[G_k^{(+)} + G_k^{(-)}] - r_{\Sigma} \sum_{j=4}^{6} \mathcal{L}_{kj}\, \widetilde{g}_j + r_{\Sigma} \sum_{j=1}^{6} \mathcal{K}_{kj} \widetilde{G}_j \in B^{-\frac{1}{p}}_{p,p}(\Sigma), \quad k = 1,2,3, \tag{6.32}$$

where $\widetilde{g}_j$ and $\widetilde{G}_j$ are given functions (see (6.25)) and the pseudodifferential operator $\mathcal{K} = [\mathcal{K}_{kj}]_{6\times 6}$ is defined by (4.7).

Let us introduce the matrix pseudodifferential operator

$$\mathfrak{L} := \big[\, \mathcal{L}_{kj} \,\big]_{3\times 3}, \quad 1 \le k,j \le 3, \tag{6.33}$$

which coincides with the first basic 3×3 block of the pseudodifferential operator $\mathcal{L} = [\mathcal{L}_{kj}]_{6\times 6}$ defined by (4.9).

Further, we rewrite the system of equation (6.31) in matrix form

$$\mathfrak{L}\, g^{(3)} = F \quad \text{ on } \quad \Sigma, \tag{6.34}$$

where

$$g^{(3)} = (g_1, g_2, g_3)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^3 \tag{6.35}$$

is the unknown vector function and

$$F = (F_1, F_2, F_3)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3 \tag{6.36}$$

is a known right hand side.

From the properties of the operator $\mathcal{L}$, described in Lemma 5.9, it follows immediately that the pseudodifferential operator $\mathfrak{L}$ is strongly elliptic as well and the principal homogeneous symbol matrix $\mathfrak{S}\big(\mathfrak{L}; x, \xi'\big) = [\mathfrak{S}_{kj}\big(\mathfrak{L}; x, \xi'\big)]_{3\times 3}$ of the operator $\mathfrak{L}$ is even and homogeneous of order $+1$ in $\xi' = (\xi_1, \xi_2)$. Therefore, we can apply Theorem B.1 and prove the following counterpart of Lemma 6.1.

Lemma 6.5 *Let $s \in \mathbb{R}$, $p > 1$, and $q \geq 1$. The operators*

$$r_{\Sigma}\,\mathfrak{L} \;:\; [\widetilde{H}^s_p(\Sigma)]^3 \to [H^{s-1}_p(\Sigma)]^3, \tag{6.37}$$

$$: \; [\widetilde{B}^s_{p,q}(\Sigma)]^3 \to [B^{s-1}_{p,q}(\Sigma)]^3, \tag{6.38}$$

are invertible if

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}. \tag{6.39}$$

Proof. By the same arguments as in the proof of Lemma 6.1 we easily derive that the operators (6.37) and (6.38) are Fredholm with zero index if the inequalities (6.39) hold. Therefore we need only to prove that the operator (6.37) has the trivial null space if $s = 1/2$, $p = q = 2$. Let $g_0 = (g_{01}, g_{02}, g_{03})^\top \in [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^3$ be a solution of the homogeneous equation $r_{\Sigma}\,\mathfrak{L}\, g_0 = 0$ on Σ.

We set $f_0 := (g_0, 0, 0, 0)^\top = (g_{01}, g_{02}, g_{03}, 0, 0, 0)^\top$ and construct the vector function $U_0 = W(f_0) \equiv W_\Sigma(f_0)$. By Theorem 4.2 we see that $U_0 = W(f_0) \in [W^1_{2,\,loc}(\mathbb{R}^3_\Sigma)]^6$ and satisfies the decay conditions (2.56). Moreover, it is also easy to see that U_0 satisfies the homogeneous crack conditions (6.21)-(6.24) due to Theorem 4.2 and the homogeneous equation for g_0 on Σ. By the uniqueness Theorem 2.2 we conclude $U_0 = 0$ in $\mathbb{R}^3_\Sigma$. Consequently, $\{U_0\}^+ - \{U_0\}^- = f_0 = 0$ on Σ which completes the proof. □

Lemma 6.5 immediately leads to the following existence and regularity results which can be proved by means of exactly the same arguments as Theorems 6.2 and 6.3.

Theorem 6.6 *Let conditions* (6.25) *be fulfilled and* $\frac{4}{3} < p < 4$. *Then the crack type BVP* (CT) *has a unique solution* $U \in [W^1_{p,\,loc}(\mathbb{R}^3_\Sigma)]^6$ *which is representable in the form*

$$U = W(g^*) + W(\widetilde{g}) - V(\widetilde{G}) \quad \text{in} \quad \mathbb{R}^3_\Sigma, \tag{6.40}$$

where $\widetilde{G} := (\widetilde{G}_1, \cdots, \widetilde{G}_6)^\top$ *and* $\widetilde{g} = (0, 0, 0, \widetilde{g}_4, \widetilde{g}_5, \widetilde{g}_6)^\top$ *are given boundary data, while the unknown vector function* $g^* = (g^{(3)}, 0, 0, 0)^\top$ *with* $g^{(3)} = (g_1, g_2, g_3)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^3$ *is defined by the uniquely solvable pseudodifferential equation* (6.34)-(6.36), *i.e., the equations* (6.31)-(6.32).

Theorem 6.7 *Let conditions* (6.25) *be fulfilled and let*

$$\frac{4}{3} < p < 4\,, \quad 1 < r < \infty, \quad 1 \leq q \leq \infty, \quad \frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}\,, \tag{6.41}$$

Further, let $U \in [W^1_p(\Omega)]^6$ *be a unique solution to the crack type BVP* (CT). *Then the following hold:*

i) *if*

$$G^{(\pm)}_k \in B^{s-1}_{r,r}(\Sigma), \;\; \widetilde{G}_k := G^{(+)}_k - G^{(-)}_k \in \widetilde{B}^{s-1}_{r,r}(\Sigma), \;\; \widetilde{g}_j \in \widetilde{B}^{s}_{r,r}(\Sigma), \;\; \widetilde{G}_j \in \widetilde{B}^{s-1}_{r,r}(\Sigma), \tag{6.42}$$

$$k = 1, 2, 3, \quad j = 4, 5, 6,$$

then $U \in [\, H^{s+\frac{1}{r}}_{r,\,loc}(\mathbb{R}^3_\Sigma)\,]^6$;

ii) *if*

$$G^{(\pm)}_k \in B^{s-1}_{r,q}(\Sigma),\ \ \widetilde{G}_k := G^{(+)}_k - G^{(-)}_k \in \widetilde{B}^{s-1}_{r,q}(\Sigma),\ \ \widetilde{g}_j \in \widetilde{B}^{s}_{r,q}(\Sigma),\ \ \widetilde{G}_j \in \widetilde{B}^{s-1}_{r,q}(\Sigma), \tag{6.43}$$
$$k = 1, 2, 3,\ \ j = 4, 5, 6,$$

then

$$U \in [\, B^{s+\frac{1}{r}}_{r,q,\,loc}(\mathbb{R}^3_\Sigma)\,]^6; \tag{6.44}$$

iii) *if* $\alpha > 0$ *is not integer and*

$$G^{(\pm)}_k \in B^{\alpha-1}_{\infty,\infty}(\Sigma),\ \ \widetilde{G}_k := G^{(+)}_k - G^{(-)}_k \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma),\ \ \widetilde{G}_j \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma),$$
$$\widetilde{g}_j \in C^{\alpha}(\Sigma),\quad r_{\partial\Sigma}\,\widetilde{g}_j = 0,\qquad k = 1, 2, 3,\ \ j = 4, 5, 6, \tag{6.45}$$

then

$$U \in \bigcap_{\alpha' < \kappa_c} [\, C^{\alpha'}(\,\overline{\Omega}\,)\,]^6,$$

with $\kappa_c = \min\{\alpha,\ \frac{1}{2}\} > 0$*; here* Ω *is either* Ω_0 *or* $\mathbb{R}^3\setminus\overline{\Omega_0}$*, where* Ω_0 *is a domain with boundary* $\Sigma_0 = \partial\Omega_0$ *whose proper part is the crack surface* Σ*.*

Remark 6.8 *If we compare the regularity results exposed in Theorems 5.21 and 6.7 for solutions of mixed* $(M)^{\pm}$ *and crack type* (CT) *BVPs near the exceptional curves, i.e., near the curve* ℓ_m *where the Dirichlet and Neumann conditions collide and near the crack edge* ℓ_c*, we see that the Hölder smoothness exponent for solution vectors at the curve* ℓ_c *is greater than the Hölder smoothness exponent at the curve* ℓ_m*, since* $\kappa_m \le \kappa_c$*, in general (cf. Remark 6.4).*

Remark 6.9 *Lemmas* 6.1 *and 6.5 with* $p = q = 2$ *and* $s = \frac{1}{2}$*, and Theorems* 6.2 *and 6.6 with* $p = 2$ *remain valid for Lipschitz domains due to Lemma 5.9 and the uniqueness Theorem 2.2.*

6.3 Crack type problem $(D)^{+}$-(CN)

Let an elastic slid occupy a bounded domain Ω^+ with boundary $S = \partial\Omega^+$ and possesses an interior crack $\Sigma \subset \Omega^+$, $S \cap \overline{\Sigma} = \varnothing$. The problem we would like to study in this subsection can be reformulated as follows (see Subsection 2.3.2): find a solution $U = (u, \varphi, \psi, \vartheta)^{\top} \in [W^1_p(\Omega^+_\Sigma)]^6$ to the equation $A(\partial, \tau)U = 0$ in $\Omega^+_\Sigma := \Omega^+ \setminus \overline{\Sigma}$ which satisfies the Dirichlet boundary condition on the exterior surface $S = \partial\Omega^+$

$$\{U\}^+ = g \quad \text{on} \quad S, \tag{6.46}$$

and (CN) type conditions on the crack faces

$$\big\{\,[\mathcal{T}(\partial, n)U]\big\}^+ + \big\{\,[\mathcal{T}(\partial, n)U]\big\}^- = G := G^{(+)} + G^{(-)} \quad \text{on} \quad \Sigma, \tag{6.47}$$

$$\big\{\,[\mathcal{T}(\partial, n)U]\big\}^+ - \big\{\,[\mathcal{T}(\partial, n)U]\big\}^- = \widetilde{G} := G^{(+)} - G^{(-)} \quad \text{on} \quad \Sigma, \tag{6.48}$$

where $g \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$ and $G^{(\pm)} = (G_1^{(\pm)}, \cdots, G_6^{(\pm)})^\top \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6$ are given vector functions on S and Σ respectively. We assume the following compatibility condition

$$\widetilde{G} = G^{(+)} - G^{(-)} \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)]^6. \tag{6.49}$$

We look for a solution vector in the form

$$U = V_S\big(\mathcal{H}_S^{-1} f\big) + W_\Sigma(h) - V_\Sigma(\widetilde{G}) \quad \text{in} \quad \Omega_\Sigma^+, \tag{6.50}$$

where V_S, V_Σ, and W_Σ are single and double layer potentials defined by (3.48) and (3.49), $\mathcal{H}_S$ is a pseudodifferential operator defined by (4.6) and $\mathcal{H}_S^{-1}$ is the inverse to the operator (5.75), $f = (f_1, \cdots, f_6)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6$ and $h = (h_1, \cdots, h_6)^\top \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$ are unknown vector functions.

It is clear that the differential equation and the crack condition (6.48) are satisfied automatically, while the conditions (6.46) and (6.47) lead to the system of pseudodifferential equations

$$f + W_\Sigma(h) = \Phi^{(1)} \quad \text{on} \quad S, \tag{6.51}$$

$$r_{_\Sigma}\, \mathcal{T}(\partial, n) V_S\big(\mathcal{H}_S^{-1} f\big) + r_{_\Sigma}\, \mathcal{L}_\Sigma\, h = \Phi^{(2)} \quad \text{on} \quad \Sigma, \tag{6.52}$$

where

$$\Phi^{(1)} := g + r_{_S} V_\Sigma(\widetilde{G}) \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \quad \Phi^{(2)} := \frac{1}{2}\, G + r_{_\Sigma}\, \mathcal{K}_\Sigma\, \widetilde{G} \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6. \tag{6.53}$$

Here $\mathcal{K}_\Sigma$ and $\mathcal{L}_\Sigma$ are pseudodifferential operators defined by (4.7) and (4.9) .

Denote the operator generated by the left hand side expressions in (6.51)-(6.52) by $\mathcal{D}$ which acts on the pair of the sought for vectors $(f, h)^\top$,

$$\mathcal{D} := \begin{bmatrix} I_6 & r_{_S}\, W_\Sigma \\ r_{_\Sigma}\, \mathcal{T}(\partial, n) V_S\big(\mathcal{H}_S^{-1}\big) & r_{_\Sigma}\, \mathcal{L}_\Sigma \end{bmatrix}_{12\times 12}. \tag{6.54}$$

Clearly, the operators $r_{_S}\, W_\Sigma$ and $r_{_\Sigma}\, \mathcal{T}(\partial, n) V_S\big(\mathcal{H}_S^{-1}\big)$ are smoothing operators, since the manifolds S and Σ are disjoint.

Set

$$\Psi = (f, h)^\top, \quad \Phi = (\Phi^{(1)}, \Phi^{(2)})^\top,$$

and rewrite equations (6.51)-(6.52) in matrix form

$$\mathcal{D}\, \Psi = \Phi. \tag{6.55}$$

Theorem 4.4 yield the following mapping properties

$$\mathcal{D} \,:\, \mathbf{X}_p^s \to \mathbf{Y}_p^s\,, \qquad \mathcal{D} \,:\, \mathbf{X}_{p,\,t}^s \to \mathbf{Y}_{p,\,t}^s, \tag{6.56}$$

$$s \in \mathbb{R}, \;\; 1 < p < \infty, \;\; 1 \le t \le \infty \;\; S, \Sigma \in C^\infty,$$

where

$$\mathbf{X}^s_p := [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^6, \quad \mathbf{Y}^s_p := [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^6 ,$$
$$\mathbf{X}^s_{p,\,t} := [B^s_{p,\,t}(S)]^6 \times [\widetilde{B}^s_{p,\,t}(\Sigma)]^6, \quad \mathbf{Y}^s_{p,\,t} := [B^s_{p,\,t}(S)]^6 \times [B^{s-1}_{p,\,t}(\Sigma)]^6,$$
$$s \in \mathbb{R}, \;\; 1 < p < \infty, \;\; 1 \le t \le \infty \;\; S, \Sigma \in C^\infty.$$

Further, let us consider the operator

$$\widetilde{\mathcal{D}} := \begin{bmatrix} I_6 & 0 \\ 0 & r_{\Sigma}\, \mathcal{L}_\Sigma \end{bmatrix}_{12\times 12} . \tag{6.57}$$

It is clear that $\widetilde{\mathcal{D}}$ has the same mapping properties as the operator $\mathcal{D}$ and the operator $\mathcal{D}-\widetilde{\mathcal{D}}$ with the same domain and range spaces as in (6.56) is a compact operator. Moreover, in view of Lemma 6.1 the operators

$$\widetilde{\mathcal{D}} \,:\, \mathbf{X}^s_p \to \mathbf{Y}^s_p\,, \qquad \widetilde{\mathcal{D}} \,:\, \mathbf{X}^s_{p,\,t} \to \mathbf{Y}^s_{p,\,t}, \tag{6.58}$$

$$\tag{6.59}$$

are invertible if

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}. \tag{6.60}$$

Therefore the operators (6.56) are Fredholm with zero index for arbitrary p and s satisfying the inequalities (6.60) hold (see Theorem B.1).

Lemma 6.10 *The operators* (6.56) *are invertible if the inequalities* (6.60) *hold.*

Proof. Since the operators (6.56) are Fredholm with zero index for arbitrary s and p satisfying (6.60), in accordance with Theorem B.1 we need only to show that the corresponding null-spaces are trivial for some particular values of the parameters s and p meeting the inequalities (6.60). Let us take $s = \frac{1}{2}$ and $p = 2$, and prove that the homogeneous system $\mathcal{D}\,\Psi = 0$, i.e., the equations (6.51)-(6.52) with $\Phi^{(1)} = \Phi^{(2)} = 0$ have only the trivial solution. Indeed, let $\Psi_0 = (f_0, h_0)^\top \in [H^{\frac{1}{2}}_2(S)]^6 \times [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6$ be a solution of the homogeneous system (6.51)-(6.52) and construct the vector $U_0(x) = V_S\big(\mathcal{H}^{-1}_S f_0\big)(x) + W_\Sigma(h_0)(x)$, $x \in \mathbb{R}^3 \setminus (S \cup \overline{\Sigma})$. One can easily show that the embedding $U_0 \in [W^1_2(\Omega^+_\Sigma)]^6$ holds and U_0 solves the homogeneous BVP $(D)^+$-(CN). By Theorem 2.1 we conclude $U_0 = 0$ in Ω^+_Σ. Hence $\{U_0\}^+_\Sigma - \{U_0\}^-_\Sigma = h_0 = 0$ on Σ follows immediately. Therefore we get $U_0 = V_S\big(\mathcal{H}^{-1}_S f_0\big) = 0$ in Ω^+_Σ which implies $f_0 = \{U_0\}^+_S = 0$ on S. Thus $\Psi_0 = (f_0, h_0)^\top = 0$ and the operator (6.56) has a trivial null space. $\square$

These invertibility properties for the operator $\mathcal{D}$ lead to the following existence results for Problem $(D)^+$-(CN).

Theorem 6.11 *Let* $4/3 < p < 4$ *and*

$$g \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6,\ \ G^{(\pm)} \in \big[B_{p,p}^{-\frac{1}{p}}(\Sigma)\big]^6,\ \ \widetilde{G} = G^{(+)} - G^{(-)} \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)]^6 .$$

Then the crack type BVP $(D)^+$*-*(CN) *possesses a unique solution* $U \in [W_p^1(\Omega_\Sigma^+)]^6$ *which can be represented by formula* (6.50)*, where the pair*

$$(f,h)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6 \times [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$$

is a unique solution of the system of boundary pseudodifferential equations(6.51)-(6.52).

Proof. Existence of solutions directly follows from Lemma 6.10 since the condition (6.60) are fulfilled for $s = 1 - \frac{1}{p}$ and $\frac{4}{3} < p < 4$. Uniqueness for $p = 2$ follows from Theorem 2.1. Let us now show uniqueness of solutions for arbitrary $p \in (\frac{4}{3}, 4)$.

Let $U \in [W_p^1(\Omega_\Sigma^+)]^6$ be a solution to the homogeneous BVP $(D)^+$-(CN).

Then by the general integral representation formula (3.54) we get

$$U(x) = -V_S(\{\mathcal{T}U\}_S^+)(x) + W_\Sigma([U]_\Sigma)(x), \quad x \in \Omega_\Sigma^+, \tag{6.61}$$

due to the homogeneous Dirichlet condition on S and homogeneous crack type conditions on Σ. Recall that $[U]_\Sigma$ stands for the jump of a vector U across the surface Σ. Note that

$$\{\mathcal{T}U\}_S^+ \in [B_{p,p}^{-\frac{1}{p}}(S)]^6, \quad [U]_\Sigma \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6 . \tag{6.62}$$

Since U solves the homogeneous BVP $(D)^+$-(CN) we arrive at the following pseudodifferential equations

$$\begin{aligned} &-\mathcal{H}_S\,\{\mathcal{T}U\}_S^+ + W_\Sigma([U]_\Sigma) = 0 \quad \text{on} \quad S,\\ &-\mathcal{T}\,V_S(\{\mathcal{T}U\}_S^+) + \mathcal{L}_\Sigma\,[U]_\Sigma = 0 \quad \text{on} \quad \Sigma, \end{aligned} \tag{6.63}$$

which can be rewritten as

$$\mathcal{D}\,\widetilde{\Psi} = 0, \tag{6.64}$$

where $\mathcal{D}$ is given by (6.54) and $\widetilde{\Psi} := (\widetilde{f},\widetilde{h})^\top$ with

$$\widetilde{f} := -\mathcal{H}_S\,\{\mathcal{T}U\}_S^+ \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \qquad \widetilde{h} = [U]_\Sigma \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6 .$$

Clearly $\widetilde{\Psi} \in \mathbf{X}_{p,p}^{1-\frac{1}{p}} := [B_{p,p}^{1-\frac{1}{p}}(S)]^6 \times [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$. Now by Lemma 6.10 we conclude that $\widetilde{\Psi} = 0$ since the conditions (6.60) are fulfilled for $s = 1 - \frac{1}{p}$ and $\frac{4}{3} < p < 4$. Consequently, $\{\mathcal{T}U\}_S^+ = 0$ on S in view of invertibility of the operator $\mathcal{H}_S$ (see Remark 5.18) and $[U]_\Sigma = 0$ on Σ. But then (6.61) yields $U = 0$ in Ω_Σ^+ which completes the proof. $\square$

Remark 6.12 *Lemma* 6.10 *and Theorem* 6.11 *with* $p = t = 2$ *and* $s = \frac{1}{2}$ *remain valid for Lipschitz domains due to Lemma 5.9 and the uniqueness Theorem 2.1.*

6.4 Crack type problem $(M)^+$-(CN)

We reformulate the problem $(M)^+$-(CN) as follows (see Subsection 2.3.2): find a solution $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_p(\Omega^+_\Sigma)]^6$ to the equation $A(\partial, \tau)U = 0$ in $\Omega^+_\Sigma := \Omega^+ \backslash \overline{\Sigma}$ which satisfies the mixed Dirichlet-Neumann type boundary conditions on the exterior surface $S = \partial\Omega^+ = \overline{S_D} \cap \overline{S_N}$

$$\{U\}^+ = g^{(D)} \quad \text{on} \quad S_D, \tag{6.65}$$

$$\{\mathcal{T}(\partial, n)U\}^+ = G^{(N)} \quad \text{on} \quad S_N, \tag{6.66}$$

and (CN) type conditions on the crack faces

$$\{[\mathcal{T}(\partial, n)U]\}^+ + \{[\mathcal{T}(\partial, n)U]\}^- = G := G^{(+)} + G^{(-)} \quad \text{on} \quad \Sigma, \tag{6.67}$$

$$\{[\mathcal{T}(\partial, n)U]\}^+ - \{[\mathcal{T}(\partial, n)U]\}^- = \widetilde{G} := G^{(+)} - G^{(-)} \quad \text{on} \quad \Sigma, \tag{6.68}$$

where $g^{(D)} \in [B^{1-\frac{1}{p}}_{p,p}(S_D)]^6$, $G^{(N)} \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^6$, and $G^{(\pm)} = (G^{(\pm)}_1, \cdots, G^{(\pm)}_6)^\top \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6$ are given vector functions on S and Σ respectively. We assume the following compatibility condition

$$\widetilde{G} = G^{(+)} - G^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6. \tag{6.69}$$

As in subsection 5.7, we denote by $g^{(e)}$ a fixed extension of the vector-function $g^{(D)}$ from S_D onto S preserving the functional space:

$$g^{(e)} \in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, \qquad r_{S_D} g^{(e)} = g^{(D)} \quad \text{on} \quad S_D. \tag{6.70}$$

Clearly, an arbitrary extension g of $g^{(D)}$ onto the whole of S, which preserves the functional space, can be then represented as

$$g = g^{(e)} + f \quad \text{with} \quad f \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6. \tag{6.71}$$

In accordance with Theorem 5.14, we can seek a solution to the BVP $(M)^+$-(CN) in the form

$$U = V_S\big(\mathcal{H}^{-1}_S\big[g^{(e)} + f - W_\Sigma(h) + V_\Sigma(\widetilde{G})\big]\big) + W_\Sigma(h) - V_\Sigma(\widetilde{G}) \quad \text{in} \quad \Omega^+_\Sigma, \tag{6.72}$$

where $\widetilde{G}$ and $g^{(e)}$ are the above introduced given vector functions with properties (6.69) and (6.70); V_S, V_Σ, and W_Σ are single and double layer potentials defined by (3.48) and (3.49), $\mathcal{H}_S$ is a pseudodifferential operator defined by (4.6) and $\mathcal{H}^{-1}_S$ is the inverse to the operator (5.75); $f = (f_1, \cdots, f_6)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6$ and $h = (h_1, \cdots, h_6)^\top \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6$ are unknown vector functions.

It is clear that the differential equation along with the Dirichlet condition (6.65) and the crack condition (6.68) are satisfied automatically, while the Neumann condition (6.66) and the crack condition (6.67) lead to the system of pseudodifferential equations

$$[-2^{-1}I_6 + \mathcal{K}_S]\,\mathcal{H}^{-1}_S\big[g^{(e)} + f - W_\Sigma(h) + V_\Sigma(\widetilde{G})\big] + \mathcal{T}W_\Sigma(h) - \mathcal{T}V_\Sigma(\widetilde{G}) = G^{(N)} \quad \text{on} \quad S_N,$$

$$2\,\mathcal{T}(\partial, n)V_S\big(\mathcal{H}^{-1}_S\big[g^{(e)} + f - W_\Sigma(h) + V_\Sigma(\widetilde{G})\big]\big) + 2\,\mathcal{L}_\Sigma\, h - 2\,\mathcal{K}_\Sigma\, \widetilde{G} = G \quad \text{on} \quad \Sigma,$$

which can be rewritten as

$$r_{_{S_N}}\mathcal{A}_S^+ f - r_{_{S_N}}\mathcal{A}_S^+ W_\Sigma(h) + r_{_{S_N}}\mathcal{T} W_\Sigma(h) = Q^{(1)} \quad \text{on} \quad S_N, \tag{6.73}$$

$$r_{_\Sigma}\mathcal{T}(\partial,n)V_S\big(\mathcal{H}_S^{-1}f\big) + r_{_\Sigma}\mathcal{L}_\Sigma h - r_{_\Sigma}\mathcal{T}V_S\big(\mathcal{H}_S^{-1}W_\Sigma(h)\big) = Q^{(2)} \quad \text{on} \quad \Sigma, \tag{6.74}$$

where $\mathcal{A}_S^+ := [-2^{-1}I_6 + \mathcal{K}_S]\,\mathcal{H}_S^{-1}$ is the Steklov-Poincaré operator and

$$Q^{(1)} := G^{(N)} - r_{_{S_N}}\mathcal{A}_S^+\big[g^{(e)} + V_\Sigma(\widetilde{G})\big] + r_{_{S_N}}\mathcal{T}V_\Sigma(\widetilde{G}) \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^6\,, \tag{6.75}$$

$$Q^{(2)} := \frac{1}{2}\,G - r_{_\Sigma}\mathcal{T}(\partial,n)V_S\big(\mathcal{H}_S^{-1}\big[g^{(e)} + V_\Sigma(\widetilde{G})\big]\big) + r_{_\Sigma}\mathcal{K}_\Sigma\,\widetilde{G} \in [B_{p,p}^{-\frac{1}{p}}(\Sigma)]^6. \tag{6.76}$$

Here $\mathcal{K}_\Sigma$ and $\mathcal{L}_\Sigma$ are pseudodifferential operators defined by (4.7) and (4.9) .

Denote by $\mathcal{M}$ the pseudodifferential matrix operator generated by the left hand side expressions in (6.73)-(6.74)

$$\mathcal{M} := \begin{bmatrix} r_{_{S_N}}\mathcal{A}_S^+ & -r_{_{S_N}}\mathcal{A}_S^+ W_\Sigma + r_{_{S_N}}\mathcal{T}W_\Sigma \\ r_{_\Sigma}\mathcal{T}(\partial,n)V_S\big(\mathcal{H}_S^{-1}\big) & r_{_\Sigma}\mathcal{L}_\Sigma - r_{_\Sigma}\mathcal{T}V_S\big(\mathcal{H}_S^{-1}W_\Sigma\big) \end{bmatrix}_{12\times 12}. \tag{6.77}$$

Clearly, the operators

$$\begin{gathered} r_{_{S_N}}\mathcal{A}_S^+ W_\Sigma, \quad r_{_{S_N}}\mathcal{A}_S^+ V_\Sigma, \quad r_{_{S_N}}\mathcal{T}W_\Sigma, \quad r_{_{S_N}}\mathcal{T}V_\Sigma, \\ r_{_\Sigma}\mathcal{T}(\partial,n)V_S\big(\mathcal{H}_S^{-1}\big), \quad r_{_\Sigma}\mathcal{T}V_S\big(\mathcal{H}_S^{-1}W_\Sigma\big), \quad r_{_\Sigma}\mathcal{T}V_S\big(\mathcal{H}_S^{-1}V_\Sigma\big) \end{gathered}$$

are smoothing operators, since the manifolds S and Σ are disjoint.

Set

$$\Psi = (f,h)^\top, \quad Q = (Q^{(1)}, Q^{(2)})^\top,$$

and rewrite equations (6.73)-(6.74) in matrix form

$$\mathcal{M}\,\Psi = Q. \tag{6.78}$$

Theorem 4.4 yield the following mapping properties

$$\begin{gathered} \mathcal{M}\,:\,\mathbb{X}_p^s \to \mathbb{Y}_p^{s-1}\,, \qquad \mathcal{M}\,:\,\mathbb{X}_{p,\,t}^s \to \mathbb{Y}_{p,\,t}^{s-1}, \\ s\in\mathbb{R},\ \ 1<p<\infty,\ \ 1\le t\le\infty\ \ S,\,\Sigma\in C^\infty, \end{gathered} \tag{6.79}$$

where

$$\begin{gathered} \mathbb{X}_p^s := [\widetilde{H}_p^s(S_N)]^6 \times [\widetilde{H}_p^s(\Sigma)]^6, \quad \mathbb{Y}_p^s := [H_p^s(S_N)]^6 \times [H_p^s(\Sigma)]^6\,, \\ \mathbb{X}_{p,\,t}^s := [\widetilde{B}_{p,\,t}^s(S_N)]^6 \times [\widetilde{B}_{p,\,t}^s(\Sigma)]^6, \quad \mathbb{Y}_{p,\,t}^s := [B_{p,\,t}^s(S_N)]^6 \times [B_{p,\,t}^s(\Sigma)]^6, \\ s\in\mathbb{R},\ \ 1<p<\infty,\ \ 1\le t\le\infty\ \ S,\,\Sigma\in C^\infty. \end{gathered}$$

Further, let us consider the operator

$$\widetilde{\mathcal{M}} := \begin{bmatrix} r_{_{S_N}}\mathcal{A}_S^+ & 0 \\ 0 & r_{_\Sigma}\mathcal{L}_\Sigma \end{bmatrix}_{12\times 12}. \tag{6.80}$$

It is clear that $\widetilde{\mathcal{M}}$ has the same mapping properties as $\mathcal{M}$ and the operator $\mathcal{M}-\widetilde{\mathcal{M}}$ with the same domain and range spaces as in (6.79) is a compact operator. Moreover, in view of Lemmas 5.19 and 6.1 the operators

$$\widetilde{\mathcal{M}}\,:\,\mathbb{X}^s_p \to \mathbb{Y}^{s-1}_p\,, \qquad \widetilde{\mathcal{M}}\,:\,\mathbb{X}^s_{p,\,t} \to \mathbb{Y}^{s-1}_{p,\,t}\,, \tag{6.81}$$

$$\tag{6.82}$$

are invertible if the following inequalities

$$\frac{1}{p}-\frac{1}{2}<s<\frac{1}{p}+\frac{1}{2}, \quad \frac{1}{p}-\frac{1}{2}+a_2<s<\frac{1}{p}+\frac{1}{2}+a_1, \tag{6.83}$$

are satisfied, where a_1 and a_2 are defined by relations (5.105).

Therefore the operators (6.79) are Fredholm with zero index if the inequalities (6.83) hold.

Lemma 6.13 *The operators* (6.79) *are invertible if the inequalities* (6.83) *hold.*

Proof. As in the case of Lemma 6.10, we need only to show that the corresponding null-spaces are trivial for some particular values of the parameters s and p meeting the inequalities (6.83). We again take $s=\frac{1}{2}$ and $p=2$, and prove that the homogeneous system $\mathcal{M}\,\Psi=0$, i.e., the equations (6.73)-(6.74) with $Q^{(1)}=Q^{(2)}=0$ have only the trivial solution. Indeed, let $\Psi_0=(f_0,h_0)^\top\in[\widetilde{H}^{\frac{1}{2}}_2(S)]^6\times[\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6$ be a solution of the homogeneous system (6.73)-(6.74) and construct the vector

$$U_0(x)=V_S\big(\mathcal{H}_S^{-1}[f_0-W_\Sigma(h_0)]\big)(x)+W_\Sigma(h_0)(x), \quad x\in\mathbb{R}^3\setminus(S\cup\overline{\Sigma}).$$

It is easy to see that $U_0\in[W^1_2(\Omega^+_\Sigma)]^6$ and U_0 solves the homogeneous BVP $(M)^+$-(CN). By Theorem 2.1 we then have $U_0=0$ in Ω^+_Σ. Hence $\{U_0\}^+_\Sigma-\{U_0\}^-_\Sigma=h_0=0$ on Σ follows immediately. Therefore we get $U_0=V_S\big(\mathcal{H}_S^{-1}f_0\big)=0$ in Ω^+_Σ which implies $f_0=\{U_0\}^+_S=0$ on S. Thus $\Psi_0=(f_0,h_0)^\top=0$. $\square$

From Lemma 6.13 the following existence results follow directly.

Theorem 6.14 *Let a_1 and a_2 be defined by relations* (5.105) *and*

$$\frac{4}{3}<p<4, \qquad \frac{4}{3-2\,a_2}<p<\frac{4}{1-2\,a_1}\,, \tag{6.84}$$

$$g^{(D)}\in\big[B^{1-\frac{1}{p}}_{p,\,p}(S_D)\big]^6, \quad G^{(N)}\in\big[B^{-\frac{1}{p}}_{p,\,p}(S_N)\big]^6,$$

$$G^{(\pm)}\in\big[B^{-\frac{1}{p}}_{p,\,p}(\Sigma)\big]^6, \;\; \widetilde{G}=G^{(+)}-G^{(-)}\in[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^6.$$

Then the crack type BVP $(M)^+$-(CN) possesses a unique solution $U\in[W^1_p(\Omega^+_\Sigma)]^6$ which can be represented by formula (6.72), *where the pair*

$$(f,h)^\top\in[\widetilde{B}^{1-\frac{1}{p}}_{p,\,p}(S_N)]^6\times[\widetilde{B}^{1-\frac{1}{p}}_{p,\,p}(\Sigma)]^6$$

is a unique solution of the system of boundary pseudodifferential equations(6.73)-(6.74).

Proof. Existence of solutions directly follows from Lemma 6.13 since the conditions (6.83) are fulfilled for $s = 1 - 1/p$ where p is restricted by the inequalities (6.84). Uniqueness for $p = 2$ follows from Theorem 2.1. Note that for $p = 2$ both relations in (6.84) hold, since $-\frac{1}{2} < a_1 \le a_2 < \frac{1}{2}$.

Now, let p satisfy the inequalities (6.84) and let $U_0 \in [W^1_p(\Omega^+_\Sigma)]^6$ be a solution to the homogeneous BVP $(M)^+$-(CN). We have to show that U_0 vanishes identically in Ω^+_Σ. We proceed as follows.

Using the general integral representation formula (3.54) and keeping in mind the homogeneous crack type conditions (6.67)-(6.68) on Σ we get

$$U_0(x) = W_S\big(\{U_0\}^+_S\big)(x) - V_S\big(\{\mathcal{T}U_0\}^+_S\big)(x) + W_\Sigma\big([U_0]_\Sigma\big)(x), \quad x \in \Omega^+_\Sigma. \tag{6.85}$$

Recall that here $[U_0]_\Sigma$ stands for the jump of the vector U_0 across the crack surface Σ and

$$h_0 := [U_0]_\Sigma \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)\big]^6. \tag{6.86}$$

Note also that due to the homogeneous mixed boundary conditions (6.65)-(6.66)

$$g_0 := \{U_0\}^+_S \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^6, \quad G_0 := \{\mathcal{T}U_0\}^+_S \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)]^6\,. \tag{6.87}$$

With the help of Theorem 5.14 and the relation $\mathcal{N}_S\mathcal{H}_S = \mathcal{H}_S\mathcal{K}_S$ (see Theorem 4.3) we conclude that the double layer potential $W_S(\{U_0\}^+_S)$ can be represented in the form of a single layer potential uniquely

$$W_S\big(\{U_0\}^+_S\big) = W_S\big(g_0\big) = V_S\big(\mathcal{A}^-_S g_0\big) \quad \text{in} \quad \Omega^+_\Sigma, \tag{6.88}$$

where $\mathcal{A}^-_S := [2^{-1}I_6 + \mathcal{K}_S]\,\mathcal{H}^{-1}_S$ is the Steklov-Poincaré operator (see Subsection 4.3). Indeed, one can easily check that the layer potentials $W_S(g_0)$ and $V_S(\mathcal{A}^-_S g_0)$ have the same Dirichlet data on the boundary S,

$$\begin{aligned}\{W_S(g_0)\}^+ &= \big[\frac{1}{2}\,I_6 + \mathcal{N}_S\big]\,g_0 = \big[\frac{1}{2}\,I_6 + \mathcal{N}_S\big]\,\mathcal{H}_S\,\mathcal{H}^{-1}_S\,g_0\\ &= \mathcal{H}_S\,\big[\frac{1}{2}\,I_6 + \mathcal{K}_S\big]\mathcal{H}^{-1}_S\,g_0 = \mathcal{H}_S\,\mathcal{A}^-_S g_0 = \{V_S\big(\mathcal{A}^-_S g_0\big)\}^+.\end{aligned}$$

Therefore $W_S(g_0) = V_S(\mathcal{A}^-_S g_0)$ in Ω^+ by the uniqueness Theorem 2.1. Consequently, from (6.85) it follows that U_0 is representable in the form

$$U_0 = V_S\big(\chi\big) + W_\Sigma\big([U_0]_\Sigma\big) \quad \text{in} \ \ \Omega^+_\Sigma \quad \text{with} \ \ \chi := \mathcal{A}^-_S\{U_0\}^+_S - \{\mathcal{T}U_0\}^+_S\,. \tag{6.89}$$

In turn, (6.89) yields

$$\{U_0\}^+_S = \mathcal{H}_S\,\chi + W_\Sigma\big([U_0]_\Sigma\big) \quad \text{on} \ \ S.$$

Whence

$$\chi = \mathcal{H}^{-1}_S\,\big[\{U_0\}^+_S - W_\Sigma\big([U_0]_\Sigma\big)\big] \quad \text{on} \ \ S$$

and finally, in view of (6.89), we arrive at the representation (cf. (6.72))

$$
\begin{aligned}
U_0 &= V_S\big(\mathcal{H}_S^{-1}\left[\{U_0\}_S^+ - W_\Sigma\big([U_0]_\Sigma\big)\right]\big) + W_\Sigma\big([U_0]_\Sigma\big) \\
&= V_S\big(\mathcal{H}_S^{-1}\left[g_0 - W_\Sigma\big(h_0\big)\right]\big) + W_\Sigma\big(h_0\big) \quad \text{ in } \quad \Omega_\Sigma^+,
\end{aligned} \tag{6.90}
$$

where g_0 and h_0 are given by relations (6.87) and (6.86).

Now recall that by assumption U_0 solves the homogeneous BVP $(M)^+$-(CN). As we see form the representation (6.90), the vector U_0 satisfies the homogeneous boundary conditions (6.65) and (6.68) with $g^{(D)} = 0$ and $\widetilde{G} = 0$, while the homogeneous conditions (6.66) and (6.67) with $G^{(N)} = 0$ and $G = 0$ give the following relations (cf. (6.73), (6.74)):

$$
r_{_{S_N}} \mathcal{A}_S^+ g_0 - r_{_{S_N}} \mathcal{A}_S^+ W_\Sigma(h_0) + r_{_{S_N}} \mathcal{T} W_\Sigma(h_0) = 0 \quad \text{on} \quad S_N, \tag{6.91}
$$

$$
r_{_\Sigma} \mathcal{T}(\partial, n) V_S\big(\mathcal{H}_S^{-1} g_0\big) + r_{_\Sigma} \mathcal{L}_\Sigma\, h_0 - r_{_\Sigma} \mathcal{T} V_S\big(\mathcal{H}_S^{-1} W_\Sigma(h_0)\big) = 0 \quad \text{on} \quad \Sigma, \tag{6.92}
$$

where $\mathcal{A}_S^+ := [-2^{-1} I_6 + \mathcal{K}_S]\, \mathcal{H}_S^{-1}$ is the Steklov-Poincaré operator (see Subsection 4.3).

It is easy to see that this system is equivalent to the homogeneous equation

$$
\mathcal{M}\, \widetilde{\Psi}_0 = 0, \tag{6.93}
$$

where $\mathcal{M}$ is given by (6.77) and $\widetilde{\Psi}_0 := (g_0, h_0)^\top \in \mathbf{X}_{p,p}^{1-\frac{1}{p}} := [B_{p,p}^{1-\frac{1}{p}}(S_N)]^6 \times [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)]^6$. By Lemma 6.13 we then conclude that $\widetilde{\Psi}_0 = 0$ since the conditions (6.83) are fulfilled if p satisfies inequalities (6.84) and $s = 1 - \frac{1}{p}$. Consequently, $g_0 := \{U_0\}_S^+ = 0$ and $h_0 := [U_0]_\Sigma = 0$. But then (6.90) yields $U_0 = 0$ in Ω_Σ^+ which completes the proof. $\square$

Remark 6.15 *Lemma* 6.13 *and Theorem* 6.14 *with $p = t = 2$ and $s = \frac{1}{2}$ remain valid for Lipschitz domains due to Lemmas 5.9 and 4.12, and the uniqueness Theorem 2.1.*

Remark 6.16 *Note that smoothness results for solutions of the BVPs $(D)^+$-(CN) and $(M)^+$-(CN) near the boundaries and near the exceptional curves are described by Theorems 5.2, 5.21 and 6.7.*

7 Basic BVPs of statics

We demonstrate our approach for the interior and exterior Neumann-type boundary-value problems of statics (see [MN], [MNT], [Mrev]). The Dirichlet and mixed type BVPs of statics can be studied quite analogously.

7.1 Formulation of Problems

The basic differential equations of statics read as follows (cf. (2.31)):

$$\begin{aligned}
&c_{rjkl}\partial_j\partial_l u_k(x) + e_{lrj}\partial_j\partial_l\varphi(x) + q_{lrj}\partial_j\partial_l\psi(x) - \lambda_{rj}\partial_j\vartheta(x) = \\
&\qquad\qquad\qquad\qquad = -X_r(x), \qquad r = 1,2,3, \\
&-e_{jkl}\partial_j\partial_l u_k(x) + \varkappa_{jl}\partial_j\partial_l\varphi(x) + a_{jl}\partial_j\partial_l\psi(x) - p_j\partial_j\vartheta(x) = -\varrho_e(x), \\
&-q_{jkl}\partial_j\partial_l u_k(x) + a_{jl}\partial_j\partial_l\varphi(x) + \mu_{jl}\partial_j\partial_l\psi(x) - m_j\partial_j\vartheta(x) = 0, \\
&\eta_{jl}\partial_j\partial_l\vartheta(x) = -Q(x).
\end{aligned} \tag{7.1}$$

In matrix form these equations can be written as

$$A(\partial)U(x) = \Phi(x),$$

where

$$U = (u_1, u_2, u_3, u_4, u_5, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top,$$
$$\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top,$$

and $A(\partial)$ is the matrix differential operator generated by equations (2.31) (cf. (2.33))

$$A(\partial) = A(\partial, 0) = [A_{pq}(\partial)]_{6\times 6} :=$$
$$:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}. \tag{7.2}$$

Neumann problems $(N)^\pm$: *Find a regular solution vector*

$$U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6 \quad (\ \textit{resp.}\ U \in [C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6),$$

to the system of equations

$$A(\partial)U = \Phi \quad in \quad \Omega^\pm,$$

satisfying the Neumann-type boundary conditions

$$\{\mathcal{T}U\}^\pm = f \quad on \quad S,$$

where $A(\partial)$ is a nonselfadjoint strongly elliptic matrix partial differential operator generated by the equations of statics of the theory of thermo-electro-magneto-elasticity defined in (7.2), *while $\mathcal{T}(\partial, n)$ is the matrix boundary operator defined in* (2.26).

Let us introduce the following class of vector functions (see Subsection 3.5.2).

Definition 7.1 *We say that a continuous vector $U = (u, \varphi, \psi, \vartheta)^\top \equiv (U_1, \cdots, U_6)^\top$ in the domain Ω^- has the property $Z(\Omega^-)$ if the following conditions are satisfied*

$$\widetilde{U}(x) := (u(x), \varphi(x), \psi(x))^\top = \mathcal{O}(1) \quad as \quad |x| \to \infty,$$

$$U_6(x) = \vartheta(x) = \mathcal{O}(|x|^{-1}) \quad as \quad |x| \to \infty,$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma_R} U_k(x)\, d\Sigma_R = 0, \quad k = \overline{1,5},$$

where Σ_R is a sphere centered at the origin and radius R.

In what follows we always assume that in the case of exterior boundary-value problem a solution possesses $Z(\Omega^-)$ property.

7.2 Potentials of Statics and their properties

From the results obtained in Subsection 3.3 it follows that the fundamental matrix $\Gamma(x) = \Gamma(x, 0) = [\Gamma_{kj}(x)]_{6\times 6}$ which solves the equation $A(\partial)\Gamma(x) = I_6\, \delta(x)$, where $\delta(\cdot)$ is the Dirac's delta distribution and I_6 stands for the unit 6×6 matrix, can be represented in the form (cf. (3.40), (3.46))

$$\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\,\xi)]\,, \tag{7.3}$$

where $\mathcal{F}^{-1}$ is the generalized inverse Fourier transform and $A^{-1}(-i\,\xi)$ is the matrix inverse to $A(-i\,\xi)$. Moreover, the entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in x and at the origin and at infinity the following asymptotic relations hold

$$\Gamma(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}. \tag{7.4}$$

From the relations (7.4) and (3.45) it easily follows that the columns of the matrix $\Gamma(x)$ possess the property $Z(\mathbb{R}^3 \setminus \{0\})$.

With the help of this fundamental matrix we construct the generalized single and double layer potentials, and the Newton-type volume potentials of statics (the potentials of statics are equipped with the subscript "zero" showing that they correspond to the above introduced pseudo-oscillation potentials with $\tau = 0$)

$$V_0(h)(x) = V_{S,0}(h)(x) = \int_S \Gamma(x-y)\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

$$W_0(h)(x) = W_{S,0}(h)(x) = \int_S [\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)]^\top\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S,$$

$$N_{\Omega^\pm,0}(g)(x) = \int_{\Omega^\pm} \Gamma(x-y)\, g(y)\, dy, \quad x \in \mathbb{R}^3,$$

where $h = (h_1, \cdots, h_6)^\top$ and $g = (g_1, \cdots, g_6)^\top$ are density vector-functions defined respectively on S and in $\Omega^\pm$; the so called *generalized stress operator* $\mathcal{P}(\partial, n)$, associated with the adjoint differential operator $A^*(\partial) = A^\top(-\partial)$, reads as (cf. (2.38))

$$\mathcal{P}(\partial, n) = \mathcal{P}(\partial, n, 0) = \big[\mathcal{P}_{pq}(\partial, n)\big]_{6\times 6} =$$

$$= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & 0 \\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}. \tag{7.5}$$

The following properties of layer potentials of statics immediately follow from their definition and the results exposed in Sections 3 and 4.

Theorem 7.2 *The generalized single and double layer potentials solve the homogeneous differential equation $A(\partial)U = 0$ in $\mathbb{R}^3 \setminus S$ and possess the property $Z(\Omega^-)$.*

With the help of Green's formulas, one can derive general integral representation formulas of solutions to the homogeneous equation $A(\partial)U = 0$ in $\Omega^\pm$. In particular, the following theorems hold.

Theorem 7.3 *Let $S = \partial\Omega^+ \in C^{1,\kappa}$ with $0 < \kappa \le 1$ and U be a regular solution to the homogeneous equation $A(\partial)U = 0$ in Ω^+ of the class $[C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$. Then there holds the integral representation formula*

$$W_0(\{U\}^+)(x) - V_0(\{\mathcal{T}U\}^+)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases}$$

Theorem 7.4 *Let $S = \partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0 < \kappa \le 1$ and let U be a regular solution to the homogeneous equation $A(\partial)U = 0$ in Ω^- of the class $[C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$ having the property $Z(\Omega^-)$. Then there holds the integral representation formula*

$$-W_0(\{U\}^-)(x) + V_0(\{\mathcal{T}U\}^-)(x) = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-. \end{cases}$$

By standard limiting procedure, these formulas can be extended to Lipschitz domains and to solution vectors from the spaces $[W^1_p(\Omega^+)]^6$ and $[W^1_{p,loc}(\Omega^-)]^6 \cap Z(\Omega^-)$ with $1 < p < \infty$ (cf., [HW], [Mc1], [NCS1]).

The qualitative and mapping properties of the layer potentials are described by the following theorems (cf. [BCN3], [Du1], [KGBB], [Mc1]).

Theorem 7.5 *Let $S = \partial\Omega^\pm \in C^{m,\kappa}$ with integers $m \geq 1$ and $k \leq m-1$, and $0 < \kappa' < \kappa \leq 1$. Then the operators*

$$V_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6, \quad W_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(\overline{\Omega^\pm})]^6 \tag{7.6}$$

are continuous.

For any $g \in [C^{0,\kappa'}(S)]^6$, $h \in [C^{1,\kappa'}(S)]^6$, and any $x \in S$ we have the following jump relations:

$$\{V_0(g)(x)\}^\pm = V_0(g)(x) = \mathcal{H}_0 g(x), \tag{7.7}$$

$$\big\{\mathcal{T}(\partial_x, n(x))V_0(g)(x)\big\}^\pm = \big[\mp 2^{-1} I_6 + \mathcal{K}_0\big] g(x), \tag{7.8}$$

$$\{W_0(g)(x)\}^\pm = [\pm 2^{-1} I_6 + \mathcal{N}_0] g(x), \tag{7.9}$$

$$\begin{aligned} &\big\{\mathcal{T}(\partial_x, n(x))W_0(h)(x)\big\}^+ = \\ &= \{\mathcal{T}(\partial_x, n(x))W_0(h)(x)\}^- = \mathcal{L}_0 h(x), \quad m \geq 2, \end{aligned} \tag{7.10}$$

where $\mathcal{H}$ is a weakly singular integral operator, $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, and $\mathcal{L}$ is a singular integro-differential operator,

$$\begin{aligned} \mathcal{H}_0 g(x) &:= \int_S \Gamma(x-y) g(y)\, dS_y, \\ \mathcal{K}_0 g(x) &:= \int_S \mathcal{T}(\partial_x, n(x))\Gamma(x-y)\; g(y)\, dS_y, \\ \mathcal{N}_0 g(x) &:= \int_S \big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)\big]^\top g(y)\, dS_y, \\ \mathcal{L}_0 h(x) &:= \lim_{\Omega^\pm \ni z \to x \in S} \mathcal{T}(\partial_z, n(x)) \int_S \big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(z-y)\big]^\top h(y)\, dS_y. \end{aligned} \tag{7.11}$$

Theorem 7.6 *Let S be a Lipschitz surface. The operators V and W can be extended to the continuous mappings*

$$\begin{aligned} &V_0 : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6, \quad V_0 \;:\; [H_2^{-\frac{1}{2}}(S)]^6 \to [H^1_{2,loc}(\Omega^-)]^6 \cap Z(\Omega^-), \\ &W_0 : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6, \quad W_0 : [H_2^{\frac{1}{2}}(S)]^6 \to [H^1_{2,loc}(\Omega^-)]^6 \cap Z(\Omega^-). \end{aligned}$$

The jump relations (7.7)–(7.10) on S remain valid for the extended operators in the corresponding function spaces.

Theorem 7.7 *Let S, m, κ, κ' and k be as in Theorem 7.5. Then the operators*

$$\mathcal{H}_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6, \quad m \geq 1, \tag{7.12}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6, \quad m \geq 1, \tag{7.13}$$

$$\mathcal{K}_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \quad m \geq 1, \tag{7.14}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6, \quad m \geq 1, \tag{7.15}$$

$$\mathcal{N}_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \quad m \geq 1, \tag{7.16}$$

$$: [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6, \quad m \geq 1, \tag{7.17}$$

$$\mathcal{L}_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k-1,\kappa'}(S)]^6, \quad m \geq 2, \quad k \geq 1, \tag{7.18}$$

$$: [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6, \quad m \geq 2, \tag{7.19}$$

are continuous. The operators (7.13), (7.15), (7.17), *and* (7.19) *are bounded if S is a Lipschitz surface.*

Proofs of the above formulated theorems are word for word proofs of the similar theorems in [Co1], [DNS1], [DNS2], [JN1], [JN2], [KGBB], [Na1], [NDS1].

From Corollary 4.9 and the uniqueness Theorem 2.3 for the Dirichlet static problem we can deduce the following assertion.

Theorem 7.8 *Let S, $m \geq 1$, κ, κ' and k be as in Theorem 7.5. Then the operators*

$$\mathcal{H}_0 : [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6, \tag{7.20}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6, \tag{7.21}$$

are invertible.

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [Ag1], [BCNS1], [Du1], [HW], [Se1], and the references therein).

Theorem 7.9 *Let V_0, W_0, $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{N}_0$ and $\mathcal{L}_0$ be as in Theorems 7.5 and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $S \in C^\infty$. The layer potential operators* (7.6) *and the boundary integral* (*pseudodifferential*) *operators* (7.12)–(7.19) *can be extended to the following continuous operators*

$$\begin{aligned}
&V_0 : [B^s_{p,p}(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6, && W_0 : [B^s_{p,p}(S)]^6 \to [H_p^{s+\frac{1}{p}}(\Omega^+)]^6,\\
&V_0 : [B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6, && W_0 : [B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)]^6,\\
&\mathcal{H}_0 : [H^s_p(S)]^6 \to [H^{s+1}_p(S)]^6, && \mathcal{K}_0 : [H^s_p(S)]^6 \to [H^s_p(S)]^6,\\
&\mathcal{N}_0 : [H^s_p(S)]^6 \to [H^s_p(S)]^6, && \mathcal{L}_0 : [H^{s+1}_p(S)]^6 \to [H^s_p(S)]^6.
\end{aligned}$$

The jump relations (7.7)–(7.10) *remain valid for arbitrary $g \in [B^s_{p,q}(S)]^6$ with $s \in \mathbb{R}$ if the limiting values* (*traces*) *on S are understood in the sense described in* [Se1].

Remark 7.10 *Let either $\Phi \in [L_p(\Omega^+)]^6$ or $\Phi \in [L_{p,comp}(\Omega^-)]^6$, $p > 1$. Then the Newtonian volume potentials $N_{\Omega^\pm}(\Phi)$ possess the following properties (see, e.g., [MP]):*

$$N_{\Omega^+,0}(\Phi) \in [W_p^2(\Omega^+)]^6, \quad N_{\Omega^-,0}(\Phi) \in [W_{p,loc}^2(\Omega^-)]^6,$$
$$A(\partial)N_{\Omega^\pm,0}(\Phi) = \Phi \quad \textit{almost everywhere in} \quad \Omega^\pm.$$

Therefore, without loss of generality, we can assume that in the formulation of the Neumann-type problems the right hand side function in the differential equations vanishes, $\Phi(x) = 0$ in $\Omega^\pm$.

7.3 Investigation of the Exterior Neumann BVP

We start with the exterior Neumann-type BVP for the domain Ω^-:

$$A(\partial)U(x) = 0, \quad x \in \Omega^-, \tag{7.22}$$

$$\big\{\mathcal{T}(\partial, n)U(x)\big\}^- = F(x), \quad x \in S. \tag{7.23}$$

We assume that $S \in C^{1,\kappa}$ and $F \in C^{0,\kappa'}(S)$ with $0 < \kappa' < \kappa \leq 1$. We investigate this problem in the space of regular vector functions $[C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$. As we have shown in Subsection 3.5.2 the homogeneous version of the exterior Neumann-type problem possesses at most one solution.

To prove the existence result, we look for a solution of the problem (7.22)–(7.23) as the single layer potential

$$U(x) \equiv V_0(h)(x) = \int\limits_S \Gamma(x-y)h(y)\, dS_y, \tag{7.24}$$

where Γ is defined by (7.3) and $h = (h_1, \dots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is unknown density. By Theorem 7.5 and in view of the boundary condition (7.23), we get the following integral equation for the density vector h,

$$[2^{-1}I_6 + \mathcal{K}_0]h = F \quad \text{on} \quad S, \tag{7.25}$$

where $\mathcal{K}_0$ is a singular integral operator defined by (7.11). Note that the operator $2^{-1}I_6 + \mathcal{K}_0$ has the following mapping properties

$$2^{-1}I_6 + \mathcal{K}_0 : [C^{0,\kappa'}(S)]^6 \to [C^{0,\kappa'}(S)]^6, \tag{7.26}$$

$$: [L_2(S)]^6 \to [L_2(S)]^6. \tag{7.27}$$

These operators are compact perturbations of their counterpart operators associated with the pseudo-oscillation equations which are studied in Section 5. Therefore we see that $2^{-1}I_6 + \mathcal{K}_0$ is a singular integral operator of normal type (i.e., its principal homogeneous symbol matrix is non-degenerate) and its index equals to zero.

Let us show that the operators (7.26) and (7.27) have trivial null spaces. To this end, it suffices to prove that the corresponding homogeneous integral equation

$$[2^{-1}I_6 + \mathcal{K}_0]h = 0 \quad \text{on} \quad S, \tag{7.28}$$

has only the trivial solution in the appropriate space. Let $h^{(0)} \in [L_2(S)]^6$ be a solution to equation (7.28). By the embedding theorems (see, e.g., [KGBB], Ch.4), we actually have that $h^{(0)} \in [C^{0,\kappa'}(S)]^6$. Now we construct the single layer potential $U_0(x) = V_0(h^{(0)})(x)$. Evidently, $U_0 \in [C^{1,\kappa'}(\overline{\Omega^{\pm}})]^6 \cap [C^2(\Omega^{\pm})]^6 \cap Z(\Omega^-)$ and the equation $A(\partial)U_0 = 0$ in $\Omega^{\pm}$ is automatically satisfied. Since $h^{(0)}$ solves equation (7.28), we have

$$\{\mathcal{T}(\partial, n)U_0\}^- = [2^{-1}I_6 + \mathcal{K}_0]h^{(0)} = 0 \quad \text{on} \quad S.$$

Therefore U_0 is a solution to the homogeneous exterior Neumann problem satisfying the property $Z(\Omega^-)$. Consequently, due to the uniqueness Theorem 3.10, $U_0 = 0$ in Ω^-. Applying the continuity property of the single layer potential we find $0 = \{U_0\}^- = \{U_0\}^+$ on S, yielding that the vector $U_0 = V_0(h^{(0)})$ represents a solution to the homogeneous interior Dirichlet problem. Now by the uniqueness Theorem 2.3 for the Dirichlet problem, we deduce that $U_0 = 0$ in Ω^+. Thus $U_0 = 0$ in $\Omega^{\pm}$. By virtue of the jump formula

$$\big\{\mathcal{T}(\partial, n)U_0\big\}^+ - \big\{\mathcal{T}(\partial, n)U_0\big\}^- = -h^{(0)} = 0 \quad \text{on} \quad S,$$

whence it follows that the null space of the operator $2^{-1}I_6 + \mathcal{K}_0$ is trivial and the operators (7.26) and (7.27) are invertible. As a ready consequence, we finally conclude that the non-homogeneous integral equation (7.25) is solvable for arbitrary right hand side vector $F \in [C^{0,\kappa'}(S)]^6$, which implies the following existence result.

Theorem 7.11 *Let $m \geq 0$ be a nonnegative integer and $0 < \kappa' < \kappa \leq 1$. Further, let $S \in C^{m+1,\kappa}$ and $F \in [C^{m,\kappa'}(S)]^6$. Then the exterior Neumann-type BVP (7.22)–(7.23) is uniquely solvable in the space of regular vector functions, $[C^{m+1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$, and the solution is representable by the single layer potential $U(x) = V_0(h)(x)$ with the density $h = (h_1, \dots, h_6)^\top \in [C^{m,\kappa'}(S)]^6$ being a unique solution of the integral equation (7.25).*

Remark 7.12 *Let S be Lipschitz and $F \in \big[H^{-1/2}(S)\big]^6$. Then by the same approach as in the reference [Mc1], the following propositions can be established:*

(i) the integral equation (7.25) is uniquely solvable in the space $[H^{-1/2}(S)]^6$;

(ii) the exterior Neumann-type BVP (7.22)–(7.23) is uniquely solvable in the space of vector -functions $[H^1_{2,loc}(\Omega^-)]^6 \cap Z(\Omega^-)$ and the solution is representable by the single layer potential (7.24), where the density vector $h \in [H^{-1/2}(S)]^6$ solves the integral equation (7.25).

7.4 Investigation of the Interior Neumann BVP

Before we go over to the interior Neumann problem we prove some preliminary assertions needed in our analysis.

7.4.1 Some auxiliary results

Let us consider the adjoint operator $A^*(\partial)$ to the operator $A(\partial)$

$$A^*(\partial) :=$$

$$:= \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times 3} & [-e_{jkl}\partial_j\partial_l]_{3\times 1} & [-q_{jkl}\partial_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{lrj}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0 \\ [q_{lrj}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0 \\ [\lambda_{rj}\partial_j]_{1\times 3} & p_j\partial_j & m_j\partial_j & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}. \tag{7.29}$$

The corresponding matrix of fundamental solutions $\Gamma^*(x-y) = [\Gamma(y-x)]^\top$ has the following property at infinity

$$\Gamma^*(x-y) = \Gamma^\top(y-x) := \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [0]_{5\times 1} \\ [\mathcal{O}(1)]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}$$

as $|x| \to \infty$. With the help of the fundamental matrix $\Gamma^*(x-y)$ we construct the single and double layer potentials, and the Newtonian volume potentials

$$V_0^*(h^*)(x) = V_{S,0}^*(h^*)(x) = \int\limits_S \Gamma^*(x-y)h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{7.30}$$

$$W_0^*(h^*)(x) = W_{S,0}^*(h^*)(x) = \int\limits_S \left[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\right]^\top h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{7.31}$$

$$N_{\Omega^\pm,0}^*(g^*)(x) = \int\limits_{\Omega^\pm} \Gamma^*(x-y)g^*(y)\,dy, \quad x \in \mathbb{R}^3,$$

where the density vector $h^* = (h_1^*, \dots, h_6^*)^\top$ is defined on S, while $g^* = (g_1^*, ..., g_6^*)^\top$ is defined in $\Omega^\pm$. We assume that in the case of the domain Ω^- the vector g^* has a compact support.

It can be shown that the layer potentials V_0^* and W_0^* possess exactly the same mapping properties and jump relations as the potentials V_0 and W_0 (see Theorems 7.5–7.9). In particular,

$$\{V_0^*(h^*)\}^+ = \{V_0^*(h^*)\}^- = \mathcal{H}_0^* h^*,$$

$$\{W_0^*(h^*)\}^\pm = \pm\, 2^{-1}\, h^* + \mathcal{K}_0^* h^*, \tag{7.32}$$

$$\left\{\mathcal{P}V_0^*(h^*)\right\}^\pm = \mp\, 2^{-1}\, h^* + \mathcal{N}_0^* h^*, \tag{7.33}$$

where $\mathcal{H}_0^*$ is a weakly singular integral operator, while $\mathcal{K}_0^*$ and $\mathcal{N}_0^*$ are singular integral operators,

$$\begin{aligned}
\mathcal{H}_0^* h^*(x) &:= \int\limits_S \Gamma^*(x-y) h^*(y)\, dS_y, \\
\mathcal{K}_0^* h^*(x) &:= \int\limits_S \big[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\big]^\top h^*(y)\, dS_y, \\
\mathcal{N}_0^* h^*(x) &:= \int\limits_S [\mathcal{P}(\partial_x, n(x))\Gamma^*(x-y)] h^*(y)\, dS_y.
\end{aligned} \tag{7.34}$$

Now we introduce a special class of vector functions which is a counterpart of the class $Z(\Omega^-)$.

Definition 7.13 *We say that a continuous vector function* $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^\top$ *has the property* $Z^*(\Omega^-)$ *in the domain* Ω^-, *if the following conditions are satisfied*

$$\begin{aligned}
&\widetilde{U}^*(x) = \big(u^*(x), \varphi^*(x), \psi^*(x)\big)^\top = \mathcal{O}(|x|^{-1}) \quad \textit{as} \quad |x| \to \infty, \\
&\vartheta^*(x) = \mathcal{O}(1) \quad \textit{as} \ \ |x| \to \infty, \\
&\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma_R} \vartheta^*(x)\, d\Sigma_R = 0,
\end{aligned}$$

where Σ_R *is a sphere centered at the origin and radius* R.

The following theorem holds.

Theorem 7.14 *The generalized single and double layer potentials, defined by* (7.30) *and* (7.31), *solve the homogeneous differential equation* $A^*(\partial)U^* = 0$ *in* $\mathbb{R}^3 \setminus S$ *and possess the property* $Z^*(\Omega^-)$.

For an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in Ω^+ one can derive the following integral representation formula

$$W_0^*(\{U^*\}^+)(x) - V_0^*\big(\{\mathcal{P}U^*\}^+\big)(x) = \begin{cases} U^*(x) & \text{for } \ x \in \Omega^+, \\ 0 & \text{for } \ x \in \Omega^-. \end{cases} \tag{7.35}$$

Similar representation formula holds also for an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in Ω^- which possesses the property $Z^*(\Omega^-)$:

$$-W_0^*\big(\{U^*\}_S^-\big)(x) + V_0^*\big(\{\mathcal{P}U^*\}_S^-\big)(x) = \begin{cases} U^*(x), & x \in \Omega^-, \\ 0, & x \in \Omega^+. \end{cases} \tag{7.36}$$

To derive this representation we denote $\Omega_R^- := B(0,R)\backslash\overline{\Omega^+}$, where $B(0,R)$ is a ball centered at the origin and radius R. Then in view of (7.35) we have

$$U^*(x) = -W^*_{S,0}\big(\{U^*\}_S^-\big)(x) + V^*_{S,0}\big(\{\mathcal{P}U^*\}_S^-\big)(x) + \Phi^*_R(x), \ \ x \in \Omega_R^-, \tag{7.37}$$

$$0 = -W^*_{S,0}\big(\{U^*\}_S^-\big)(x) + V^*_{S,0}\big(\{\mathcal{P}U^*\}_S^-\big)(x) + \Phi^*_R(x), \ \ x \in \Omega^+, \tag{7.38}$$

where

$$\Phi^*_R(x) := W^*_{\Sigma_R,0}\big(U^*\big)(x) - V^*_{\Sigma_R,0}\big(\mathcal{P}U^*\big)(x). \tag{7.39}$$

Here $V^*_{\mathcal{M},0}$ and $W^*_{\mathcal{M},0}$ denote the single and double layer potential operators (7.30) and (7.31) with integration surface $\mathcal{M}$. Evidently

$$A^*(\partial)\Phi^*_R(x) = 0, \ \ |x| < R. \tag{7.40}$$

In turn, from (7.37) and (7.38) we get

$$\begin{aligned} \Phi^*_R(x) &= U^*(x) + W^*_{S,0}\big(\{U^*\}_S^-\big)(x) - V^*_{S,0}\big(\{\mathcal{P}U^*\}_S^-\big)(x), \ \ x \in \Omega_R^-, \\ \Phi^*_R(x) &= W^*_{S,0}\big(\{U^*\}_S^-\big)(x) - V^*_{S,0}\big(\{\mathcal{P}U^*\}_S^-\big)(x), \ \ x \in \Omega^+, \end{aligned} \tag{7.41}$$

whence the equality $\Phi^*_{R_1}(x) = \Phi^*_{R_2}(x)$ follows for $|x| < R_1 < R_2$. We assume that R_1 and R_2 are sufficiently large numbers. Therefore, for an arbitrary fixed point $x \in \mathbb{R}^3$ the following limit exists

$$\Phi^*(x) := \lim_{R\to\infty} \Phi^*_R(x) =$$

$$= \begin{cases} U^*(x) + W^*_{S,0}\big(\{U^*\}_S^-\big)(x) - V^*_{S,0}\big(\{\mathcal{P}U^*\}_S^-\big)(x), & x \in \Omega^-, \\ W^*_{S,0}\big(\{U^*\}_S^-\big)(x) - V^*_{S,0}\big(\{\mathcal{P}U^*\}_S^-\big)(x), & x \in \Omega^+, \end{cases} \tag{7.42}$$

and $A^*(\partial)\Phi^*(x) = 0$ for all $x \in \Omega^+ \cup \Omega^-$. On the other hand, for arbitrary fixed point $x \in \mathbb{R}^3$ and a number R_1, such that $|x| < R_1$ and $\overline{\Omega^+} \subset B(0,R_1)$, from (7.41) we have

$$\Phi^*(x) = \lim_{R\to\infty} \Phi^*_R(x) = \Phi^*_{R_1}(x).$$

Now from (7.39)–(7.40) we deduce

$$A^*(\partial)\Phi^*(x) = 0 \ \ \forall\, x \in \mathbb{R}^3. \tag{7.43}$$

Since $U^*_{S,0}$, $W^*_{S,0}$, $V^*_{S,0} \in Z^*(\Omega^-)$ we conclude from (7.42) that $\Phi^*(x) \in Z^*(\mathbb{R}^3)$. In particular, we have

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma_R} \Phi^*(x)\, d\Sigma_R = 0. \tag{7.44}$$

Our goal is to show that

$$\Phi^*(x) = 0 \ \ \forall\, x \in \mathbb{R}^3.$$

Applying the generalized Fourier transform to equation (7.43) we get

$$A^*(-i\xi)\widehat{\Phi}^*(\xi) = 0, \ \ \xi \in \mathbb{R}^3,$$

where $\widehat{\Phi}^*(\xi)$ is the Fourier transform of Φ^*. Taking into account that $\det A^*(-i\xi) \neq 0$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, we conclude that the support of the generalized functional $\widehat{\Phi}^*(\xi)$ is the origin and consequently

$$\widehat{\Phi}^*(\xi) = \sum_{|\alpha| \leq M} c_\alpha \delta^{(\alpha)}(\xi),$$

where α is a multi-index, c_α are arbitrary constant vectors and M is some nonnegative integer. Then it follows that $\Phi^*(x)$ is polynomial in x and due to the inclusion $\Phi^* \in Z^*(\Omega^-)$, $\Phi^*(x)$ is bounded at infinity, i.e., $\Phi^*(x) = const$ in $\mathbb{R}^3$. Therefore (7.44) implies that $\Phi^*(x)$ vanishes identically in $\mathbb{R}^3$. This proves that the formula (7.36) holds.

Theorem 7.15 *Let $S \in C^{2,\kappa}$ and $h \in \big[C^{1,\kappa'}(S)\big]^6$ with $0 < \kappa' < \kappa \leq 1$. Then for the double layer potential W_0^* defined by* (7.31) *there holds the following formula* (*generalized Lyapunov–Tauber relation*)

$$\big\{\mathcal{P}W_0^*(h)\big\}^+ = \big\{\mathcal{P}W_0^*(h)\big\}^- \quad on \quad S, \tag{7.45}$$

where the operator $\mathcal{P}$ is given by (7.5).

For $h \in [H_2^{\frac{1}{2}}(S)]^6$ the relation (7.45) *also holds true in the space $[H_2^{-\frac{1}{2}}(S)]^6$.*

Proof. Since $h \in \big[C^{1,\kappa'}(S)\big]^6$, evidently $U^* := W_0^*(h) \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6$. It is clear that the vector U^* is a solution of the homogeneous equation $A^*(\partial)U^*(x) = 0$ in $\Omega^+ \cup \Omega^-$, where the operator $A^*(\partial)$ is defined by (7.29). With the help of (7.35) and (7.36), for the vector function U^* we derive the following representation formula

$$U^*(x) = W_0^*([U^*]_S)(x) - V_0^*\big([\mathcal{P}U^*]_S\big)(x), \ \ x \in \Omega^+ \cup \Omega^-, \tag{7.46}$$

where

$$[U^*]_S \equiv \{U^*\}^+ - \{U^*\}^- \quad \text{and} \quad [\mathcal{P}U^*]_S \equiv \{\mathcal{P}U^*\}^+ - \{\mathcal{P}U^*\}^- \quad \text{on} \quad S.$$

In view of the equality $U^* = W^*(h)$, from (7.46) we get

$$W_0^*(h)(x) = W^*([W_0^*(h)]_S)(x) - V_0^*([\mathcal{P}W_0^*(h)]_S)(x), \ \ x \in \Omega^+ \cup \Omega^-.$$

Using the jump relation (7.32), we find

$$[U^*]_S = [W^*(h)]_S = \{W_0^*(h)\}^+ - \{W_0^*(h)\}^- = h.$$

Therefore

$$W_0^*(h)(x) = W_0^*(h)(x) - V_0^*([\mathcal{P}W_0^*(h)]_S)(x), \ \ x \in \Omega^+ \cup \Omega^-,$$

i.e., $V_0^*(\Phi^*)(x) = 0$ in $\Omega^+ \cup \Omega^-$, where $\Phi^* := [\mathcal{P}W_0^*(h)]_S$. With the help of the jump relation (7.33) finally we arrive at the equation

$$0 = \{\mathcal{P}V_0^*(\Phi^*)\}^- - \{\mathcal{P}V_0^*(\Phi^*)\}^+ = \Phi^* = [\mathcal{P}W_0^*(h)]_S = \{\mathcal{P}W_0^*(h)\}^+ - \{\mathcal{P}W_0^*(h)\}^-$$

on S, which completes the proof for the regular case.

The second part of the theorem can be proved by standard limiting procedure. $\square$

Let us consider the interior and exterior homogeneous Dirichlet BVPs for the adjoint operator $A^*(\partial)$

$$A^*(\partial)U^* = 0 \quad \text{in} \quad \Omega^\pm, \tag{7.47}$$

$$\{U^*\}^\pm = 0 \quad \text{on} \quad S. \tag{7.48}$$

In the case of the interior problem, we assume that either U^* is a regular vector of the class $[C^{1,\kappa'}(\overline{\Omega^+})]^6$ or $U^* \in [W_2^1(\Omega^+)]^6$, while in the case of the exterior problem, we assume that either $U^* \in [C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap Z^*(\Omega^-)$ or $U^* \in [W^1_{2,loc}(\Omega^-)]^6 \cap Z^*(\Omega^-)$.

Theorem 7.16 *The interior and exterior homogeneous Dirichlet type BVPs* (7.47)–(7.48) *have only the trivial solution in the appropriate spaces.*

Proof. First we treat the exterior Dirichlet problem. In view of the structure of the operator $A^*(\partial)$, it is easy to see that we can consider separately the BVP for the vector function $\widetilde{U}^* = (u^*, \varphi^*, \psi^*)^\top$, constructed by the first five components of the solution vector U^*,

$$\widetilde{A}^*(\partial)\widetilde{U}^*(x) = 0, \ \ x \in \Omega^-, \tag{7.49}$$

$$\{\widetilde{U}^*(x)\}^- = 0, \ \ x \in S, \tag{7.50}$$

where $\widetilde{A}^*(\partial)$ is the 5×5 matrix differential operator, obtained from $A^*(\partial)$ by deleting the sixth column and the sixth row,

$$\widetilde{A}^*(\partial) := \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times3} & [-e_{jkl}\partial_j\partial_l]_{3\times1} & [-q_{jkl}\partial_j\partial_l]_{3\times1} \\ [e_{lrj}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l \\ [q_{lrj}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times5}. \tag{7.51}$$

With the help of Green's identity in $\Omega_R^- = B(0,R)\setminus\overline{\Omega^+}$, we have

$$\int\limits_{\Omega_R^-} \Big[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\Big]\, dx =$$

$$= -\int\limits_S \{\widetilde{U}^*\}^- \cdot \{\widetilde{P}(\partial,n)\widetilde{U}^*\}^-\, dS + \int\limits_{\Sigma_R} \widetilde{U}^* \cdot \widetilde{P}(\partial,n)\widetilde{U}^*\, d\Sigma_R, \tag{7.52}$$

where

$$\widetilde{\mathcal{P}}(\partial,n):=\begin{bmatrix}[c_{rjkl}n_j\partial_l]_{3\times3} & [-e_{lrj}n_j\partial_l]_{3\times1} & [-q_{lrj}n_j\partial_l]_{3\times1}\\ [e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l\\ [q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l\end{bmatrix}_{5\times5}, \tag{7.53}$$

and

$$\begin{aligned}\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*)=c_{rjkl}\partial_l u_k^*\partial_j u_r^*+\varkappa_{jl}\partial_l\varphi^*\partial_j\varphi^*+\\ +a_{jl}(\partial_l\varphi^*\partial_j\psi^*+\partial_j\psi^*\partial_l\varphi^*)+\mu_{jl}\partial_l\psi^*\partial_j\psi^*.\end{aligned} \tag{7.54}$$

Due to the fact that U^* possesses the property $Z^*(\Omega^-)$, it follows that $\widetilde{U}^*=\mathcal{O}(|x|^{-1})$ and $\partial_j\widetilde{U}^*=\mathcal{O}(|x|^{-2})$ as $|x|\to\infty$, $j=1,2,3$. Therefore,

$$\begin{aligned}&\Big|\int\limits_{\Sigma_R}\widetilde{U}^*\cdot\widetilde{P}(\partial,n)\widetilde{U}^*\,d\Sigma_R\Big|\leq\\ &\leq\int\limits_{\Sigma_R}\frac{C}{R^3}\,d\Sigma_R=\frac{C}{R^3}\,4\pi R^2=\frac{4\pi C}{R}\to0\quad\text{as}\quad R\to\infty.\end{aligned} \tag{7.55}$$

Taking into account that $\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*)\geq0$, applying the relations (7.49), (7.50), and (7.55), from (7.52) we conclude that $\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*)=0$. Hence in view of (2.11)-(2.13) it follows that $\widetilde{U}^*=(a\times x+b,b_4,b_5)$, where a and b are arbitrary constant vectors, and b_4 and b_5 are arbitrary scalar constants. Here the symbol $\times$ denotes the cross product operation. Due to the boundary condition (7.50) we get then $a=b=0$ and $b_4=b_5=0$, from which we derive that $\widetilde{U}^*=0$. Since $\widetilde{U}^*$ vanishes in Ω^-, from (7.47)–(7.48) we arrive at the following boundary-value problem for ϑ^*,

$$\begin{aligned}\eta_{kj}\partial_k\partial_j\vartheta^*=0\quad&\text{in}\quad\Omega^-,\\ \{\vartheta^*\}^-=0\quad&\text{on}\quad S.\end{aligned} \tag{7.56}$$

From boundedness of ϑ^* at infinity and from (7.56) one can derive that $\vartheta^*(x)=C+\mathcal{O}(|x|^{-1})$, where C is an arbitrary constant. In view of $U^*\in Z^*(\Omega^-)$ we have $C=0$ and $\vartheta^*(x)=\mathcal{O}(|x|^{-1})$, $\partial_j\vartheta^*(x)=\mathcal{O}(|x|^{-2})$, $j=1,2,3$. Therefore we can apply Green's formula

$$\begin{aligned}&\int\limits_{\Omega_R^-}\Big[\vartheta^*\,\eta_{kj}\partial_k\partial_j\vartheta^*+\eta_{kj}\partial_k\vartheta^*\,\partial_j\vartheta^*\Big]\,dx=\\ &=-\int\limits_S\{\vartheta^*\}^-\big\{\eta_{kj}n_k\partial_j\vartheta^*\big\}^-\,dS+\int\limits_{\Sigma_R}\vartheta^*\,\eta_{kj}n_k\partial_j\vartheta^*\,d\Sigma_R.\end{aligned}$$

Passing to the limit as $R\to\infty$, we get

$$\int\limits_{\Omega^-}\eta_{kj}\partial_k\vartheta^*\partial_j\vartheta^*\,dx=0.$$

Using the fact that the matrix $[\eta_{kj}]_{3\times 3}$ is positive definite, we conclude that $\vartheta^* = C_1 = const$ and since $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$, finally we get that $\vartheta^* = 0$ in Ω^-. Thus $U^* = 0$ in Ω^- which completes the proof for the exterior problem.

The interior problem can be treated quite similarly. □

7.5 Investigation of the interior Neumann BVP

First let us treat the uniqueness question. To this end we consider the homogeneous interior Neumann-type BVP

$$A(\partial)U(x) = 0, \quad x \in \Omega^+, \tag{7.57}$$

$$\big\{\mathcal{T}(\partial, n)U(x)\big\}^+ = 0, \quad x \in S = \partial\Omega^+. \tag{7.58}$$

As it is shown in Subsection 2.4, a general solution to the problem (7.57)-(7.58) can be represented in the form (see Theorem 2.4)

$$U = \sum_{k=1}^{9} C_k U^{(k)} \quad \text{in} \quad \Omega^+, \tag{7.59}$$

where C_k are arbitrary scalar constants and $\{U^{(k)}\}_{k=1}^{9}$ is the basis in the space of solution vectors of the homogeneous problem (7.57)–(7.58). They can be constructed explicitly and read as

$$U^{(k)} = \big(\widetilde{V}^{(k)}, 0\big)^\top, \quad k = \overline{1,8}, \quad U^{(9)} = \big(\widetilde{V}^{(9)}, 1\big)^\top, \tag{7.60}$$

where $U^{(k)} = (u^{(k)}, \varphi^{(k)}, \psi^{(k)}, \vartheta^{(k)})^\top$, $\widetilde{V}^{(k)} = (u^{(k)}, \varphi^{(k)}, \psi^{(k)})^\top$,

$$\begin{aligned}
&\widetilde{V}^{(1)} = (0, -x_3, x_2, 0, 0)^\top, && \widetilde{V}^{(2)} = (x_3, 0, -x_1, 0, 0)^\top,\\
&\widetilde{V}^{(3)} = (-x_2, x_1, 0, 0, 0)^\top, && \widetilde{V}^{(4)} = (1, 0, 0, 0, 0)^\top,\\
&\widetilde{V}^{(5)} = (0, 1, 0, 0, 0)^\top, && \widetilde{V}^{(6)} = (0, 0, 1, 0, 0)^\top,\\
&\widetilde{V}^{(7)} = (0, 0, 0, 1, 0)^\top, && \widetilde{V}^{(8)} = (0, 0, 0, 0, 1)^\top,
\end{aligned}$$

and $\widetilde{V}^{(9)}$ is defined as

$$\begin{gathered}
\widetilde{V}^{(9)} = (u^{(9)}, \varphi^{(9)}, \psi^{(9)})^\top, \quad u_k^{(9)} = b_{kq}x_q, \quad k = 1, 2, 3,\\
\varphi^{(9)} = c_q x_q, \quad \psi^{(9)} = d_q x_q,
\end{gathered}$$

with the twelve coefficients $b_{kq} = b_{qk}$, c_q and d_q, $k, q = 1, 2, 3$, defined by the uniquely solvable linear algebraic system of equations

$$\begin{aligned}
&c_{rjkl}b_{kl} + e_{lrj}c_l + q_{lrj}d_l = \lambda_{rj}, \quad r, j = 1, 2, 3,\\
&-e_{jkl}b_{kl} + \varkappa_{jl}c_l + a_{jl}d_l = p_j, \quad j = 1, 2, 3,\\
&-q_{jkl}b_{kl} + a_{jl}c_l + \mu_{jl}d_l = m_j, \quad j = 1, 2, 3.
\end{aligned}$$

We have shown in the proof of Theorem 2.4 that the vector U given by (7.59) can be rewritten as

$$U=(\widetilde{V},0)^\top+b_6(\widetilde{V}^{(9)},1)^\top$$

where $\widetilde{V}=(a\times x+b,b_4,b_5)^\top$, and $a=(a_1,a_2,a_3)^\top$ and $b=(b_1,b_2,b_3)^\top$ are arbitrary constant vectors, while b_4, b_5, b_6 are arbitrary scalar constants.

Now, let us consider the non-homogeneous interior Neumann-type BVP

$$A(\partial)U(x)=0,\ \ x\in\Omega^+, \tag{7.61}$$

$$\big\{\mathcal{T}(\partial,n)U(x)\big\}^+=F(x),\ \ x\in S, \tag{7.62}$$

where $U\in[C^{1,\kappa'}(\overline{\Omega^+})]^6\cap[C^2(\Omega^+)]^6$ is a sought for vector and $F\in[C^{0,\kappa'}(S)]^6$ is a given vector-function. It is clear that if the problem (7.61)–(7.62) is solvable, then a solution is defined within a summand vector of type (7.59).

We look for a solution to the problem (7.61)–(7.62) in the form of the single layer potential,

$$U(x)=V_0(h)(x),\ \ x\in\Omega^+, \tag{7.63}$$

where $h=(h_1,\dots,h_6)^\top\in[C^{0,\kappa'}(S)]^6$ is an unknown density. From the boundary condition (7.62) and by virtue of the jump relation (7.8) (see Theorem 7.5) we get the following integral equation for the density vector h

$$[-2^{-1}I_6+\mathcal{K}_0]h=F \quad \text{on} \quad S, \tag{7.64}$$

where $\mathcal{K}_0$ is a singular integral operator defined by (7.11). Note that $-2^{-1}I_6+\mathcal{K})_0$ is a singular integral operator of normal type with index zero. Now we investigate the null space $\mathrm{Ker}(-2^{-1}I_6+\mathcal{K}_0)$. To this end, we consider the homogeneous equation

$$[-2^{-1}I_6+\mathcal{K}_0]h=0 \quad \text{on} \quad S \tag{7.65}$$

and assume that a vector $h^{(0)}$ is a solution to (7.65), i.e., $h^{(0)}\in\mathrm{Ker}(-2^{-1}I_6+\mathcal{K}_0)$. Since $h^{(0)}\in[C^{0,\kappa'}(S)]^6$, it is evident that the corresponding single layer potential $U_0(x)=V(h^{(0)})(x)$ belongs to the space of regular vector functions and solves the homogeneous equation

$$A(\partial)U_0(x)=0 \quad \text{in} \quad \Omega^+.$$

Moreover, $\{\mathcal{T}(\partial,n)U_0(x)\}^+=-2^{-1}h^{(0)}+\mathcal{K}_0h^{(0)}=0$ on S due to (7.65), i.e., $U_0(x)$ solves the homogeneous interior Neumann problem. Therefore, in accordance to the above results, we can write $U_0(x)=\sum\limits_{k=1}^{9}C_kU^{(k)}(x)$ in Ω^+, where C_k, $k=\overline{1,9}$, are some constants, and the vectors $U^{(k)}(x)$ are defined by (7.60). Hence we have

$$V_0(h^{(0)})(x)=\sum_{k=1}^{9}C_kU^{(k)}(x),\ \ x\in\Omega^+.$$

If we take into account the jump relation (7.7), we derive that

$$\big\{V_0(h^{(0)})(x)\big\}^+ \equiv \mathcal{H}_0(h^{(0)})(x) = \sum_{k=1}^{9} C_k U^{(k)}(x), \quad x \in S. \tag{7.66}$$

Keeping in mind that the operators

$$\begin{aligned} \mathcal{H}_0 &: [H^{-\frac{1}{2}}(S)]^6 \to [H^{\frac{1}{2}}(S)]^6, \\ &: [C^{0,\kappa'}(S)]^6 \to [C^{1,\kappa'}(S)]^6 \end{aligned}$$

are invertible, from (7.66) we obtain

$$h^{(0)} = \sum_{k=1}^{9} C_k h^{(k)}(x), \quad x \in S,$$

with

$$h^{(k)} := \mathcal{H}_0^{-1}(U^{(k)}), \quad k = \overline{1,9}. \tag{7.67}$$

Further, we show that the system of vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent. Let us assume the opposite. Then there exist constants c_k, $k = \overline{1,9}$, such that $\sum\limits_{k=1}^{9} |c_k| \neq 0$ and the following relation

$$\sum_{k=1}^{9} c_k h^{(k)} = 0 \quad \text{on} \quad S$$

holds, i.e., $\sum\limits_{k=1}^{9} c_k \mathcal{H}_0^{-1}(U^{(k)}) = 0$ on S. Hence we get

$$\mathcal{H}_0^{-1}\Big(\sum_{k=1}^{9} c_k U^{(k)}\Big) = 0 \quad \text{on} \quad S,$$

and, consequently,

$$\sum_{k=1}^{9} c_k U^{(k)}(x) = 0, \quad x \in S. \tag{7.68}$$

Now consider the vector

$$U^*(x) \equiv \sum_{k=1}^{9} c_k U^{(k)}(x), \quad x \in \Omega^+.$$

Since the vectors $U^{(k)}$ are solutions of the homogeneous equation (7.61), in view of (7.68) we have

$$A(\partial)U^*(x) = 0, \quad x \in \Omega^+,$$

$$\{U^*(x)\}^+ = \bigg\{\sum_{k=1}^{9} c_k U^{(k)}(x)\bigg\}^+ = 0, \ \ x \in S.$$

That is, U^* is a solution of the homogeneous interior Dirichlet problem and in accordance with the uniqueness theorem for the interior Dirichlet BVP we conclude $U^*(x) = 0$ in Ω^+, i.e.,

$$\sum_{k=1}^{9} c_k U^{(k)}(x) = 0, \quad x \in \Omega^+.$$

This contradicts to linear independence of the system $\{U^{(k)}\}_{k=1}^{9}$. Thus, the system of the vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent which implies that

$$\dim \operatorname{Ker}(-2^{-1} I_6 + \mathcal{K}_0) \geq 9.$$

Next we show that

$$\dim \operatorname{Ker}(-2^{-1} I_6 + \mathcal{K}_0) \leq 9.$$

Let the equation $(-2^{-1} I_6 + \mathcal{K}_))h = 0$ have a solution $h^{(10)}$ which is not representable in the form of a linear combination of the system $\{h^{(k)}\}_{k=1}^{9}$. Then the system $\{h^{(k)}\}_{k=1}^{10}$ is linearly independent. It is easy to show that the system of the corresponding single layer potentials $V^{(k)}(x) := V_0(h^{(k)})(x)$, $k = \overline{1,10}$, $x \in \Omega^+$, is linearly independent as well. Indeed, let us assume the opposite. Then there are constants a_k, such that

$$U(x) := \sum_{k=1}^{10} a_k V^{(k)}(x) = 0, \ \ x \in \Omega^+, \tag{7.69}$$

with $\sum\limits_{k=1}^{10} |a_k| \neq 0$. From (7.69) we then derive that $\{U(x)\}^+ = 0$, $x \in S$. Therefore,

$$\{U\}^+ = \sum_{k=1}^{10} a_k \{V^{(k)}\}^+ = \sum_{k=1}^{10} a_k \mathcal{H}_0(h^{(k)}) = \mathcal{H}_0\Big(\sum_{k=1}^{10} a_k h^{(k)}\Big) = 0 \quad \text{on} \quad S.$$

Whence, due to the invertibility of the operator $\mathcal{H}_0$, we get

$$\sum_{k=1}^{10} a_k h^{(k)} = 0 \quad \text{on} \quad S.$$

which contradicts to the linear independence of the system $\{h^{(k)}\}_{k=1}^{10}$.

Thus the system $\{V_0(h^{(k)})(x)\}_{k=1}^{10}$ is linearly independent.

On the other hand, we have

$$A(\partial) V^{(k)}(x) = 0, \ \ x \in \Omega^+,$$
$$\big\{\mathcal{T} V^{(k)}\big\}^+ = (-2^{-1} I_6 + \mathcal{K}_0) h^{(k)} = 0, \ \ x \in S,$$

since $h^{(k)}$, $k=\overline{1,10}$, are solutions to the homogeneous equation (7.65). Therefore, the vectors $V^{(k)}$, $k=\overline{1,10}$, are solutions to the homogeneous interior Neumann-type BVP and they can be expressed by linear combinations of the vectors $U^{(j)}$, $j=\overline{1,9}$, defined in (7.60). Whence it follows that the system $\{V^{(k)}\}_{k=1}^{10}$ is linearly dependent and so is the system $\{h^{(k)}\}_{k=1}^{10}$ for an arbitrary solution $h^{(10)}$ of the equation (7.65). Consequently, $\dim\operatorname{Ker}(-2^{-1}I_6+\mathcal{K}_0)\le 9$ implying that $\dim\operatorname{Ker}(-2^{-1}I_6+\mathcal{K}_0)=9$. We can consider the system $h^{(1)},\dots,h^{(9)}$ defined in (7.67) as basis vectors of the null space of the operator $-2^{-1}I_6+\mathcal{K}_0$. If h_0 is a particular solution to the nonhomogeneous integral equation (7.64), then a general solution of the same equation is represented as

$$h=h_0+\sum_{k=1}^{9}c_kh^{(k)},$$

where c_k are arbitrary constants.

For our further analysis we need also to study the homogeneous interior Neumann-type BVP for the adjoint operator $A^*(\partial)$,

$$A^*(\partial)U^*=0 \quad \text{in} \quad \Omega^+, \tag{7.70}$$

$$\{\mathcal{P}U^*\}^+=0 \quad \text{on} \quad S=\partial\Omega^+; \tag{7.71}$$

here the adjoint operator $A^*(\partial)$ and the boundary operator $\mathcal{P}$ are defined by (7.29) and (7.5) respectively.

Note that in the case of the problem (7.70)–(7.71) we get also two separated problems:

a) For the vector function $\widetilde{U}^*\equiv(u^*,\varphi^*,\psi^*)^\top$,

$$\widetilde{A}^*(\partial)\widetilde{U}^*=0 \quad \text{in} \quad \Omega^+, \tag{7.72}$$

$$\big\{\widetilde{\mathcal{P}}\widetilde{U}^*\big\}^+=0 \quad \text{on} \quad S, \tag{7.73}$$

where $\widetilde{A}^*$ and $\widetilde{\mathcal{P}}$ are defined by (7.51) and (7.53) respectively, and

b) For the function $U_6^*\equiv\vartheta^*$

$$\lambda_{rj}\partial_ju_r^*+p_j\partial_j\varphi^*+m_j\partial_j\psi^*+\eta_{jl}\partial_j\partial_l\vartheta^*=0 \quad \text{in} \quad \Omega^+, \tag{7.74}$$

$$\eta_{jl}n_j\partial_l\vartheta^*=0 \quad \text{on} \quad S. \tag{7.75}$$

For a regular solution vector $\widetilde{U}^*$ of the problem (7.72)–(7.73) we can write the following Green's identity

$$\int\limits_{\Omega^+}\Big[\widetilde{U}^*\cdot\widetilde{A}^*(\partial)\widetilde{U}^*+\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*)\Big]\,dx=\int\limits_{\partial\Omega^+}\{\widetilde{U}^*\}^+\cdot\big\{\widetilde{\mathcal{P}}(\partial,n)\widetilde{U}^*\big\}^+\,dS, \tag{7.76}$$

where $\widetilde{\mathcal{E}}$ is given by (7.54). If we take into account the conditions (7.72)–(7.73), from (7.76) we get

$$\int\limits_{\Omega^+}\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*)\,dx=0.$$

Hence we have that $\partial_j \varphi^* = 0$, $\partial_j \psi^* = 0$, $j = 1, 2, 3$, and $\partial_l u_k^* + \partial_j u_r^* = 0$ in Ω^+. Therefore, $u^*(x) = a \times x + b$ is a rigid displacement vector, $\varphi^* = b_4$ and $\psi^* = b_5$ are arbitrary constants in Ω^+. It is evident that

$$\lambda_{rj} \partial_j u_r^* = \frac{1}{2}\, \lambda_{rj} (\partial_j u_r^* + \partial_r u_j^*) = 0$$

and $p_j \partial_j \varphi^* = m_j \partial_j \psi^* = 0$. Then from (7.74)–(7.75) we get the following BVP for the scalar function ϑ^*,

$$\eta_{jl} \partial_j \partial_l \vartheta^* = 0 \quad \text{in} \quad \Omega^+,$$

$$\eta_{jl} n_j \partial_l \vartheta^* = 0 \quad \text{on} \quad S.$$

Using the following Green's identity

$$\int\limits_{\Omega^+} \eta_{jl} \partial_j \partial_l \vartheta^* \, \vartheta^* \, dx = -\int\limits_{\Omega^+} \eta_{jl} \partial_l \vartheta^* \, \partial_j \vartheta^* \, dx + \int\limits_{\partial\Omega^+} \{\eta_{jl} n_j \partial_l \vartheta^*\}^+ \{\partial_j \vartheta^*\}^+ \, dS,$$

we find

$$\int\limits_{\Omega^+} \eta_{jl} \partial_l \vartheta^* \, \partial_j \vartheta^* \, dx = 0,$$

and by the positive definiteness of the matrix $[\eta_{jl}]_{3\times 3}$ we get $\partial_j \vartheta^* = 0$, $j = \overline{1,3}$, in Ω^+, i.e., $\vartheta^* = b_6 = const$ in Ω^+. Consequently, a general solution $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^\top$ of the adjoint homogeneous BVP (7.70)–(7.71) can be represented as

$$U^*(x) = \sum_{k=1}^{9} C_k U^{*(k)}(x), \quad x \in \Omega^+,$$

where C_k are arbitrary scalar constants and

$$\begin{aligned}
&U^{*(1)} = (0, -x_3, x_2, 0, 0, 0)^\top, && U^{*(2)} = (x_3, 0, -x_1, 0, 0, 0)^\top,\\
&U^{*(3)} = (-x_2, x_1, 0, 0, 0, 0)^\top, && U^{*(4)} = (1, 0, 0, 0, 0, 0)^\top,\\
&U^{*(5)} = (0, 1, 0, 0, 0, 0)^\top, && U^{*(6)} = (0, 0, 1, 0, 0, 0)^\top,\\
&U^{*(7)} = (0, 0, 0, 1, 0, 0)^\top, && U^{*(8)} = (0, 0, 0, 0, 1, 0)^\top,\\
&U^{*(9)} = (0, 0, 0, 0, 0, 1)^\top.
\end{aligned} \tag{7.77}$$

As we see, $U^{*(k)} = U^{(k)}$, $k = \overline{1,8}$, where $U^{(k)}$, $k = \overline{1,8}$, is given in (7.60). One can easily check that the system $\{U^{*(k)}\}_{k=1}^{9}$ is linearly independent. As a result we get the following

Proposition 7.17 *The space of solutions of the adjoint homogeneous BVP* (7.70)–(7.71) *is nine dimensional and an arbitrary solution can be represented as a linear combination of the vectors* $\{U^{*(k)}\}_{k=1}^{9}$*, i.e., the system* $\{U^{*(k)}\}_{k=1}^{9}$ *is a basis in the space of solutions to the homogeneous BVP* (7.70)–(7.71).

Now, we return to equation (7.64) and consider the corresponding homogeneous adjoint equation

$$(-2^{-1}I_6 + \mathcal{K}_0^*)h^* = 0 \quad \text{on} \quad S,$$

where $\mathcal{K}_0^*$ is the adjoint operator to $\mathcal{K}_0$ defined by the duality relation,

$$(\mathcal{K}_0 h, h^*)_{L_2(S)} = (h, \mathcal{K}_0^* h^*)_{L_2(S)}, \quad \forall h, h^* \in [L_2(S)]^6.$$

It is easy to show that the operator $\mathcal{K}_0^*$ is the same as the operator given by (7.34). In what follows we prove that $\dim \operatorname{Ker}\big(-\frac{1}{2}\, I_6 + \mathcal{K}_0^*\big) = 9$.

Indeed, in accordance with Proposition 7.17 we have that $A^*(\partial)U^{*(k)} = 0$ in Ω^+ and $\{\mathcal{P}U^{*(k)}\}^+ = 0$ on S. Therefore from (7.35) we have

$$U^{*(k)}(x) = W_0^*\big(\{U^{*(k)}\}^+\big)(x), \;\; x \in \Omega^+. \tag{7.78}$$

By the jump relations (7.32) we get

$$h^{*(k)} = 2^{-1}\, h^{*(k)} + \mathcal{K}_0^* h^{*(k)} \quad \text{on} \quad S,$$

where

$$h^{*(k)} := \{U^{*(k)}\}^+, \;\; k = \overline{1,9}. \tag{7.79}$$

Whence it follows that

$$\big(-2^{-1}\, I_6 + \mathcal{K}_0^*\big)h^{*(k)} = 0, \;\; k = \overline{1,9}.$$

By Theorem 7.16 and the relations (7.78) and (7.79) we conclude that the system $\big\{h^{*(k)}\big\}_{k=1}^{9}$ is linearly independent, and therefore

$$\dim \operatorname{Ker}\big(-2^{-1}\, I_6 + \mathcal{K}_0^*\big) \geq 9.$$

Now, let $h^{*(0)} \in \operatorname{Ker}\big(-2^{-1}\, I_6 + \mathcal{K}_0^*\big)$, i.e., $\big(-2^{-1}\, I_6 + \mathcal{K}_0^*\big)h^{*(0)} = 0$. The corresponding double layer potential $U_0^*(x) := W_0^*(h^{*(0)})(x)$ is a solution to the homogeneous equation $A^*(\partial)U_0^* = 0$ in Ω^+. Moreover, $\{W_0^*(h^{*(0)})\}^- = -2^{-1}\, h^{*(0)} + \mathcal{K}_0^* h^{*(0)} = 0$ on S. Consequently, U_0^* is a solution of the homogeneous exterior Dirichlet BVP possessing the property $Z^*(\Omega^-)$. With the help of the uniqueness Theorem 7.16 we conclude that $W_0^*(h^{*(0)}) = 0$ in Ω^-. Further, $\big\{\mathcal{P}W_0^*(h^{*(0)})\big\}^+ = \{\mathcal{P}W_0^*(h^{*(0)})\}^- = 0$ due to Theorem 7.15, and for the vector function U_0^* we arrive at the following BVP,

$$A^*(\partial)U_0^* = 0 \quad \text{in} \quad \Omega^+,$$
$$\big\{\mathcal{P}U_0^*\big\}^+ = 0 \quad \text{on} \quad S.$$

Using Proposition 7.17 we can write

$$U_0^*(x) = W_0^*(h^{*(0)})(x) = \sum_{k=1}^{9} c_k U^{*(k)}(x), \;\; x \in \Omega^+,$$

where c_k are some constants. The jump relation for the double layer potential then gives

$$\begin{aligned}&\big\{W_0^*(h^{*(0)})(x)\big\}^+ - \big\{W_0^*(h^{*(0)})(x)\big\}^-\\ &= h^{*(0)}(x) = \sum_{k=1}^{9} c_k\big\{U^{*(k)}(x)\big\}^+ = \sum_{k=1}^{9} c_k h^{*(k)}(x), \quad x\in S,\end{aligned}$$

which implies that the system $\big\{h^{*(k)}\big\}_{k=1}^{9}$ represents a basis of the null space $\mathrm{Ker}\,\big(-2^{-1}\,I_6+\mathcal{K}^*\big)$. Whence it follows that $\dim\mathrm{Ker}\,\big(-2^{-1}\,I_6+\mathcal{K}_0^*\big)=9$.

Now we can formulate the following basic existence theorem for the integral equation (7.64) and the interior Neumann-type BVP.

Theorem 7.18 *Let $m\geq 0$ be a nonnegative integer and $0<\kappa'<\kappa\leq 1$. Further, let $S\in C^{m+1,\kappa}$ and $F\in[C^{m,\kappa'}(S)]^6$. The necessary and sufficient conditions for the integral equation* (7.64) *and the interior Neumann-type BVP* (7.61)–(7.62) *to be solvable read as*

$$\int_S F(x)\cdot h^{*(k)}(x)\,dS = 0, \quad k=\overline{1,9}, \tag{7.80}$$

where the system $\{h^{*(k)}\}_{k=1}^{9}$ is defined explicitly by (7.79) *and* (7.77).

If these conditions are satisfied, then a solution vector to the interior Neumann-type BVP is representable by the single layer potential (7.63)*, where the density vector $h\in[C^{m,\kappa'}(S)]^6$ is defined by the integral equation* (7.64).

A solution vector function $U\in[C^{m+1,\kappa'}(\overline{\Omega^+})]^6$ is defined modulo a linear combination of the vector functions $\{U^{(k)}\}_{k=1}^{9}$ given by (7.60).

Remark 7.19 *Similar to the exterior problem, if S is a Lipschitz surface, $F\in\big[H^{-1/2}(S)\big]^6$, and the conditions* (7.80) *is fulfilled, then*

(i) the integral equation (7.64) *is solvable in the space $\big[H^{-1/2}(S)\big]^6$;*

(ii) the interior Neumann-type BVP (7.61)-(7.62) *is solvable in the space $\big[H_2^1(\Omega^+)\big]^6$ and solutions are representable by the single layer potential* (7.63)*, where the density vector $h\in\big[H^{-1/2}(S)\big]^6$ solves the integral equation* (7.64)*;*

(iii) A solution $U\in\big[H_2^1(\Omega^+)\big]^6$ to the interior Neumann-type BVP (7.61)-(7.62) *is defined modulo a linear combination of the vector functions $\big\{U^{(k)}\big\}_{k=1}^{9}$ given by* (7.60).

8 Appendix A: Structural properties of bounded solutions in exterior domains

Here we prove several technical lemmas.

Lemma A.1 *Let $U = (u_1, u_2, \cdots, u_N)^\top$ be a bounded solution to the homogeneous differential equation*

$$L(\partial)U(x) = 0, \quad x \in \Omega^-, \tag{A.1}$$

where $\Omega^- \subset \mathbb{R}^3$ is a complement of a bounded region $\overline{\Omega^+}$ with a compact boundary and $L(\partial) = [L_{kj}(\partial)]_{N\times N}$ is a strongly elliptic second order matrix differential operator with constant coefficients,

$$L_{kj}(\partial) = \sum_{p,q=1}^{3} a_{pq}^{kj}\partial_p\partial_q, \quad k,j = \overline{1,N}.$$

Then

$$U(x) = C + \mathcal{O}(|x|^{-1}) \ \ \text{as} \ \ |x| \to +\infty\,, \tag{A.2}$$

where $C = (C_1, \cdots, C_N)^\top$ is a constant vector.

Proof. Let U be a bounded solution to equation (A.1) and $B(O,R)$ be a ball centered at the origin and radius R, such that $\overline{\Omega^+} \subset B(O,R)$. Clearly, $U \in [C^\infty(\Omega^-)]^N$ due to the ellipticity of the operator $L(\partial)$. Let $V = (v_1, \cdots, v_N)^\top \in [C^\infty(\mathbb{R}^3)]^N$ be a vector whose restriction on $\Omega_R^- := \Omega^- \setminus \overline{B(O,R)}$ coincides with U, i.e,

$$V(x) = U(x) \quad \text{for} \quad x \in \Omega_R^-\,. \tag{A.3}$$

Due to (A.1) and (A.3) the vector V solves the nonhomogeneous differential equation

$$L(\partial)V(x) = \Phi(x), \quad x \in \mathbb{R}^3, \tag{A.4}$$

with $\Phi = (\Phi_1, \cdots, \Phi_N)^\top \in [C^\infty_{comp}(\mathbb{R}^3)]^N$ having a compact support, $\operatorname{supp}\Phi \subset \overline{B(O,R)}$. Keeping in mind that V is bounded, we can apply the generalized Fourier transform to equation (A.4) to obatin

$$L(-i\,\xi)\,\widehat{V}(\xi) = \widehat{\Phi}(\xi), \quad \xi \in \mathbb{R}^3, \tag{A.5}$$

where $\widehat{V} = \mathcal{F}[\,V\,]$ and $\widehat{\Phi} = \mathcal{F}[\,\Phi\,] \in C^\infty(\mathbb{R}^3)$. This equation is understood in the sense of tempered distributions. Since $\det L(-i\,\xi) \neq 0$ for $\xi \neq 0$ and the entries of the inverse matrix $[\,L(-i\,\xi)\,]^{-1}$ are C^∞-smooth homogeneous functions of order -2 in $\mathbb{R}^3 \setminus \{0\}$, from (A.5) we conclude

$$\widehat{V}(\xi) = [\,L(-i\,\xi)\,]^{-1}\,\widehat{\Phi}(\xi) + \sum_{|\alpha|\leq M} C_\alpha\,\delta^{(\alpha)}(\xi), \tag{A.6}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $C_\alpha = (C_{\alpha,1}, \cdots, C_{\alpha,N})^\top$ are arbitrary constant vectors, M is a nonnegative integer, $\delta(\cdot)$ is Dirac's distribution and $\delta^{(\alpha)} = \partial^\alpha\delta$.

By applying the inverse Fourier transform to (A.6) we get

$$V(x) = \mathcal{F}^{-1}_{\xi\to x}\Big([\,L(-i\,\xi)\,]^{-1}\,\widehat{\Phi}(\xi)\Big) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha. \tag{A.7}$$

Denote by $\Gamma_L(x)$ the fundamental matrix of the operator $L(\partial)$ whose entries are homogeneous functions of order -1,

$$\Gamma_L(x) := \mathcal{F}^{-1}_{\xi\to x}\Big([\,L(-i\,\xi)\,]^{-1}\Big), \quad \Gamma_L \in C^\infty(\mathbb{R}^3\setminus\{0\}), \quad L(\partial)\Gamma_L(x) = \delta(x)\, I_N. \tag{A.8}$$

Then (A.7) can be rewritten as follows

$$V(x) = \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}\Big(\Gamma_L * \Phi\Big) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha = \Big(\Gamma_L * \Phi\Big)(x) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha, \tag{A.9}$$

where $*$ denotes the convolution operator. Therefore,

$$V(x) = \int\limits_{\mathbb{R}^3} \Gamma_L(x-y)\,\Phi(y)\,dy + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha. \tag{A.10}$$

Since $\operatorname{supp}\Phi \subset \overline{B(O,R)}$ is compact, the first summand in the right hand side in (A.10) decays at infinity as $\mathcal{O}(|x|^{-1})$. Then it follows that $C_\alpha = 0$ for $|\alpha| \geq 1$ due to boundedness of V at infinity. Finally, we get

$$V(x) = \int\limits_{B(O,R)} \Gamma_L(x-y)\,\Phi(y)\,dy + C = C + \mathcal{O}(|x|^{-1}), \tag{A.11}$$

where $C = C_{(0,\cdots,0)} =: (C_1,\cdots,C_N)^\top$ is an arbitrary constant vector. $\square$

Lemma A.2 *Let $L(\partial)$ be as in Lemma A.1 and $P = (P_1, P_2, \cdots, P_N)^\top \in [C^\infty(\mathbb{R}^3\setminus\{0\})]^N$ be an odd homogeneous vector function of order -2. Then the equation*

$$L(\partial)U(x) = P(x), \quad x \in \mathbb{R}^3\setminus\{0\}, \tag{A.12}$$

has a unique homogeneous solution $U^{(0)} \in [C^\infty(\mathbb{R}^3\setminus\{0\})]^N$ of zero order satisfying the condition

$$\int_{|x|=1} U^{(0)}(x)\,dS = 0. \tag{A.13}$$

Proof. From (A.12) by the Fourier transform we get

$$L(-i\,\xi)\widehat{U}(\xi) = \widehat{P}(\xi), \quad x \in \mathbb{R}^3, \tag{A.14}$$

where $\widehat{P}(\xi)$ is an odd homogeneous vector function of order -1, $\det L(-i\,\xi) \neq 0$ for $\xi \neq 0$ and the entries of the inverse matrix $[\,L(-i\,\xi)\,]^{-1}$ are even, C^∞-smooth homogeneous functions of order -2.

The equation (A.14) is understood in the sense of the space of tempered distributions and as in the proof of Lemma A.1 we have

$$\widehat{U}(\xi) = [L(-i\,\xi)]^{-1}\,\widehat{P}(\xi) + \sum_{|\alpha|\le M} C_\alpha\,\delta^{(\alpha)}(\xi) \tag{A.15}$$

with the same α, C_α and M as in (A.6).

Note that the first summand in the right hand side is an odd homogeneous function of order -3 satisfying the condition

$$\int_{|\xi|=1} [L(-i\,\xi)]^{-1}\,\widehat{P}(\xi)\,dS = 0. \tag{A.16}$$

Therefore, we can regularize this summand and consider it in the Principal Value (v.p.) sense. Then the corresponding inverse Fourier transform

$$U^{(0)}(x) := \mathcal{F}^{-1}_{\xi\to x}\Big(\text{v.p.}\,[L(-i\,\xi)]^{-1}\,\widehat{P}(\xi)\Big) \tag{A.17}$$

is a homogeneous vector function of order zero satisfying the condition

$$\int_{|x|=1} U^{(0)}(x)\,dS = 0. \tag{A.18}$$

Moreover, $U^{(0)} \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ (see, e.g., [Miz], Assertion 2.13 and Theorem 2.16, pp. 127-128).

Now, from (A.15) by the inverse Fourier transform we get

$$U(x) = U^{(0)}(x) + \sum_{|\alpha|\le M} C_\alpha\,x^\alpha.$$

Since U should be a homogeneous vector function of order zero satisfying condition (A.13) we conclude that $C_\alpha = 0$ for all α in view of (A.18), and

$$U(x) = U^{(0)}(x), \tag{A.19}$$

which completes the proof. $\square$

Lemma A.3 *Let $L(\partial)$ be as in Lemma A.1, $\Gamma_L(x)$ be the fundamental solution of the operator $L(\partial)$ defined by* (A.8)*, and $Q = (Q_1, Q_2, \cdots, Q_N)^\top \in [C^\infty(\overline{\Omega^-})]^N$ with*

$$\partial^\alpha Q_j(x) = \mathcal{O}(|x|^{-3-|\alpha|}) \quad as \quad |x| \to \infty, \quad j = \overline{1, N},$$

for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Then the vector

$$V(x) = \int\limits_{\Omega^-} \Gamma_L(x-y)\,Q(y)\,dy, \tag{A.20}$$

is a particular solution of the equation

$$L(\partial)U(x) = Q(x), \quad x \in \Omega^-. \tag{A.21}$$

Moreover, $V \in [C^\infty(\Omega^-)]^N \cap [C^2(\overline{\Omega^-})]^N$ *and*

$$\partial^\alpha V(x) = \mathcal{O}(|x|^{-1-|\alpha|} \ln |x|\,) \quad as \quad |x| \to \infty \tag{A.22}$$

for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Proof. To verify the equation $L(\partial)V = Q$ in Ω^- and the inclusion $V \in [C^\infty(\Omega^-)]^N \cap [C^2(\overline{\Omega^-})]^N$ is standard. The estimate (A.22) follows from the relation

$$|\partial^\alpha V(x)| \leq c_1 \int\limits_{\Omega^-} \frac{1}{|x-y|\,|y|^{3+|\alpha|}}\, dy \leq c_1 \sum_{k=1}^{4} \int\limits_{\Omega_k} \frac{1}{|x-y|\,|y|^{3+|\alpha|}}\, dy\,, \tag{A.23}$$

where c_1 is a positive constant, $r = |x|$ is sufficiently large and

$$\Omega_1 = \Omega^- \cap B(O, \tfrac{r}{2}), \qquad \Omega_2 = \Omega^- \cap B(x, \tfrac{r}{2}),$$
$$\Omega_3 = B(O, \tfrac{3r}{2}) \setminus [B(x, \tfrac{r}{2}) \cup \Omega_2], \quad \Omega_4 = \Omega^- \setminus B(O, \tfrac{3r}{2}).$$

We recall that the origin of the coordinate system belongs to the domain Ω^+. □

Corollary A.4 *Let* $L(\partial)$, Ω^-, P, *and* Q *be as in Lemmas A.1-A.3 and* $\Phi \in [L_{2,\,comp}(\Omega^-)]^N$. *Further, let* $U \in [W^1_{2,\,loc}(\Omega^-)]^N$ *be a solution of the equation*

$$L(\partial)U(x) = P(x) + Q(x) + \Phi(x), \quad x \in \Omega^- \tag{A.24}$$

satisfying the condition $U(x) = \mathcal{O}(1)$ *as* $|x| \to \infty$.

Then U *can be represented as*

$$U(x) = C + U^{(0)}(x) + U^{(1)}(x),$$

where $C = (C_1, \cdots, C_N)^\top$ *is a constant vector,* $U^{(0)}$ *is given by* (A.17) *and*

$$U^{(1)} \in [W^1_{2,\,loc}(\Omega^-)]^N \cap [C^\infty(\mathbb{R}^3 \setminus \text{supp}\ \Phi)]^N$$

possesses the following asymptotic at infinity

$$\partial^\alpha U^{(1)}(x) = \mathcal{O}(|x|^{-1-|\alpha|} \ln |x|\,) \quad as \quad |x| \to \infty$$

for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Proof. Let $\Gamma_L(x)$ be the fundamental matrix of the operator $L(\partial)$ defined by (A.8). Note that the Newtonian potential

$$N_{\Omega^-}(\Phi)(x) := \int\limits_{\Omega^-} \Gamma_L(x-y)\, \Phi(y)\, dy = \int\limits_{\Omega^- \cap \text{supp}\ \Phi} \Gamma_L(x-y)\, \Phi(y)\, dy$$

belongs to $[W^2_{2,loc}(\Omega^-)]^N \cap [C^\infty(\mathbb{R}^3 \setminus \text{supp}\,\Phi)]^N$, solves the equation $L(\partial)N_{\Omega^-}(\Phi) = \Phi$ in Ω^-, and at infinity has the property $\partial^\alpha N_{\Omega^-}(\Phi)(x) = \mathcal{O}(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Then it is clear that the vector

$$U^*(x) := U^{(0)}(x) + N_{\Omega^-}(Q)(x) + N_{\Omega^-}(\Phi)(x)$$

is bounded at infinity and solves the nonhomogeneous equation (A.24) due to Lemmas A.2-A.3. Now, the proof follows from Lemma A.1. $\square$

9 Appendix B: Some results on pseudodifferential equations on manifolds with boundary

Here we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary, in both Bessel potential and Besov spaces. These are the main tools for proving existence theorems for mixed boundary–transmission and crack problems using potential methods. They can be found e.g. in [Esk1], [Grb1], [Sh1].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n–dimensional, non-self-intersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$, and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\sigma_{\mathcal{A}}(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \cdots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\sigma_{\mathcal{A}}(x,0,\cdots,0,+1)\,]^{-1}[\,\sigma_{\mathcal{A}}(x,0,\cdots,0,-1)\,], \quad x \in \partial\overline{\mathcal{M}}.$$

Introduce the notation

$$\delta_j(x) = \Re\,\big[\,(2\pi\, i)^{-1}\ln\lambda_j(x)\,\big],\ j = 1,\cdots,N,$$

where $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty,0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$, $j = \overline{1,N}$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. Further, note that in the particular case when $\sigma_{\mathcal{A}}(x,\xi)$ is a positive definite matrix, for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $\delta_j(x) = 0$ for $j = 1,\cdots,N$, since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1,N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary are characterized by the following theorem.

Theorem B.1 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\Re\,\sigma_{\mathcal{A}}(x,\xi)\,\eta\,\cdot\,\eta \ge c_0\,|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\mathcal{A} \;:\; \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N, \tag{B.1}$$

$$\;:\; \big[\widetilde{B}^s_{p,t}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,t}(\mathcal{M})\big]^N, \tag{B.2}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\le j\le N}\delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\le j\le N}\delta_j(x). \tag{B.3}$$

Moreover, the null–spaces and indices of the operators (B.1) *and* (B.2) *are the same* (*for all values of the parameter $t \in [1,+\infty]$*) *provided p and s satisfy the inequality* (B.3).

10 Appendix C: Explicit expressions for symbol matrices.

Here we present the explicit expressions for the homogeneous principal symbol matrices of the pseudodifferential operators introduced in the main body of the paper, in Section 4. With the help of the results exposed in Subsection 3.1 and, in particular, using formula (2.79), we derive the following formulas for the principal homogeneous symbol matrices:

$$\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2) = [\,M_{pq}(x,\xi_1,\xi_2)\,]_{6\times 6} = \begin{bmatrix} [M_{kj}(x,\xi_1,\xi_2)]_{5\times 5} & [\,0\,]_{5\times 1} \\ [\,0\,]_{1\times 5} & M_{66}(x,\xi_1,\xi_2) \end{bmatrix}_{6\times 6}$$
$$:= -\frac{1}{2\pi}\int_{\ell_\pm} [\,A^{(0)}(\,B\,\xi\,)\,]^{-1}\,d\xi_3, \qquad \xi = (\xi_1,\xi_2,\xi_3), \tag{C.1}$$

$$\mathfrak{S}(\pm 2^{-1}I_6+\mathcal{K};x,\xi_1,\xi_2) = [\,K^{(\pm)}_{pq}(x,\xi_1,\xi_2)\,]_{6\times 6} = \begin{bmatrix} [K^{(\pm)}_{kj}(x,\xi_1,\xi_2)]_{5\times 5} & [\,0\,]_{5\times 1} \\ [\,0\,]_{1\times 5} & \pm 2^{-1} \end{bmatrix}_{6\times 6}$$
$$:= \frac{i}{2\pi}\int_{\ell_\mp} \mathcal{T}^{(0)}(\,B\,\xi,n)\,[\,A^{(0)}(\,B\,\xi\,)\,]^{-1}\,d\xi_3, \tag{C.2}$$

$$\mathfrak{S}(\pm 2^{-1}I_6+\mathcal{N};x,\xi_1,\xi_2) = [\,N^{(\pm)}_{pq}(x,\xi_1,\xi_2)\,]_{6\times 6} = \begin{bmatrix} [N^{(\pm)}_{kj}(x,\xi_1,\xi_2)]_{5\times 5} & [\,0\,]_{5\times 1} \\ [\,0\,]_{1\times 5} & \pm 2^{-1} \end{bmatrix}_{6\times 6}$$
$$:= -\frac{i}{2\pi}\int_{\ell_\pm} [\,A^{(0)}(\,B\,\xi\,)\,]^{-1}\,[\mathcal{P}^{(0)}(\,B\,\xi,n)]^\top\,d\xi_3, \tag{C.3}$$

$$\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2) = [\,L_{pq}(x,\xi_1,\xi_2)\,]_{6\times 6} = \begin{bmatrix} [L_{kj}(x,\xi_1,\xi_2)]_{5\times 5} & [\,0\,]_{5\times 1} \\ [\,0\,]_{1\times 5} & L_{66}(x,\xi_1,\xi_2) \end{bmatrix}_{6\times 6}$$
$$:= -\frac{1}{2\pi}\int_{\ell_\pm} \mathcal{T}^{(0)}(\,B\,\xi,n)\,[\,A^{(0)}(\,B\,\xi\,)\,]^{-1}\,[\mathcal{P}^{(0)}(\,B\,\xi,n)]^\top\,d\xi_3, \tag{C.4}$$

where the matrices $A^{(0)}(\,\cdot\,)$, $\mathcal{T}^{(0)}(\,\cdot\,,\,\cdot\,)$ and $\mathcal{P}^{(0)}(\,\cdot\,,\,\cdot\,)$ are defined by (2.35), (2.28) and (2.39) respectively,

$$B = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{C.5}$$

is an orthogonal matrix with $\det B(x) = 1$ for $x \in \partial\Omega^\pm = S$; here $n(x)$ is the exterior unit normal vector to S, while $l(x)$ and $m(x)$ are orthogonal unit vectors in the tangential plane associated with some local chart; ℓ_- (ℓ_+) is a closed contours in the lower (upper) complex $\xi_3 = \xi_3' + i\,\xi_3''$ half-plane, orientated clockwise (counterclockwise) and circumventing all roots with negative (positive) imaginary parts of the equation $\det A^{(0)}(\,B\,\xi\,) = 0$ with respect to ξ_3, while $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$ is to be considered as parameter.

In (C.2) and (C.3) we employed that $\mathcal{N}_{66}$ and $\mathcal{K}_{66}$ are weakly singular integral operators since their kernel functions, the co-normal derivatives $\eta_{jl}\,n_j(y)\,\partial_l\Gamma_{66}(x-y)$ and $\eta_{jl}\,n_j(x)\,\partial_l\Gamma_{66}(x-y)$ are weakly singular functions of type $\mathcal{O}(|x-y|^{-\gamma})$ on S with $\gamma < 2$.

The principal homogeneous symbol matrices (C.1)-(C.4) are elliptic. Moreover, the matrices $-\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2)$ and $\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)$ are strongly elliptic, i.e., there is a positive constant c depending on the material parameters such that

$$\Re\,[-\mathfrak{S}(\mathcal{H}; x, \xi_1, \xi_2)\eta \cdot \eta] \geq c\,|\xi|^{-1}\,|\eta|^2 \quad \text{for all} \;\; x \in S,\; (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\},\; \eta \in \mathbb{C}^6, \tag{C.6}$$

$$\Re\,[\mathfrak{S}(\mathcal{L}; x, \xi_1, \xi_2)\eta \cdot \eta] \geq c\,|\xi|\,|\eta|^2 \quad \text{for all} \;\; x \in S,\; (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\},\; \eta \in \mathbb{C}^6, \tag{C.7}$$

The entries of the matrices (C.1) and (C.4) are even functions in (ξ_1, ξ_2).

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