Contents

Preface

The present work is devoted to construction and investigation of high degree precision decomposition schemes for evolution problem, on the basis of approximation of its solving operator (semigroup).

In the first chapter there are constructed the third order accuracy decomposition schemes. In the first and second sections there are constructed third order precision exponential and rational splittings for two-dimensional evolution problem. In the third section there is constructed decomposition scheme for multi-dimensional evolution problem. Fourth and fifth sections are devoted to construction of third order precision sequential type decomposition schemes for two and multi-dimensional cases.

In the second chapter there is constructed the fourth order accuracy decomposition schemes for evolution problem. In the sixth and seventh sections there are constructed fourth order precision exponential and rational splittings for two-dimensional evolution problem, and in the eighth section is constructed fourth order precision decomposition scheme for multi-dimensional case. Ninth section is devoted to construction of fourth order precision sequential type decomposition scheme.

In the third chapter there are constructed the third order accuracy decomposition schemes for an evolution problem with variable operator. Namely, there is considered the case when main operator is a product of scalar function depending on t and constant operator.

In the tenth and eleventh sections there are constructed third order precision differential and rational splittings for an evolution problem with variable operator.

In the appendix there are given results of numerical calculations for heat transfer equation. These calculations are carried out using existing first and second order and constructed in this work third order accuracy decomposition schemes. Comparative analysis of numerical calculations for different order decomposition schemes is carried out.

Authors

Introduction

It is known that mathematical simulation of processes taking place in the nature frequently leads to consideration of boundary-value problems for partial-differential evolution (nonstationary) equation. These kind of problems can be considered as a Cauchy abstract problem in a Banach space for an evolution equation with an unbounded operator.

Study of approximated schemes for solution of evolution problems leads to the conclusion that a certain operator (solution operator of the discrete problem) corresponds to each approximated scheme. This operator approximates the solution operator (semigroup) of the initial continuous problem. For example, if we use the Rotte scheme for the solution of an evolution problem, then the solution operator of the difference problem thus obtained will be a discrete semigroup and we will have the approximation of a continuous semigroup by discrete semigroups (see [40], Ch. IX). On the other hand, on the basis of the approximation of a continuous semigroup, we can construct an approximated scheme for solution of an evolution problem.

Decomposition formulas approximate a continuous semigroup by means of the combination of the semigroups generated by the addends of the operator generating this semigroup.

The first decomposition formula for an exponential matrix function was constructed by Lie in 1875. Trotter generalized this formula for an exponential operator function-semigroup in 1959 (see [60]). The resolvent analogue of this formula for the first time was constructed by Chernoff in 1968 (see [8],[9]). At the same time, in the sixties of the XX century, in order to elaborate numerical methods for solution of multi-dimensional problems of mathematical physics, the subject of construction of decomposition schemes has naturally raised. Decomposition schemes allow to reduce a solution of multi-dimensional problems to the solution of one-dimensional problems.

First works dedicated to construction and investigation of decomposition schemes were published in the fifties and sixties of the XX century (see V. B. Andreev [2], G. A. Baker [3], G. A. Baker, T. A. Oliphant [4], G. Birkhoff, R. S. Varga [6], G. Birkhoff, R. S. Varga, D. Young [7], J. Douglas [13], J. Douglas, H. Rachford [14], E. G. Diakonov [10],[11], M. Dryja [15], G. Fairweather, A. R. Gourlay, A. R. Mitchell [17], I. V. Fryazinov [18]), D. G. Gordeziani [28], A. R. Gourlay, A. R. Mitchell [32], N. N. Ianenko [33], [34], N. N. Ianenko, G. V. Demidov [35], A. N. Konovalov [41], G. I. Marchuk, N. N. Ianenko [45], G. I. Marchuk, U. M. Sultangazin [46], D. Peaceman, H. Rachford [47], V. P. Ilin [38], A. A. Samarskii [54]-[56], R. Temam [59]). These works became a basis of the further investigation of decomposition schemes.

We can show that the split problem, obtained by means of a decomposition method, generates the Trotter formula (see Trotter H. [60]), or the Chernoff formula (see Chernoff P. R. [8], [9]), or a formula which is a combination of these formulas. Thus, an estimate of decomposition method is equivalent to the study of approximation of continuous semigroup by Lie-Trotter and Lie-Chernoff type formulas. The works of T. Ichinose and S. Takanobu [36], T. Ichinose and H. Tamura [37], J. Rogava [49], (see also [50], T. II) are devoted to estimate of error of Lie-Trotter and Lie-Chernoff type formulas.

We call Lie-Trotter type formulas the formulas which approximate a semigroup by a combination of semigroups generated by the addends of the operator generating this semigroup.

We call Lie-Chernoff type formulas the formulas which are obtained from Lie-Trotter type formulas if we replace semigroups by the corresponding rational operator functions (resolvents).

Decomposition schemes conditionally can be divided into two groups - differential and difference. Lie-Trotter type formulas correspond to differential decomposition schemes and Lie-Chernoff type formulas - to difference schemes.

Decomposition schemes, associated with the Lie and Trotter formulas, allow to split a Cauchy problem for an evolution equation with the operator $A = A_1 + A_2$ into two problems, respectively with the operators A_1 and A_2, which are solved sequentially on the time interval with the length t/n.

Decomposition schemes associated with the Chernoff formula are known as the fractional-step method (see N. N. Ianenko [34]).

Decomposition schemes in view of numerical calculation can be divided into two groups: schemes of sequential account (see for example G. I. Marchuk [44]) and schemes of parallel account (D. G. Gordeziani, H. V. Meladze [30], [31], D. G. Gordeziani, A. A. Samarskii [29], A. M. Kuzyk, V. L. Makarov [43]). In [50] (see chapter II), there are obtained explicit a priori estimations for decomposition schemes of parallel account considered in [30]. There exist a lot of works devoted to decomposition schemes. For example, see [34], [44], [57] and the references therein.

In the above-stated works the considered schemes are of the first or second precision order. As it is known to us, the high accuracy order decomposition schemes in case of two addends ($A = A_1 + A_2$) for the first time were obtained by B. O. Dia and M. Schatzman (see [12]). Note that the formulas constructed in these works are not automatically stable decomposition formulas. Decomposition formula is called automatically stable if a sum of the absolute values of split coefficients is equal to one. Q. Sheng has proved (see [58]) that, on the real number field, there does not exist such automatically stable splitting of $exp\,(-tA)$, the accuracy order of which is higher than two.

The present work is devoted to construction and investigation of the high order accuracy decomposition schemes for an evolution problem.

In this work, by introducing a complex parameter, the third and fourth order accuracy decomposition schemes are constructed for a two and multi-dimensional evolution problems. The main operator of the evolution problem conditionally is called the m-dimensional split operator if it represents a sum of m (> 1) addends $(A = A_1 + ... + A_m)$. The formulas, corresponding to the constructed schemes, are automatically stable decomposition formulas. For the considered schemes, there are obtained explicit a priori estimations. Under the explicit estimation we mean such a priori estimation for the solution error, where constants in the right-hand side do not depend on the solution of the initial continuous problem, i.e. are absolute constants.

In the works [19]-[27], [51]-[53],[61],[62] we have constructed the third and fourth order accuracy decomposition schemes for two and multi-dimensional homogeneous and inhomogeneous evolution problems. In the present work these schemes are discussed on the basis of conception that any decomposition formula generates the certain decomposition scheme and, vice versa, every decomposition scheme generates certain decomposition formula, which approximates the solving operator (semigroup) of evolution problem.

In Banach space there are constructed the third and fourth order accuracy decomposition schemes for evolution problem with operator $A = A_1 + A_2 + ... + A_m$ $(m \geq 2)$, which generates strongly continuous semigroup $U(t, A) = \exp(-tA)$. These schemes are based on the following decomposition formulas of semigroup approximation:

$$
\begin{aligned}
V_1(t) &= \frac{1}{2}\left[T(t,\overline{\alpha})\,\overline{T}(t,\alpha) + \overline{T}(t,\overline{\alpha})\,T(t,\alpha)\right], \\
V_2(t) &= T\left(t,\frac{\alpha}{2}\right)\overline{T}\left(t,\frac{\alpha}{2}\right)T\left(t,\frac{\overline{\alpha}}{2}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{2}\right), \\
V_3(t) &= \frac{1}{2}\left[T\left(t,\frac{\alpha}{2}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{2}\right)T\left(t,\frac{\overline{\alpha}}{2}\right)\overline{T}\left(t,\frac{\alpha}{2}\right)\right. \\
&\quad \left. + \overline{T}\left(t,\frac{\alpha}{2}\right)T\left(t,\frac{\overline{\alpha}}{2}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{2}\right)T\left(t,\frac{\alpha}{2}\right)\right], \\
V_4(t) &= T\left(t,\frac{\overline{\alpha}}{4}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{4}\right)T\left(t,\frac{\alpha}{4}\right)\overline{T}\left(t,\frac{\alpha}{4}\right) \\
&\quad \times T\left(t,\frac{\alpha}{4}\right)\overline{T}\left(t,\frac{\alpha}{4}\right)T\left(t,\frac{\overline{\alpha}}{4}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{4}\right),
\end{aligned}
$$

where $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ $\quad \left(i = \sqrt{-1}\right)$,

$$
\begin{aligned}
T(t,\alpha) &= U(t,\alpha A_1)\,...U(t,\alpha A_{m-1})\,U(t,\alpha A_m), \\
\overline{T}(t,\alpha) &= U(t,\alpha A_m)\,...U(t,\alpha A_2)\,U(t,\alpha A_1).
\end{aligned}
$$

Here upper index defines dimension (Number of addends of the operator A) of evolution problem. It is meant that the operators $(-\gamma A_j)$, $\gamma = 1,\ \alpha,\ \overline{\alpha}$ $(j = 1, ..., m)$ generate strongly continuous semigroups and $\|U(t, \gamma A_j)\| \leq e^{\omega t}$, $\quad \omega = const > 0$.

For the above stated formulas the following estimations are true:

$$
\begin{aligned}
\left\|\left(U(t_k, A) - [V_j(\tau)]^k\right)\varphi\right\| &= O\left(\tau^3\right), \quad \varphi \in D\left(A^4\right), \quad j = 1, 2, \\
\left\|\left(U(t_k, A) - [V_j(\tau)]^k\right)\varphi\right\| &= O\left(\tau^4\right), \quad \varphi \in D\left(A^5\right), \quad j = 3, 4,
\end{aligned}
$$

where $t_k = k\tau$, $\tau > 0$ is a time step.

In case of homogeneous evolution problem works the obvious rule, according to which can be constructed decomposition schemes corresponding to the above-mentioned formulas. For instance, to $V_1(t)$ (in case of $m = 2$) corresponds the following decomposition scheme:

$$\begin{aligned}
\frac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) &= 0, \quad v_k^1(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^2(t)}{dt} + A_2 v_k^2(t) &= 0, \quad v_k^2(t_{k-1}) = v_k^1(t_k), \\
\frac{dv_k^3(t)}{dt} + \overline{\alpha} A_1 v_k^3(t) &= 0, \quad v_k^3(t_{k-1}) = v_k^2(t_k);
\end{aligned}$$

$$\begin{aligned}
\frac{dw_k^1(t)}{dt} + \alpha A_2 w_k^1(t) &= 0, \quad w_k^1(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dw_k^2(t)}{dt} + A_1 w_k^2(t) &= 0, \quad w_k^2(t_{k-1}) = w_k^1(t_k), \\
\frac{dw_k^3(t)}{dt} + \overline{\alpha} A_2 w_k^3(t) &= 0, \quad w_k^3(t_{k-1}) = w_k^2(t_k),
\end{aligned}$$

$$u_k(t) = \frac{1}{2}\left[v_k^3(t) + w_k^3(t)\right], \quad t \in [t_{k-1}, t_k], \quad k = 1, 2, \dots,$$

where $u_0(0)$ is initial value of the $u(t)$ exact solution of evolution problem. As an approximate value of $u(t)$ at point $t = t_k$ we declare $u_k(t_k)$. Precision of the above-mentioned scheme is $O(\tau^3)$.

This estimations remain true if $U(t, A)$ will be changed corresponding by the third and the fourth order accuracy rational operator functions:

$$\begin{aligned}
W(t, A) &= \left(I - \frac{1}{3}tA\right)(I + \lambda tA)^{-1}\left(I + \overline{\lambda} tA\right)^{-1}, \quad \lambda = \frac{1}{3} \pm i\frac{1}{3\sqrt{2}}, \\
W(t, A) &= \left(I - \frac{\alpha}{2}tA\right)\left(I + \frac{\overline{\alpha}}{2}tA\right)^{-1}\left(I - \frac{\overline{\alpha}}{2}tA\right)\left(I + \frac{\alpha}{2}tA\right)^{-1}.
\end{aligned}$$

There is shown the stability of the constructed decomposition scheme and explicit a prior error estimations for approximate solutions are obtained. In the present work there is also considered case when the main operator is depending on t, namely it is a product of scalar function dependant on t and the constant operator.

Chapter I

The Third Order Accuracy Decomposition Schemes

§1. The third order accuracy decomposition scheme for non-homogeneous evolution problem

1. Decomposition scheme and theorem on error estimation

Let us consider Cauchy abstract problem in Banach space X

$$\frac{du\,(t)}{dt} + Au\,(t) = f\,(t)\,, \qquad t > 0, \qquad u\,(0) = \varphi. \tag{1.1}$$

Here A is a closed linear operator with the domain $D\,(A)$, which is everywhere dense in X, φ is a given element from $D\,(A)$, $f\,(t) \in C^1\,(X;[0;\infty))$.

Suppose that $(-A)$ operator generates a strongly continuous semigroup, then solution of the problem (1.1) is given by the formula (see [39],[42]):

$$u(t) = U(t, A)\varphi + \int\limits_0^t U(t-s, A)f(s)ds, \tag{1.2}$$

where $U\,(t, A) = exp(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are compactly defined, closed linear operators in X.

Let us introduce a difference net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Along with problem (1.1) we consider two sequences of the following problems on each $[t_{k-1}, t_k]$ interval.

$$\begin{aligned}
\frac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) &= \frac{\alpha}{2} f(t_k) - 2\sigma_0 (t_k - t) f'(t_k), \\
v_k^1(t_{k-1}) &= u_{k-1}(t_{k-1}), \\
\frac{dv_k^2(t)}{dt} + A_2 v_k^2(t) &= \frac{1}{2} f(t_k) - 2\sigma_1 (t_k - t) f'(t_k), \\
v_k^2(t_{k-1}) &= v_k^1(t_k), \\
\frac{dv_k^3(t)}{dt} + \overline{\alpha} A_1 v_k^3(t) &= \frac{\overline{\alpha}}{2} f(t_k) - 2\sigma_2 (t_k - t) f'(t_k) + \frac{(t_k - t)^2}{2} f''(t_k), \\
v_k^3(t_{k-1}) &= v_k^2(t_k);
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
\frac{dw_k^1(t)}{dt} + \alpha A_2 w_k^1(t) &= \frac{\alpha}{2} f(t_k) - 2\sigma_0 (t_k - t) f'(t_k), \\
w_k^1(t_{k-1}) &= u_{k-1}(t_{k-1}), \\
\frac{dw_k^2(t)}{dt} + A_1 w_k^2(t) &= \frac{1}{2} f(t_k) - 2\sigma_1 (t_k - t) f'(t_k), \\
w_k^2(t_{k-1}) &= w_k^1(t_k), \\
\frac{dw_k^3(t)}{dt} + \overline{\alpha} A_2 w_k^3(t) &= \frac{\overline{\alpha}}{2} f(t_k) - 2\sigma_2 (t_k - t) f'(t_k) + \frac{(t_k - t)^2}{2} f''(t_k), \\
w_k^3(t_{k-1}) &= w_k^2(t_k).
\end{aligned} \tag{1.4}$$

Here $\sigma_0,\ \sigma_1,\ \sigma_2$ and α are numerical complex parameters with $Re(\alpha) > 0$, $u_0(0) = \varphi$. Suppose that $(-\gamma A_j),\ \gamma = 1,\ \alpha,\ \overline{\alpha} \quad (j = 1, 2)$ operators generate strongly continuous semigroups.

On each $[t_{k-1}, t_k]$ $(k = 1, 2, ...)$ interval $u_k(t)$ are defined as follows:

$$u_k(t) = \frac{1}{2}\left[v_k^3(t) + w_k^3(t)\right]. \tag{1.5}$$

We declare $u_k(t)$ function as approached solution of the problem (1.1) on $[t_{k-1}, t_k]$ interval.

We shall need natural degrees of the operator $A = A_1 + A_2$. Usually they are defined as follows:

$$\begin{aligned}
A^2 &= \left(A_1^2 + A_2^2\right) + (A_1A_2 + A_2A_1), \\
A^3 &= \left(A_1^3 + A_2^3\right) + \left(A_1^2A_2 + ... + A_2^2A_1\right) + (A_1A_2A_1 + A_2A_1A_2),
\end{aligned}$$

Analogously are defined A^s, $s > 3$. Obviously, the domain $D(A^s)$ of the operator A^s is the intersection of the domains of its addends.

Let us introduce the following definitions:

$$\begin{aligned}
\|\varphi\|_A &= \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D(A); \\
\|\varphi\|_{A^2} &= \left\|A_1^2\varphi\right\| + \left\|A_2^2\varphi\right\| + \|A_1A_2\varphi\| + \|A_2A_1\varphi\|, \quad \varphi \in D\left(A^2\right),
\end{aligned}$$

where $\|\cdot\|$ is a norm in X. Analogously are defined $\|\varphi\|_{A^s}$, $s > 2$.

Theorem 1.1 *Let the following conditions be satisfied:*

(a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$;

(b) $(-\gamma A_j)$, $\gamma = 1,\ \alpha,\ \overline{\alpha}\ \ (j = 1, 2)$ *and* $(-A)$ *operators generate strongly continuous semigroups;*

(c) There exist such real number ω*, that*

$$\begin{aligned} \|U(t, A)\| &\leq Me^{\omega t}, \quad M = const > 0, \\ \|U(t, \gamma A_j)\| &\leq e^{\omega t} \quad (j = 1, 2;\ \gamma = 1, \alpha, \overline{\alpha}); \end{aligned}$$

(d) $U(s, A)\varphi \in D(A^4)$ *for every fixed* $s \geq 0$;

(e) $f(t) \in C^3([0, \infty); X)$; $\quad f(t) \in D(A^3)$, $f^{(k)}(t) \in D\left(A^{3-k}\right)$, $k = 1, 2$ *and* $U(s, A) f(t) \in D(A^4)$ *for every fixed* t *and* s $(t, s \geq 0)$;

(f) $\sigma_0 = \frac{2-\overline{\alpha}}{4+\alpha} - \frac{2+\overline{\alpha}}{4+\alpha}\sigma_1, \quad \sigma_2 = \frac{1+\overline{\alpha}}{2(4+\alpha)} - \frac{3-2\overline{\alpha}}{4+\alpha}\sigma_1$ *(σ_1 is any complex number).*

Then the following estimation holds:

$$\begin{aligned} \|u_k(t_k) - u(t_k)\| \leq\ & ce^{\omega_0 t_k} t_k \tau^3 \Bigg(\sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^4} \\ & + t_k \sup_{s,t\in[0,t_k]} \|U(s, A) f(t)\|_{A^4} + \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} \\ & + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} + \sup_{t\in[0,t_k]} \|f''(t)\|_{A} + \sup_{t\in[0,t_k]} \|f'''(t)\| \Bigg), \end{aligned}$$

where c, ω_0 *are absolute positive constants.*

2. Third order accuracy exponential splitting of semigroup

The solving operator of the homogeneous evolution problem corresponding to the decomposed problem (1.3)-(1.4) is $V^k(\tau)$, where

$$V(\tau) = \frac{1}{2}\left[U(\tau, \overline{\alpha}A_1)U(\tau, A_2)U(\tau, \alpha A_1) + U(\tau, \overline{\alpha}A_2)U(\tau, A_1)U(\tau, \alpha A_2)\right].$$

It is clear that operator $V^k(\tau)$ must approximate solving operator of the homogeneous evolution problem - semigroup.

The following theorem takes place.

Theorem 1.2 *If the conditions (a), (b), (c) and (d) of the Theorem 1.1 are satisfied, then for every natural* k *the following estimation holds:*

$$\left\|\left[U(t_k, A) - V^k(\tau)\right]\varphi\right\| \leq ce^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^4}, \qquad (1.6)$$

where c, ω_0 *are positive constants.*

Proof. According to the formula (see Kato. T. [40], p. 603):

$$A\int_r^t U(s,A)\,ds = U(r,A) - U(t,A)\,, \quad 0 \le r \le t,$$

we can get the following expansion:

$$U(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t,A)\,, \tag{1.7}$$

where

$$R_k(t,A) = (-A)^k \int_0^t \int_0^{s_1} \ldots \int_0^{s_{k-1}} U(s,A)\, ds ds_{k-1} \ldots ds_{1.} \tag{1.8}$$

Let us consider $V(\tau)$ and decompose both its items from the right to left according to the formula (1.7) so that each residual member is of the fourth order. Then, using elementary algebraic transformations, we shall get:

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \widetilde{R}_4(\tau)\,, \tag{1.9}$$

where

$$\widetilde{R}_4(\tau) = \frac{1}{2}\left[R_{1,2}(\tau) + R_{2,1}(\tau)\right], \tag{1.10}$$

and where

$$\begin{aligned}
R_{i,j}(\tau) &= R_4(\tau,\overline{\alpha}A_i) - \tau R_3(\tau,\overline{\alpha}A_i)\,A_j + \frac{1}{2}\tau^2 R_2(\tau,\overline{\alpha}A_i)\,A_j^2 \\
&\quad -\frac{1}{6}\tau^3 R_1(\tau,\overline{\alpha}A_i)\,A_j^3 + U(\tau,\overline{\alpha}A_i)\,R_4(\tau,A_j) \\
&\quad -\alpha\tau R_3(\tau,\overline{\alpha}A_i)\,A_i + \alpha\tau^2 R_2(\tau,\overline{\alpha}A_i)\,A_j A_i \\
&\quad -\frac{1}{2}\alpha\tau^3 R_1(\tau,\overline{\alpha}A_i)\,A_j^2 A - \alpha\tau U(\tau,\overline{\alpha}A_i)\,R_3(\tau,A_j)\,A_i \\
&\quad +\frac{1}{2}\alpha^2\tau^2 R_2(\tau,\overline{\alpha}A_i)\,A_i^2 - \frac{1}{2}\alpha^2\tau^3 R_1(\tau,\overline{\alpha}A_i)\,A_j A \\
&\quad +\frac{1}{2}\alpha^2\tau^2 U(\tau,\overline{\alpha}A_i)\,R_2(\tau,A_j)\,A_i^2 - \frac{1}{6}\alpha^3\tau^3 R_1(t,\overline{\alpha}A_i)\,A_i^3 \\
&\quad -\frac{1}{6}\alpha^3\tau^3 U(\tau,\overline{\alpha}A_i)\,R_1(\tau,A_j)\,A_i^3 \\
&\quad +U(\tau,\overline{\alpha}A_i)\,U(\tau,A_j)\,R_4(\tau,\alpha A_i)\,, \\
i,j &= 1,2.
\end{aligned}$$

According to the formula (1.7) we have:

$$U(\tau,A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_4(\tau,A)\,. \tag{1.11}$$

From equalities (1.9) and (1.11) we have:

$$U(\tau, A) - V(\tau) = R_4(\tau, A) - \widetilde{R}_4(\tau).$$

Hence, according to the formula (1.11) and condition (c) of the theorem we get the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le ce^{\omega_0 \tau}\tau^4 \|\varphi\|_{A^4}, \quad \varphi \in D(A^4). \tag{1.12}$$

The following decomposition is obvious:

$$\begin{aligned}\left[U(t_k, A) - V^k(\tau)\right]\varphi &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ &= \sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau, A) - V(\tau)\right]U^{i-1}(\tau, A).\end{aligned}$$

Hence, according to the formula (1.11) and condition (c) of the theorem we get the sought estimation. □

Remark 1.3. *It is obvious that according to the condition of the Theorem* ($\|U(t, \gamma A_j)\| \le e^{\omega t}$) *the norm of the operator* $V^k(\tau)$ *is less or equal to* $e^{\omega_0 t_k}$. *From here follows stability of the above-stated decomposition schema on each finite time interval.*

3. Error estimation for approximate solution

Let us prove the auxiliary **Lemmas** on which the proof of the **Theorem 1.1** is based**.**

Lemma 1.4. *If the conditions (a), (b) and (c) of the Theorem 1.1 are satisfied, then the following estimation holds:*

$$\left\|\int_0^\tau \left[U((t, A)) - \left(\frac{\alpha}{2}V_0(\tau, t) + \frac{1}{2}V_1(\tau, t) + \frac{\overline{\alpha}}{2}V_2(t)\right)\right]\varphi dt\right\| \le$$

$$\le ce^{\omega_0 \tau}\tau^4 \|\varphi\|_{A^3}, \ \varphi \in D(A^3), \tag{1.13}$$

where

$$\begin{aligned}V_0(\tau, t) &= \frac{1}{2}\left[U(\tau, \overline{\alpha}A_1)U(\tau, A_2)U(t, \alpha A_1) + U(\tau, \overline{\alpha}A_2)U(\tau, A_1)U(t, \alpha A_2)\right], \\ V_1(\tau, t) &= \frac{1}{2}\left[U(\tau, \overline{\alpha}A_1)U(t, A_2) + U(\tau, \overline{\alpha}A_2)U(t, A_1)\right], \\ V_2(t) &= \frac{1}{2}\left[U(t, \overline{\alpha}A_1) + U(t, \overline{\alpha}A_2)\right].\end{aligned}$$

Here c and ω_0 *are positive constants.*

Proof. Let us consider $V_0(\tau, t)$ and decompose both its items from the right to left according to the formula (1.7) so that each residual member is of the third order. Then, using elementary algebraic transformations, we shall get:

$$
\begin{aligned}
V_0(\tau,t) &= \frac{1}{2}\left[2I-\tau\left(\left(\overline{\alpha}+\alpha\frac{t}{\tau}+1\right)A_1+\left(\overline{\alpha}+\alpha\frac{t}{\tau}+1\right)A_2\right)\right.\\
&+\frac{1}{2}\tau^2\left(\overline{\alpha}^2+2\overline{\alpha}\alpha\frac{t}{\tau}+\alpha^2\frac{t^2}{\tau^2}+1\right)A_1^2\\
&+\left(\overline{\alpha}^2+2\overline{\alpha}\alpha\frac{t}{\tau}+\alpha^2\frac{t^2}{\tau^2}+1\right)A_2^2\\
&+\left.\left(2\alpha\frac{t}{\tau}+2\overline{\alpha}\right)A_1A_2+\left(2\alpha\frac{t}{\tau}+2\overline{\alpha}\right)A_2A_1\right)\Bigg]\\
&+R_{1,0}(\tau,t),
\end{aligned}
\tag{1.14}
$$

where

$$
\begin{aligned}
R_{1,0}(\tau,t) &= \frac{1}{2}\Bigg[R_3(\tau,\overline{\alpha}A_1)-\tau R_2(\tau,\overline{\alpha}A_1)A_2+\frac{1}{2}\tau^2R_1(\tau,\overline{\alpha}A_1)A_2^2\\
&+U(\tau,\overline{\alpha}A_1)R_3(\tau,A_2)-\alpha tR_2(\tau,\overline{\alpha}A_1)A_1-\alpha tR_2(\tau,\overline{\alpha}A_1)A_1\\
&+\alpha\tau tR_1(\tau,\overline{\alpha}A_1)A_2A_1-\alpha tU(\tau,\overline{\alpha}A_1)R_2(\tau,A_2)A_1\\
&+\frac{1}{2}\alpha^2t^2R_1(\tau,\overline{\alpha}A_1)A_1^2+\frac{1}{2}\alpha^2t^2U(\tau,\overline{\alpha}A_1)R_1(\tau,A_2)A_1^2\\
&+U(\tau,\overline{\alpha}A_1)U(\tau,A_2)R_3(t,\alpha A_1)+R_3(\tau,\overline{\alpha}A_2)\\
&-\tau R_2(\tau,\overline{\alpha}A_2)A_1+\tau^2R_1(\tau,\overline{\alpha}A_2)A_1^2+U(\tau,\overline{\alpha}A_2)R_3(\tau,A_1)\\
&-\alpha tR_2(\tau,\overline{\alpha}A_2)A_2+\alpha\tau tR_1(\tau,\overline{\alpha}A_2)A_1A_2\\
&-\alpha tU(\tau,\overline{\alpha}A_2)R_2(\tau,A_1)A_2+\frac{1}{2}\alpha^2t^2R_1(\tau,\overline{\alpha}A_2)A_2^2\\
&+\frac{1}{2}\alpha^2t^2U(\tau,\overline{\alpha}A_2)R_1(\tau,A_1)A_2^2\\
&+U(\tau,\overline{\alpha}A_2)U(\tau,A_1)R_3(t,\alpha A_2)\Bigg].
\end{aligned}
$$

Let us similarly decompose $V_1(\tau,t)$:

$$
\begin{aligned}
V_1(\tau,t) &= \frac{1}{2}\left[2I-\tau\left(\left(\overline{\alpha}+\frac{t}{\tau}\right)A_1+\left(\overline{\alpha}+\frac{t}{\tau}\right)A_2\right)\right.\\
&+\frac{1}{2}\tau^2\left(\left(\overline{\alpha}^2+\frac{t^2}{\tau^2}\right)A_1^2+\left(\overline{\alpha}^2+\frac{t^2}{\tau^2}\right)A_2^2\right. && (1.15)\\
&\left.\left.+2\overline{\alpha}\frac{t}{\tau}A_1A_2+2\overline{\alpha}\frac{t}{\tau}A_2A_1\right)\right]+R_{1,1}(\tau,t), && (1.16)
\end{aligned}
$$

where

$$
\begin{aligned}
R_{1,1}(\tau,t) &= \frac{1}{2}\Bigg[R_3(\tau,\overline{\alpha}A_1)-tR_2(\tau,\overline{\alpha}A_1)A_2+\frac{1}{2}t^2R_1(\tau,\overline{\alpha}A_1)A_2^2\\
&+U(\tau,\overline{\alpha}A_1)R_3(t,A_2)+R_3(\tau,\overline{\alpha}A_2)-tR_2(\tau,\overline{\alpha}A_2)A_1\\
&+\frac{1}{2}t^2R_1(\tau,\overline{\alpha}A_2)A_1^2+U(\tau,\overline{\alpha}A_2)R_3(t,A_1)\Bigg].
\end{aligned}
$$

Finally for $V_2(t)$, we have:

$$\begin{aligned} V_2(t) &= \frac{1}{2}\left[2I - \tau\left(\overline{\alpha}\frac{t}{\tau}A_1 + \overline{\alpha}\frac{t}{\tau}A_2\right)\right. \\ &\left. + \frac{1}{2}\tau^2\left(\overline{\alpha}^2\frac{t^2}{\tau^2}A_1^2 + \overline{\alpha}^2\frac{t^2}{\tau^2}A_2^2\right)\right] + R_{1,2}(t), \end{aligned} \quad (1.17)$$

where

$$R_{1,2}(t) = \frac{1}{2}\left[R_3(t, \overline{\alpha}A_1) + R_3(t, \overline{\alpha}A_2)\right].$$

Finally using decompositions (1.14),(1.16) and (1.17) we have:

$$\begin{aligned} &\frac{\alpha}{2}V_0(\tau,t) + \frac{1}{2}V_1(\tau,t) + \frac{\overline{\alpha}}{2}V_2(t) \\ = &\ I - \tau\left[\left(\frac{1}{3} + \frac{1}{3\tau}t\right)A_1 + \left(\frac{1}{3} + \frac{1}{3\tau}t\right)\right]A_2 \\ &+ \frac{1}{2}\tau^2\left[\left(\frac{1}{12}\overline{\alpha} + \frac{1}{4}\overline{\alpha}^2 + \frac{1}{4}\alpha + \frac{1}{6\tau}\alpha t + \frac{1}{4\tau^2}t^2\right)A_1^2 + \right. \\ &+ \left(\frac{1}{12}\overline{\alpha} + \frac{1}{4}\overline{\alpha}^2 + \frac{1}{4}\alpha + \frac{1}{6\tau}\alpha t + \frac{1}{4\tau^2}t^2\right)A_2^2 \\ &\left. + \left(\frac{1}{6} + \frac{1}{3\tau}t\right)A_1A_2 + \left(\frac{1}{6} + \frac{1}{3\tau}t\right)A_2A_1\right] + R^{(1)}(\tau,t), \end{aligned} \quad (1.18)$$

where

$$R^{(1)}(\tau,t) = \frac{\alpha}{2}R_{1,0}(\tau,t) + \frac{1}{2}R_{1,1}(\tau,t) + \frac{\overline{\alpha}}{2}R_{1,2}(t).$$

Let us integrate equality (1.18) from 0 to τ and group together similar members:

$$\begin{aligned} \int_0^\tau \left[\frac{\alpha}{2}V_0(\tau,t) + \frac{1}{2}V_1(\tau,t) + \frac{\overline{\alpha}}{2}V_2(t)\right]\varphi dt &= \left(\tau I - \frac{1}{2}\tau^2 A + \frac{1}{6}\tau^3 A^2\right)\varphi \\ &+ \int_0^\tau R^{(1)}(\tau,t)\varphi dt. \end{aligned} \quad (1.19)$$

According to the formula (1.7) we have:

$$\int_0^\tau U(t,A)\varphi dt = \left(\tau I - \frac{1}{2}\tau^2 A + \frac{1}{6}\tau^3 A^2\right)\varphi + \int_0^\tau R_3(t,A)\varphi dt. \quad (1.20)$$

From equalities (1.19) and (1.20) we have:

$$\begin{aligned} &\int_0^\tau \left[U(t,A) - \left(\frac{\alpha}{2}V_0(\tau,t) + \frac{1}{2}V_1(\tau,t) + \frac{\overline{\alpha}}{2}V_2(t)\right)\right]\varphi dt \\ = &\int_0^\tau R_3(t,A)\varphi dt - \int_0^\tau R^{(1)}(\tau,t)\varphi dt. \end{aligned} \quad (1.21)$$

Let us consider the second addend of a right member of the formula (1.21) and use the formula (1.8). We shall get the following estimation:

$$\left\| \int_0^\tau R^{(1)}(\tau,t)\,\varphi dt \right\| \leq c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^3}, \quad \varphi \in D\left(A^3\right). \tag{1.22}$$

Similarly for the first addend we have:

$$\left\| \int_0^\tau R_3(t,A)\,\varphi dt \right\| \leq c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^3}, \quad \varphi \in D\left(A^3\right). \tag{1.23}$$

From equality (1.21), using inequalities (1.22) and (1.23) we get estimation (1.13). □

Lemma 1.5. *If the conditions (a), (b) and (c) of the Theorem are satisfied, then the following estimation holds:*

$$\left\| \int_0^\tau \left[U(s,A) - 2\left(\sigma_0 V_0(\tau,s) + \sigma_1 V_1(\tau,s) + \sigma_2 V_2(s)\right)\right] s\varphi ds \right\|$$
$$\leq c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^2}, \quad \varphi \in D\left(A^2\right), \tag{1.24}$$

where $V_0(\tau,t)$, $V_1(\tau,t)$, $V_2(t)$ *are defined in Lemma 1.4;* c, ω_0 *are positive constants.*

Proof. Let us consider $V_0(\tau,t)$ and decompose it similarly as in **Lemma 1.4** with the difference that each residual member is of the second order. Then we get:

$$V_0(\tau,t) = I - \frac{\overline{\alpha}\tau + \tau + \alpha t}{2} A + R_{2,0}(\tau,t), \tag{1.25}$$

where

$$\begin{aligned} R_{2,0}(\tau,t) = & \frac{1}{2}\left[R_2(\tau,\overline{\alpha}A_1) - \tau R_1(\tau,\overline{\alpha}A_1) A_2 + U(\tau,\overline{\alpha}A_1) R_2(\tau,A_2)\right. \\ & -\alpha t R_1(\tau,\overline{\alpha}A_1) A_1 - \alpha t U(\tau,\overline{\alpha}A_1) R_1(\tau,A_2) A_1 \\ & + U(\tau,\overline{\alpha}A_1) U(\tau,A_2) R_2(t,\alpha A_1) + R_2(\tau,\overline{\alpha}A_2) - \\ & \tau R_1(\tau,\overline{\alpha}A_2) A_1 + U(\tau,\overline{\alpha}A_2) R_2(\tau,A_1) - \alpha t R_1(\tau,\overline{\alpha}A_2) A_2 \\ & \left. -\alpha t U(\tau,\overline{\alpha}A_2) R_1(\tau,A_1) A_2 + U(\tau,\overline{\alpha}A_2) U(\tau,A_1) R_2(t,\alpha A_2)\right]. \end{aligned}$$

Similarly for $V_1(\tau,t)$ we have:

$$V_1(\tau,t) = I - \frac{\overline{\alpha}\tau + t}{2} A + R_{2,1}(\tau,t), \tag{1.26}$$

where

$$\begin{aligned} R_{2,1}(\tau,t) = & \frac{1}{2}\left[R_2(\tau,\overline{\alpha}A_1) - t R_1(\tau,\overline{\alpha}A_1) A_2 + U(\tau,\overline{\alpha}A_1) R_2(t,A_2)\right. \\ & \left. + R_2(\tau,\overline{\alpha}A_2) - t R_1(\tau,\overline{\alpha}A_2) A_1 + U(\tau,\overline{\alpha}A_2) R_2(t,A_1)\right]. \end{aligned}$$

Similarly for $V_2(t)$ we have:

$$V_2(t) = I - \frac{1}{2}\overline{\alpha} tA + R_{2,2}(t), \tag{1.27}$$

where

$$R_{2,2}(t) = \frac{R_2(t, \overline{\alpha}A_1) + R_2(t, \overline{\alpha}A_2)}{2}.$$

Taking into account equalities (1.25),(1.26) and (1.27) we get:

$$\begin{aligned}
& 2(\sigma_0 V_0(\tau, s) + \sigma_1 V_1(\tau, s) + \sigma_2 V_2(s)) \\
= \; & 2\left[\sigma_0 \left(I - \frac{\overline{\alpha}\tau + \tau + \alpha t}{2} A + R_{2,0}(\tau, t)\right)\right. \\
& \left. + \sigma_1 \left(I - \frac{\tau\overline{\alpha} + t}{2} A + R_{2,1}(\tau, t)\right) + \sigma_2 \left(I - \frac{1}{2}\overline{\alpha} tA + R_{2,2}(\tau, t)\right)\right) \\
= \; & 2(\sigma_0 + \sigma_1 + \sigma_2) I - [\sigma_0(\overline{\alpha}\tau + \tau + \alpha t) + \sigma_1(\overline{\alpha}\tau + t) + \sigma_2 \overline{\alpha} t] A \\
& + R^{(2)}(\tau, t),
\end{aligned} \tag{1.28}$$

where

$$R^{(2)}(\tau, t) = 2(\sigma_0 R_{2,0}(\tau, t) + \sigma_1 R_{2,1}(\tau, t) + \sigma_2 R_{2,2}(t)).$$

Let us multiply (1.28) on s, integrate it from 0 to τ and group together similar members, then we get:

$$\begin{aligned}
& \int_0^\tau 2(\sigma_0 V_0(\tau, s) + \sigma_1 V_1(\tau, s) + \sigma_2 V_2(s)) s\varphi ds \\
= \; & \left[\tau^2 (\sigma_0 + \sigma_1 + \sigma_2) I - \tau^3 \left(\left(\frac{1}{2}\overline{\alpha} + \frac{1}{2} + \frac{1}{3}\alpha\right)\sigma_0 + \right.\right. \\
& \left.\left. + \left(\frac{1}{2}\overline{\alpha} + \frac{1}{3}\right)\sigma_1 + \frac{1}{3}\overline{\alpha}\sigma_2\right) A\right]\varphi + \int_0^\tau sR^{(2)}(\tau, s)\varphi ds.
\end{aligned}$$

According to relations between parameters $\sigma_0, \sigma_1, \sigma_2$ we have:

$$\begin{aligned}
\sigma_0 + \sigma_1 + \sigma_2 &= \frac{1}{2}, \\
\left(\frac{1}{2}\overline{\alpha} + \frac{1}{2} + \frac{1}{3}\alpha\right)\sigma_0 + \left(\frac{1}{2}\overline{\alpha} + \frac{1}{3}\right)\sigma_1 + \frac{1}{3}\overline{\alpha}\sigma_2 &= \frac{1}{3}.
\end{aligned}$$

Taking into account these equalities we get

$$
\begin{aligned}
&\int_0^\tau 2\left(\sigma_0 V_0(\tau,s)+\sigma_1 V_1(\tau,s)+\sigma_2 V_2(s)\right)s\varphi ds\\
=\;&\left(\frac{1}{2}\tau^2 I-\frac{1}{3}\tau^3 A\right)\varphi\\
&+\int_0^\tau sR^{(2)}(\tau,s)\varphi ds. \qquad (1.29)
\end{aligned}
$$

According to formula (1.7) we have:

$$
\int_0^\tau U(s,A)\,s\varphi ds=\left(\frac{1}{2}\tau^2 I-\frac{1}{3}\tau^3 A\right)\varphi+\int_0^\tau R_2(s,A)\,s\varphi ds. \qquad (1.30)
$$

From (1.29) and (1.30) we get:

$$
\begin{aligned}
&\int_0^\tau \left[U(s,A)-2\left(\sigma_0 V_0(\tau,s)+\sigma_1 V_1(\tau,s)+\sigma_2 V_2(s)\right)\right]s\varphi ds\\
=\;&\int_0^\tau \left(R_2(s,A)+R^{(2)}(\tau,s)\right)s\varphi ds. \qquad (1.31)
\end{aligned}
$$

Let us take into account the formula (1.8) and estimate second addend of a right side of equality (1.31), we obtain:

$$
\left\|\int_0^\tau sR^{(2)}(\tau,s)\,\varphi ds\right\|\le ce^{\omega_0\tau}\tau^4\left\|\varphi\right\|_{A^2}, \quad \varphi\in D\left(A^2\right). \qquad (1.32)
$$

Also the following estimation holds:

$$
\left\|\int_0^\tau sR_2(s,A)\,\varphi ds\right\|\le ce^{\omega_0\tau}\tau^4\left\|\varphi\right\|_{A^2}, \quad \varphi\in D\left(A^2\right). \qquad (1.33)
$$

From the equality (1.31), using inequalities (1.32) and (1.33) we obtain estimation (1.24). □

Lemma 1.7. *If the conditions a)), b)) and c)) of the Theorem are satisfied, then the following estimation holds:*

$$
\left\|\int_0^\tau \left[U(s,A)-V_2(s)\right]\frac{s^2}{2}\varphi ds\right\|\le ce^{\omega_0\tau}\tau^4\left\|\varphi\right\|_A, \quad \varphi\in D(A),
$$

where $V_2(s)$ is defined in the Lemma1.4; c_0, ω_0 are positive constants.

Proof. According to the formulas (1.7), (1.8) we obtain the following estimation:

$$
\begin{aligned}
& \left\| \int_0^\tau [U(s,A) - V_2(s)] \frac{s^2}{2} \varphi ds \right\| \\
= & \left\| \int_0^\tau \left[U(s,A) - \frac{1}{2} \left(U(s,\overline{\alpha}A_1) + U(s,\overline{\alpha}A_2) \right) \right] \frac{s^2}{2} \varphi ds \right\| \\
= & \left\| \int_0^\tau \left[I + R_1(s,A) - \frac{1}{2} \left(I + R_1(s,\overline{\alpha}A_1) + I + R_1(s,\overline{\alpha}A_2) \right) \right] \frac{s^2}{2} \varphi ds \right\| \\
= & \left\| \left[\int_0^\tau \left[R_1(s,A) - \frac{R_1(s,\overline{\alpha}A_1) + R_1(s,\overline{\alpha}A_2)}{2} \right] \frac{s^2}{2} \varphi ds \right] \right\| \\
\leq & \frac{1}{2} \int_0^\tau \int_0^s s^2 \left\| U(s_1,A) \right\| ds_1 ds \left\| A\varphi \right\| \\
& + \frac{1}{4} \int_0^\tau \int_0^s s^2 \left\| U(s_1,\overline{\alpha}A_1) \right\| ds_1 ds \left| \overline{\alpha} \right| \left\| A_1\varphi \right\| \\
& + \frac{1}{4} \int_0^\tau \int_0^s s^2 \left\| U(s_1,\overline{\alpha}A_2) \right\| ds_1 ds \left| \overline{\alpha} \right| \left\| A_2\varphi \right\| \\
\leq & \frac{1}{2} M \int_0^\tau \int_0^s s^2 e^{\omega s_1} ds_1 ds \left\| A\varphi \right\| + \frac{1}{4} \left| \overline{\alpha} \right| \int_0^\tau \int_0^s s^2 e^{\omega s_1} ds_1 ds \left\| A_1\varphi \right\| \\
& + \frac{1}{4} \left| \overline{\alpha} \right| \int_0^\tau \int_0^s s^2 e^{\omega s_1} ds_1 ds \left\| A_2\varphi \right\| \\
\leq & c e^{\omega_0 \tau} \tau^4 \left\| \varphi \right\|_A, \quad \varphi \in D(A). \qquad \square
\end{aligned}
$$

Let us return to the proof of the **Theorem 1.1.**

Proof of Theorem 1.1. According to the property of a semigroup, the solution of the problem (1.1) in $t = t_k$ point can be written as follows:

$$
\begin{aligned}
u(t_k) &= U(t_k, A)\varphi + \int_0^{t_k} U(t_k - s, A) f(s)\, ds \\
&= U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A) F_i^{(1)},
\end{aligned}
$$

where

$$
\begin{aligned}
F_i^{(1)} &= \int_{t_{i-1}}^{t_i} U(t_i - s, A) f(s)\, ds = \\
&= \int_{t_{i-1}}^{t_i} U(t_i - s, A) \\
&\quad \times \left[f(t_i) - (t_i - s) f'(t_i) + \frac{(t_i - s)^2}{2} f''(t_i) + \widetilde{R}_3(f, t_i, s) \right] ds \\
&= \int_{t_{i-1}}^{t_i} U(t_i - s, A) f(t_i)\, ds - \int_{t_{i-1}}^{t_i} U(t_i - s, A)(t_i - s) f'(t_i)\, ds \\
&\quad + \int_{t_{i-1}}^{t_i} U(t_i - s, A) \frac{(t_i - s)^2}{2} f''(t_i)\, ds \\
&\quad + \int_{t_{i-1}}^{t_i} U(t_i - s, A) \widetilde{R}_3(f, t_i, s)\, ds, \qquad (1.34)
\end{aligned}
$$

where

$$
\widetilde{R}_3(f, t_i, s) = -\int_s^{t_i} \int_{\xi_1}^{t_i} \int_{\xi_2}^{t_i} f'''(\xi)\, d\xi d\xi_2 d\xi_1.
$$

Similarly $u_k(t_k)$ can be written as follows:

$$
\begin{aligned}
u_k(t_k) &= V(\tau)\, u_{k-1}(t_{k-1}) \\
&+ \int\limits_{t_{k-1}}^{t_k} V_0(\tau, t_k - s) \left[\frac{\alpha}{2} f(t_k) - 2\sigma_0 (t_k - s) f'(t_k)\right] ds \\
&+ \int\limits_{t_{k-1}}^{t_k} V_1(\tau, t_k - s) \left[\frac{1}{2} f(t_k) - 2\sigma_1 (t_k - s) f'(t_k)\right] ds \\
&+ \int\limits_{t_{k-1}}^{t_k} V_2(t_k - s) \\
&\times \left[\frac{\overline{\alpha}}{2} f(t_k) - 2\sigma_2 (t_k - s) f'(t_k) + \frac{(t_k - s)^2}{2} f''(t_k)\right] ds \\
&= V(\tau)\, u_{k-1}(t_{k-1}) \\
&+ \int\limits_{t_{k-1}}^{t_k} \left[\frac{\alpha}{2} V_0(\tau, t_k - s) + \frac{1}{2} V_1(\tau, t_k - s) + \frac{\overline{\alpha}}{2} V_2(t_k - s)\right] f(t_k)\, ds \\
&- \int\limits_{t_{k-1}}^{t_k} \left[\sigma_0 V_0(\tau, t_k - s) + \sigma_1 V_1(\tau, t_k - s) + \sigma_2 V_2(t_k - s)\right] \\
&\times 2(t_k - s) f'(t_k)\, ds \\
&+ \int\limits_{t_{k-1}}^{t_k} V_2(t_k - s) \frac{(s - t_k)^2}{2} f''(t_k)\, ds \\
&= V^k(\tau)\, \varphi + \sum_{i=1}^{k} V^{k-i}(\tau)\, F_i^{(2)}, \qquad (1.35)
\end{aligned}
$$

where

$$
\begin{aligned}
F_i^{(2)} &= \int\limits_{t_{i-1}}^{t_i} \left[\frac{\alpha}{2} V_0(\tau, t_i - s) + \frac{1}{2} V_1(\tau, t_i - s) + \frac{\overline{\alpha}}{2} V_2(t_i - s)\right] f(t_i) \\
&- \int\limits_{t_{i-1}}^{t_i} \left[\sigma_0 V_0(\tau, t_i - s) + \sigma_1 V_1(\tau, t_i - s) + \sigma_2 V_2(t_i - s)\right] 2(t_i - s) f'(t_i)\, ds \\
&+ \int\limits_{t_{i-1}}^{t_i} V_2(t_i - s) \frac{(t_i - s)^2}{2} f''(t_i)\, ds.
\end{aligned}
$$

From (1.34) and (1.35) we obtain:

$$\begin{aligned} u_k(t_k) - u(t_k) &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ &\quad + \sum_{i=0}^{k} \left[U^{k-i}(\tau, A) F_i^{(1)} - V^{k-i}(\tau) F_i^{(2)}\right] \\ &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ &\quad + \sum_{i=1}^{k} \Big(\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right) F_i^{(1)} \\ &\quad + V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\Big). \end{aligned} \tag{1.36}$$

Let us consider the following difference:

$$\begin{aligned} & F_i^{(1)} - F_i^{(2)} \\ = & \int_{t_{i-1}}^{t_i} \Big[U(t_i - s, A) \\ & - \left(\frac{\alpha}{2} V_0(\tau, t_i - s) + \frac{1}{2} V_1(\tau, t_i - s) + \frac{\overline{\alpha}}{2} V_2(t_i - s)\right)\Big] f(t_i)\, ds \\ & - \int_{t_{i-1}}^{t_i} \big[U(t_i - s, A) \\ & -2\left(\sigma_0 V_0(\tau, t_i - s) + \sigma_1 V_1(\tau, t_i - s) + \sigma_2 V_2(t_i - s)\right)(t_i - s)\big] f'(t_i)\, ds \\ & + \int_{t_{i-1}}^{t_i} \left[U(t_i - s, A) - V_2(t_i - s)\right] \frac{(t_i - s)^2}{2} f''(t_i)\, ds \\ & + \int_{t_{i-1}}^{t_i} U(t_i - s, A) \widetilde{R}_3(f, t_i, s)\, ds \\ = & \int_0^{\tau} \left[U(s, A) - \left(\frac{\alpha}{2} V_0(\tau, s) + \frac{1}{2} V_1(\tau, s) + \frac{\overline{\alpha}}{2} V_2(s)\right)\right] f(t_i)\, ds \\ & - \int_0^{\tau} \left[U(s, A) - 2\left(\sigma_0 V_0(\tau, s) + \sigma_1 V_1(\tau, s) + \sigma_2 V_2(s)\right)\right] s f'(t_i)\, ds \\ & + \int_0^{\tau} \left[U(s, A) - V_2(s)\right] \frac{s^2}{2} f''(t_k)\, ds \\ & + \int_0^{\tau} U(s, A) \widetilde{R}_3(f, t_i, t_i - s)\, ds. \end{aligned} \tag{1.37}$$

Hence, according to the **Lemma 1.5, Lemma 1.6** and **Lemma 1.7** we obtain the following estimation:

$$\left\|F_k^{(1)}-F_k^{(2)}\right\| \leq ce^{\omega_0\tau}\tau^4 \tag{1.38}$$
$$\times\left(\|f(t_k)\|_{A^3}+\|f'(t_k)\|_{A^2}+\|f''(t_k)\|_A+\sup_{t\in[0,t_k]}\|f'''(t)\|\right)$$

According to the **Theorem 1.2** the following inequality holds:

$$\begin{aligned}
&\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau,A)-V^{k-i}(\tau)\right)F_i^{(1)}\right\| \\
= & \left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau,A)-V^{k-i}(\tau)\right)\int\limits_{t_{i-1}}^{t_i}U(t_i-s,A)f(s)ds\right\| \\
= & \left\|\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}\left(U^{k-i}(\tau,A)-V^{k-i}(\tau)\right)U(t_i-s,A)f(s)ds\right\| \\
\leq & \; ce^{\omega_0\tau}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}t_{k-i}\tau^3\left\|U(t_i-s,A)f(s)ds\right\|_{A^4} \\
\leq & \; ce^{\omega_0 t_k}t_k^2\tau^3\sup_{s,t\in[0,t_k]}\|U(s,A)f(t)\|_{A^4}.
\end{aligned} \tag{1.39}$$

From the equality (1.36) according to the estimations (1.38), (1.39), condition (c) of the **Theorem 1.1** and **Theorem 1.2** we get:

$$\begin{aligned}
&\|u_k(t_k)-u(t_k)\| \\
\leq & \left\|\left[U^k(\tau,A)-V^k(\tau)\right]\varphi\right\| \\
& +\sum_{i=1}^{k}\left[\left\|\left[U^{k-i}(\tau,A)-V^{k-i}(\tau)\right]F_i^{(1)}\right\|+\left\|V^{k-i}(\tau)\right\|\left\|\left(F_i^{(1)}-F_i^{(2)}\right)\right\|\right] \\
\leq & \; ce^{\omega_0 t_k}t_k\tau^3\left(\sup_{s\in[0,t_k]}\|U(s,A)\varphi\|_{A^4}+t_k\sup_{s,t\in[0,t_k]}\|U(s,A)f(t)\|_{A^4}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k} e^{\omega(k-i)\tau} c e^{\omega_0 \tau} \tau^4 \left(\sup_{t\in[0,t_i]} \|f(t)\|_{A^3} \right. \\
&\left. + \sup_{t\in[0,t_i]} \|f'(t)\|_{A^2} + \sup_{t\in[0,t_i]} \|f''(t)\|_{A} + \sup_{t\in[0,t_i]} \|f'''(t)\| \right) \\
\leq\ & c e^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4} + t_k \sup_{s,t\in[0,t_k]} \|U(s,A) f(t)\|_{A^4} + \right. \\
&\left. + \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} + \sup_{t\in[0,t_k]} \|f''(t)\|_{A} + \sup_{t\in[0,t_k]} \|f'''(t)\| \right). \qquad \square
\end{aligned}
$$

Remark 1.8. *In case operators A_1 and A_2 are matrixes, it is obvious, that conditions of the theorem are automatically met. Also conditions of the Theorem are met, if A_1, A_2 and A are self-adjoint, positive definite operators.*

Remark 1.9. *Third degree precision is reached by introducing complex parameter. Because of this, the each equation of the given decomposed system is changed by a pair of real equations, unlike lower order precision schemes. To solve the specific problem, for example the matrix factorization may be used, where the coefficients are the matrixes of the second order, unlike lower order precision schemes, where may be used common factorization.*

Remark 1.10. *The sum of absolute values of coefficients of $V(\tau)$ transition operator equals to one. Because of this, the considered scheme is stable for any A_1 and A_2 bounded operators.*

§2. Third order accuracy rational splitting

1. Construction of rational splitting algorithm and theorem on error estimation

Let us consider (1.1) evolution problem. Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are compactly defined, closed, linear operators in X.

In the previous paragraph there is constructed the following decomposition formula with the local precision of fourth order:

$$V(\tau) = \frac{1}{2}\left[U(\tau, \overline{\alpha}A_1)\, U(\tau, A_2)\, U(\tau, \alpha A_1) + U(\tau, \overline{\alpha}A_2)\, U(\tau, A_1)\, U(\tau, \alpha A_2)\right], \tag{2.1}$$

where $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$.

In means that that:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^4\right),$$

where $O_p(\tau^4)$ is the operator, norm of which is of the fourth order with respect to τ (more precisely, in the case of the unbounded operator $\|O_p(\tau^4)\varphi\| = O(\tau^4)$ for any φ from the definition domain of $O_p(\tau^4)$). At the same time, we will constructed the semigroup approximations with the local precision of the fourth order using the following rational approximations:

$$\begin{aligned} W(\tau, A) &= aI + b\,(I + \lambda\tau A)^{-1} + c\,(I + \lambda\tau A)^{-2}, \qquad (2.2)\\ W(\tau, A) &= \left(I - \frac{1}{3}\tau A\right)(I + \lambda\tau A)^{-1}\left(I + \overline{\lambda}\tau A\right)^{-1}, \end{aligned}$$

where in the first formula $\lambda = \frac{1}{2} + \frac{1}{2\sqrt{3}}$, $a = 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}$, $b = \frac{3}{\lambda} - \frac{1}{\lambda^2}$, $c = \frac{1}{2\lambda^2} - \frac{1}{\lambda}$, and in the second $\lambda = \frac{1}{3} \pm i\frac{1}{3\sqrt{2}}$ $\left(i = \sqrt{-1}\right)$.

The approximations defined by formulas (2.2) in the scalar case represent the Pade approximations for exponential functions (see [5]).

Using simple transformation, we can show that the operator $W(\tau, A)$ defined by formula (2.2) coincides with the transition operator of the Calahan scheme (see [63]). The stability of the Calahan scheme for an abstract parabolic equation is investigated in [1].

On the basis of formulas (2.1) and (2.2) we can construct the following decomposition formula (Analogously we can construct a decomposition formula for another rational approximations):

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left[W(\tau, \overline{\alpha}A_1)\, W(\tau, A_2)\, W(\tau, \alpha A_1)\right.\\ &\quad \left.+ W(\tau, \overline{\alpha}A_2)\, W(\tau, A_1)\, W(\tau, \alpha A_2)\right]. \qquad (2.3) \end{aligned}$$

Below we shall show that this formula has the precision of the fourth order:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^4\right).$$

In the present paragraph, on the basis of formula (2.3), a decomposition scheme with the third order precision will be constructed for the solution of problem (1.1).

From formula (1.2) we have:

$$u(t_k) = U(\tau, A)u\left(t_{k-1}\right) + \int\limits_{t_{k-1}}^{t_k} U(t_k - s, A)f(s)ds.$$

Let us rewrite this formula in the following form:

$$\begin{aligned} u\left(t_k\right) &= U(\tau, A)u\left(t_{k-1}\right) \\ &\quad + \frac{\tau}{4}\left(3U\left(\tau, \frac{1}{3}A\right) f\left(t_{k-1/3}\right) + U\left(\tau, A\right) f\left(t_{k-1}\right)\right) + R_{k,4}\left(\tau\right), \\ u\left(t_0\right) &= \varphi \quad \left(k = 1, 2, ...\right), \end{aligned} \tag{2.4}$$

where $R_{k,4}\left(\tau\right)$ is the residual member of the quadrature formula

$$\begin{aligned} R_{k,4}\left(\tau\right) &= \int\limits_{t_{k-1}}^{t_k} U(t_k - s, A)f(s)ds \\ &\quad - \frac{\tau}{4}\left(3U\left(\tau, \frac{1}{3}A\right) f\left(t_{k-1/3}\right) + U\left(\tau, A\right) f\left(t_{k-1}\right)\right) \cdot \end{aligned} \tag{2.5}$$

For the sufficiently smooth function f the following estimation is true (see. Lemma 2.3):

$$\left\|R_{k,4}\left(\tau\right)\right\| = O\left(\tau^4\right).$$

On the basis of formula (2.4) let us construct the following scheme:

$$\begin{aligned} u_k &= V(\tau)u_{k-1} \\ &\quad + \frac{\tau}{4}\left(3S\left(\frac{1}{3}\tau\right) f\left(t_{k-1/3}\right) + S\left(\tau\right) f\left(t_{k-1}\right)\right), \\ u_0 &= \varphi \quad \left(k = 1, 2, ...\right), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left[W\left(\tau, \overline{\alpha}A_1\right) W\left(\tau, A_2\right) W\left(\tau, \alpha A_1\right)\right. \\ &\quad \left. + W\left(\tau, \overline{\alpha}A_2\right) W\left(\tau, A_1\right) W\left(\tau, \alpha A_2\right)\right], \\ S(\tau) &= K\left(\tau, \frac{1}{2}A_1\right) K\left(\tau, A_2\right) K\left(\tau, \frac{1}{2}A_1\right), \\ K(\tau, A) &= \left(I - \frac{1}{2}\tau A\right)\left(I + \frac{1}{2}\tau A\right)^{-1}. \end{aligned}$$

and $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}, \quad \lambda = \frac{1}{2} + \frac{1}{2\sqrt{3}}, \quad a = 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}, \quad b = \frac{3}{\lambda} - \frac{1}{\lambda^2}, \quad c = \frac{1}{2\lambda^2} - \frac{1}{\lambda}.$
Let us note that the operator $K(\tau, A)$ is the transition operator of the Krank-Nickolson scheme.

Let us perform the computation of the scheme (2.6) by the following algorithm:

$$u_k = u_k^{(0)} + \frac{\tau}{4}\left(3u_k^{(1)} + u_k^{(2)}\right),$$

where $u_{k,0}$ is calculated by the scheme:

$$\begin{aligned}
v_{k-2/3} &= W(\tau, \alpha A_1)\, u_{k-1}, \quad w_{k-2/3} = W(\tau, \alpha A_2)\, u_{k-1},\\
v_{k-1/3} &= W(\tau, A_2)\, v_{k-2/3}, \quad w_{k-1/3} = W(\tau, A_1)\, w_{k-2/3},\\
v_k &= W(\tau, \overline{\alpha} A_1)\, v_{k-1/3}, \quad w_k = W(\tau, \overline{\alpha} A_2)\, w_{k-1/3},\\
u_k^{(0)} &= \frac{1}{2}[v_k + w_k], \quad u_0 = \varphi,
\end{aligned} \tag{2.7}$$

and u_k^s $(s = 1, 2)$ - by the scheme:

$$\begin{aligned}
u_{k-2/3}^{(s)} &= K\left(\tau, \frac{1}{2}\gamma_s A_1\right) f\left(t_k - \gamma_s \tau\right),\\
u_{k-1/3}^{(s)} &= K\left(\tau, \gamma_s A_2\right) u_{k-2/3}^{(s)},\\
u_k^{(s)} &= K\left(\tau, \frac{1}{2}\gamma_s A_1\right) u_{k-1/3}^{(s)},
\end{aligned}$$

with $\gamma_1 = \frac{1}{3}$ and $\gamma_2 = 1$.

The following theorem takes place.

Theorem 2.1 *Let the following conditions be satisfied:*

(a) There exists such $\tau_0 > 0$ that for any $0 < \tau \leq \tau_0$ there exist operators $(I + \gamma\lambda\tau A_j)^{-1}$, $j = 1, 2$, $\gamma = 1, \alpha, \overline{\alpha}$ and they are bounded. Besides, the following inequalities are true:

$$\|W(\tau, \gamma A_j)\| \leq e^{\omega\tau}, \quad \omega = const > 0;$$

(b) The operator $(-A)$ generates the strongly continuous semigroup $U(t, A) = \exp(-tA)$, for which the following inequality is true:

$$\|U(t, A)\| \leq Me^{\omega t}, \quad M, \omega = const > 0;$$

(c) $U(s, A)\varphi \in D[A^4]$ for any $s \geq 0$;

(d) $f(t) \in C^3([0, \infty); X)$; $f(t) \in D[A^3]$, $f'(t) \in D[A^2]$, $f''(t) \in D[A]$ and $U(s, A) f(t) \in D[A^4]$ for any fixed t and s $(t, s \geq 0)$.

Then the following estimation holds:

$$\begin{aligned}\|u(t_k)-u_k\| \ \leq \ & ce^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4} \right. \\ & + t_k \sup_{s,t\in[0,t_k]} \|U(s,A)f(t)\|_{A^4} \\ & + \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} \\ & \left. + \sup_{t\in[0,t_k]} \|f''(t)\|_A + \sup_{t\in[0,t_k]} \|f'''(t)\| \right), \qquad (2.8)\end{aligned}$$

where c and ω_0 are positive constants.

2. Third order accuracy rational splitting of semigroup

The following theorem takes place.

Theorem 2.2 *If the conditions (a), (b) and (c) of the Theorem 2.1 are satisfied, then the following estimation holds:*

$$\left\| \left[U\left(t_k, A\right) - V^k\left(\tau\right) \right] \varphi \right\| \leq ce^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4}, \qquad (2.9)$$

where c and ω_0 are positive constants.

For the proof of this theorem we need the following lemma.

Lemma 2.3 *If the condition (a) of the Theorem 2.1 is satisfied, then for the operator $W(t,A)$ the following decomposition is true:*

$$W(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_{W,k}(t,A), \quad k = 1,2,3,4, \qquad (2.10)$$

where, for the residual member, the following estimation holds:

$$\|R_{W,k}(t,A)\varphi\| \leq c_0 e^{\omega_0 t} t^k \left\|A^k \varphi\right\|, \quad \varphi \in D\left[A^k\right], \quad c_0, \omega_0 = const > 0. \quad (2.11)$$

Proof. We obviously have:

$$\begin{aligned}(I+\gamma A)^{-1} &= I - I + (I+\gamma A)^{-1} = I - (I+\gamma A)^{-1}(I+\gamma A - I) \\ &= I - \gamma A (I+A)^{-1}.\end{aligned}$$

From this for any natural k we can get the following expansion:

$$(I+\gamma A)^{-1} = \sum_{i=0}^{k-1} (-1)^i \gamma^i A^i + \gamma^k A^k (I+\gamma A)^{-1}. \qquad (2.12)$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.12) up to the first order, we obtain:

$$\begin{aligned} W(\tau, A) &= aI + b(I + \lambda\tau A)^{-1} + c(I + \lambda\tau A)^{-2} \\ &= (a + b + c)I + R_{W,1}(\tau, A), \end{aligned} \tag{2.13}$$

where

$$R_{W,1}(\tau, A) = -(b + c)\lambda\tau A(I + \lambda\tau A)^{-1} - c\lambda\tau A(I + \lambda\tau A)^{-2}.$$

Since $(I + \lambda\tau A)^{-1}$ is bounded according to the condition *(a)* of the Theorem 2.1, therefore:

$$\|R_{W,1}(\tau, A)\varphi\| \leq c_0 e^{\omega_0\tau}\tau\|A\varphi\|, \quad \varphi \in D[A]. \tag{2.14}$$

Substituting the values of the parameters a, b and c in (2.13), we obtain:

$$W(\tau, A) = I + R_{W_1,1}(\tau, A). \tag{2.15}$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.12) up to the second order:

$$W(\tau, A) = (a + b + c)I - (b + 2c)\lambda\tau A + R_{W,2}(\tau, A), \tag{2.16}$$

where

$$R_{W,2}(\tau, A) = (b + 2c)\lambda^2\tau^2A^2(I + \lambda\tau A)^{-1} + \lambda^2\tau^2(I + \lambda\tau A)^{-2}A^2.$$

According to the condition *(a)* of the Theorem 2.1 we have:

$$\|R_{W,2}(\tau, A)\varphi\| \leq c_0 e^{\omega_0\tau}\tau^2\left\|A^2\varphi\right\|, \quad \varphi \in D\left[A^2\right]. \tag{2.17}$$

If we substitute the values of the parameters a, b and c in (2.16), we obtain:

$$W(\tau, A) = I - \tau A + R_{W,2}(\tau, A). \tag{2.18}$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.12) up to the third order:

$$\begin{aligned} W(\tau, A) &= (a + b + c)I - (b + 2c)\lambda\tau A + (b + 3c)\lambda^2\tau^2A^2 \\ &\quad + R_{W,3}(\tau, A), \end{aligned} \tag{2.19}$$

where

$$R_{W,3}(\tau, A) = -(b + 3c)\lambda^3\tau^3(I + \lambda\tau A)^{-1}A^3 - c\lambda^3\tau^3(I + \lambda\tau A)^{-2}A^3,$$

According to the condition *(a)* of the Theorem 2.1 we have:

$$\|R_{W,3}(\tau, A)\varphi\| \leq c_0 e^{\omega_0\tau}\tau^3\left\|A^3\varphi\right\|, \quad \varphi \in D\left[A^3\right]. \tag{2.20}$$

If we substitute the values of the parameters a, b and c in (2.19), we obtain:

$$W(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 + R_{W,3}(\tau, A). \tag{2.21}$$

Finally let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.12) up to the fourth order:

$$\begin{aligned} W(\tau, A) &= (a+b+c) I - (b+2c)\lambda\tau A + (b+3c)\lambda^2\tau^2 A^2 \\ &\quad -(b+4c)\lambda^3\tau^3 A^3 + R_{W,4}(\tau, A), \end{aligned} \tag{2.22}$$

where

$$R_{W,4}(\tau, A) = (b+4c)\lambda^4\tau^4 (I + \lambda\tau A)^{-1} A^4 + c\lambda^4\tau^4 (I + \lambda\tau A)^{-2} A^4.$$

According to the condition *(a)* of the Theorem 2.1 we have:

$$\|R_{W,4}(\tau, A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^4 \left\|A^4\varphi\right\|, \quad \varphi \in D\left[A^4\right]. \tag{2.23}$$

If we substitute the values of the parameters a, b and c in (2.22), we obtain:

$$W(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{W,4}(\tau, A). \tag{2.24}$$

Uniting formulas (2.15),(2.18),(2.21) and (2.24) we obtain formula (2.10), and uniting inequalities (2.14), (2.17), (2.20) and (2.23) we obtain estimation (2.21). □

Proof of Theorem 2.2. Let us decompose all the rational approximations in the operator $V(\tau)$ according to the formula (2.10) from right to left, so that each residual member be of the fourth order. We shall have:

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{V,4}(\tau), \tag{2.25}$$

where

$$R_{V,4}(\tau) = \frac{1}{2}\left[R_{1,2}(\tau) + R_{2,1}(\tau)\right],$$

and

$$\begin{aligned} R_{i,j}(\tau) &= R_{W,4}(\tau, \overline{\alpha}A_i) - \tau R_{W,3}(\tau, \overline{\alpha}A_i)A_j + \frac{1}{2}\tau^2 R_{W,2}(\tau, \overline{\alpha}A_i)A_j^2 \\ &\quad -\frac{1}{6}\tau^3 R_{W,1}(\tau, \overline{\alpha}A_i)A_j^3 + W(\tau, \overline{\alpha}A_i)R_{W,4}(\tau, A_j) \\ &\quad -\alpha\tau R_{W,3}(\tau, \overline{\alpha}A_i)A_i \\ &\quad +\alpha\tau^2 R_{W,2}(\tau, \overline{\alpha}A_i)A_jA_i - \frac{1}{2}\alpha\tau^3 R_{W,1}(\tau, \overline{\alpha}A_i)A_j^2A_i \end{aligned}$$

$$
\begin{aligned}
& -\alpha\tau W(\tau,\overline{\alpha}A_i)R_{W,3}(\tau,A_j)A_i \\
& +\frac{1}{2}\alpha^2\tau^2 R_{W,2}(\tau,\overline{\alpha}A_i)A_i^2-\frac{1}{2}\alpha^2\tau^3 R_{W,1}(\tau,\overline{\alpha}A_i)A_jA_i^2 \\
& +\frac{1}{2}\alpha^2\tau^2 W(\tau,\overline{\alpha}A_i)R_{W,2}(\tau,A_j)A_i^2-\frac{1}{6}\alpha^3\tau^3 R_{W,1}(t,\overline{\alpha}A_i)A_i^3 \\
& -\frac{1}{6}\alpha^3\tau^3 W(\tau,\overline{\alpha}A_i)R_{W,1}(\tau,A_j)A_i^3 \\
& +W(\tau,\overline{\alpha}A_i)W(\tau,A_j)R_{W,4}(\tau,\alpha A_i), \\
i,j \;&=\; 1,2.
\end{aligned}
$$

Hence according to the condition *(a)* of the Theorem 2.1 we have the following estimation:

$$
\|R_{V,4}(\tau)\varphi\| \leq ce^{\omega_0\tau}\tau^4\|\varphi\|_{A^4}, \quad \varphi\in D\left[A^4\right]. \tag{2.26}
$$

From the (1.7) ($k=4$) and (2.25) it follows:

$$
U(\tau,A)-V(\tau)=R_4(\tau,A)-R_{V,4}(\tau).
$$

From here according to inequalities (1.8) and (2.26) we obtain the following estimation:

$$
\|[U(\tau,A)-V(\tau)]\varphi\| \leq ce^{\omega_0\tau}\tau^4\|\varphi\|_{A^4}, \quad \varphi\in D\left[A^4\right]. \tag{2.27}
$$

The following representation is obvious:

$$
\begin{aligned}
\left[U(t_k,A)-V^k(\tau)\right]\varphi &= \left[U^k(\tau,A)-V^k(\tau)\right]\varphi \\
&= \sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau,A)-V(\tau)\right]U^{i-1}(\tau,A)\varphi.
\end{aligned}
$$

Hence, according to the conditions *(a), (b), (c)* of the Theorem 2.1 and inequality (2.27), we have the sought estimation. □

3. Error estimation for approximate solution

Let us prove the auxiliary **Lemmas** on which the proof of the **Theorem 2.1** is based**.**

Lemma 2.4 *Let the following conditions be satisfied:*

(a) The operator A satisfies the conditions of the Theorem 2.1;

(b) $f(t) \in C^3([0,\infty);X)$, and $f(t) \in D[A^3]$ for every fixed t, $f^{(k)}(t) \in D\left[A^{3-k}\right]$, $k=1,2$.

Then the following estimation holds

$$\begin{aligned}
&\left\| \int_0^\tau U(\tau-s,A)\, f(s)\, ds - \frac{\tau}{4}\left[U(\tau,A)\, f(0) + 3U\left(\frac{1}{3}\tau, A\right) f\left(\frac{2}{3}\tau\right)\right] \right\| \\
\leq\ & c e^{\omega_0 \tau} \tau^4 \left[\left\| A^3 f\left(\frac{2}{3}\tau\right)\right\| + \sup_{\xi\in[o,\tau]} \left\| A^2 f'(\xi)\right\| \right. \\
&\left. + \sup_{\xi\in[o,\tau]} \left\| A f''(\xi)\right\| + \sup_{\xi\in[o,\tau]} \left\| f'''(\xi)\right\| \right], \qquad (2.28)
\end{aligned}$$

where c and ω_0 are positive constants.

Proof. Using the simple transformation, we will obtain the following representation:

$$\begin{aligned}
&\int_0^\tau U(\tau-s,A) f(s)\, ds - \frac{\tau}{4}\left[U(\tau,A)\, f(0) + 3U\left(\frac{1}{3}\tau, A\right) f\left(\frac{2}{3}\tau\right)\right] \\
=\ & r(\tau) - U(\tau,A)\, z(\tau) - R(\tau,A)\, f\left(\frac{2}{3}\tau\right). \qquad (2.29)
\end{aligned}$$

where

$$\begin{aligned}
z(\tau) &= \frac{1}{4}\int_0^\tau f(0)\, ds + \frac{3}{4}\int_0^\tau f\left(\frac{2}{3}\tau\right) ds - \int_0^\tau f(s)\, ds, \\
R(\tau,A) &= \frac{3}{4}\int_0^\tau U\left(\frac{1}{3}\tau, A\right) ds + \frac{1}{4}\int_0^\tau U(\tau,A)\, ds - \int_0^\tau U(\tau-s,A) ds
\end{aligned}$$

and

$$r(\tau) = \int_0^\tau \left[U(\tau-s,A) - U(\tau,A)\right]\left[f(s) - f\left(\frac{2}{3}\tau\right)\right] ds.$$

According to formula (1.7) for $r(\tau)$ we can obtain the following representation:

$$\begin{aligned} r(\tau) &= A\int_0^\tau \left[\int_0^s A\int_0^\xi U(\tau-\eta,A)d\eta d\xi \int_{\frac{2}{3}\tau}^s f'(\xi)\,d\xi\right] ds \\ &\quad -A\int_0^\tau\left[\int_0^s U(\tau,A)\,d\xi \int_{\frac{2}{3}\tau}^s \int_0^\xi f''(\eta)\,d\eta d\xi\right] ds. \end{aligned}$$

Hence we obtain the following estimation:

$$\|r(\tau)\| \le ce^{\omega\tau}\tau^4\left[\sup_{\xi\in[o,\tau]}\left\|A^2 f'(\xi)\right\| + \sup_{\xi\in[o,\tau]}\|Af''(\xi)\|\right]. \tag{2.30}$$

For the function $(-z(\tau))$ the following representation is valid:

$$-z(\tau) = \frac{1}{4}\int_0^\tau\int_0^s\int_0^\xi\int_0^\eta f'''(\zeta)d\zeta d\eta d\xi ds + \frac{3}{4}\int_0^\tau\int_{\frac{2}{3}\tau}^s\int_0^\xi\int_0^\eta f'''(\zeta)d\zeta d\eta d\xi ds.$$

Hence we obtain the following estimation:

$$\|U(\tau,A)\,z(\tau)\| \le ce^{\omega\tau}\tau^4\sup_{s\in[o,\tau]}\|f'''(s)\|. \tag{2.31}$$

And finally let us transform the integral $R(\tau,A)$ according to formula (1.7):

$$\begin{aligned} R(\tau,A) &= \frac{3}{4}A^3\int_0^\tau\int_{\frac{2}{3}\tau}^s\int_0^\xi\int_0^\eta U(\tau-\zeta,A)d\zeta d\eta d\xi ds \\ &\quad \frac{1}{4}A^3\int_0^\tau\int_0^s\int_0^\xi\int_0^\eta U(\tau-\zeta,A)d\zeta d\eta d\xi ds. \end{aligned}$$

Hence we obtain the following estimation:

$$\left\|R(\tau,A)\,f\left(\frac{2}{3}\tau\right)\right\| \le ce^{\omega\tau}\tau^4\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\|. \tag{2.32}$$

From equality (2.29) according to inequalities (2.30), (2.31) and (2.32) we obtain the sought estimation.

According to the Lemma 2.3 for $R_{k,4}(\tau)$ (see formula (1.7)), the following estimation holds:

$$\begin{aligned}\|R_{k,4}(\tau)\| &\leq ce^{\omega_0\tau}\tau^4\left[\left\|A^3 f\left(\frac{2}{3}\tau\right)\right\| + \sup_{\xi\in[t_{k-1},t_k]}\left\|A^2 f'(\xi)\right\|\right.\\ &\left. + \sup_{\xi\in[t_{k-1},t_k]}\|Af''(\xi)\| + \sup_{\xi\in[t_{k-1},t_k]}\|f'''(\xi)\|\right]. \qquad (2.33)\end{aligned}$$

Let us return to the proof of the **Theorem 2.1.**

Proof of Theorem 2.1. Let us write formula (2.4) in the following form:

$$u(t_k) = U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A)\left(F_i^{(1)} + R_{k,4}(\tau)\right), \qquad (2.34)$$

where

$$F_i^{(1)} = \frac{\tau}{4}\left(3U\left(\frac{1}{3}\tau, A\right) f\left(t_{i-1/3}\right) + U(\tau, A) f(t_{i-1})\right). \qquad (2.35)$$

Analogously let us present u_k as follows:

$$u_k = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau)F_i^{(2)}, \qquad (2.36)$$

where

$$F_i^{(2)} = \frac{\tau}{4}\left(3S\left(\frac{1}{3}\tau\right) f\left(t_{i-1/3}\right) + S(\tau) f(t_{i-1})\right). \qquad (2.37)$$

From equalities (2.34) and (2.36) it follows:

$$\begin{aligned}u(t_k) - u_k &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi\\ &+\sum_{i=0}^{k}\left[U^{k-i}(\tau, A)F_i^{(1)} - V^{k-i}(\tau)F_i^{(2)}\right]\\ &+\sum_{i=0}^{k} U^{k-i}(\tau, A)R_{k,4}(\tau) = \left[U^k(\tau, A) - V^k(\tau)\right]\varphi\\ &+\sum_{i=1}^{k}\left[\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right)F_i^{(1)}\right.\\ &\left.+V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right]\\ &+\sum_{i=0}^{k} U^{k-i}(\tau, A)R_{k,4}(\tau). \qquad (2.38)\end{aligned}$$

From formulas (2.35) and (2.37) we have:

$$F_i^{(1)} - F_i^{(2)} = \frac{\tau}{4}\left(3\left(U\left(\frac{1}{3}\tau, A\right) - S\left(\frac{1}{3}\tau\right)\right) f\left(t_{i-1/3}\right) + \right.$$
$$\left. + \left(U\left(\tau, A\right) - S\left(\frac{1}{3}\tau\right)\right) f\left(t_{i-1}\right)\right). \qquad (2.39)$$

The following inequality can be easily obtained:

$$\left\|\left[U\left(\tau, A\right) - K\left(\tau\right)\right]\varphi\right\| \leq ce^{\omega_0\tau}\tau^3 \left\|\varphi\right\|_{A^3}, \quad \varphi \in D\left[A^3\right].$$

Hence analogously to estimation (2.27) we obtain:

$$\left\|\left[U\left(\tau, A\right) - S\left(\tau\right)\right]\varphi\right\| \leq ce^{\omega_0\tau}\tau^3 \left\|\varphi\right\|_{A^3}, \quad \varphi \in D\left[A^3\right].$$

According to this inequality, from equality (2.39) we obtain the following estimation:

$$\left\|F_k^{(1)} - F_k^{(2)}\right\| \leq ce^{\omega_0\tau}\tau^4 \sup_{t\in[t_{k-1},t_k]} \left\|f(t)\right\|_{A^3}. \qquad (2.40)$$

According to the Lemma 2.1 we have:

$$\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right) F_i^{(1)}\right\| \leq ce^{\omega_0 t_k} t_k^2 \tau^3 \sup_{s,t\in[0,t_k]} \left\|U(s, A) f\left(t\right)\right\|_{A^4}. \qquad (2.41)$$

From equality (2.38) according to inequalities (2.40), (2.41), (2.9), (2.41) and the condition *(b)* of the Theorem 2.1 we obtain:

$$\begin{aligned}\left\|u(t_k) - u_k\right\| \leq \; & ce^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s\in[0,t_k]} \left\|U(s, A)\varphi\right\|_{A^4}\right. \\ & \left. + t_k \sup_{s,t\in[0,t_k]} \left\|U(s, A) f\left(t\right)\right\|_{A^4} + \sup_{t\in[0,t_k]} \left\|f(t)\right\|_{A^3}\right) \\ & + \sup_{t\in[0,t_k]} \left\|f'(t)\right\|_{A^2} + \sup_{t\in[0,t_k]} \left\|f''(t)\right\|_{A} + \sup_{t\in[0,t_k]} \left\|f'''(t)\right\|. \qquad \square\end{aligned}$$

Remark 2.5. *The operator $V^k(\tau)$ is the solution operator of the above-considered decomposed problem. It is obvious that, according to the condition of the Theorem 2.1 ($\|W(t, \gamma A_j)\| \leq e^{\omega t}$), the norm of the operator $V^k(\tau)$ is less than or equal to $e^{\omega_0 t_k}$. From this follows the stability of the above-stated decomposition scheme on each finite time interval.*

Remark 2.6. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint non negative operators, in estimation (2.8) ω_0 will be replaced by* 0*. Alongside with this, for the transition operator of the splitted problem, the estimation $\left\|V^k(\tau)\right\| \leq 1$ will be true.*

Remark 2.7. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint, positive definite operators, in estimation (2.8) ω_0 will be replaced* *by* $-\alpha_0$, $\alpha_0 > 0$. *Alongside with this, for the transition operator of the splitted problem, the estimation* $\left\|V^k(\tau)\right\| \leq e^{-\alpha_1 t_k}$, $\alpha_1 > 0$ *will be true.*

Remark 2.8. *According to the classical theorem of Hille-Philips-Iosida ([48]), if the operator* $(-A)$ *generates a strongly continuous semigroup, then the inequality in the condition (b) of the Theorem 2.1 is automatically satisfied. The proof of this inequality is based on the uniform boundedness principle, according to which the constants* M *and* ω *exist, but generally can not be explicitly constructed (according to the method of the proof). That is why we demand satisfying of the inequality in the condition (b) of the Theorem 2.1.*

4. Stability of the splitted problem

In this paragraph we state the sufficient conditions, from which follows the inequality:

$$\left\|V^k(\tau)\right\| \leq c, \quad c = const > 0 \quad (k = 1, 2, ...).$$

Fulfilment of this inequality means the stability of splitted problem.

Let us examine first the stability of non split problem. Below we will prove the theorems, concerning the stability of non split problems with the transition operators given by formulas (2.2). These theorems obviously have an independent value, and the proof of the stability of split problem is based on them.

Theorem 2.9 *Assume that* A *is a linear, closed, densely defined operator in the Banach space* X. *Assume the sector* $S = \{z : |\arg z| < \varphi_0, \ z \neq 0, 0 < \varphi_0 < \frac{\pi}{2}\}$ *completely includes the spectrum of the operator* A *and for any* $z \notin S$ $(z \neq 0)$ *the following inequality holds:*

$$\left\|(zI - A)^{-1}\right) \| \leq \frac{c}{|z|}, \quad c = const > 0. \tag{2.42}$$

Then, for any $\tau > 0$ *and natural* k, *the following estimation is valid:*

$$\left\|W^k(\tau, A)\right\| \leq c, \quad c = const > 0,$$

where

$$W(\tau, A) = \left(I - \frac{1}{3}\tau A\right)(I + \lambda\tau A)^{-1}\left(I + \overline{\lambda}\tau A\right)^{-1}, \quad \lambda = \frac{1}{3} \pm i\frac{1}{3\sqrt{2}}.$$

The proof of the Theorem 2.9 is based on the following lemma.

Lemma 2.10 *Assume that the operator* A *satisfies conditions of the Theorem 1.1.*

Then for any $\tau > 0$ *and natural* k *the following inequality is valid:*

$$\left\|(I + \tau A)^{-k}\right\| \leq c, \quad c = const > 0.$$

Proof. Let us compare the operator $(I+\tau A)^{-k}$ to the operator $(I+(t_k/2)\,A)^{-2}$ ($t_k = k\tau$). With this purpose we present their difference by means of the Danford-Taylor integral (see [16] Ch. VII):

$$\begin{aligned}(I+\tau A)^{-k} - \left(I+\frac{t_k}{2}A\right)^{-2} &= \frac{1}{2\pi i}\int\limits_{\Gamma}\left((1+\tau z)^{-k} - \left(1+\frac{t_k}{2}z\right)^{-2}\right) \\ &\quad \times (zI-A)^{-1}\,dz, \qquad (2.43)\end{aligned}$$

where Γ is a bound of the sector $\left\{|\arg z| \le \varphi, \quad \varphi_0 \le \varphi < \frac{\pi}{2}\right\}$. Let us estimation the absolute value of the integrand scalar function. With this purpose we use the following representation:

$$\begin{aligned}(1+\tau z)^{-k} - \left(1+\frac{t_k}{2}z\right)^{-2} &= \int\limits_0^{t_k}\frac{d}{ds}\left[\left(1+\frac{t_k-s}{2}z\right)^{-2}\left(1+\frac{s}{k}z\right)^{-k}\right]ds \\ &= z^2\int\limits_0^{t_k}\left(\frac{s}{k}-\frac{t_k-s}{2}\right)\times\left(1+\frac{t_k-s}{2}z\right)^{-3} \\ &\quad \times\left(1+\frac{s}{k}z\right)^{-k-1}ds. \qquad (2.44)\end{aligned}$$

Obviously we have:

$$\begin{aligned}\left|1+\frac{s}{k}z\right|^{k+1} &= \left|1+\frac{s}{k}\rho\left(\cos\varphi+i\sin\varphi\right)\right|^{k+1} \\ &= \left(1+2\frac{s}{k}\mu\rho+\frac{s^2}{k^2}\rho^2\right)^{\frac{k+1}{2}}, \\ \mu &= \cos\varphi, \quad \varphi=\arg(z), \quad |z|=\rho.\end{aligned}$$

From here follows the inequality:

$$\begin{aligned}\left|1+\frac{s}{k}z\right|^{k+1} &\ge \left(1+\frac{s}{k}\mu\rho\right)^{k+1} \ge 1+\frac{k+1}{k}s\mu\rho \\ &\quad +\frac{k+1}{2k}s^2\mu^2\rho^2+\frac{k^2-1}{6k^2}s^3\mu^3\rho^3 \\ &\ge 1+s\mu\rho+\frac{1}{2}s^2\mu^2\rho^2+\frac{1}{8}s^3\mu^3\rho^3 \quad (k\ge 2).\end{aligned}$$

With account of this inequality we have:

$$
\begin{aligned}
&\left|1+\frac{t_k-s}{2}z\right|^3\left|1+\frac{s}{k}z\right|^{k+1}\\
\geq\ &\left(1+(t_k-s)\,\mu\rho+\frac{1}{2}(t_k-s)^2\mu^2\rho^2+\frac{1}{8}(t_k-s)^3\mu^3\rho^3\right)\\
&\times\left(1+s\mu\rho+\frac{1}{2}s^2\mu^2\rho^2+\frac{1}{8}s^3\mu^3\rho^3\right)\\
\geq\ &1+t_k\mu\rho+\frac{1}{2}\left(s^2+(t_k-s)^2\right)\mu^2\rho^2\\
&+\frac{1}{8}\left(s^3+(t_k-s)^3\right)\mu^3\rho^3\\
\geq\ &1+t_k\mu\rho+\frac{1}{4}t_k^2\mu^2\rho^2+\frac{1}{32}t_k^3\mu^3\rho^3\\
\geq\ &(1+\mu_0t_k\rho)^3,\quad \mu_0=\frac{1}{3\sqrt{2}}\mu.
\end{aligned}
\tag{2.45}
$$

From (2.44), with account of (2.45), it follows:

$$
\begin{aligned}
\left|(I+\tau z)^{-k}-\left(I+\frac{t_k}{2}z\right)^{-2}\right| &\leq \frac{\rho^2}{(1+\mu_0t_k\rho)^3}\int\limits_0^{t_k}\left(\frac{s}{k}+\frac{t_k-s}{2}\right)ds\\
&\leq \frac{(t_k\rho)^2}{(1+\mu_0t_k\rho)^3}
\end{aligned}
\tag{2.46}
$$

From (2.43), with account of (2.46) and (2.42), it follows:

$$
\left\|(I+\tau A)^{-k}-\left(I+\frac{t_k}{2}A\right)^{-2}\right\| \leq ct_k^2\int\limits_0^{\infty}\frac{\rho}{(1+\mu_0t_k\rho)^3}d\rho=c. \tag{2.47}
$$

Due to inequality (2.42) we have:

$$
\left\|\left(I+\frac{t_k}{2}A\right)^{-2}\right\|\leq c. \tag{2.48}
$$

From (2.47) and (2.48), according to the triangle inequality, the sought estimation follows. □

Proof of the Theorem 2.9.

Let us compare the operator $W^k(\tau,A)$ to the corresponding powers of the operator $W_0(\tau,A)=(I+\tau A)^{-1}$. Obviously the representation is valid:

$$
W_0^k(\tau,A)-W^k(\tau,A)=\left(W_0(\tau,A)-W(\tau,A)\right)\sum_{i=0}^{k-1}W_0^i(\tau,A)\,W^{k-i-1}(\tau,A), \tag{2.49}
$$

In order to estimate the norm of the operator in the right hand-side of this equality let us estimate the absolute values of the scalar functions $W(\tau,z)$, $W_0(\tau,z)$, and $W_0(\tau,z)-W(\tau,z)$ $(z\in\Gamma)$. We obtain:

$$W(\tau,z)=\frac{P_1(\tau z)}{P_2(\tau z)},$$

where

$$\begin{aligned} P_1(z) &= 1-\frac{1}{3}z, \\ P_2(z) &= 1+\frac{2}{3}z+\frac{1}{6}z^2. \end{aligned}$$

Let us calculate the squares of the modules of the polynomials $P_1(\tau z)$ and $P_2(\tau z)$:

$$|P_1(\tau z)|^2 = \left|1-\frac{1}{3}\tau\rho(\cos\varphi+i\sin\varphi)\right|^2 = 1-\frac{2}{3}\tau\mu\rho+\frac{1}{9}\tau^2\rho^2, \quad (2.50)$$

$$\begin{aligned} |P_2(\tau z)|^2 &= \left|1+\frac{2}{3}\tau\rho(\cos\varphi+i\sin\varphi)\right. \\ &\quad \left.+\frac{1}{6}\tau^2\rho^2(\cos(2\varphi)+i\sin(2\varphi))\right|^2 \\ &= 1+\frac{4}{3}\tau\mu\rho+\left(\frac{1}{9}+\frac{2}{3}\mu^2\right)\tau^2\rho^2 \\ &\quad +\frac{2}{9}\tau^3\mu\rho^3+\frac{1}{36}\tau^4\rho^4, \end{aligned} \quad (2.51)$$

where $\mu=\cos\varphi$, $\varphi=\arg(z)$, $|z|=\rho$.

From (2.50) and (2.51) it follows:

$$(1+\tau\mu_1\rho)^2|P_1(\tau z)|^2\le|P_2(\tau z)|^2, \quad \mu_1=\frac{1}{3}\mu.$$

From here we obtain:

$$|W(\tau,z)|=\frac{|P_1(\tau z)|}{|P_2(\tau z)|}\le\frac{1}{1+\mu_1\tau\rho}. \quad (2.52)$$

Let us estimate the absolute value of the function $W_0(\tau,z)-W(\tau,z)$. We obviously have:

$$|W_0(\tau,z)-W(\tau,z)|=\frac{\frac{1}{4}\tau^2\rho^2}{(1+2\tau\mu\rho+\tau^2\rho^2)^{\frac{1}{2}}|P_2(\tau z)|}.$$

From here, taking into account the inequality $|P_2(\tau z)|\ge(1+\tau\mu_1\rho)^2$, it follows:

$$|W_0(\tau,z)-W(\tau,z)|\le\frac{\tau^2\rho^2}{(1+\mu_1\tau\rho)^3}. \quad (2.53)$$

For the absolute value of $W_0(\tau, z)$, the following estimation holds:

$$\begin{aligned} |W_0(\tau, z)| &= \frac{1}{|1+\tau\rho(\cos\varphi + i\sin\varphi)|} \\ &= \frac{1}{(1+2\tau\mu\rho+\tau^2\rho^2)^{\frac{1}{2}}} \leq \frac{1}{1+\mu\tau\rho}. \end{aligned} \tag{2.54}$$

Let us present the operator-function $W_0^k(\tau, A) - W^k(\tau, A)$ by means of the Danford-Taylor integral:

$$W_0^k(\tau, A) - W^k(\tau, A) = \frac{1}{2\pi i}\int_\Gamma \left(W_0^k(\tau, z) - W^k(\tau, z)\right)(zI - A)^{-1}\, dz,$$

where Γ is the bound of the sector $\left\{|\arg z| \leq \varphi, \quad \varphi_0 \leq \varphi < \frac{\pi}{2}\right\}$. From here, according to (2.49), we obtain:

$$\begin{aligned} W_0^k(\tau, A) - W^k(\tau, A) &= \frac{1}{2\pi i}\int_\Gamma ((W_0(\tau, z) - W(\tau, z)) \\ &\quad \times \sum_{i=0}^{k-1} W_0^i(\tau, z)\, W^{k-i-1}(\tau, z)\Bigg)(zI - A)^{-1}\, dz, \end{aligned}$$

From here, with account of inequalities (2.42),(2.52),(2.53) and (2.54), we obtain the following estimation:

$$\begin{aligned} \left\|W_0^k(\tau, A) - W^k(\tau, A)\right\| &\leq c\int_0^\infty \Bigg(\frac{\tau^2\rho^2}{(1+\tau\mu_1\rho)^3} \\ &\quad \times \sum_{i=0}^{k-1} \frac{1}{(1+\tau\mu\rho)^i}\frac{1}{(1+\tau\mu_1\rho)^{k-i-1}}\Bigg)\frac{1}{\rho}d\rho \\ &\leq ck\tau\int_0^\infty \frac{\tau\rho d\rho}{(1+\tau\mu_1\rho)^{k+1}} \\ &= ck\int_0^\infty \frac{xdx}{(1+x)^{k+1}} = c. \end{aligned}$$

From this inequality and the estimation of Lemma 2.10, according to the triangle inequality, follows the sought estimation. □

Theorem 2.11 *Assume that the operator A satisfies conditions of the Theorem 2.1.*

Then, for any $\tau > 0$ and natural k, the following estimation holds:

$$\left\|W^k(\tau, A)\right\| \leq c, \quad c = const > 0, \tag{2.55}$$

where

$$\begin{aligned} W(\tau, A) &= aI + b(I+\lambda\tau A)^{-1} + c(I+\lambda\tau A)^{-2}, \\ \lambda &= \frac{1}{2} + \frac{1}{2\sqrt{3}}, \\ a &= 1 - \frac{2}{\lambda} + \frac{1}{2\lambda^2}, \\ b &= \frac{3}{\lambda} - \frac{1}{\lambda^2}, \\ c &= \frac{1}{2\lambda^2} - \frac{1}{\lambda}. \end{aligned}$$

Proof. Estimation (2.55) was proven by Alibekov and Sobolevskii (see [1]), for the case when the operator A, instead of condition (2.42), satisfies the following condition:

$$\|zI - A\| \leq \frac{c}{1+|z|}, \quad c = const > 0. \tag{2.56}$$

The above-mentioned authors present the operator $W^k(\tau, A)$ as the sum of the following three addends:

$$\begin{aligned} W^k(\tau, A) &= \left((a+b+c) + (2a+b)\lambda\tau A + a\lambda^2\tau^2A^2\right) \\ &\quad \times (I+\lambda\tau A)^{-2} W^{k-1}(\tau, A) \\ &= \left(1 + (2a+b)\lambda\tau A + a\lambda^2\tau^2A^2\right)(I+\lambda\tau A)^{-2} W^{k-1}(\tau, A) \\ &= J_{1,k}(\tau, A) + J_{2,k}(\tau, A) + J_{3,k}(\tau, A), \end{aligned} \tag{2.57}$$

where

$$\begin{aligned} J_{1,k}(\tau, A) &= (I+\lambda\tau A)^{-2} W^{k-1}(\tau, A), \\ J_{2,k}(\tau, A) &= 2a_0\lambda\tau A(I+\lambda\tau A)^{-2} W^{k-1}(\tau, A), \ a_0 = 2a+b, \\ J_{3,k}(\tau, A) &= a\lambda^2\tau^2A^2(I+\lambda\tau A)^{-2} W^{k-1}(\tau, A). \end{aligned}$$

It should be noted that the estimations (for any $\tau > 0$ and natural k):

$$\|J_{l,k}(\tau, A)\| \leq c, \quad l = 2,3, \quad c = const > 0 \tag{2.58}$$

are valid in the case when the operator A satisfies condition (2.42). The above-mentioned authors need rather heavier condition (2.56) to obtain for the operator $J_{1,k}(\tau, A)$ an estimation, analogous to estimation (2.58), since in this case they use fraction powers of the operator A. Below we give the estimation of the operator $J_{1,k}(\tau, A)$ in the case of condition (2.42).

Let us estimate the norm of the operator $J_{1,k}(\tau, A)$. At first we estimate the module of the scalar function $W(\tau, z)$. Obviously we have:

$$W(\tau, z) = \frac{P_3(\tau z)}{P_4(\tau z)},$$

where

$$\begin{aligned} P_3(z) &= 1 + a_0\lambda z + a\lambda^2 z^2, \\ P_4(z) &= (1+\lambda z)^2. \end{aligned}$$

Let us calculate the modules of the polynomials $P_3(\tau z)$ and $P_4(\tau z)$:

$$\begin{aligned} |P_3(\tau z)|^2 &= |1 + a_0\lambda\tau\rho(\cos\varphi + i\sin\varphi) \\ &\quad + a\lambda^2\tau^2\rho^2(\cos(2\varphi) + i\sin(2\varphi))\big|^2 \\ &= 1 + 2a_0\mu\lambda\tau\rho + 2\left(1+2a\mu^2\right)\lambda^2\tau^2\rho^2 \qquad (2.59) \\ &\quad + 2aa_0\mu\lambda^3\tau^3\rho^3 + a^2\lambda^4\tau^4\rho^4, \\ |P_4(\tau z)| &= |1 + \lambda\tau\rho(\cos\varphi + i\sin\varphi)|^2 \\ &= 1 + 2\mu\lambda\tau\rho + \lambda^2\tau^2\rho^2. \qquad (2.60) \end{aligned}$$

From (2.59) and (2.60) it follows:

$$|P_3(\tau z)|^2 \leq |P_4(\tau z)|^2.$$

From here follows the estimation:

$$|W(\tau, z)| \leq 1. \tag{2.61}$$

In order to estimate the norm of the operator $J_{1,k}(\tau, A)$, we compare it to the following operator:

$$W_1(\tau, A) = \left((I + a_0\lambda\tau A)(I + \tau A)^{-2}\right)^{k-1}(I + \lambda\tau A)^{-2},$$

Let us present the difference between the operators $J_{1,k}(\tau, A)$ and $W_1(\tau, A)$ in the form:

$$\begin{aligned} J_{1,k}(\tau, A) - W_1(\tau, A) &= (I + \lambda\tau A)^{-2} \\ &\quad \times \left(W^{k-1}(\tau, A) - \left((I + a_0\lambda\tau A)(I + \lambda\tau A)^{-2}\right)^{k-1}\right) \\ &= (I + \lambda\tau A)^{-2}\left(W(\tau, A) - (I + a_0\lambda\tau A)(I + \lambda\tau A)^{-2}\right) \\ &\quad \times \sum_{i=0}^{k-2}\left((I + a_0\lambda\tau A)(I + \lambda\tau A)^{-2}\right)^i W^{k-i-2}(\tau, A) \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{(1+\lambda\tau z)^2} \\ &\quad \times \left(\frac{1 + a_0\lambda\tau z + a\lambda^2\tau^2 z^2}{(1+\lambda\tau z)^2} - \frac{1 + a_0\lambda\tau z}{(1+\lambda\tau z)^2}\right) \\ &\quad \times \sum_{i=0}^{k-2}\left(\frac{1 + a_0\lambda\tau z}{(1+\lambda\tau z)^2}\right)^i W^{k-i-2}(\tau, z)(zI - A)^{-1}\,dz \\ &= \frac{1}{2\pi i}\sum_{i=0}^{k-2}\int_\Gamma \frac{a\lambda^2\tau^2 z}{(1+\lambda\tau z)^4}\left(\frac{1 + a_0\lambda\tau z}{(1+\lambda\tau z)^2}\right)^i \\ &\quad \times W^{k-i-2}(\tau, z)\, z\,(zI - A)^{-1}\,dz. \qquad (2.62) \end{aligned}$$

By simple calculations we obtain:

$$\begin{aligned}\left|\frac{1+a_0\lambda\tau z}{(1+\lambda\tau z)^2}\right| &= \frac{|1+a_0\lambda\tau\rho(\cos\varphi+i\sin\varphi)|}{|1+\lambda\tau\rho(\cos\varphi+i\sin\varphi)|^2}\\ &\le \frac{1}{(1+2\lambda\tau\mu\rho+\lambda^2\tau^2\rho^2)^{\frac{1}{2}}}\le\frac{1}{1+\lambda\tau\mu\rho}.\end{aligned}\tag{2.63}$$

From (2.62), with account of inequalities (2.42), (2.61) and (2.63), we obtain:

$$\begin{aligned}\|J_{1,k}(\tau,A)-W_1(\tau,A)\| &\le c\sum_{i=0}^{k-2}\int_0^\infty\frac{\tau^2\rho}{(1+\lambda\tau\mu\rho)^{i+4}}d\rho\\ &= c\sum_{i=0}^{k-2}\int_0^\infty\frac{x}{(1+x)^{i+4}}dx\\ &= c\sum_{i=0}^{k-2}\int_0^\infty\left(\frac{1}{(1+x)^{i+3}}-\frac{1}{(1+x)^{i+4}}\right)dx\\ &= c\sum_{i=0}^{k-2}\left(\frac{1}{i+2}-\frac{1}{i+3}\right)\\ &= \left(\frac{1}{2}-\frac{1}{k+1}\right)c\le c\cdot\end{aligned}\tag{2.64}$$

In order to obtain the final estimation, we need to estimate the norm of the operator $W_1(\tau,A)$. According to the Lemma 2.10 and the inequality $a_0 = 2a+b<1$, we have:

$$\begin{aligned}\|W_1(\tau,A)\| &\le \left\|\left((I+a_0\lambda\tau A)(I+\lambda\tau A)^{-2}\right)^k(I+\lambda\tau A)^{-2}\right\|\\ &\le \left\|\left((I+a_0\lambda\tau A)(I+\lambda\tau A)^{-1}\right)^k\right\|\left\|(I+\lambda\tau A)^{-(k+2)}\right\|\\ &\le c\left\|\left(a_0I+(1-a_0)(I+\lambda\tau A)^{-1}\right)^k\right\|\\ &\le c\sum_{i=0}^{k}\binom{i}{k}a_0^i(1-a_0)^{k-i}\left\|(I+\lambda\tau A)^{-(k-i)}\right\|\\ &\le c\sum_{i=0}^{k}\binom{i}{k}a_0^i(1-a_0)^{k-i}=c.\end{aligned}$$

From here and (2.64), due to the triangle inequality, it follows:

$$\|J_{1,k}(\tau,A)\|\le c,\quad c=const>0.\tag{2.65}$$

From (2.57), with account of inequalities (2.58) and (2.65), we obtain the sought estimation. □

Theorem 2.12 *Assume that the linear, closed, densely defined operators A_1 and A_2 in the Banach space X satisfy the following conditions:*

(a) The sector $S = \left\{ z : |\arg z| < \varphi_0,\ z \neq 0,\ 0 < \varphi_0 < \frac{\pi}{3} \right\}$ completely includes spectrums of the operators A_1 and A_2 and for any $z \notin S$ $(z \neq 0)$ the inequality holds:

$$\left\| (zI - A_j)^{-1} \right\| \leq \frac{c}{|z|}, \quad c = const > 0, \quad j = 1, 2;$$

(b) There exists such point $z_0 \notin S$ that the resolvents of the operators A_1 and A_2 are commutative at the point z_0.

Then, for any $\tau > 0$, for the transition operators corresponding to the decomposition schemes defined by formulas (1.4), the following estimation is valid:

$$\left\| V^k(\tau) \right\| \leq c, \quad c = const > 0 \quad (k = 1, 2, ...),$$

where

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left(V_1(\tau) + V_2(\tau) \right), \\ V_1(\tau) &= W(\tau, \alpha A_1)\, W(\tau, A_2)\, W(\tau, \overline{\alpha} A_1), \\ V_2(\tau) &= W(\tau, \alpha A_2)\, W(\tau, A_1)\, W(\tau, \overline{\alpha} A_2). \end{aligned}$$

Proof. It follows from the condition *(b)* of the theorem that the resolvents of the operators A_1 and A_2 are commutative at any points $z_1, z_2 \notin S$, respectively. From here it follows that the operators $W(\tau, A_1)$ and $W(\tau, A_2)$ are commutative. Therefore the equalities are valid:

$$V_1^k(\tau) = W^k(\tau, \alpha A_1)\, W^k(\tau, A_2)\, W^k(\tau, \overline{\alpha} A_1), \tag{2.66}$$

$$V_2^k(\tau) = W^k(\tau, \alpha A_2)\, W^k(\tau, A_1)\, W^k(\tau, \overline{\alpha} A_2). \tag{2.67}$$

It is obvious that if the operators A_1 and A_2 satisfy conditions of the Theorem 2.12, then the operators γA_1 and γA_2 $(\gamma = 1, \alpha, \overline{\alpha})$ will satisfy conditions of the Theorem 2.9. Therefore, from formulas (2.66) and (2.67), due to the Theorem 2.9 (Theorem 2.11), follow the estimations:

$$\left\| V_l^k(\tau) \right\| \leq c, \quad l = 1, 2, \quad c = const > 0. \tag{2.68}$$

From the commutativity of the operators $W(\tau, A_1)$ and $W(\tau, A_2)$ follows the commutativity of the operators $V_1(\tau)$ and $V_2(\tau)$, hence the representation is valid:

$$V^k(\tau) = \left(\frac{1}{2}\left(V_1(\tau) + V_2(\tau) \right) \right)^k = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} V_1^{k-j}(\tau)\, V_2^j(\tau).$$

From here, according to inequalities (2.68), follows the estimation:

$$\left\|V^k(\tau)\right\| \le \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left\|V_1^{k-j}(\tau)\right\|\left\|V_2^j(\tau)\right\| \le \frac{1}{2^k}c\sum_{j=0}^{k}\binom{k}{j} = c. \qquad \square$$

Theorem 2.13 *Assume that A_1 and A_2 are linear, normal, densely defined operators in the Hilbert space H. Assume further that the sector $S = \left\{z : |\arg z| < \varphi_0,\ z \neq 0,\ 0 < \varphi_0 \le \frac{\pi}{3}\right\}$ completely includes the spectrums of the operators A_1 and A_2.*

Then, for any $\tau > 0$, for the transition operators corresponding to the decomposition schemes defined by formulas (2.2), the following estimation is valid:

$$\|V(\tau)\| \le 1.$$

Proof. Since the operators A_1 and A_2 are normal, their corresponding resolvents also will be normal operators (see T. [40], Ch. 5, §3). From here it follows that $W(\tau, \gamma A_1)$ and $W(\tau, \gamma A_2)$ are also normal operators. Therefore, due to inequalities (2.52) and (2.61), the estimation is valid:

$$\|W(\tau, \gamma A_j)\| \le \sup_{z\in S}|W(\tau, \gamma z)| \le 1.$$

From here follows the sought estimation. $\square$

Remark 2.14. *Estimation (2.8) holds when the operators A_1 and A_2 satisfy the conditions of the Theorem 2.12, the operator A satisfies the conditions of the Theorem 2.9, and besides the conditions (c) and (d) of the Theorem 2.1 are valid.*

Remark 2.15. *It is obvious that if the resolvents of the operators A_1 and A_2 are commutative, then for exponential splitting we have an exact coincidence. As regards resolvent splitting, it has an essential value even for the commutative case, as the exact coincidence does not take place and therefore, it is important to construct a stable splitting with the high order precision.*

In the case when the operators A_1, A_2 are matrices, it is obvious that the conditions of the Theorem 2.1 are automatically satisfied. The conditions of Theorem 2.1 are also satisfied if A_1, A_2 and A are self-adjoint, positive definite operators. Moreover, the conditions of the Theorem 2.1 are automatically satisfied if the operators A_1, A_2 and A are normal operators. However, in this case, certain restrictions are imposed on the spectrums of this operators: the spectrum of the operator A have to be included in the right half-plane and the spectrums of the operators A_1 and A_2 have to be included in the sector with angle of 120^0, in order the spectrums of the operators A_1 and A_2 to remain in the right half-plane after turning by $\pm 30^0$ (this is caused by multiplication of the operators A_1 and A_2 on the parameters α and $\overline{\alpha}$).

The third order precision is reached by introducing a complex parameter. For this reason, each equation of the given decomposed system is replaced by a pair of real equations, unlike the lower order precision schemes. To solve the specific problem, (for example) the matrix factorization may be used, where the coefficients are the matrices of the second order, unlike the lower order precision schemes, where the common factorization may be used.

It must be noted that, unlike the high order precision decomposition schemes considered in [12], the sum of absolute values of coefficients of the addends of the transition operator $V(\tau)$ equals to one. Hence the considered scheme is stable for any bounded operators A_1, A_2.

§3. Third order accuracy decomposition scheme for multidimensional evolution problem

1. Decomposition scheme for homogeneous equation and theorem on error estimation

Let us consider the Cauchy abstract problem in the Banach space X :

$$\frac{du(t)}{dt} + Au(t) = 0, \qquad t > 0, \qquad u(0) = \varphi. \tag{3.1}$$

Here A is a closed linear operator with the domain $D(A)$, which is everywhere dense in X, φ is a given element from $D\,(A)$.

Suppose that $(-A)$ operator generates a strongly continuous semigroup $\{\exp(-tA)\}_{t\geq 0}$, then the solution of the problem (3.1) is given by the following formula (see [39],[42]):

$$u(t) = U(t, A)\varphi, \quad \varphi \in D\,(A)\,, \tag{3.2}$$

where $U(t, A) \equiv \exp(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2 + ... + A_m$, where $A_j \quad (j = 1, 2, ..., m)$ are compactly defined, closed linear operators in X.

Let us introduce a difference net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Along with the problem (3.1) we consider two sequences of the following problems on each interval $[t_{k-1}, t_k]$:

$$\begin{aligned}
&\frac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) = 0, && \frac{dw_k^1(t)}{dt} + \alpha A_m w_k^1(t) = 0,\\
&v_k^1(t_{k-1}) = u_{k-1}(t_{k-1}), && w_k^1(t_{k-1}) = u_{k-1}(t_{k-1}),\\
&\frac{dv_k^2(t)}{dt} + \alpha A_2 v_k^2(t) = 0, && \frac{dw_k^2(t)}{dt} + \alpha A_{m-1} w_k^2(t) = 0,\\
&v_k^2(t_{k-1}) = v_k^1(t_k), && w_k^2(t_{k-1}) = w_k^1(t_k),\\
&\quad . \quad . \quad . \quad . \quad . && . \quad . \quad . \quad . \quad . \quad .\\
&\frac{dv_k^{m-1}(t)}{dt} + \alpha A_{m-1} v_k^{m-1}(t) = 0, && \frac{dw_k^{m-1}(t)}{dt} + \alpha A_2 w_k^{m-1}(t) = 0,\\
&v_k^{m-1}(t_{k-1}) = v_k^{m-2}(t_k), && w_k^{m-1}(t_{k-1}) = w_k^{m-2}(t_k),
\end{aligned}$$

$$\frac{dv_k^m(t)}{dt} + A_m v_k^m(t) = 0, \qquad \frac{dw_k^m(t)}{dt} + A_1 w_k^m(t) = 0, \tag{3.3}$$
$$v_k^m(t_{k-1}) = v_k^{m-1}(t_k), \qquad w_k^m(t_{k-1}) = w_k^{m-1}(t_k),$$
$$\frac{dv_k^{m+1}(t)}{dt} + \overline{\alpha} A_{m-1} v_k^{m-1}(t) = 0, \qquad \frac{dw_k^{m+1}(t)}{dt} + \overline{\alpha} A_2 w_k^{m+1}(t) = 0,$$
$$v_k^{m+1}(t_{k-1}) = v_k^m(t_k), \qquad w_k^{m+1}(t_{k-1}) = w_k^m(t_k),$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$\frac{dv_k^{2m-2}(t)}{dt} + \overline{\alpha} A_2 v_k^{2m-2}(t) = 0, \qquad \frac{dw_k^{2m-2}(t)}{dt} + \overline{\alpha} A_{m-1} w_k^{2m-2}(t) = 0,$$
$$v_k^{2m-2}(t_{k-1}) = v_k^{2m-3}(t_k), \qquad w_k^{2m-2}(t_{k-1}) = w_k^{2m-3}(t_k),$$
$$\frac{dv_k^{2m-1}(t)}{dt} + \overline{\alpha} A_1 v_k^{2m-1}(t) = 0, \qquad \frac{dw_k^{2m-1}(t)}{dt} + \overline{\alpha} A_m w_k^{2m-1}(t) = 0,$$
$$v_k^{2m-1}(t_{k-1}) = v_k^{2m-2}(t_k), \qquad w_k^{2m-1}(t_{k-1}) = w_k^{2m-2}(t_k).$$

Here α is a numerical complex parameter with $Re\,(\alpha) > 0$, $u_0(0) = \varphi$. Suppose that $(-A_j)$, $(-\alpha A_j)$ and $(-\overline{\alpha} A_j)$ $(j = 1, 2, ..., m)$ operators generate strongly continuous semigroups.

On each $[t_{k-1}, t_k]$ $(k = 1, 2, ...)$ interval $u_k(t)$ are defined as follows:

$$u_k(t) = \frac{1}{2}[v_k^{2m-1}(t) + w_k^{2m-1}(t)]. \tag{3.4}$$

We consider the function $u_k(t)$ as an approximate solution of the problem (3.1) on the interval $[t_{k-1}, t_k]$.

We will need natural degrees of the operator $A = A_1 + A_2 + ... + A_m$ $(A^s, \quad s = 2, 3, 4)$. In case of two addends $(m = 2)$ they are defined in paragraph 1. Analogously are defined A^s $(s = 2, 3, 4)$ when $m > 2$.

Let us introduce the following definitions:

$$\|\varphi\|_A = \|A_1\varphi\| + ... + \|A_m\varphi\|, \quad \varphi \in D\,(A),$$
$$\|\varphi\|_{A^2} = \sum_{i,j=1}^{m} \|A_i A_j \varphi\|, \quad \varphi \in D\left(A^2\right),$$

where $\|\cdot\|$ is a norm in X, similarly are defined $\|\varphi\|_{A^s}$ $\quad (s = 3, 4)$.

Theorem 3.1. *Let the following conditions be satisfied:*

(a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$;

(b) $(-\gamma A_j)$, $\gamma = 1, \alpha, \overline{\alpha} \quad (j = 1, 2, ..., m)$ *and* $(-A)$ *operators generate strongly continuous semigroups, for which the following estimations hold correspondingly:*

$$\|U(t, \gamma A_j)\| \;\le\; e^{\omega t},$$
$$\|U(t, A)\| \;\le\; M e^{\omega t}, \quad M, \omega = const > 0;$$

(c) $U(s,A)\varphi \in D(A^4)$ *for every fixed* $s \geq 0$.
Then the following estimation holds:

$$\|u_k(t_k) - u(t_k)\| \leq ce^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4},$$

where c and ω_0 *are positive constants.*

2. Construction of solving operator of splitted problem

It is obvious, that according to the formula (3.2) for the system (3.3) we have:

$$\begin{aligned} v_k^j(t_k) &= U(\tau, \alpha A_j)\, v_k^{j-1}(t_k), \quad j = 1, ..., m-1, \\ v_k^m(t_k) &= U(\tau, A_m)\, v_k^{m-1}(t_k), \\ v_k^{m+j}(t_k) &= U(\tau, \overline{\alpha} A_{m-j})\, v_k^{m+j-1}(t_k), \quad j = 1, ..., m-1, \end{aligned}$$

where $k = 1, 2, ...,$

$$v_k^0(t_k) = u_{k-1}(t_{k-1}), \quad u_0(0) = \varphi.$$

Hence we have:

$$v_k^{2m-1}(t_k) = V_1(\tau)\, u_{k-1}(t_{k-1}),$$

where

$$V_1(\tau) = U(\tau, \overline{\alpha} A_1) ... U(\tau, \overline{\alpha} A_{m-1})\, U(\tau, A_m)\, U(\tau, \alpha A_{m-1}) ... U(\tau, \alpha A_1).$$

Analogously we obtain that:

$$w_k^{2m-1}(t_k) = V_2(\tau)\, u_{k-1}(t_{k-1}),$$

where

$$V_2(\tau) = U(\tau, \overline{\alpha} A_m) ... U(\tau, \overline{\alpha} A_2)\, U(\tau, A_1)\, U(\tau, \alpha A_2) ... U(\tau, \alpha A_m).$$

So according to the formula (3.4) we obtain:

$$u_k(t_k) = V(\tau)\, u_{k-1}(t_{k-1}) = V^k(\tau)\, \varphi, \tag{3.5}$$

where

$$V(\tau) = \frac{1}{2}\left(V_1(\tau) + V_2(\tau)\right).$$

Remark 3.2: *The operator* $V^k(\tau)$ *is a solving operator of the above considered decomposed problem. It is obvious that according to the condition of the* **Theorem 3.1** ($U(t, \gamma A_i) \leq e^{\omega t}$)

$$\left\|V^k(\tau)\right\| \leq e^{\omega_1 t_k}, \tag{3.6}$$

where $\omega_1 = (2m-1)\omega$. From here it follows the stability of the above-stated decomposition schema on each finite time interval.

Let us suppose that $W(\tau)$ is a combination (sum, product) of semigroups, generated by operators $(-\gamma A_i)$ $(i = 1, 2, ..., m)$. Let us decompose all semigroups including in the operator $W(\tau)$ according to the formula (1.7), multiply these decompositions, group together the similar members and define the coefficients of the members $(-\tau A_i), (\tau^2 A_i A_j)$ and $(\tau^3 A_i A_j A_k)$ $(i, j, k = 1, 2, ..., m)$ to be correspondingly $[W(\tau)]_i, [W(\tau)]_{i,j}$ and $[W(\tau)]_{i,j,k}$ in the obtained decomposition.

3. Error estimation for approximate solution

Proof of the Theorem 3.1.

If we decompose all semigroups in the $V(\tau)$ from right to left according to the formula (1.7) so that each residual member is of the fourth degree, we get the following formula:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{m} [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^{m} [V(\tau)]_{i,j} A_i A_j \\ &\quad - \tau^3 \sum_{i,j,k=1}^{m} [V(\tau)]_{i,j,k} A_i A_j A_k + R_4^{(m)}(\tau). \end{aligned} \tag{3.7}$$

Similarly to $R_4^{(2)}$, according to the first inequality of the condition (b) of the **Theorem 3.1** the following estimation is true for $R_4^{(m)}(\tau)$ $(m > 2)$:

$$\left\| R_4^{(m)}(\tau)\varphi \right\| \le c e^{\omega_2 \tau} \tau^4 \|\varphi\|_{A^4}, \quad \varphi \in D\left(A^4\right), \tag{3.8}$$

where c and ω_2 are positive constants.

It is obvious that:

$$\begin{aligned} [V(\tau)]_i &= \frac{1}{2}\left([V_1(\tau)]_i + [V_2(\tau)]_i\right), & i = 1, 2, ..., m, \\ [V(\tau)]_{i,j} &= \frac{1}{2}\left([V_1(\tau)]_{i,j} + [V_2(\tau)]_{i,j}\right), & i, j = 1, 2, ..., m, \\ [V(\tau)]_{i,j,k} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k} + [V_2(\tau)]_{i,j,k}\right), & i, j, k = 1, 2, ..., m. \end{aligned}$$

Let us compute coefficients $[V_1(\tau)]_i$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$ which are generated by operators $(-\gamma A_i)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V_1(\tau)]_i = [U(\tau, A_i)]_i = 1.$$

Analogously

$$[V_2(\tau)]_i = [U(\tau, A_i)]_i = 1.$$

So we have

$$[V(\tau)]_i = 1, \qquad i = 1, 2, ..., m.$$

Let us compute coefficients $[V_1(\tau)]_{i,j}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$ which are generated by operators $(-\gamma A_i)$and $(-\gamma A_j)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V_1(\tau)]_{i,j} = [U(\tau, \overline{\alpha} A_{i_1}) U(\tau, A_{i_2}) U(\tau, \alpha A_{i_1})]_{i,j} .$$

Analogously

$$[V_2(\tau)]_{i,j} = [U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_1}) U(\tau, \alpha A_{i_2})]_{i,j} ,$$

where (i_1, i_2) is a pair of i and j indices, arranged in an increasing order. According to the (1.9) we have:

$$\frac{1}{2}\left([U(\tau, \alpha A_{i_1}) U(\tau, A_{i_2}) U(\tau, \overline{\alpha} A_{i_1})]_{i,j} + [U(\tau, \alpha A_{i_2}) U(\tau, A_{i_1}) U(\tau, \overline{\alpha} A_{i_2})]_{i,j} \right) = \frac{1}{2}.$$

So we have

$$[V(\tau)]_{i,j} = \frac{1}{2}, \qquad i, j = 1, 2, ..., m.$$

Let us compute coefficients $[V_1(\tau)]_{i,j,k}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$, which are generated by operators $(-\gamma A_i), (-\gamma A_j)$ and $(-\gamma A_k)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_1}) U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_3}) U(\tau, \alpha A_{i_2}) U(\tau, \alpha A_{i_1})]_{i,j,k} .$$

Analogously

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_3}) U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_1}) U(\tau, \alpha A_{i_2}) U(\tau, \alpha A_{i_3})]_{i,j,k} ,$$

where (i_1, i_2, i_3) is a triple of i, j and k indices, arranged in an increasing order.

Firstly let us consider the case when $i = j = k$,we have:

$$[V_1(\tau)]_{i,j,k} = [U(\tau, A_i)]_{i,i,i} = \frac{1}{6}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, A_i)]_{i,i,i} = \frac{1}{6}.$$

Now let us consider the case when only two of i, j, k indices are different. In this case we have:

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_1}) U(\tau, A_{i_2}) U(\tau, \alpha A_{i_1})]_{i,j,k}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_1}) U(\tau, \alpha A_{i_2})]_{i,j,k}.$$

where (i_1, i_2) is pair of different indices of i, j and k triple, arranged in an increasing order. According to the (1.9) we have**:**

$$[V(\tau)]_{i,j,k} = \frac{1}{6}.$$

Now let us consider the case when i, j, k indices are different. We have six variants. Let us consider each one separately:

Case 1. If $i < j < k$, then

$$\begin{aligned}
[V_1(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha} A_i) U(\tau, \overline{\alpha} A_j) U(\tau, A_k) U(\tau, \alpha A_j) U(\tau, \alpha A_i)]_{i,j,k} \\
&= [U(\tau, \overline{\alpha} A_i)]_i [U(\tau, \overline{\alpha} A_j)]_j [U(\tau, A_k)]_k = \overline{\alpha}^2
\end{aligned}$$

and

$$\begin{aligned}
[V_2(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha} A_k) U(\tau, \overline{\alpha} A_j) U(\tau, A_i) U(\tau, \alpha A_j) U(\tau, \alpha A_k)]_{i,j,k} \\
&= [U(\tau, A_i)]_i [U(\tau, \alpha A_j)]_j [U(\tau, \alpha A_k)]_k = \alpha^2.
\end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left(\alpha^2 + \overline{\alpha}^2\right) = \frac{1}{6}.$$

Case 2. If $i < k < j$, then

$$\begin{aligned}
[V_1(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha} A_i) U(\tau, \overline{\alpha} A_k) U(\tau, A_j) U(\tau, \alpha A_k) U(\tau, \alpha A_i)]_{i,j,k} \\
&= [U(\tau, \overline{\alpha} A_i)]_i [U(\tau, A_j)]_j [U(\tau, \alpha A_k)]_k = \alpha\overline{\alpha}
\end{aligned}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_j) U(\tau, \overline{\alpha} A_k) U(\tau, A_i) U(\tau, \alpha A_k) U(\tau, \alpha A_j)]_{i,j,k} = 0.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 3. If $j < i < k$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_j) U(\tau, \overline{\alpha} A_i) U(\tau, A_k) U(\tau, \alpha A_i) U(\tau, \alpha A_j)]_{i,j,k} = 0$$

and

$$\begin{aligned}
[V_2(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha} A_k) U(\tau, \overline{\alpha} A_i) U(\tau, A_j) U(\tau, \alpha A_i) U(\tau, \alpha A_k)]_{i,j,k} \\
&= [U(\tau, \overline{\alpha} A_i)]_i [U(\tau, A_j)]_j [U(\tau, \alpha A_k)]_k = \alpha\overline{\alpha}.
\end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 4. If $j < k < i$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha}A_j)\, U(\tau, \overline{\alpha}A_k)\, U(\tau, A_i)\, U(\tau, \alpha A_k)\, U(\tau, \alpha A_j)]_{i,j,k} = 0$$

and

$$\begin{aligned} [V_2(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha}A_i)\, U(\tau, \overline{\alpha}A_k)\, U(\tau, A_j)\, U(\tau, \alpha A_k)\, U(\tau, \alpha A_i)]_{i,j,k} \\ &= [U(\tau, \overline{\alpha}A_i)]_i\, [U(\tau, A_j)]_j\, [U(\tau, \alpha A_k)]_k = \alpha\overline{\alpha}. \end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 5. If $k < i < j$, then

$$\begin{aligned} [V_1(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha}A_k)\, U(\tau, \overline{\alpha}A_i)\, U(\tau, A_j)\, U(\tau, \alpha A_i)\, U(\tau, \alpha A_k)]_{i,j,k} \\ &= [U(\tau, \overline{\alpha}A_i)]_i\, [U(\tau, A_j)]_j\, [U(\tau, \alpha A_k)]_k = \alpha\overline{\alpha} \end{aligned}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha}A_j)\, U(\tau, \overline{\alpha}A_i)\, U(\tau, A_k)\, U(\tau, \alpha A_i)\, U(\tau, \alpha A_j)]_{i,j,k} = 0.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 6. If $k < j < i$, then

$$\begin{aligned} [V_1(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha}A_k)\, U(\tau, \overline{\alpha}A_j)\, U(\tau, A_i)\, U(\tau, \alpha A_j)\, U(\tau, \alpha A_k)]_{i,j,k} \\ &= [U(\tau, A_i)]_i\, [U(\tau, \alpha A_j)]_j\, [U(\tau, \alpha A_k)]_k = \alpha^2 \end{aligned}$$

and

$$\begin{aligned} [V_2(\tau)]_{i,j,k} &= [U(\tau, \overline{\alpha}A_i)\, U(\tau, \overline{\alpha}A_j)\, U(\tau, A_k)\, U(\tau, \alpha A_j)\, U(\tau, \alpha A_i)]_{i,j,k} \\ &= [U(\tau, \overline{\alpha}A_i)]_i\, [U(\tau, \overline{\alpha}A_j)]_j\, [U(\tau, A_k)]_k = \overline{\alpha}^2. \end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left(\alpha^2 + \overline{\alpha}^2\right) = \frac{1}{6}.$$

Finally, for any triple (i, j, k) we have:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}.$$

Inserting in (3.7) the obtained coefficients, we will get:

$$\begin{aligned}
V(\tau) &= I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \sum_{i,j=1}^{m} A_i A_j - \frac{1}{6}\tau^3 \sum_{i,j,k=1}^{m} A_i A_j A_k + R_4^{(m)}(\tau) \\
&= I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \left(\sum_{i=1}^{m} A_i\right)^2 - \frac{1}{6}\tau^3 \left(\sum_{i=1}^{m} A_i\right)^3 + R_4^{(m)}(\tau) \\
&= I - \tau A + \frac{1}{2}\tau^2 A - \frac{1}{6}\tau^3 A^3 + R_4^{(m)}(\tau). \qquad (3.9)
\end{aligned}$$

According to the formula (1.7) we have:

$$U(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A - \frac{1}{6}\tau^3 A^3 + R_4(\tau, A). \qquad (3.10)$$

According to the second inequality of the condition (b) of the **Theorem 3.1** the following estimation is true for $R_4(\tau, A)$:

$$\|R_4(\tau, A)\varphi\| \leq c e^{\omega\tau}\tau^4 \left\|A^4\varphi\right\| \leq c e^{\omega\tau}\tau^4 \|\varphi\|_{A^4}. \qquad (3.11)$$

According to the formulas (3.9) and (3.10) we have:

$$U(\tau, A) - V(\tau) = R_4(\tau, A) - R_4^{(m)}(\tau).$$

Hence using inequalities (3.8) and (3.11) we can get the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \leq c e^{\omega_2\tau}\tau^4 \|\varphi\|_{A^4}. \qquad (3.12)$$

According to the formulas (3.2) and (3.5) we have:

$$\begin{aligned}
u(t_k) - u_k(t_k) &= \left[U(t_k, A) - V^k(\tau)\right]\varphi = \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\
&= \sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau, A) - V(\tau)\right] U((i-1)\tau, A)\varphi.
\end{aligned}$$

Hence according to the inequalities (3.6) and (3.12) we can obtain the following estimation:

$$\begin{aligned}
\|u(t_k) - u_k(t_k)\| &\leq \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \|[U(\tau, A) - V(\tau)] U((i-1)\tau, A)\varphi\| \\
&\leq \sum_{i=1}^{k} e^{\omega_1(k-i)\tau} c e^{\omega_2\tau}\tau^4 \|U((i-1)\tau, A)\varphi\|_{A^4} \\
&\leq c e^{\omega_0 t_k}\tau^4 \sum_{i=1}^{k} \|U((i-1)\tau, A)\varphi\|_{A^4} \\
&\leq c e^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^4}. \qquad \square
\end{aligned}$$

§4. Sequential type third order accuracy decomposition scheme

Let us consider the problem (3.1). Let $A = A_1 + A_2$, where $A_i \ \ (i = 1, 2)$ are closed operators, densely defined in X.

Together with problem (3.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(2)}(t)}{dt} + \overline{\alpha} A_2 v_k^{(2)}(t) &= 0, \quad v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k), \\
\frac{dv_k^{(3)}(t)}{dt} + \frac{1}{2} A_1 v_k^{(3)}(t) &= 0, \quad v_k^{(3)}(t_{k-1}) = v_k^{(2)}(t_k), \\
\frac{dv_k^{(4)}(t)}{dt} + \alpha A_2 v_k^{(4)}(t) &= 0, \quad v_k^{(4)}(t_{k-1}) = v_k^{(3)}(t_k), \\
\frac{dv_k^{(5)}(t)}{dt} + \frac{\alpha}{2} A_1 v_k^{(5)}(t) &= 0, \quad v_k^{(5)}(t_{k-1}) = v_k^{(4)}(t_k),
\end{aligned}$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Suppose that the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j), \ j = 1, 2$ generate strongly continuous semigroups.

$u_k(t), \ k = 1, 2, .., $ is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = v_k^{(5)}(t). \tag{4.1}$$

We declare function $u_k(t)$ as an approximated solution of problem (3.1) on each interval $[t_{k-1}, t_k]$.

Theorem 4.1. *Let the conditions (a), (b) and (c) of Theorem 1.1 be fulfilled. Then the following estimation holds:*

$$\|u(t_k) - u_k(t_k)\| \le c e^{\omega_0 t_k} t_k \tau^3 \sup_{s \in [0, t_k]} \|U(s, A)\varphi\|_{A^4},$$

where c and ω_0 are positive constants.

Proof. From formula (4.1) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{4.2}$$

where

$$V(\tau) = U\left(\tau, \frac{\alpha}{2} A_1\right) U(\tau, \alpha A_2)\, U\left(\tau, \frac{1}{2} A_1\right) U(\tau, \overline{\alpha} A_2)\, U\left(\tau, \frac{\overline{\alpha}}{2} A_1\right).$$

Remark 4.2. *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem 1.1. In this case, for the solving operator, the following estimation holds:*

$$\left\|V^k(\tau)\right\| \le e^{\omega_1 t_k}, \tag{4.3}$$

where ω_1 is a positive constant.

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i=1,2)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (1.7), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $(\tau^2 A_i A_j)$ and $(-\tau^3 A_i A_j A_k)$ $(i,j,k=1,2)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$ and $[T(\tau)]_{i,j,k}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (1.7) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{2} [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^{2} [V(\tau)]_{i,j} A_i A_j \\ &\quad - \tau^3 \sum_{i,j,k=1}^{2} [V(\tau)]_{i,j,k} A_i A_j A_k + \widetilde{R}_4(\tau) . \end{aligned} \tag{4.4}$$

According to the first inequality of the condition *(b)* of the Theorem, for $\widetilde{R}_4(\tau)$, the following estimation holds:

$$\left\| \widetilde{R}_4(\tau)\varphi \right\| \le c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^4} , \quad \varphi \in D\left(A^4\right), \tag{4.5}$$

where c and ω_0 are positive constants.

Let us calculate the coefficients $[V(\tau)]_i$ corresponding to the first order members in formula (4.4). It is obvious that the members, corresponding to these coefficients, are obtained from the decomposition of only those factors (semigroups) of the operator $V(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identical operators) will participate.

On the whole, we have two cases: $i=1$ and $i=2$. Let us consider the case $i=1$. We obviously have:

$$[V(\tau)]_1 = [U(\tau, A_1)]_1 = 1. \tag{4.6}$$

Analogously for $i=2$ we have:

$$[V(\tau)]_2 = [U(\tau, A_2)]_2 = 1. \tag{4.7}$$

By combining formulas (4.6) and (4.7), we will obtain:

$$[V(\tau)]_i = 1, \quad i=1,2. \tag{4.8}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i,j=1,2)$ corresponding to the second order members included in formula (4.4). On the whole we have two cases: $(i,j) = (1,1)$, $(1,2)$, $(2,1)$, $(2,2)$. Let us consider the case $(i,j) = (1,1)$. We obviously have:

$$[V(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}. \tag{4.9}$$

Analogously for $(i,j) = (2,2)$ we have:

$$[V(\tau)]_{2,2} = [U(\tau, A_2)]_{2,2} = \frac{1}{2}. \tag{4.10}$$

Let us consider the case $(i,j) = (1,2)$, we obviously have:

$$\begin{aligned} [V(\tau)]_{1,2} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 [U(\tau, \alpha A_2)]_2 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 [U(\tau, \overline{\alpha} A_2)]_2 \\ &\quad + \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_1 [U(\tau, \overline{\alpha} A_2)]_2 \\ &= \frac{\alpha}{2}\alpha + \frac{\alpha}{2}\overline{\alpha} + \overline{\alpha}\frac{1}{2} = \frac{\alpha(\alpha + \overline{\alpha}) + \overline{\alpha}}{2} = \frac{1}{2}. \end{aligned} \tag{4.11}$$

For $(i,j) = (2,1)$ we have:

$$\begin{aligned} [V(\tau)]_{2,1} &= [U(\tau, \alpha A_2)]_2 \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_1 \\ &\quad + [U(\tau, \alpha A_2)]_2 \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right)\right]_1 \\ &\quad + [U(\tau, \overline{\alpha} A_2)]_2 \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right)\right]_1 \\ &= \alpha\frac{1}{2} + \alpha\frac{\overline{\alpha}}{2} + \overline{\alpha}\frac{\overline{\alpha}}{2} = \frac{\alpha + \overline{\alpha}(\alpha + \overline{\alpha})}{2} = \frac{1}{2}. \end{aligned} \tag{4.12}$$

Here we used the identity $\alpha + \overline{\alpha} = 1$.

By combining formulas (4.9) - (4.12), we will obtain:

$$[V(\tau)]_{i,j} = \frac{1}{2}, \quad i,j = 1,2. \tag{4.13}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i,j,k = 1,2)$ corresponding to the third order members in formula (4.4). On the whole we have eight cases: $(i,j,k) = (1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,1)$, $(2,2,2)$. Let us consider the case $(i,j,k) = (1,1,1)$. We obviously have:

$$[V(\tau)]_{1,1,1} = [U(\tau, A_1)]_{1,1,1} = \frac{1}{6}. \tag{4.14}$$

Analogously for $(i,j) = (2,2,2)$ we have:

$$[V(\tau)]_{2,2,2} = [U(\tau, A_2)]_{2,2,2} = \frac{1}{6}. \tag{4.15}$$

Thus Let us calculate the case $(i,j,k)=(1,1,2)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,1,2} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}[U(\tau,\alpha A_2)]_2 \\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}[U(\tau,\overline{\alpha}A_2)]_2 \\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1}\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1}[U(\tau,\overline{\alpha}A_2)]_2 \\
&+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}[U(\tau,\overline{\alpha}A_2)]_2 \\
&= \frac{\alpha^2}{8}\alpha+\frac{\alpha^2}{8}\overline{\alpha}+\frac{\alpha}{2}\frac{1}{2}\overline{\alpha}+\frac{1}{8}\overline{\alpha} \\
&= \frac{\alpha^2(\alpha+\overline{\alpha})+2\alpha\overline{\alpha}+\overline{\alpha}}{8}=\frac{\alpha^2+\alpha\overline{\alpha}+\alpha\overline{\alpha}+\overline{\alpha}}{8} \\
&= \frac{\alpha(\alpha+\overline{\alpha})+\alpha\overline{\alpha}+\overline{\alpha}}{8}=\frac{(\alpha+\overline{\alpha})+\alpha\overline{\alpha}}{8}=\frac{1}{6}. \qquad (4.16)
\end{aligned}$$

For $(i,j,k)=(2,2,1)$ we have:

$$\begin{aligned}
[V(\tau)]_{2,2,1} &= [U(\tau,\alpha A_2)]_{2,2}\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1} \\
&+[U(\tau,\alpha A_2)]_{2,2}\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_{1} \\
&+[U(\tau,\alpha A_2)]_{2}[U(\tau,\overline{\alpha}A_2)]_2\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_{1} \\
&+[U(\tau,\overline{\alpha}A_1)]_{2,2}\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_{1} \\
&= \frac{\alpha^2}{2}\frac{1}{2}+\frac{\alpha^2}{2}\frac{\overline{\alpha}}{2}+\alpha\overline{\alpha}\frac{\overline{\alpha}}{2}+\frac{\overline{\alpha}^2}{2}\frac{\overline{\alpha}}{2} \\
&= \frac{\alpha^2+\overline{\alpha}(\alpha^2+\overline{\alpha}^2)+2\alpha\overline{\alpha}^2}{4} \\
&= \frac{\alpha(1-\overline{\alpha})+\frac{1}{3}\overline{\alpha}+\frac{2}{3}\overline{\alpha}}{4}=\frac{\alpha-\frac{1}{3}+\overline{\alpha}}{4}=\frac{1}{6}. \qquad (4.17)
\end{aligned}$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus Let us calculate the case $(i,j,k)=(1,2,2)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,2,2} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\alpha A_2)]_{2,2} \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha}A_2)]_{2,2} \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\alpha A_2)]_2 [U(\tau,\overline{\alpha}A_2)]_2 \\
&\quad + \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha}A_2)]_{2,2} \\
&= \frac{\alpha}{2}\frac{\alpha^2}{2}+\frac{\alpha}{2}\frac{\overline{\alpha}^2}{2}+\frac{\alpha}{2}\alpha\overline{\alpha}+\frac{1}{2}\frac{\overline{\alpha}^2}{2} \\
&= \frac{\alpha\left(\alpha^2+\overline{\alpha}^2\right)+2\alpha^2\overline{\alpha}+\overline{\alpha}^2}{4} \\
&= \frac{\frac{1}{3}\alpha+\frac{2}{3}\alpha+\overline{\alpha}(1-\alpha)}{4} = \frac{(\alpha+\overline{\alpha})-\alpha\overline{\alpha}}{4} = \frac{1}{6}.
\end{aligned} \tag{4.18}$$

For $(i,j,k)=(2,1,1)$ we have:

$$\begin{aligned}
[V(\tau)]_{2,1,1} &= [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1} \\
&\quad + [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_{1,1} \\
&\quad + [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right]_1 \\
&\quad + [U(\tau,\overline{\alpha}A_1)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_{1,1} \\
&= \alpha\frac{1}{8}+\alpha\frac{\overline{\alpha}^2}{8}+\alpha\frac{1}{2}\frac{\overline{\alpha}}{2}+\overline{\alpha}\frac{\overline{\alpha}^2}{8} \\
&= \frac{\alpha+\overline{\alpha}^2(\alpha+\overline{\alpha})+2\alpha\overline{\alpha}}{8} \\
&= \frac{\alpha+\overline{\alpha}(\alpha+\overline{\alpha})+\alpha\overline{\alpha}}{4} = \frac{\alpha+\overline{\alpha}-\frac{1}{3}}{4} = \frac{1}{6}.
\end{aligned} \tag{4.19}$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus Let us calculate the case $(i,j,k)=(1,2,1)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,2,1} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha} A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha} A_2)]_2 \left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_1 \\
&= \frac{\alpha}{2}\alpha\frac{1}{2}+\frac{\alpha}{2}\alpha\frac{\overline{\alpha}}{2}+\frac{\alpha}{2}\overline{\alpha}\frac{\overline{\alpha}}{2}+\frac{1}{2}\overline{\alpha}\frac{\overline{\alpha}}{2} \\
&= \frac{(\alpha^2+\overline{\alpha}^2)+\alpha\overline{\alpha}(\alpha+\overline{\alpha})}{4}=\frac{1}{6}. \qquad (4.20)
\end{aligned}$$

For $(i,j,k)=(2,1,2)$ we have:

$$\begin{aligned}
[V(\tau)]_{2,1,2} &= [U(\tau,\alpha A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha} A_2)]_2 \\
&= \alpha\frac{1}{2}\overline{\alpha}=\frac{1}{6}. \qquad (4.21)
\end{aligned}$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

By combining formulas (4.14) - (4.21), we will obtain:

$$[V(\tau)]_{i,j,k}=\frac{1}{6},\quad i,j,k=1,2. \qquad (4.22)$$

From equality (4.4), taking into account formulas (4.8), (4.13) and (4.22), we will obtain:

$$\begin{aligned}
V(\tau) &= I-\tau\sum_{i=1}^{2}A_i+\frac{1}{2}\tau^2\sum_{i,j=1}^{2}A_iA_j-\frac{1}{6}\tau^3\sum_{i,j,k=1}^{2}A_iA_jA_k+\widetilde{R}_4(\tau) \\
&= I-\tau\sum_{i=1}^{2}A_i+\frac{1}{2}\tau^2\left(\sum_{i=1}^{2}A_i\right)^2-\frac{1}{6}\tau^3\left(\sum_{i=1}^{2}A_i\right)^3+\widetilde{R}_4(\tau) \\
&= I-\tau A+\frac{1}{2}\tau^2A^2-\frac{1}{6}\tau^3A^3+R_4(\tau). \qquad (4.23)
\end{aligned}$$

According to formula (1.7) we have:

$$U(\tau,A)=I-\tau A+\frac{1}{2}\tau^2A^2-\frac{1}{6}\tau^3A^3+R_4(\tau,A), \qquad (4.24)$$

where $R_4(\tau,A)$ is defined form (1.8).

According to condition *(b)* of the second inequality of the Theorem, for $R_4(\tau, A)$, the following estimation holds:

$$\|R_4(\tau, A)\varphi\| \leq ce^{\omega\tau}\tau^4 \left\|A^4\varphi\right\| \leq ce^{\omega\tau}\tau^4 \|\varphi\|_{A^4}. \qquad (4.25)$$

According to equalities (4.23) and (4.24):

$$U(\tau, A) - V(\tau) = R_4(\tau, A) - \widetilde{R}_4(\tau).$$

From here, taking into account (4.5) and (4.25), we will obtain the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \leq ce^{\omega_2\tau}\tau^4 \|\varphi\|_{A^4}. \qquad (4.26)$$

From equalities (3.2) and (4.2), taking into account inequalities (4.3) and (4.26), we will obtain:

$$\begin{aligned}
\|u(t_k) - u_k(t_k)\| &= \left\|\left[U(t_k, A) - V^k(\tau)\right]\varphi\right\| = \left\|\left[U^k(\tau, A) - V^k(\tau)\right]\varphi\right\| \\
&= \left\|\left[\sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau, A) - V(\tau)\right]U((i-1)\tau, A)\right]\varphi\right\| \\
&\leq \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \|[U(\tau, A) - V(\tau)]U((i-1)\tau, A)\varphi\| \\
&\leq \sum_{i=1}^{k} e^{\omega_1(k-i)\tau} ce^{\omega_2\tau}\tau^4 \|U((i-1)\tau, A)\varphi\|_{A^4} \\
&\leq ce^{\omega_0 t_k}\tau^4 \sum_{i=1}^{k} \|U((i-1)\tau, A)\varphi\|_{A^4} \\
&\leq kce^{\omega_0 t_k}\tau^4 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^4} \\
&\leq ce^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^4}. \qquad \square
\end{aligned}$$

§5. Sequential type third order accuracy decomposition scheme for multidimensional evolution problem

Let us consider the problem (3.1). Let $A = A_1 + ... + A_m$, where $A_i \quad (i = 1,..,m)$ are closed operators, densely defined in X.

Together with problem (3.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$
\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_i v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dv_k^{(m)}(t)}{dt} + \overline{\alpha} A_m v_k^{(m)}(t) &= 0, \quad v_k^{(m)}(t_{k-1}) = v_k^{(m-1)}(t_k), \\
\frac{dv_k^{(2m-i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_2 v_k^{(2m-i)}(t) &= 0, \quad v_k^{(2m-i)}(t_{k-1}) = v_k^{(2m-i-1)}(t_k), \\
i &= m-1, ..., 2, \\
\frac{dv_k^{(2m-1)}(t)}{dt} + \frac{1}{2} A_1 v_k^{(2m-1)}(t) &= 0, \quad v_k^{(2m-1)}(t_{k-1}) = v_k^{(2m-2)}(t_k), \\
\frac{dv_k^{(2m-2+i)}(t)}{dt} + \frac{\alpha}{2} A_i v_k^{(2m-2+i)}(t) &= 0, \quad v_k^{(2m-2+i)}(t_{k-1}) = v_k^{(2m-3+i)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dv_k^{(3m-2)}(t)}{dt} + \alpha A_m v_k^{(3m-2)}(t) &= 0, \quad v_k^{(3m-2)}(t_{k-1}) = v_k^{(3m-3)}(t_k), \\
\frac{dv_k^{(4m-2-i)}(t)}{dt} + \frac{\alpha}{2} A_i v_k^{(4m-2-i)}(t) &= 0, \quad v_k^{(4m-2-i)}(t_{k-1}) = v_k^{(4m-3-i)}(t_k), \\
i &= m-1, ..., 1.
\end{aligned}
$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Suppose that the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j)$, $j = 1, ..., m$ generate strongly continuous semigroups.

$u_k(t)$, $k = 1, 2, ..,$ is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = v_k^{(4m-3)}(t). \tag{5.1}$$

We declare function $u_k(t)$ as an approximated solution of problem (3.1) on each interval $[t_{k-1}, t_k]$.

Theorem 5.1. *Let the conditions of Theorem 3.1 be fulfilled. Then the following estimation holds:*

$$\|u(t_k) - u_k(t_k)\| \leq ce^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4},$$

where c and ω_0 are positive constants.

Proof. From formula (5.1) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{5.2}$$

where

$$\begin{aligned} V(\tau) &= U\left(\tau, \frac{\alpha}{2}A_1\right)...U\left(\tau, \frac{\alpha}{2}A_{m-1}\right)U(\tau, \alpha A_m) \\ &\times U\left(\tau, \frac{\alpha}{2}A_{m-1}\right)...U\left(\tau, \frac{\alpha}{2}A_2\right)U\left(\tau, \frac{1}{2}A_1\right) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)...U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right)U(\tau, \overline{\alpha}A_m) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right)...U\left(\tau, \frac{\overline{\alpha}}{2}A_1\right). \end{aligned}$$

Remark 5.2. *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem 3.1. In this case, for the solving operator, the following estimation holds:*

$$\left\|V^k(\tau)\right\| \leq e^{\omega_1 t_k}, \tag{5.3}$$

where ω_1 is a positive constant.

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1, ..., m)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (1.7), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $(\tau^2 A_i A_j)$ and $(-\tau^3 A_i A_j A_k)$ $(i, j, k = 1, ..., m)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$ and $[T(\tau)]_{i,j,k}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (1.7) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau\sum_{i=1}^{m}[V(\tau)]_i A_i + \tau^2\sum_{i,j=1}^{m}[V(\tau)]_{i,j} A_i A_j \\ &-\tau^3\sum_{i,j,k=1}^{m}[V(\tau)]_{i,j,k} A_i A_j A_k + \widetilde{R}_4(\tau). \end{aligned} \tag{5.4}$$

According to the first inequality of the condition *(b)* of the Theorem, for $\widetilde{R}_4(\tau)$, the following estimation holds:

$$\left\|\widetilde{R}_4(\tau)\varphi\right\| \leq ce^{\omega_0\tau}\tau^4\|\varphi\|_{A^4}, \quad \varphi \in D\left(A^4\right), \tag{5.5}$$

where c and ω_0 are positive constants.

Let us compute coefficients $[V(\tau)]_i$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V(\tau)$ which are generated by operators $(-\gamma A_i)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V(\tau)]_i = 1, \qquad i = 1, ..., m.$$

Let us compute coefficients $[V(\tau)]_{i,j}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$ which are generated by operators $(-\gamma A_i)$and $(-\gamma A_j)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$[V(\tau)]_{i,j} = \left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \alpha A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \overline{\alpha}A_{i_2}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j},$$

where (i_1, i_2) is a pair of i and j indices, arranged in an increasing order. According to the **Theorem 4.1** we have**:**

$$\left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \alpha A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \overline{\alpha}A_{i_2}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j} = \frac{1}{2}.$$

So we have

$$[V(\tau)]_{i,j} = \frac{1}{2}, \qquad i, j = 1, 2, ..., m.$$

Let us compute coefficients $[V(\tau)]_{i,j,k}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V(\tau)$, which are generated by operators $(-\gamma A_i), (-\gamma A_j)$ and $(-\gamma A_k)$. From decomposition of other semigroups only first addends (identical operators) will be used. So we have:

$$\begin{aligned}[V(\tau)]_{i,j,k} \;\; = \;\; & U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)U\left(\tau, \alpha A_{i_3}\right)U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right) \\ & U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_2}\right)U\left(\tau, \overline{\alpha}A_{i_3}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_2}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right),\end{aligned}$$

where (i_1, i_2, i_3) is a triple of i, j and k indices, arranged in an increasing order.

Firstly let us consider the case when $i = j = k$,we have:

$$[V(\tau)]_{i,j,k} = [U(\tau, A_i)]_{i,i,i} = \frac{1}{6}.$$

Now let us consider the case when only two of i, j, k indices are different. In this case we have:

$$[V(\tau)]_{i,j,k} = \left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \alpha A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \overline{\alpha}A_{i_2}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j,k},$$

where (i_1, i_2) is pair of different indices of i, j and k triple, arranged in an increasing order. According to the **Theorem 4.1** we have**:**

$$\left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \alpha A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \overline{\alpha}A_{i_2}\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_{i_1}\right)\right]_{i,j,k} = \frac{1}{6}.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}, \qquad i, j, k = 1, 2, ..., m.$$

Now let us consider the case when i, j, k indices are different. We have six variants. Let us consider each one separately:

Case 1. If $i < j < k$, then

$$\begin{aligned}[V_1(\tau)]_{i,j,k} &= \left[U\left(\tau, \overline{\alpha}A_i\right)U\left(\tau, \overline{\alpha}A_j\right)U\left(\tau, A_k\right)U\left(\tau, \alpha A_j\right)U\left(\tau, \alpha A_i\right)\right]_{i,j,k} \\ &= \left[U\left(\tau, \overline{\alpha}A_i\right)\right]_i\left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j\left[U\left(\tau, A_k\right)\right]_k = \overline{\alpha}^2\end{aligned}$$

and

$$\begin{aligned}[V_2(\tau)]_{i,j,k} &= \left[U\left(\tau, \overline{\alpha}A_k\right)U\left(\tau, \overline{\alpha}A_j\right)U\left(\tau, A_i\right)U\left(\tau, \alpha A_j\right)U\left(\tau, \alpha A_k\right)\right]_{i,j,k} \\ &= \left[U\left(\tau, A_i\right)\right]_i\left[U\left(\tau, \alpha A_j\right)\right]_j\left[U\left(\tau, \alpha A_k\right)\right]_k = \alpha^2.\end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left(\alpha^2 + \overline{\alpha}^2\right) = \frac{1}{6}.$$

Case 2. If $i < k < j$, then

$$\begin{aligned}[V_1(\tau)]_{i,j,k} &= \left[U\left(\tau, \overline{\alpha}A_i\right)U\left(\tau, \overline{\alpha}A_k\right)U\left(\tau, A_j\right)U\left(\tau, \alpha A_k\right)U\left(\tau, \alpha A_i\right)\right]_{i,j,k} \\ &= \left[U\left(\tau, \overline{\alpha}A_i\right)\right]_i\left[U\left(\tau, A_j\right)\right]_j\left[U\left(\tau, \alpha A_k\right)\right]_k = \alpha\overline{\alpha}\end{aligned}$$

and

$$[V_2(\tau)]_{i,j,k} = \left[U\left(\tau, \overline{\alpha}A_j\right)U\left(\tau, \overline{\alpha}A_k\right)U\left(\tau, A_i\right)U\left(\tau, \alpha A_k\right)U\left(\tau, \alpha A_j\right)\right]_{i,j,k} = 0.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 3. If $j < i < k$, then

$$[V_1(\tau)]_{i,j,k} = \left[U\left(\tau, \overline{\alpha}A_j\right)U\left(\tau, \overline{\alpha}A_i\right)U\left(\tau, A_k\right)U\left(\tau, \alpha A_i\right)U\left(\tau, \alpha A_j\right)\right]_{i,j,k} = 0$$

and

$$\begin{aligned}[V_2(\tau)]_{i,j,k} &= [U(\tau,\overline{\alpha}A_k)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_j)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_k)]_{i,j,k} \\ &= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha\overline{\alpha}.\end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 4. If $j < k < i$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_j)\,U(\tau,\overline{\alpha}A_k)\,U(\tau,A_i)\,U(\tau,\alpha A_k)\,U(\tau,\alpha A_j)]_{i,j,k} = 0$$

and

$$\begin{aligned}[V_2(\tau)]_{i,j,k} &= [U(\tau,\overline{\alpha}A_i)\,U(\tau,\overline{\alpha}A_k)\,U(\tau,A_j)\,U(\tau,\alpha A_k)\,U(\tau,\alpha A_i)]_{i,j,k} \\ &= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha\overline{\alpha}.\end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 5. If $k < i < j$, then

$$\begin{aligned}[V_1(\tau)]_{i,j,k} &= [U(\tau,\overline{\alpha}A_k)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_j)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_k)]_{i,j,k} \\ &= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha\overline{\alpha}\end{aligned}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_j)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_k)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_j)]_{i,j,k} = 0.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 6. If $k < j < i$, then

$$\begin{aligned}[V_1(\tau)]_{i,j,k} &= [U(\tau,\overline{\alpha}A_k)\,U(\tau,\overline{\alpha}A_j)\,U(\tau,A_i)\,U(\tau,\alpha A_j)\,U(\tau,\alpha A_k)]_{i,j,k} \\ &= [U(\tau,A_i)]_i\,[U(\tau,\alpha A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha^2\end{aligned}$$

and

$$\begin{aligned}[V_2(\tau)]_{i,j,k} &= [U(\tau,\overline{\alpha}A_i)\,U(\tau,\overline{\alpha}A_j)\,U(\tau,A_k)\,U(\tau,\alpha A_j)\,U(\tau,\alpha A_i)]_{i,j,k} \\ &= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,\overline{\alpha}A_j)]_j\,[U(\tau,A_k)]_k = \overline{\alpha}^2.\end{aligned}$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left(\alpha^2 + \overline{\alpha}^2\right) = \frac{1}{6}.$$

Finally, for any triple (i, j, k) we have:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}.$$

Inserting in (5.4) the obtained coefficients, we will get:

$$\begin{aligned}
V(\tau) &= I - \tau\sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \sum_{i,j=1}^{m} A_i A_j - \frac{1}{6}\tau^3 \sum_{i,j,k=1}^{m} A_i A_j A_k + R_4^{(m)}(\tau) \\
&= I - \tau\sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \left(\sum_{i=1}^{m} A_i\right)^2 - \frac{1}{6}\tau^3 \left(\sum_{i=1}^{m} A_i\right)^3 + R_4^{(m)}(\tau) \\
&= I - \tau A + \frac{1}{2}\tau^2 A - \frac{1}{6}\tau^3 A^3 + R_4^{(m)}(\tau). \qquad (5.6)
\end{aligned}$$

According to the second inequality of the condition (b) of the **Theorem 3.1** the following estimation is true for $R_4(\tau, A)$:

$$\|R_4(\tau, A)\varphi\| \le ce^{\omega\tau}\tau^4 \left\|A^4\varphi\right\| \le ce^{\omega\tau}\tau^4 \|\varphi\|_{A^4}. \qquad (5.7)$$

According to the formulas (1.7) and (5.6) we have:

$$U(\tau, A) - V(\tau) = R_4(\tau, A) - R_4^{(m)}(\tau).$$

Hence using inequalities (5.7) and (5.5) we can get the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le ce^{\omega_2\tau}\tau^4 \|\varphi\|_{A^4}. \qquad (5.8)$$

Analogously of two-dimensional case, from equalities (3.2) and (5.1), taking into account inequalities (4.3) and (5.8), we will obtain sought estimation. □

Chapter II

The Fourth Order Accuracy Decomposition Schemes

§6. The fourth order accuracy decomposition scheme for evolution problem

1. Differential splitting and error estimation of approximate solution

Let us consider the problem (3.1). Let $A = A_1 + A_2$, where $A_i \ \ (i = 1, 2)$ are closed operators, densely defined in X.

Together with problem (3.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\alpha}{2} A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(2)}(t)}{dt} + \frac{1}{2} A_2 v_k^{(2)}(t) &= 0, \quad v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k), \\
\frac{dv_k^{(3)}(t)}{dt} + \overline{\alpha} A_1 v_k^{(3)}(t) &= 0, \quad v_k^{(3)}(t_{k-1}) = v_k^{(2)}(t_k), \\
\frac{dv_k^{(4)}(t)}{dt} + \frac{1}{2} A_2 v_k^{(4)}(t) &= 0, \quad v_k^{(4)}(t_{k-1}) = v_k^{(3)}(t_k), \\
\frac{dv_k^{(5)}(t)}{dt} + \frac{\alpha}{2} A_1 v_k^{(5)}(t) &= 0, \quad v_k^{(5)}(t_{k-1}) = v_k^{(4)}(t_k),
\end{aligned}$$

$$\begin{aligned}
\frac{dw_k^{(1)}(t)}{dt} + \frac{\alpha}{2} A_2 w_k^{(1)}(t) &= 0, \quad w_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dw_k^{(2)}(t)}{dt} + \frac{1}{2} A_1 w_k^{(2)}(t) &= 0, \quad w_k^{(2)}(t_{k-1}) = w_k^{(1)}(t_k), \\
\frac{dw_k^{(3)}(t)}{dt} + \overline{\alpha} A_2 w_k^{(3)}(t) &= 0, \quad w_k^{(3)}(t_{k-1}) = w_k^{(2)}(t_k), \\
\frac{dw_k^{(4)}(t)}{dt} + \frac{1}{2} A_1 w_k^{(4)}(t) &= 0, \quad w_k^{(4)}(t_{k-1}) = w_k^{(3)}(t_k), \\
\frac{dw_k^{(5)}(t)}{dt} + \frac{\alpha}{2} A_2 w_k^{(5)}(t) &= 0, \quad w_k^{(5)}(t_{k-1}) = w_k^{(4)}(t_k),
\end{aligned}$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Suppose that the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j),\ j = 1,2$ generate strongly continuous semigroups.

$u_k(t),\ k = 1,2,..,$ is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = \frac{1}{2}[v_k^{(5)}(t) + w_k^{(5)}(t)]. \tag{6.1}$$

We declare function $u_k(t)$ as an approximated solution of problem (3.1) on each interval $[t_{k-1}, t_k]$.

Theorem 6.1. *Let the conditions (a) and (b) of Theorem 1.1 be fulfilled and $U(s,A)\varphi \in D(A^5)$ for each fixed $s \geq 0$. Then the following estimation holds:*

$$\|u(t_k) - u_k(t_k)\| \leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [0,t_k]} \|U(s,A)\varphi\|_{A^5},$$

where c and ω_0 are positive constants.

Proof. From formula (6.1) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{6.2}$$

where

$$V(\tau) = \frac{1}{2}[V_1(\tau) + V_2(\tau)],$$

and where

$$\begin{aligned} V_1(\tau) &= U\left(\tau, \frac{\alpha}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U(\tau, \overline{\alpha}A_1) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right), \\ V_2(\tau) &= U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) U(\tau, \overline{\alpha}A_2) U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right). \end{aligned}$$

Remark 6.2. *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem1.1. In this case, for the solving operator, the following estimation holds:*

$$\left\|V^k(\tau)\right\| \leq e^{\omega_1 t_k}, \tag{6.3}$$

where ω_1 is a positive constant.

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1,2)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (1.7), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $(\tau^2 A_i A_j)$, $(-\tau^3 A_i A_j A_k)$ and $(\tau^4 A_i A_j A_k A_l)$ $(i,j,k,l = 1,2)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$, $[T(\tau)]_{i,j,k}$ and $[T(\tau)]_{i,j,k,l}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (1.7) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned}V(\tau) &= I-\tau\sum_{i=1}^{2}[V(\tau)]_i A_i+\tau^2\sum_{i,j=1}^{2}[V(\tau)]_{i,j} A_iA_j\\ &-\tau^3\sum_{i,j,k=1}^{2}[V(\tau)]_{i,j,k} A_iA_jA_k\\ &+\tau^4\sum_{i,j,k,l=1}^{2}[V(\tau)]_{i,j,k,l} A_iA_jA_kA_l+\widetilde{R}_5(\tau).\end{aligned} \tag{6.4}$$

According to the first inequality of the condition *(b)* of the Theorem, for $\widetilde{R}_5(\tau)$, the following estimation holds:

$$\left\|\widetilde{R}_5(\tau)\varphi\right\| \le ce^{\omega_0\tau}\tau^5\|\varphi\|_{A^5}, \quad \varphi\in D\left(A^5\right), \tag{6.5}$$

where c and ω_0 are positive constants.

It is obvious that, for the coefficients in formula (6.4), we have:

$$\begin{aligned}[V(\tau)]_i &= \frac{1}{2}\left([V_1(\tau)]_i+[V_2(\tau)]_i\right), \quad i=1,2,\\ [V(\tau)]_{i,j} &= \frac{1}{2}\left([V_1(\tau)]_{i,j}+[V_2(\tau)]_{i,j}\right), \quad i,j=1,2,\\ [V(\tau)]_{i,j,k} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k}+[V_2(\tau)]_{i,j,k}\right), \quad i,j,k=1,2,\\ [V(\tau)]_{i,j,k,l} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k,l}+[V_2(\tau)]_{i,j,k,l}\right), \quad i,j,k,l=1,2.\end{aligned}$$

Let us make two remarks which will simplify a calculation of coefficients in decomposition (6.4):

Remark 6.3. *Operator $V(\tau)$ will not change if we replace with each other the operators A_1 and A_2 in its expression, as in this case $V_1(\tau)$ will coincide with $V_2(\tau)$, and $V_2(\tau)$ - with $V_1(\tau)$. Therefore we have:*

$$\begin{aligned}[V(\tau)]_i &= [V(\tau)]_{3-i}, \quad i=1,2;\\ [V(\tau)]_{i,j} &= [V(\tau)]_{3-i,3-j}, \quad i,j=1,2;\\ [V(\tau)]_{i,j,k} &= [V(\tau)]_{3-i,3-j,3-k}, \quad i,j,k=1,2;\\ [V(\tau)]_{i,j,k,l} &= [V(\tau)]_{3-i,3-j,3-k,3-l}, \quad i,j,k,l=1,2.\end{aligned}$$

Remark 6.4. *Operators $V_1(\tau)$ and $V_2(\tau)$ are symmetrical in the sense that in their expressions the factors (semigroups) equally remote from the ends coincide with each other. Therefore we have:*

$$\begin{aligned}[V(\tau)]_{i,j} &= [V(\tau)]_{j,i}, \quad i,j=1,2;\\ [V(\tau)]_{i,j,k} &= [V(\tau)]_{k,j,i}, \quad i,j,k=1,2;\\ [V(\tau)]_{i,j,k,l} &= [V(\tau)]_{l,k,j,i}, \quad i,j,k,l=1,2.\end{aligned}$$

Let us calculate the coefficients $[V(\tau)]_i$ corresponding to the first order members in formula (6.4). It is obvious that the members, corresponding to these coefficients, are obtained from the decomposition of only those factors (semigroups) of the operator $V(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identical operators) will participate.

On the whole, we have two cases: $i = 1$ and $i = 2$. Let us consider the case $i = 1$. We obviously have:

$$[V_1(\tau)]_1 = [U(\tau, A_1)]_1 = 1$$

and

$$[V_2(\tau)]_1 = [U(\tau, A_1)]_1 = 1.$$

Thus

$$[V(\tau)]_1 = \frac{1}{2}\left([V_1(\tau)]_1 + [V_2(\tau)]_1\right) = 1.$$

According to **Remark 6.3**:

$$[V(\tau)]_2 = [V(\tau)]_1 = 1. \tag{6.6}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i, j = 1, 2)$ corresponding to the second order members included in formula (6.4). On the whole we have two cases: $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2)$. Let us consider the case $(i, j) = (1, 1)$. We obviously have:

$$[V_1(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}$$

and

$$[V_2(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}.$$

Therefore

$$[V(\tau)]_{1,1} = \frac{1}{2}\left([V_1(\tau)]_{1,1} + [V_2(\tau)]_{1,1}\right) = \frac{1}{2}.$$

According to **Remark 6.3**:

$$[V(\tau)]_{2,2} = [V(\tau)]_{1,1} = \frac{1}{2}. \tag{6.7}$$

Let us consider the case $(i, j) = (1, 2)$, we obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,2} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \end{aligned}$$

$$
\begin{aligned}
&+\left[U\left(\tau,\overline{\alpha}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2\\
=\;&\frac{\alpha}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{1}{2}+\overline{\alpha}\frac{1}{2}=\frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
[V_2(\tau)]_{1,2} \;=\;& \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2
\end{aligned}
$$

$$
\begin{aligned}
&+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\\
=\;&\overline{\alpha}\frac{1}{2}+\frac{\alpha}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{1}{2}=\frac{1}{2}.
\end{aligned}
$$

Thus

$$
[V(\tau)]_{1,2}=\frac{1}{2}\left([V_1(\tau)]_{1,2}+[V_2(\tau)]_{1,2}\right)=\frac{1}{2}.
$$

According to **Remark 6.3**:

$$
[V(\tau)]_{2,1}=[V(\tau)]_{1,2}=\frac{1}{2}. \tag{6.8}
$$

Here we used the identity $\alpha+\overline{\alpha}=1$.

By combining formulas (6.7) and (6.8), we will obtain:

$$
[V(\tau)]_{i,j}=\frac{1}{2},\quad i,j=1,2. \tag{6.9}
$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i,j,k=1,2)$ corresponding to the third order members in formula (6.4). On the whole we have eight cases: $(i,j,k)=(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,1)$, $(2,2,2)$. Let us consider the case $(i,j,k)=(1,1,1)$. We obviously have:

$$
[V_1(\tau)]_{1,1,1}=\left[U\left(\tau,A_1\right)\right]_{1,1,1}=\frac{1}{6}
$$

and

$$
[V_2(\tau)]_{1,1,1}=\left[U\left(\tau,A_1\right)\right]_{1,1,1}=\frac{1}{6}.
$$

Thus:

$$
[V(\tau)]_{1,1,1}=\frac{1}{2}\left([V_1(\tau)]_{1,1,1}+[V_2(\tau)]_{1,1,1}\right)=\frac{1}{6}.
$$

According to **Remark 6.3**:

$$
[V(\tau)]_{2,2,2}=[V(\tau)]_{1,1,1}=\frac{1}{6}. \tag{6.10}
$$

Let us calculate the case $(i,j,k)=(1,1,2)$. We obviously have:

$$\begin{aligned}
[V_1(\tau)]_{1,1,2} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \\
&\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \\
&\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \\
&\quad+\left[U\left(\tau,\overline{\alpha}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \\
&= \frac{1}{2}\frac{\alpha^2}{8}+\frac{1}{2}\frac{\alpha^2}{8}+\frac{\alpha}{2}\overline{\alpha}\frac{1}{2}+\frac{\overline{\alpha}^2}{2}\frac{1}{2}=\frac{1+\overline{\alpha}^2}{8}
\end{aligned}$$

and

$$\begin{aligned}
[V_2(\tau)]_{1,1,2} &= \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2 \\
&\quad+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
&\quad+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
&\quad+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
&= \frac{1}{8}\overline{\alpha}+\frac{1}{8}\frac{\alpha}{2}+\frac{1}{2}\frac{1}{2}\frac{\alpha}{2}+\frac{1}{8}\frac{\alpha}{2}=\frac{1+\alpha}{8}.
\end{aligned}$$

Thus

$$[V(\tau)]_{1,1,2}=\frac{1}{2}\Big([V_1(\tau)]_{1,1,2}+[V_2(\tau)]_{1,1,2}\Big)=\frac{2+\overline{\alpha}^2+\alpha}{16}=\frac{1}{6},$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha+\overline{\alpha}^2=\frac{2}{3}$.

According to **Remark 6.3** and **Remark 6.4**:

$$[V(\tau)]_{1,1,2}=[V(\tau)]_{2,1,1}=[V(\tau)]_{2,2,1}=[V(\tau)]_{1,2,2}=\frac{1}{6}. \tag{6.11}$$

Let us consider the case $(i,j,k)=(1,2,1)$. We obviously have:

$$\begin{aligned}[V_1(\tau)]_{1,2,1} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2[U(\tau,\overline{\alpha}A_1)]_1 \\ &\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\ &\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\ &\quad+[U(\tau,\overline{\alpha}A_1)]_1\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\ &= \frac{\alpha}{2}\frac{1}{2}\overline{\alpha}+\frac{\alpha}{2}\frac{1}{2}\frac{\alpha}{2}+\frac{\alpha}{2}\frac{1}{2}\frac{\alpha}{2}+\overline{\alpha}\frac{1}{2}\frac{\alpha}{2}=\frac{1}{6}+\frac{\alpha^2}{4}\end{aligned}$$

and

$$[V_2(\tau)]_{1,2,1}=\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1[U(\tau,\overline{\alpha}A_2)]_2\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1=\frac{\overline{\alpha}}{4}.$$

Thus

$$[V(\tau)]_{1,2,1}=\frac{1}{2}\Big([V_1(\tau)]_{1,1,2}+[V_2(\tau)]_{1,1,2}\Big)=\frac{1}{12}+\frac{\alpha^2+\overline{\alpha}}{8}=\frac{1}{6}.$$

Here we used the identity $\alpha^2+\overline{\alpha}=\frac{2}{3}$.

According to **Remark 6.3**:

$$[V(\tau)]_{2,1,2}=[V(\tau)]_{1,2,1}=\frac{1}{6}. \tag{6.12}$$

By combining formulas (6.10), (6.11) and (6.12), we will obtain:

$$[V(\tau)]_{i,j,k}=\frac{1}{6},\quad i,j,k=1,2. \tag{6.13}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k,l}$ $(i,j,k,l=1,2)$ corresponding to the fourth order members in formula (6.4). On the whole we have sixteen cases: $(i,j,k,l)=(1,1,1,1)$, $(1,1,1,2)$, ..., $(2,2,2,1)$, $(2,2,2,2)$. Let us consider the case $(i,j,k,l)=(1,1,1,1)$. We obviously have:

$$[V_1(\tau)]_{1,1,1,1}=[U(\tau,A_1)]_{1,1,1,1}=\frac{1}{24}$$

and

$$[V_2(\tau)]_{1,1,1,1}=[U(\tau,A_1)]_{1,1,1,1}=\frac{1}{24}.$$

Thus:

$$[V(\tau)]_{1,1,1,1}=\frac{1}{2}\Big([V_1(\tau)]_{1,1,1,1}+[V_2(\tau)]_{1,1,1,1}\Big)=\frac{1}{24}.$$

According to **Remark 6.3**:

$$[V(\tau)]_{2,2,2,2} = [V(\tau)]_{1,1,1,1} = \frac{1}{24}. \tag{6.14}$$

Let us consider the case $(i,j,k,l) = (1,1,1,2)$, we obviously have:

$$\begin{aligned}
[V_1(\tau)]_{1,1,1,2} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau, \overline{\alpha}A_1\right)\right]_1\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1}\left[U\left(\tau, \overline{\alpha}A_1\right)\right]_{1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \overline{\alpha}A_1\right)\right]_{1,1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 \\
&= \frac{\alpha^3}{48}\frac{1}{2} + \frac{\alpha^3}{48}\frac{1}{2} + \frac{\alpha^2}{8}\overline{\alpha}\frac{1}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}^2}{2}\frac{1}{2} + \frac{\overline{\alpha}^3}{6}\frac{1}{2} \\
&= \frac{\alpha^3 + \alpha + 4\overline{\alpha}^3 + 2\overline{\alpha}}{48} = \frac{1 + 3\overline{\alpha}^3 + \overline{\alpha}}{48}
\end{aligned}$$

and

$$\begin{aligned}
[V_2(\tau)]_{1,1,1,2} &= \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1,1,1}\left[U\left(\tau, \overline{\alpha}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1,1,1}\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau, \frac{1}{2}A_1\right)\right]_1\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1}\left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2 \\
&\quad + \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1,1,1}\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2 \\
&= \frac{1}{48}\overline{\alpha} + \frac{1}{48}\frac{\alpha}{2} + \frac{1}{8}\frac{1}{2}\frac{\alpha}{2} + \frac{1}{2}\frac{1}{8}\frac{\alpha}{2} + \frac{1}{48}\frac{\alpha}{2} = \frac{1 + 3\alpha}{48}.
\end{aligned}$$

Thus

$$\begin{aligned}
[V(\tau)]_{1,1,1,2} &= \frac{1}{2}\Big([V_1(\tau)]_{1,1,1,2} + [V_2(\tau)]_{1,1,1,2}\Big) \\
&= \frac{2 + 3\left(\overline{\alpha}^3 + \alpha\right) + \overline{\alpha}}{96} = \frac{1}{24}.
\end{aligned}$$

Here we used the identities $3\left(\overline{\alpha}^3 + \alpha\right) = 2 - \overline{\alpha}, \quad \overline{\alpha}^3 + \alpha^3 = 0$.

According to **Remark 6.3** and **Remark 6.4**:

$$[V(\tau)]_{1,1,1,2} = [V(\tau)]_{2,1,1,1} = [V(\tau)]_{1,2,2,2} = [V(\tau)]_{2,2,2,1} = \frac{1}{24}. \tag{6.15}$$

Let us consider the case $(i, j, k, l) = (1, 1, 2, 1)$. We obviously have:

$$\begin{aligned}
[V_1(\tau)]_{1,1,2,1} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2 [U(\tau, \overline{\alpha}A_1)]_1 \\
&+ \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2\left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \\
&+ \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2\left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \\
&+ \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 [U(\tau, \overline{\alpha}A_1)]_1\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2\left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \\
&+ [U(\tau, \overline{\alpha}A_1)]_{1,1}\left[U\left(\tau, \frac{1}{2}A_2\right)\right]_2\left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \\
&= \frac{\alpha^2}{8}\frac{1}{2}\overline{\alpha} + \frac{\alpha^2}{8}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha^2}{8}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha}{2}\overline{\alpha}\frac{1}{2}\frac{\alpha}{2} + \frac{\overline{\alpha}^2}{2}\frac{1}{2}\frac{\alpha}{2} \\
&= \frac{3\alpha^3 + 3\alpha + 2\overline{\alpha}}{48} = \frac{3\alpha^3 + \alpha + 2}{48}
\end{aligned}$$

and

$$\begin{aligned}
[V_2(\tau)]_{1,1,2,1} &= \left[U\left(\tau, \frac{1}{2}A_1\right)\right]_{1,1} [U(\tau, \overline{\alpha}A_2)]_2\left[U\left(\tau, \frac{1}{2}A_1\right)\right]_1 \\
&= \frac{1}{8}\overline{\alpha}\frac{1}{2} = \frac{1}{16}\overline{\alpha}.
\end{aligned}$$

Thus

$$\begin{aligned}
[V(\tau)]_{1,1,2,1} &= \frac{1}{2}\Big([V_1(\tau)]_{1,1,2,1} + [V_2(\tau)]_{1,1,2,1}\Big) \\
&= \frac{3\alpha^3 + \alpha + 2 + 3\overline{\alpha}}{96} = \frac{3\left(\alpha^3 + \overline{\alpha}\right) + \alpha + 2}{96} = \frac{1}{24}.
\end{aligned}$$

Here we used the identity $\overline{\alpha}^3 + \alpha^3 = 0$.

According to **Remark 6.3** and **Remark 6.4**:

$$[V(\tau)]_{1,1,2,1} = [V(\tau)]_{2,2,1,2} = [V(\tau)]_{1,2,1,1} = [V(\tau)]_{2,1,2,2} = \frac{1}{24}. \tag{6.16}$$

Let us consider the case $(i,j,k,l)=(1,1,2,2)$. We obviously have:

$$
\begin{aligned}
[V_1(\tau)]_{1,1,2,2} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \\
&\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2} \\
&\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \\
&\quad+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1}\left[U\left(\tau,\overline{\alpha}A_1\right)\right]_{1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \\
&\quad+\left[U\left(\tau,\overline{\alpha}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \\
&= \frac{\alpha^2}{8}\frac{1}{8}+\frac{\alpha^2}{8}\frac{1}{2}\frac{1}{2}+\frac{\alpha^2}{8}\frac{1}{8}+\frac{\alpha}{2}\overline{\alpha}\frac{1}{8}+\frac{\overline{\alpha}^2}{2}\frac{1}{8} \\
&= \frac{\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2}{16}=\frac{1}{24}
\end{aligned}
$$

and

$$
\begin{aligned}
[V_2(\tau)]_{1,1,2,2} &= \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_{2,2}\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1} \\
&\quad\times\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_{2}\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_{2} \\
&\quad+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_{2,2} \\
&\quad+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1}\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1}\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_{2,2} \\
&\quad+\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_{2,2} \\
&= \frac{1}{8}\frac{\overline{\alpha}^2}{2}+\frac{1}{8}\overline{\alpha}\frac{\alpha}{2}+\frac{1}{8}\frac{\alpha^2}{8}+\frac{1}{2}\frac{1}{2}\frac{\alpha^2}{8}+\frac{1}{8}\frac{\alpha^2}{8} \\
&= \frac{\overline{\alpha}^2+\overline{\alpha}\alpha+\alpha^2}{16}=\frac{1}{24}.
\end{aligned}
$$

Thus

$$[V(\tau)]_{1,1,2,2}=\frac{1}{2}\Big([V_1(\tau)]_{1,1,2,2}+[V_2(\tau)]_{1,1,2,2}\Big)=\frac{1}{24}.$$

According to **Remark 6.3**:

$$[V(\tau)]_{1,1,2,2}=[V(\tau)]_{2,2,1,1}=\frac{1}{24}. \tag{6.17}$$

Let us consider the case $(i,j,k,l)=(1,2,2,1)$. We obviously have:

$$\begin{aligned}
[V_1(\tau)]_{1,2,2,1} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} [U(\tau,\overline{\alpha}A_1)]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \\
&\quad \times \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\
&\quad + \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\
&\quad + [U(\tau,\overline{\alpha}A_1)]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_{2,2} \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \\
&= \frac{\alpha}{2}\frac{1}{8}\overline{\alpha} + \frac{\alpha}{2}\frac{1}{8}\frac{\alpha}{2} + \frac{\alpha}{2}\frac{1}{2}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha}{2}\frac{1}{8}\frac{\alpha}{2} + \overline{\alpha}\frac{1}{8}\frac{\alpha}{2} \\
&= \frac{\alpha^2+\alpha\overline{\alpha}}{8}
\end{aligned}$$

and

$$\begin{aligned}
[V_2(\tau)]_{1,2,2,1} &= \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha}A_2)]_{2,2} \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 \\
&= \frac{1}{2}\frac{\overline{\alpha}^2}{2}\frac{1}{2} = \frac{\overline{\alpha}^2}{8}.
\end{aligned}$$

Thus

$$\begin{aligned}
[V(\tau)]_{1,2,2,1} &= \frac{1}{2}\Big([V_1(\tau)]_{1,2,2,1} + [V_2(\tau)]_{1,2,2,1}\Big) \\
&= \frac{\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2}{16} = \frac{1}{24}.
\end{aligned}$$

According to **Remark 6.3**:

$$[V(\tau)]_{1,2,2,1} = [V(\tau)]_{2,1,1,2} = \frac{1}{24}. \tag{6.18}$$

Let us consider the case $(i,j,k,l)=(1,2,1,2)$. We obviously have:

$$\begin{aligned}
[V_1(\tau)]_{1,2,1,2} &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 [U(\tau,\overline{\alpha}A_1)]_1 \left[U\left(\tau,\frac{1}{2}A_2\right)\right]_2 \\
&= \frac{\alpha}{2}\frac{1}{2}\overline{\alpha}\frac{1}{2} = \frac{1}{24}
\end{aligned}$$

and

$$\begin{aligned}[V_2(\tau)]_{1,2,1,2} &= \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 [U(\tau,\overline{\alpha}A_2)]_2 \left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1 \left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_2 \\ &= \frac{1}{2}\overline{\alpha}\frac{1}{2}\frac{\alpha}{2} = \frac{1}{24}.\end{aligned}$$

Thus

$$[V(\tau)]_{1,2,1,2} = \frac{1}{2}\Big([V_1(\tau)]_{1,2,1,2} + [V_2(\tau)]_{1,2,1,2}\Big) = \frac{1}{24},$$

According to **Remark 6.3**:

$$[V(\tau)]_{1,2,1,2} = [V(\tau)]_{2,1,2,1} = \frac{1}{24}. \qquad (6.19)$$

By combining formulas (6.14)-(6.19), we will obtain:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \quad i,j,k,l = 1,2. \qquad (6.20)$$

From equality (6.4), taking into account formulas (6.6), (6.9), (6.13) and (6.20), we will obtain:

$$\begin{aligned}V(\tau) &= I - \tau\sum_{i=1}^{2}A_i + \frac{1}{2}\tau^2\sum_{i,j=1}^{2}A_iA_j - \frac{1}{6}\tau^3\sum_{i,j,k=1}^{2}A_iA_jA_k \\ &\quad + \frac{1}{24}\tau^4\sum_{i,j,k,l=1}^{2}A_iA_jA_kA_l + \widetilde{R}_5(\tau) \\ &= I - \tau\sum_{i=1}^{2}A_i + \frac{1}{2}\tau^2\left(\sum_{i=1}^{2}A_i\right)^2 - \frac{1}{6}\tau^3\left(\sum_{i=1}^{2}A_i\right)^3 \\ &\quad + \frac{1}{24}\tau^4\left(\sum_{i=1}^{2}A_i\right)^4 + \widetilde{R}_5(\tau) \\ &= I - \tau A + \frac{1}{2}\tau^2A^2 - \frac{1}{6}\tau^3A^3 + \frac{1}{24}\tau^4A^4 + \widetilde{R}_5(\tau). \qquad (6.21)\end{aligned}$$

According to formula (1.7):

$$U(\tau,A) = I - \tau A + \frac{1}{2}\tau^2A^2 - \frac{1}{6}\tau^3A^3 + \frac{1}{24}\tau^4A^4 + R_5(\tau,A). \qquad (6.22)$$

According to condition *(b)* of the second inequality of the Theorem, for $R_5(\tau,A)$, the following estimation holds:

$$\|R_5(\tau,A)\varphi\| \le ce^{\omega\tau}\tau^5\left\|A^5\varphi\right\| \le ce^{\omega\tau}\tau^5\|\varphi\|_{A^5}. \qquad (6.23)$$

According to equalities (6.21) and (6.22):

$$U(\tau,A) - V(\tau) = R_5(\tau,A) - \widetilde{R}_5(\tau).$$

From here, taking into account inequalities (6.5) and (6.23), we will obtain the following estimation:

$$\left\| \left[U\left(\tau, A\right) - V\left(\tau\right) \right] \varphi \right\| \leq c e^{\omega_2 \tau} \tau^5 \left\| \varphi \right\|_{A^5}. \tag{6.24}$$

From equalities (3.2) and (6.2), taking into account inequalities (6.3) and (6.24), we will obtain:

$$\begin{aligned}
\left\| u(t_k) - u_k(t_k) \right\| &= \left\| \left[U(t_k, A) - V^k\left(\tau\right) \right] \varphi \right\| = \left\| \left[U^k\left(\tau, A\right) - V^k\left(\tau\right) \right] \varphi \right\| \\
&= \left\| \left[\sum_{i=1}^{k} V^{k-i}\left(\tau\right) \left[U\left(\tau, A\right) - V\left(\tau\right) \right] U\left(\left(i-1\right)\tau, A\right) \right] \varphi \right\| \\
&\leq \sum_{i=1}^{k} \left\| V\left(\tau\right) \right\|^{k-i} \left\| \left[U\left(\tau, A\right) - V\left(\tau\right) \right] U\left(\left(i-1\right)\tau, A\right) \varphi \right\| \\
&\leq \sum_{i=1}^{k} e^{\omega_1 (k-i)\tau} c e^{\omega_2 \tau} \tau^5 \left\| U\left(\left(i-1\right)\tau, A\right) \varphi \right\|_{A^5} \\
&\leq c e^{\omega_0 t_k} \tau^5 \sum_{i=1}^{k} \left\| U\left(\left(i-1\right)\tau, A\right) \varphi \right\|_{A^5} \\
&\leq k c e^{\omega_0 t_k} \tau^5 \sup_{s \in [o, t_k]} \left\| U\left(s, A\right) \varphi \right\|_{A^5} \\
&\leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [o, t_k]} \left\| U\left(s, A\right) \varphi \right\|_{A^5}.
\end{aligned}$$

Remark 6.5. *In case of a Hilbert space, if* A_1, A_2 *and* $A_1 + A_2$ *are self adjoint nonnegative operators, then* ω_0 *will be replaced by* 0 *in the estimation of the Theorem. In addition, for the solution operator of the split problem, the following estimation holds:* $\left\| V^k\left(\tau\right) \right\| \leq 1$.

Remark 6.6. *In case of a Hilbert space, if* A_1, A_2 *and* $A_1 + A_2$ *are self adjoint positive defined operators, then* ω_0 *will be replaced by* $(-\alpha_0)$, $\alpha_0 > 0$ *in the estimation of the Theorem. In addition, for the solution operator of the split problem, the following estimation holds:* $\left\| V^k\left(\tau\right) \right\| \leq e^{-\alpha_1 t_k}$, $\alpha_1 > 0$.

2. Connection between decomposition formulas with different accuracies

It is interesting if there exists a certain regularity, on the basis of which it is available to construct automatically stable decomposition formulas with accuracy of any order. Concerning the above-mentioned let us consider the concrete first and second order accuracy decomposition formulas and see whether there exists a connection between them.

$$V^{(1)}\left(\tau\right) = U\left(\tau, A_1\right) U\left(\tau, A_2\right), \tag{6.25}$$

$$V^{(2)}\left(\tau\right) = U\left(\tau, \frac{1}{2} A_1\right) U\left(\tau, A_2\right) U\left(\tau, \frac{1}{2} A_1\right). \tag{6.26}$$

In this formula and the formulas given below, the upper indices of the operator V denote the order of the corresponding decomposition formula. Formula (6.25) represents the first order accuracy decomposition formula (see [60]), while formula (6.26) represents the second order accuracy decomposition formula (see [4]). In order to show more clearly the connection between them, let us rewrite formula (6.26) in the following form:

$$\begin{aligned} V^{(2)}(\tau) &= \left[U\left(\tau,\frac{1}{2}A_1\right)U\left(\tau,\frac{1}{2}A_2\right)\right]\left[U\left(\tau,\frac{1}{2}A_2\right)\left(\tau,\frac{1}{2}A_1\right)\right] \\ &= V^{(1)}\left(\frac{1}{2}\tau\right)\overline{V^{(1)}}\left(\frac{1}{2}\tau\right). \end{aligned}$$

In this formula and the formulas given below, we denote by $\overline{V}$ the multiplication of factors of the operator V in the reverse order.

The regularity of the same type exists between the third and fourth order accuracy decomposition formulas, constructed by us (see [19]-[62]). In order to show this, let us introduce the following notations:

$$\begin{aligned} V^{(3)}(\tau) &= \frac{1}{2}\left[V_1^{(3)}(\tau)+V_2^{(3)}(\tau)\right], \qquad (6.27) \\ V_1^{(3)}(\tau) &= U(\tau,\alpha A_1)\,U(\tau,A_2)\,U(\tau,\overline{\alpha}A_1), \\ V_2^{(3)}(\tau) &= U(\tau,\alpha A_2)\,U(\tau,A_1)\,U(\tau,\overline{\alpha}A_2), \end{aligned}$$

and

$$\begin{aligned} V^{(4)}(\tau) &= \frac{1}{2}\left[V_1^{(4)}(\tau)+V_2^{(4)}(\tau)\right], \qquad (6.28) \\ V_1^{(4)}(\tau) &= U\left(\tau,\frac{\alpha}{2}A_1\right)U\left(\tau,\frac{1}{2}A_2\right)U(\tau,\overline{\alpha}A_1)\,U\left(\tau,\frac{1}{2}A_2\right)U\left(\tau,\frac{\alpha}{2}A_1\right), \\ V_2^{(4)}(\tau) &= U\left(\tau,\frac{\alpha}{2}A_2\right)U\left(\tau,\frac{1}{2}A_1\right)U(\tau,\overline{\alpha}A_2)\,U\left(\tau,\frac{1}{2}A_1\right)U\left(\tau,\frac{\alpha}{2}A_2\right). \end{aligned}$$

In order to reveal the connection between formulas (6.27) and (6.28), let rewrite the addends of formula (6.28) in the following form:

$$\begin{aligned} V_1^{(4)}(\tau) &= \left[U\left(\tau,\frac{\alpha}{2}A_1\right)U\left(\tau,\frac{1}{2}A_2\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)\right] \\ &\quad\times\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)U\left(\tau,\frac{1}{2}A_2\right)U\left(\tau,\frac{\alpha}{2}A_1\right)\right] \\ &= V_1^{(3)}\left(\frac{1}{2}\tau\right)\overline{V_1^{(3)}}\left(\frac{1}{2}\tau\right), \\ V_2^{(4)}(\tau) &= \left[U\left(\tau,\frac{\alpha}{2}A_2\right)U\left(\tau,\frac{1}{2}A_1\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right] \\ &\quad\times\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)U\left(\tau,\frac{1}{2}A_1\right)U\left(\tau,\frac{\alpha}{2}A_2\right)\right] \\ &= V_2^{(3)}\left(\frac{1}{2}\tau\right)\overline{V_2^{(3)}}\left(\frac{1}{2}\tau\right), \end{aligned}$$

Finally we obtain:

$$V^{(4)}(\tau) = \frac{1}{2}\left[V_1^{(3)}\left(\frac{1}{2}\tau\right)\overline{V_1^{(3)}}\left(\frac{1}{2}\tau\right) + V_2^{(3)}\left(\frac{1}{2}\tau\right)\overline{V_2^{(3)}}\left(\frac{1}{2}\tau\right)\right].$$

Unfortunately, the following formula constructed by the same rule:

$$\begin{aligned} V^{(5)}(\tau) &= \frac{1}{2}\left[V_1^{(4)}\left(\frac{1}{2}\tau\right)\overline{V_1^{(4)}}\left(\frac{1}{2}\tau\right) + V_2^{(4)}\left(\frac{1}{2}\tau\right)\overline{V_2^{(4)}}\left(\frac{1}{2}\tau\right)\right] \\ &= \frac{1}{2}\left[U\left(\tau,\frac{\alpha}{4}A_1\right)U\left(\tau,\frac{1}{4}A_2\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)U\left(\tau,\frac{1}{4}A_2\right)U\left(\tau,\frac{\alpha}{2}A_1\right)\right. \\ &\quad \times U\left(\tau,\frac{1}{4}A_2\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_1\right)U\left(\tau,\frac{1}{4}A_2\right)U\left(\tau,\frac{\alpha}{4}A_1\right) \\ &\quad + U\left(\tau,\frac{\alpha}{4}A_2\right)U\left(\tau,\frac{1}{4}A_1\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)U\left(\tau,\frac{1}{4}A_1\right)U\left(\tau,\frac{\alpha}{2}A_2\right) \\ &\quad \left.\times U\left(\tau,\frac{1}{4}A_1\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)U\left(\tau,\frac{1}{4}A_1\right)U\left(\tau,\frac{\alpha}{4}A_2\right)\right]. \end{aligned}$$

does not represent the fifth order accuracy decomposition formula. To check this out, it is sufficient to calculate, for example, the coefficients $\left[V^{(5)}(\tau)\right]_{1,2,1,2,1}$. We see that

$$\left[V^{(5)}(\tau)\right]_{1,2,1,2,1} \neq \frac{1}{5!}.$$

In our opinion, it is interesting and important to find the general regularity, by means of which it will be available to construct recurrently an automatically stable decomposition formula with accuracy of any order, or to prove that, on the complex number field, there does not exist an automatically stable decomposition formula with accuracy of order more than four (as well as on the real number field there does not exist an automatically stable decomposition formula with accuracy of order more than two). In addition, it is not excluded that, to obtain the higher order accuracy, it will be necessary to use as split parameters, for example, quaternions instead of complex numbers,

In our opinion, these questions are very interesting and difficult, and we work in this direction, but we have not yet obtain actual results.

§7. The fourth order accuracy rational splitting

1. Construction of rational splitting algorithm

Let us consider (1.1) evolution problem. Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are compactly defined, closed, linear operators in X.

In the previous paragraph there is constructed the following decomposition formula with the local precision of fifth order:

$$\begin{aligned} T(\tau) &= \frac{1}{2}\left[T_1(\tau) + T_2(\tau)\right], \qquad (7.1)\\ T_1(\tau) &= U\left(\tau, \frac{\alpha}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U(\tau, \overline{\alpha}A_1) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right),\\ T_2(\tau) &= U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) U(\tau, \overline{\alpha}A_2) U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right). \end{aligned}$$

where $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$.

In the above-mentioned work it is shown that:

$$U(\tau, A) - T(\tau) = O_p\left(\tau^5\right),$$

where $O_p\left(\tau^5\right)$ is the operator, norm of which is of the fifth order with respect to τ (more precisely, in the case of the unbounded operator $\|O_p\left(\tau^5\right)\varphi\| = O\left(\tau^5\right)$ for any φ from the definition domain of $O_p\left(\tau^5\right)$). In the present work (see Section 2) we construct the semigroup approximations with the local precision of the fifth order using the following rational approximation:

$$W(\tau, A) = \left(I - \frac{\alpha}{2}\tau A\right)\left(I + \frac{\overline{\alpha}}{2}\tau A\right)^{-1}\left(I - \frac{\overline{\alpha}}{2}\tau A\right)\left(I + \frac{\alpha}{2}\tau A\right)^{-1}. \qquad (7.2)$$

The approximation defined by formula (7.2) in the scalar case represent the Pade approximations for exponential functions (see [5]).

On the basis of formulas (7.1) and (7.2) we can construct the following decomposition formula:

$$\begin{aligned} V(\tau) &= \frac{1}{2}\left[V_1(\tau) + V_2(\tau)\right], \qquad (7.3)\\ V_1(\tau) &= W\left(\tau, \frac{\alpha}{2}A_1\right) W\left(\tau, \frac{1}{2}A_2\right) W(\tau, \overline{\alpha}A_1) W\left(\tau, \frac{1}{2}A_2\right) W\left(\tau, \frac{\alpha}{2}A_1\right),\\ V_2(\tau) &= W\left(\tau, \frac{\alpha}{2}A_2\right) W\left(\tau, \frac{1}{2}A_1\right) W(\tau, \overline{\alpha}A_2) W\left(\tau, \frac{1}{2}A_1\right) W\left(\tau, \frac{\alpha}{2}A_2\right) \end{aligned}$$

Below we shall show that this formula has the precision of the fifth order:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^5\right).$$

In the present paragraph, on the basis of formula (7.3), a decomposition scheme with the fourth order precision will be constructed for the solution of problem (1.1).

According to formula (7.2), we have:

$$u(t_k) = U(\tau, A)u(t_{k-1}) + \int\limits_{t_{k-1}}^{t_k} U(t_k - s, A)f(s)ds.$$

Let us use Simpson's formula and rewrite this formula in the following form:

$$\begin{aligned} u(t_k) &= U(\tau, A)\, u(t_{k-1}) + \frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right)\right. \\ &\quad \left. + U(\tau, A)\, f(t_{k-1})\right) + R_{5,k}(\tau), \\ u(t_0) &= \varphi, \quad k = 1, 2, \dots\ . \end{aligned} \tag{7.4}$$

For the sufficiently smooth function f the following estimation is true (see. Lemma 2.3):

$$\|R_{k,5}(\tau)\| = O\left(\tau^5\right). \tag{7.5}$$

On the basis of formula (7.4) let us construct the following scheme:

$$\begin{aligned} u_k &= V(\tau)\, u_{k-1} \\ &\quad + \frac{\tau}{6}\left(f(t_k) + 4V\left(\frac{\tau}{2}\right) f\left(t_{k-1/2}\right) + V(\tau)\, f(t_{k-1})\right), \\ u_0 &= \varphi, \quad k = 1, 2, \dots.\ . \end{aligned} \tag{7.6}$$

Let us perform the computation of the scheme (7.5) by the following algorithm:

$$u_k = u_k^{(0)} + \frac{2\tau}{3} u_k^{(1)} + \frac{\tau}{6} f(t_k),$$

where $u_{k,0}$ is calculated by the scheme:

$$\begin{aligned} v_{k-4/5}^{(0)} &= W\left(\tau, \frac{\alpha}{2} A_1\right)\left(u_{k-1} + \frac{\tau}{6} f(t_{k-1})\right) \quad w_{k-4/5}^{(0)} = W\left(\tau, \frac{\alpha}{2} A_2\right) u_{k-1}, \\ v_{k-3/5}^{(0)} &= W\left(\tau, \frac{1}{2} A_2\right) v_{k-4/5}^{(0)}, \quad w_{k-3/5}^{(0)} = W\left(\tau, \frac{1}{2} A_1\right) w_{k-4/5}^{(0)}, \\ v_{k-2/5}^{(0)} &= W(\tau, \overline{\alpha} A_1)\, v_{k-3/5}^{(0)}, \qquad w_{k-2/5}^{(0)} = W(\tau, \overline{\alpha} A_2)\, w_{k-3/5}^{(0)}, \\ v_{k-1/5}^{(0)} &= W\left(\tau, \frac{1}{2} A_2\right) v_{k-2/5}^{(0)}, \qquad w_{k-1/5} = W\left(\tau, \frac{1}{2} A_1\right) w_{k-2/5}^{(0)}, \\ v_k^{(0)} &= W\left(\tau, \frac{\alpha}{2} A_1\right) v_{k-1/5}^{(0)}, \qquad w_k^{(0)} = W\left(\tau, \frac{\alpha}{2} A_2\right) w_{k-1/5}^{(0)}, \\ u_k^{(0)} &= \frac{1}{2}[v_k^{(0)} + w_k^{(0)}], \quad u_0 = \varphi + \frac{\tau}{6} f(0), \end{aligned} \tag{7.7}$$

and $u_k^{(1)}$ - by the scheme:

$$\begin{aligned}
v_{k-4/5}^{(1)} &= W\left(\frac{\tau}{2},\frac{\alpha}{2}A_1\right)f\left(t_{k-1/2}\right), \quad w_{k-4/5}^{(1)} = W\left(\frac{\tau}{2},\frac{\alpha}{2}A_2\right)f\left(t_{k-1/2}\right),\\
v_{k-3/5}^{(1)} &= W\left(\frac{\tau}{2},\frac{1}{2}A_2\right)v_{k-2/3}^{(1)}, \quad w_{k-3/5}^{(1)} = W\left(\frac{\tau}{2},\frac{1}{2}A_1\right)w_{k-4/5}^{(1)},\\
v_{k-2/5}^{(1)} &= W\left(\frac{\tau}{2},\overline{\alpha}A_1\right)v_{k-1/3}^{(1)}, \quad w_{k-2/5}^{(1)} = W\left(\frac{\tau}{2},\overline{\alpha}A_2\right)w_{k-3/5}^{(1)},\\
v_{k-1/5}^{(1)} &= W\left(\frac{\tau}{2},\frac{1}{2}A_2\right)v_{k-1/3}^{(1)}, \quad w_{k-1/5}^{(1)} = W\left(\frac{\tau}{2},\frac{1}{2}A_1\right)w_{k-2/5}^{(1)},\\
v_k^{(1)} &= W\left(\frac{\tau}{2},\frac{\alpha}{2}A_1\right)v_{k-1/3}^{(1)}, \quad w_k^{(1)} = W\left(\frac{\tau}{2},\frac{\alpha}{2}A_2\right)w_{k-1/5}^{(1)},\\
u_k^{(1)} &= \frac{1}{2}[v_k^{(1)} + w_k^{(1)}]. \qquad (7.8)
\end{aligned}$$

2. Theorem on error estimation

The following theorem takes place.

Theorem 7.1. *Let the following conditions be satisfied:*

(a) There exists such $\tau_0 > 0$ that for any $0 < \tau \leq \tau_0$ there exist operators $(I + \tau\lambda\gamma A_j)^{-1}$, $j = 1, 2,\ \gamma = 1, \alpha, \overline{\alpha},\ \lambda = \alpha, \overline{\alpha}$ and they are bounded. Besides, the following inequalities are true:

$$\|W(\tau, \gamma A_j)\| \leq e^{\omega\tau}, \quad \omega = const > 0;$$

(b) The operator $(-A)$ generates the strongly continuous semigroup $U(t, A) = \exp(-tA)$, for which the following inequality is true:

$$\|U(t, A)\| \leq Me^{\omega t}, \quad M, \omega = const > 0;$$

(c) $U(s, A)\varphi \in D(A^5)$ for any $s \geq 0$;

(d) $f(t) \in C^4([0,\infty); X)$; $f(t) \in D(A^4)$, $f'(t) \in D(A^3)$, $f''(t) \in D(A^2)$, $f'''(t) \in D(A)$ and $U(s, A)f(t) \in D(A^4)$ for any fixed t and s $(t, s \geq 0)$.

Then the following estimation holds:

$$\begin{aligned}
\|u(t_k) - u_k\| \leq\ & ce^{\omega_0 t_k}t_k\tau^4\Bigg(\sup_{s\in[0,t_k]}\|U(s,A)\varphi\|_{A^5}\\
&+t_k\sup_{s,t\in[0,t_k]}\|U(s,A)f(t)\|_{A^5} + \sup_{t\in[0,t_k]}\|f(t)\|_{A^4}\\
&+\sup_{t\in[0,t_k]}\|f'(t)\|_{A^3} + \sup_{t\in[0,t_k]}\|f''(t)\|_{A^2}\\
&+\sup_{t\in[0,t_k]}\|f'''(t)\|_{A} + \sup_{t\in[0,t_k]}\left\|f^{(IV)}(t)\right\|\Bigg), \qquad (7.9)
\end{aligned}$$

where c and ω_0 are positive constants.

Let us prove the auxiliary lemmas on which the proof of the Theorem 7.1 is based.

Lemma 7.2. *If the condition (a) of the Theorem 7.1 is satisfied, then for the operator* $W(t,A)$ *the following decomposition is true:*

$$W(t,A)=\sum_{i=0}^{k-1}(-1)^i\frac{t^i}{i!}A^i+R_{W,k}(t,A),\quad k=1,...,5, \tag{7.10}$$

where, for the residual member, the following estimation holds:

$$\begin{aligned}\|R_{W,k}(t,A)\varphi\| &\le c_0e^{\omega_0 t}t^k\left\|A^k\varphi\right\|,\quad \varphi\in D\left(A^k\right),\\ c_0,\ \omega_0 &= const>0.\end{aligned} \tag{7.11}$$

proof. We obviously have:

$$\begin{aligned}(I+\gamma A)^{-1} &= I-I+(I+\gamma A)^{-1}=I-(I+\gamma A)^{-1}(I+\gamma A-I)\\ &= I-\gamma A(I+A)^{-1}.\end{aligned}$$

From this for any natural k we can get the following expansion:

$$(I+\gamma A)^{-1}=\sum_{i=0}^{k-1}(-1)^i\gamma^iA^i+\gamma^kA^k(I+\gamma A)^{-1}. \tag{7.12}$$

Let us rewrite $W(\tau,A)$ in the following form:

$$W(\tau,A)=S(\tau,A)-\frac{1}{2}\tau AS(\tau,A)+\frac{1}{12}\tau^2A^2S(\tau,A)$$

where

$$S(\tau,A)=\left(I+\frac{\overline{\alpha}}{2}\tau A\right)^{-1}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}.$$

Let us decompose $S(\tau,A)$ by means of the formula (7.12), we obtain the following recurrent relation:

$$S(\tau,A)=I-\frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}}{2}\tau AS(\tau,A). \tag{7.13}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (7.13) up to the first order, we obtain:

$$W(\tau,A)=I-R_{W,1}(\tau,A), \tag{7.14}$$

where

$$\begin{aligned}R_{W,1}(\tau,A) &= \tau A\left(\frac{\alpha}{2}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}+1}{2}S(\tau,A)\right)\\ &\quad +\frac{1}{12}\tau^2A^2S(\tau,A).\end{aligned}$$

Since $(I+\lambda\tau A)^{-1}$ is bounded according to the condition *(a)* of the Theorem 7.1, therefore:

$$\|R_{W,1}(\tau,A)\varphi\| \le c_0 e^{\omega_0\tau}\tau\,\|A\varphi\|, \quad \varphi\in D(A). \tag{7.15}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (7.13) up to the second order:

$$\begin{aligned} W(\tau,A) &= I-\tau A\left(\frac{\alpha}{2}I-\frac{\alpha^2}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{1+\overline{\alpha}}{2}I\right.\\ &\quad \left.-\frac{\alpha+\alpha\overline{\alpha}}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}+\overline{\alpha}^2}{4}\tau AS(\tau,A)\right)\\ &\quad +\frac{1}{12}\tau^2A^2S(\tau,A)\\ &= I-\tau A+R_{W,2}(\tau,A) \end{aligned}$$

where

$$\begin{aligned} R_{W,2}(\tau,A) &= \frac{\alpha^2+\alpha+\alpha\overline{\alpha}}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\\ &\quad +\frac{3\overline{\alpha}+3\overline{\alpha}^2+1}{12}S(\tau,A)\\ &= \frac{\alpha-\frac{1}{3}+\alpha+\frac{1}{3}}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\\ &\quad +\frac{3\overline{\alpha}+3\overline{\alpha}-1+1}{12}S(\tau,A)\\ &= \tau^2A^2\left(\frac{\alpha}{2}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{\overline{\alpha}}{2}S(\tau,A)\right). \end{aligned}$$

According to the condition *(a)* of the Theorem 7.1 we have:

$$\|R_{W,2}(\tau,A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^2\left\|A^2\varphi\right\|, \quad \varphi\in D\left(A^2\right). \tag{7.16}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (7.13) up to the third order:

$$\begin{aligned} W(\tau,A) &= I-\tau A+\tau^2A^2\left(\frac{\alpha}{2}I-\frac{\alpha^2}{4}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\right.\\ &\quad \left.+\frac{\overline{\alpha}}{2}\left(I-\frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}-\frac{\overline{\alpha}}{2}\tau AS(\tau,A)\right)\right)\\ &= I-\tau A+\frac{1}{2}\tau^2A^2+R_{W,3}(\tau,A), \end{aligned} \tag{7.17}$$

where

$$\begin{aligned} R_{W,3}(\tau,A) &= -\tau^3A^3\left(\frac{1+3\alpha^2}{12}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{\overline{\alpha}^2}{4}R(\tau,A)\right)\\ &= -\tau^3A^3\left(\frac{\alpha}{4}\left(I+\frac{\alpha}{2}\tau A\right)^{-1}+\frac{\overline{\alpha}^2}{4}R(\tau,A)\right). \end{aligned}$$

According to the condition *(a)* of the Theorem 7.1 we have:

$$\|R_{W,3}(\tau,A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^3 \left\|A^3\varphi\right\|, \quad \varphi \in D\left(A^3\right). \tag{7.18}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (7.13) up to the fourth order:

$$\begin{aligned} W(\tau,A) &= I - \tau A + \frac{1}{2}\tau^2A^2 - \tau^3A^3\left(\frac{\alpha}{4}I - \frac{\alpha^2}{8}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\right. \\ &\left. +\frac{\overline{\alpha}^2}{4}\left(I - \frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1} - \frac{\overline{\alpha}}{2}\tau A S(\tau,A)\right)\right) \\ &= I - \tau A + \frac{1}{2}\tau^2A^2 - \frac{1}{6}\tau^3A^3 + R_{W,4}(\tau,A), \end{aligned} \tag{7.19}$$

where

$$\begin{aligned} R_{W,4}(\tau,A) &= \tau^4A^4\left(\frac{\alpha^2+\alpha\overline{\alpha}^2}{8}\left(I+\frac{\alpha}{2}\tau A\right)^{-1} + \frac{\overline{\alpha}^3}{8}S(\tau,A)\right) \\ &= \tau^4A^4\left(\frac{\alpha}{12}\left(I+\frac{\alpha}{2}\tau A\right)^{-1} + \frac{\overline{\alpha}^3}{8}S(\tau,A)\right) \end{aligned}$$

According to the condition *(a)* of the Theorem 7.1 we have:

$$\|R_{W,4}(\tau,A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^4 \left\|A^4\varphi\right\|, \quad \varphi \in D\left(A^4\right). \tag{7.20}$$

Let us decompose the rational approximation $W(\tau,A)$ according to the formula (7.13) up to the fifth order:

$$\begin{aligned} W(\tau,A) &= I - \tau A + \frac{1}{2}\tau^2A^2 - \frac{1}{6}\tau^3A^3 \\ &-\tau^4A^4\left(\frac{\alpha}{12} - \frac{\alpha^2}{24}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1}\right. \\ &\left. +\frac{\overline{\alpha}^3}{8}\left(I - \frac{\alpha}{2}\tau A\left(I+\frac{\alpha}{2}\tau A\right)^{-1} - \frac{\overline{\alpha}}{2}\tau A S(\tau,A)\right)\right) \\ &= I - \tau A + \frac{1}{2}\tau^2A^2 - \frac{1}{6}\tau^3A^3 \\ &-\frac{1}{24}\tau^4A^4 + R_{W,5}(\tau,A), \end{aligned} \tag{7.21}$$

where

$$\begin{aligned} R_{W,5}(\tau,A) &= \tau^5A^5\left(\frac{2\alpha^2+3\overline{\alpha}^3\alpha}{48}\left(I+\frac{\alpha}{2}\tau A\right)^{-1} + \frac{\overline{\alpha}^4}{16}S(\tau,A)\right) \\ &= \tau^5A^5\left(\frac{\alpha}{24}\left(I+\frac{\alpha}{2}\tau A\right)^{-1} + \frac{\overline{\alpha}^4}{16}S(\tau,A)\right) \end{aligned}$$

According to the condition *(a)* of the Theorem 7.1 we have:

$$\|R_{W,5}(\tau, A)\varphi\| \le c_0 e^{\omega_0\tau}\tau^5 \left\|A^5\varphi\right\|, \quad \varphi \in D\left(A^5\right). \qquad \square \tag{7.22}$$

Lemma 7.3. *If the conditions (a), (b) and (c) of the Theorem 7.1 are satisfied, then the following estimation holds:*

$$\left\|\left[U^k(\tau, A) - V^k(\tau)\right]\varphi\right\| \le c e^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^5}, \tag{7.23}$$

where c *and* ω_0 *are positive constants.*

Proof.

Let us decompose $W(\tau, A)$ operators in the expression of $V(\tau)$ according to the formula (7.10) from right to left, so that each residual member be of the fifth order. We shall have:

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_{V,5}(\tau), \tag{7.24}$$

where for the residual member according to the condition *(a)* of the Theorem 7.1 we have the following estimation:

$$\|R_{V,5}(\tau)\varphi\| \le c e^{\omega_0\tau}\tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \tag{7.25}$$

From the (1.7) and (7.24) it follows:

$$U(\tau, A) - V(\tau) = R_5(\tau, A) - R_{V,5}(\tau).$$

From here according to (1.8) and (7.25) we obtain the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le c e^{\omega_0\tau}\tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \tag{7.26}$$

The following representation is obvious:

$$\left[U^k(\tau, A) - V^k(\tau)\right]\varphi = \sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau, A) - V(\tau)\right]U^{i-1}(\tau, A)\varphi.$$

Hence, according to the conditions *(a), (b), (c)* of the Theorem 7.1 and inequality (7.26), we have the sought estimation. $\square$

Lemma 7.4. *Let the following conditions be satisfied:*

(a) The operator A *satisfies the conditions of the Theorem 7.1;*

(b) $f(t) \in C^4([0,\infty); X)$, $f(t) \in D\left(A^4\right)$ *and* $f^{(k)}(t) \in D\left(A^{4-k}\right)$ $(k = 1, 2, 3)$ *for every fixed* $t \ge 0$.

Then the following estimation holds

$$\|R_{5,k}(\tau)\| \le c e^{\omega_0\tau}\tau^5 \sum_{i=0}^{4} \max_{s\in[t_{k-1},t_k]} \left\|f^{(i)}(s)\right\|_{A^{4-i}}, \tag{7.27}$$

where

$$
\begin{aligned}
R_{5,k}(\tau) &= \int_{t_{k-1}}^{t_k} U(t_k - s, A) f(s)\, ds \\
&\quad - \frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) \right. \\
&\quad \left. + U(\tau, A) f(t_{k-1}) \right) \qquad (7.28)
\end{aligned}
$$

and where c and ω_0 are positive constants, and $f^{(0)}(s) = f(s)$.

Proof. By means of changing variables, the integral in the equality (7.28) takes the following form:

$$
\int_{t_{k-1}}^{t_k} U(t_k - s, A) f(s)\, ds = \int_0^{\tau} U(\tau - s, A) f(t_{k-1} + s)\, ds.
$$

If we decompose the function $f(t_{k-1} + s)$ into the Taylor series, and expand the semigroup $U(\tau - s, A)$ according to formula (1.7), we obtain:

$$
U(\tau - s, A) f(t_{k-1} + s) = P_{3,k}(s) + \widetilde{R}_{4,k}(\tau, s), \qquad (7.29)
$$

where

$$
\begin{aligned}
P_{3,k}(s) &= \left(I - (\tau - s) A + \frac{(\tau - s)^2}{2} A^2 - \frac{(\tau - s)^3}{6} A^3 \right) f(t_{k-1}) \\
&\quad + s\left(I - (\tau - s) A + \frac{(\tau - s)^2}{2} A^2 \right) f'(t_{k-1}) \\
&\quad + \frac{s^2}{2} (I - (\tau - s) A) f''(t_{k-1}) + \frac{s^3}{6} f'''(t_{k-1}), \\
\widetilde{R}_{4,k}(\tau, s) &= \frac{1}{6} U(\tau - s, A) \int_0^s (s - \xi)^3 f^{(IV)}(t_{k-1} + \xi)\, d\xi \\
&\quad + R_4(\tau - s, A) f(t_{k-1}) \\
&\quad + (\tau - s) A R_3(\tau - s, A) f'(t_{k-1}) \\
&\quad + \frac{(\tau - s)^2}{2} A^2 R_2(\tau - s, A) f''(t_{k-1}) \\
&\quad + \frac{(\tau - s)^3}{6} A^3 R_1(\tau - s, A) f'''(t_{k-1}).
\end{aligned}
$$

Hence according condition *(b)* and *(d)* of the Theorem 7.1 we obtain the following estimation:

$$
\widetilde{R}_{4,k}(\tau, s) \le c e^{\omega_0 \tau} \tau^4 \sum_{i=0}^{4} \max_{s \in [t_{k-1}, t_k]} \left\| f^{(i)}(s) \right\|_{A^{4-i}}. \qquad (7.30)
$$

From equality (7.28) with account of formula (7.29), we have:

$$\begin{aligned}
R_{5,k}(\tau) &= \int_0^\tau U(\tau - s, A) f(t_{k-1}+s)\, ds \\
&\quad -\frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + U(\tau, A) f(t_{k-1})\right) \\
&= \int_0^\tau P_{3,k}(s)\, ds + \int_0^\tau \widetilde{R}_{4,k}(\tau, s)\, ds \\
&\quad -\frac{\tau}{6}\left(P_{3,k}(\tau) + 4P_{3,k}\left(\frac{\tau}{2}\right) + P_{3,k}(0)\right) \\
&\quad -\frac{\tau}{6}\widetilde{R}_{4,k}(\tau, 0) + 4\widetilde{R}_{4,k}\left(\tau, \frac{\tau}{2}\right) + \widetilde{R}_{4,k}(\tau, \tau), \qquad (7.31)
\end{aligned}$$

Because of Simpson's formula is exact for polynomial of the third order, for $R_{5,k}(\tau)$ we have:

$$R_{5,k}(\tau) = \int_0^\tau \widetilde{R}_{4,k}(\tau, s)\, ds - \frac{\tau}{6}\left(\widetilde{R}_{4,k}(\tau, 0) + 4\widetilde{R}_{4,k}\left(\tau, \frac{\tau}{2}\right) + \widetilde{R}_{4,k}(\tau, \tau)\right).$$

hence according to inequality (7.29), we have:

$$\|R_{5,k}(\tau)\| \le c e^{\omega_0 \tau} \tau^5 \sum_{i=0}^{4} \max_{s \in [t_{k-1}, t_k]} \left\| f^{(i)}(s) \right\|_{A^{4-i}}. \qquad \square \qquad (7.32)$$

Let us return to the proof of the Theorem 7.1.

Let us write formula (7.4) in the following form:

$$u(t_k) = U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A)\left(F_i^{(1)} + R_{5,k}(\tau)\right), \qquad (7.33)$$

where

$$F_k^{(1)} = \frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + U(\tau, A) f(t_{k-1})\right). \qquad (7.34)$$

Analogously let us present u_k as follows:

$$u_k = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau) F_i^{(2)}, \qquad (7.35)$$

where

$$F_i^{(2)} = \frac{\tau}{6}\left(f(t_k) + 4V\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + V(\tau, A) f(t_{k-1})\right). \qquad (7.36)$$

From equalities (7.33) and (7.35) it follows:

$$\begin{aligned} u(t_k) - u_k &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ &\quad + \sum_{i=0}^{k}\left[U^{k-i}(\tau, A)F_i^{(1)} - V^{k-i}(\tau)F_i^{(2)}\right] \\ &\quad + \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{5,k}(\tau) \\ &= \left[U^k(\tau, A) - V^k(\tau)\right]\varphi + \sum_{i=1}^{k}\left[\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right)F_i^{(1)}\right. \\ &\quad \left. + V^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right] + \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{5,k}(\tau). \end{aligned} \tag{7.37}$$

From formulas (7.34) and (7.36) we have:

$$\begin{aligned} F_k^{(1)} - F_k^{(2)} &= \frac{\tau}{6}\left(4\left(U\left(\frac{\tau}{2}, A\right) - V\right) f\left(t_{k-1/2}\right)\right. \\ &\quad \left. + \left(U(\tau, A) - V(\tau, A)\right) f\left(t_{k-1}\right)\right) \end{aligned} \tag{7.38}$$

From here, according to inequality (7.24) and Lemma 7.2 we obtain the following estimation:

$$\left\|F_k^{(1)} - F_k^{(2)}\right\| \le c e^{\omega_0 \tau}\tau^5 \sup_{t\in[t_{k-1}, t_k]} \|f(t)\|_{A^4}. \tag{7.39}$$

According to the Lemma 7.2 we have:

$$\begin{aligned} &\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau, A) - V^{k-i}(\tau)\right)F_i^{(1)}\right\| \\ &\le c e^{\omega_0 t_k} t_k^2 \tau^4 \sup_{s,t\in[0, t_k]} \|U(s, A) f(t)\|_{A^5}. \end{aligned} \tag{7.40}$$

From equality (7.37) according to inequalities (7.39), (7.40), (7.27) and the condition *(b)* of the Theorem 7.1 we obtain sought estimation. □

Remark 7.5. *The operator $V^k(\tau)$ is the solution operator of the above-considered decomposed problem. It is obvious that, according to the condition of the Theorem 7.1 ($\|W(t, \gamma A_j)\| \le e^{\omega t}$), the norm of the operator $V^k(\tau)$ is less than or equal to $e^{\omega_0 t_k}$. From this follows the stability of the above-stated decomposition scheme on each finite time interval.*

Remark 7.6. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint non negative operators, in estimation (7.7) ω_0 will be replaced by 0. Alongside with this, for the transition operator of the split problem, the estimation $\left\|V^k(\tau)\right\| \le 1$ will be true.*

Remark 7.7. *In the case of the Hilbert space, when A_1, A_2 and $A_1 + A_2$ are self-adjoint, positive definite operators, in estimation (7.7) ω_0 will be replaced by $-\alpha_0$, $\alpha_0 > 0$. Alongside with this, for the transition operator of the split problem, the estimation $\left\|V^k(\tau)\right\| \le e^{-\alpha_1 t_k}$, $\alpha_1 > 0$ will be true.*

§8. The fourth order accuracy decomposition scheme for a multi-dimensional evolution problem

1. Differential splitting and error estimation of approximate solution

Let us consider the problem (3.1). Let $A = A_1 + A_2 + ... + A_m$, $m \geq 2$, where A_i $(i = 1, ..., m)$ are closed operators, densely defined in X.

Together with problem (3.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\alpha}{2} A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\alpha}{2} A_i v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dv_k^{(m)}(t)}{dt} + \frac{1}{2} A_m v_k^{(m)}(t) &= 0, \quad v_k^{(m)}(t_{k-1}) = v_k^{(m-1)}(t_k), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_{2m-i} v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= m+1, ..., 2m-2, \\
\frac{dv_k^{(2m-1)}(t)}{dt} + \overline{\alpha} A_1 v_k^{(2m-1)}(t) &= 0, \quad v_k^{(2m-1)}(t_{k-1}) = v_k^{(2m-2)}(t_k), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_{i-2m+2} v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 2m, ..., 3m-3, \\
\frac{dv_k^{(3m-2)}(t)}{dt} + \frac{1}{2} A_m v_k^{(3m-2)}(t) &= 0, \quad v_k^{(3m-2)}(t_{k-1}) = v_k^{(3m-3)}(t_k), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\alpha}{2} A_{4m-i-2} v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 3m-1, ..., 4m-4, \\
\frac{dv_k^{(4m-4)}(t)}{dt} + \frac{\alpha}{2} A_1 v_k^{(4m-4)}(t) &= 0, \quad v_k^{(4m-4)}(t_{k-1}) = v_k^{(4m-3)}(t_k),
\end{aligned}$$

$$
\begin{aligned}
\frac{dw_k^{(1)}(t)}{dt} + \frac{\alpha}{2} A_m w_k^{(1)}(t) &= 0, \quad w_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\alpha}{2} A_{m-i+1} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dw_k^{(m)}(t)}{dt} + \frac{1}{2} A_1 w_k^{(m)}(t) &= 0, \quad w_k^{(m)}(t_{k-1}) = w_k^{(m-1)}(t_k), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_{i-m+1} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= m+1, ..., 2m-2, \\
\frac{dw_k^{(2m-1)}(t)}{dt} + \overline{\alpha} A_m w_k^{(2m-1)}(t) &= 0, \quad w_k^{(2m-1)}(t_{k-1}) = w_k^{(2m-2)}(t_k), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_{3m-i-1} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= 2m, ..., 3m-3, \\
\frac{dw_k^{(3m-2)}(t)}{dt} + \frac{1}{2} A_1 w_k^{(3m-2)}(t) &= 0, \quad w_k^{(3m-2)}(t_{k-1}) = w_k^{(3m-3)}(t_k), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\alpha}{2} A_{i-3m+3} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= 3m-1, ..., 4m-4, \\
\frac{dw_k^{(4m-4)}(t)}{dt} + \frac{\alpha}{2} A_m w_k^{(4m-4)}(t) &= 0, \quad w_k^{(4m-4)}(t_{k-1}) = w_k^{(4m-3)}(t_k),
\end{aligned}
$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Let the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j)$, $j = 1, ..., m$ generate strongly continuous semigroups.

$u_k(t)$, $k = 1, 2, ..,$ is defined on each interval $[t_{k-1}, t_k]$, as follows:

$$
u_k(t) = \frac{1}{2}\left[v_k^{(4m-4)}(t) + w_k^{(4m-4)}(t)\right]. \tag{8.1}
$$

We declare function $u_k(t)$ as an approximated solution of problem (3.1) on each interval $[t_{k-1}, t_k]$.

The following theorem takes place.

Theorem 8.1 *Let the following conditions be fulfilled:*

(a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$;

(b) Let the operators $(-\gamma A_j)$, $\gamma = 1,\ \alpha,\ \overline{\alpha}$ $(j = 1, ..., m, \quad m \geq 2)$ *and* $(-A)$ *generate strongly continuous semigroups, for which the following estimations are true:*

$$
\begin{aligned}
\|U(t, \gamma A_j)\| &\leq e^{\omega t}, \\
\|U(t, A)\| &\leq Me^{\omega t}, \quad M, \omega = const > 0;
\end{aligned}
$$

(c) $U(s,A)\varphi \in D(A^5)$ *for each fixed* $s \geq 0$ *.*

Then the following estimation holds:

$$\|u(t_k) - u_k(t_k)\| \leq ce^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^5},$$

where c *and* ω_0 *are positive constants.*

Proof. From formula (8.1) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{8.2}$$

where

$$V(\tau) = \frac{1}{2}\left[V_1(\tau) + V_2(\tau)\right], \tag{8.3}$$

and

$$\begin{aligned} V_1(\tau) &= U\left(\tau, \frac{\alpha}{2}A_1\right) ... U\left(\tau, \frac{\alpha}{2}A_{m-1}\right) U\left(\tau, \frac{1}{2}A_m\right) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right) ... U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \overline{\alpha}A_1\right) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) ... U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right) U\left(\tau, \frac{1}{2}A_m\right) \\ &U\left(\tau, \frac{\alpha}{2}A_{m-1}\right) ... U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right), \end{aligned} \tag{8.4}$$

$$\begin{aligned} V_2(\tau) &= U\left(\tau, \frac{\alpha}{2}A_m\right) ... U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) ... U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right) U\left(\tau, \overline{\alpha}A_m\right) \\ &\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right) ... U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) \\ &\times U\left(\tau, \frac{\alpha}{2}A_2\right) ... U\left(\tau, \frac{\alpha}{2}A_{m-1}\right) U\left(\tau, \frac{\alpha}{2}A_m\right). \end{aligned} \tag{8.5}$$

Remark 8.2 *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem 8.1. In this case, for the solving operator, the following estimation holds:*

$$\left\|V^k(\tau)\right\| \leq e^{\omega_1 t_k}, \tag{8.6}$$

where ω_1 *is positive constant.*

Let us introduce the following notations for combinations (sum, product) of semigroups: Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1, ..., m)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (1.7), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$,

$(\tau^2 A_i A_j)$, $(-\tau^3 A_i A_j A_k)$ and $(\tau^4 A_i A_j A_k A_l)$ $(i,j,k,l = 1,...,m)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$, $[T(\tau)]_{i,j,k}$ and $[T(\tau)]_{i,j,k,l}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (1.7) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{m} [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^{m} [V(\tau)]_{i,j} A_i A_j \\ & -\tau^3 \sum_{i,j,k=1}^{m} [V(\tau)]_{i,j,k} A_i A_j A_k \\ & +\tau^4 \sum_{i,j,k,l=1}^{m} [V(\tau)]_{i,j,k,l} A_i A_j A_k A_l + \widetilde{R}_5(\tau). \end{aligned} \tag{8.7}$$

According to the first inequality of the condition *(b)* of the Theorem 8.1, for $\widetilde{R}_5(\tau)$, the following estimation holds:

$$\left\| \widetilde{R}_5(\tau)\varphi \right\| \le c e^{\omega_0 \tau} \tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right), \tag{8.8}$$

where c and ω_0 are positive constants. It is obvious that, for the coefficients in formula (8.7), according to formula (8.3), we have:

$$\begin{aligned} [V(\tau)]_i &= \frac{1}{2}\left([V_1(\tau)]_i + [V_2(\tau)]_i\right), & (8.9)\\ i &= 1,...,m, \\ [V(\tau)]_{i,j} &= \frac{1}{2}\left([V_1(\tau)]_{i,j} + [V_2(\tau)]_{i,j}\right), & (8.10)\\ i,j &= 1,...,m, \\ [V(\tau)]_{i,j,k} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k} + [V_2(\tau)]_{i,j,k}\right), & (8.11)\\ i,j,k &= 1,...,m, \\ [V(\tau)]_{i,j,k,l} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k,l} + [V_2(\tau)]_{i,j,k,l}\right), & (8.12)\\ i,j,k,l &= 1,...,m. \end{aligned}$$

Let us state the auxiliary lemma, which will be basis of the proof of the Theorem 8.1.

If conditions (a) and (b) of the Theorem 8.1 are fulfilled and $m = 2$, then the following expansion is true (see 7.24):

$$V(\tau) = I - \tau A + \frac{\tau^2}{2}A^2 - \frac{\tau^3}{6}A^3 + \frac{\tau^4}{24}A^4 + \widetilde{R}_5(\tau), \tag{8.13}$$

where for the remainder term $\widetilde{R}_5(\tau)$, the following estimation takes place:

$$\left\| \widetilde{R}_5(\tau)\varphi \right\| \le c e^{\omega_0 \tau} \tau^5 \sup_{s\in[0,\tau]} \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right).$$

Let us make a remark which will simplify a calculation of coefficients in decomposition (8.7).

Remark 8.3 The operators $V_1(\tau)$ and $V_2(\tau)$ are symmetric in the sense that in their expressions the factors equally remote from the ends coincide with each other. Therefore we have:

$$\begin{aligned}
[V(\tau)]_{i,j} &= [V(\tau)]_{j,i}\,, \quad i,j=1,...,m;\\
[V(\tau)]_{i,j,k} &= [V(\tau)]_{k,j,i}\,, \quad i,j,k=1,...,m;\\
[V(\tau)]_{i,j,k,l} &= [V(\tau)]_{l,k,j,i}\,, \quad i,j,k,l=1,...,m.
\end{aligned}$$

Let us calculate the coefficients $[V(\tau)]_i$ $(i=1,...,m)$ corresponding to the first order members in formula (8.7). It is obvious that the members, corresponding to these coefficients, can be obtained from the decomposition of only those factors (semigroups) of the operators $V_1(\tau)$ and $V_2(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identity operators) will participate.

According to formulas (8.4) and (8.5), for any i have:

$$[V_1(\tau)]_i = [U(\tau, A_i)]_i = 1, \quad [V_2(\tau)]_i = [U(\tau, A_i)]_i = 1.$$

From here, according to formula (8.9), we obtain:

$$[V(\tau)]_i = 1. \tag{8.14}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i,j=1,...,m)$ corresponding to the second order members in formula (8.7). It is obvious that the members, corresponding to these coefficients, can be obtained from the decomposition of only those factors (semigroups) of the operators $V_1(\tau)$ and $V_2(\tau)$, which are generated by the operators $(-\gamma A_i)$ and $(-\gamma A_j)$, and from the decomposition of other semigroups only first addends (the members with identity operators) will participate. Let $i_1 = \min(i,j)$ and $i_2 = \max(i,j)$, then from formula (8.10), with account of (8.4) and (8.5), we obtain:

$$\begin{aligned}
[V(\tau)]_{i,j} = \frac{1}{2}\Bigg(&\bigg[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \frac{1}{2}A_{i_2}\right)U\left(\tau, \overline{\alpha}A_{i_1}\right)\\
&\times U\left(\tau, \frac{1}{2}A_{i_2}\right)U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)\bigg]_{i,j}\\
&+U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \overline{\alpha}A_{i_2}\right)\bigg]\\
&\times U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)\bigg]_{i,j}\Bigg).
\end{aligned}$$

From here, according to (8.13), we obtain:

$$[V(\tau)]_{i,j} = \frac{1}{2}. \tag{8.15}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i,j,k=1,...,m)$ corresponding to the third order members in formula (8.7). For $i=j=k$, according to formulas (8.4) and (8.5), we have:

$$
\begin{aligned}
[V_1(\tau)]_{i,i,i} &= [U(\tau,A_i)]_{i,i,i} = \frac{1}{6},\\
[V_2(\tau)]_{i,i,i} &= [U(\tau,A_i)]_{i,i,i} = \frac{1}{6}.
\end{aligned}
$$

From here, according to formula (8.11), we obtain:

$$
[V(\tau)]_{i,i,i} = \frac{1}{6}. \tag{8.16}
$$

Let us consider the case when only two of the indices i,j and k differ from each other. Let $i_1=\min(i,j,k)$ and $i_2=\max(i,j,k)$, then from formula (8.11), with account of (8.4) and (8.5), we obtain:

$$
\begin{aligned}
[V(\tau)]_{i,j,k} &= \frac{1}{2}\left(U\left(\tau,\frac{\alpha}{2}A_{i_1}\right)U\left(\tau,\frac{1}{2}A_{i_2}\right)U\left(\tau,\overline{\alpha}A_{i_1}\right)\right.\\
&\quad U\left(\tau,\frac{1}{2}A_{i_2}\right)U\left(\tau,\frac{\alpha}{2}A_{i_1}\right)\Big]_{i,j,k}\\
&\quad +\left[U\left(\tau,\frac{\alpha}{2}A_{i_2}\right)U\left(\tau,\frac{1}{2}A_{i_1}\right)U\left(\tau,\overline{\alpha}A_{i_2}\right)\right.\\
&\quad \left.\left.+\,U\left(\tau,\frac{1}{2}A_{i_1}\right)U\left(\tau,\frac{\alpha}{2}A_{i_2}\right)\right]_{i,j,k}\right).
\end{aligned}
$$

From here, according to (8.13), we obtain:

$$
[V(\tau)]_{i,j,k} = \frac{1}{6}, \tag{8.17}
$$

for any indices i,j and k, where only two of them differ from each other.

Let us consider the case when the indices i,j and k differ from each other.

If $i < j < k$ then, according to formula (8.4), the representation is valid:

$$
\begin{aligned}
[V_1(\tau)]_{i,j,k} &= \Bigg[U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{1}{2}A_k\right) \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) U\left(\tau, \overline{\alpha}A_i\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) \\
&\quad \times U\left(\tau, \frac{1}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_i\right)\Bigg]_{i,j,k} \\
&= \left[U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_i \left[U\left(\tau, \frac{\alpha}{2}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_i \left[U\left(\tau, \frac{\alpha}{2}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_i \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_i \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \overline{\alpha}A_i\right)\right]_i \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&= \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2} + \overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2} \\
&= \frac{\alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2}{4} = \frac{1}{6}. \qquad (8.18)
\end{aligned}
$$

Here we used the identities: $\alpha^2 + \overline{\alpha}^2 = \frac{1}{3}$, $\alpha\overline{\alpha} = \frac{1}{3}$. Analogously from (8.20) we obtain:

$$[V_2(\tau)]_{i,j,k} = \frac{1}{6}, \quad i < j < k. \qquad (8.19)$$

From formula (8.11), with account of formulas (8.18) and (8.19), we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i < j < k. \qquad (8.20)$$

From here, due to **Remark 8.3**, we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad k < j < i. \qquad (8.21)$$

Now consider the case $j < i < k$. Due to formula (8.4), the representation is valid:

$$
\begin{aligned}
[V_1(\tau)]_{i,j,k} &= \left[U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{1}{2}A_k\right) \right. \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right) U\left(\tau, \overline{\alpha}A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right) \\
&\quad \left. \times U\left(\tau, \frac{1}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right) \right]_{i,j,k} \\
&= \left[U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&= \frac{\alpha}{2}\overline{\alpha}\frac{1}{2} + \frac{\overline{\alpha}}{2}\overline{\alpha}\frac{1}{2} = \frac{\alpha\overline{\alpha} + \overline{\alpha}^2}{4} = \frac{\overline{\alpha}}{4}.
\end{aligned}
\tag{8.22}
$$

Analogously, from (8.20) we obtain:

$$
[V_2(\tau)]_{i,j,k} = \frac{\alpha\overline{\alpha} + \alpha}{4}, \quad j < i < k. \tag{8.23}
$$

From formula (8.11), with account of formulas (8.22) and (8.23), we obtain:

$$
[V(\tau)]_{i,j,k} = \frac{\overline{\alpha} + \alpha\overline{\alpha} + \alpha}{8} = \frac{1}{6}, \quad j < i < k. \tag{8.24}
$$

From here, due to **Remark 8.2**, we obtain:

$$
[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad k < i < j. \tag{8.25}
$$

Now consider the case $j < k < i$. According to formula (8.4), the representation is valid:

$$
\begin{aligned}
[V_1(\tau)]_{i,j,k} &= \left[U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_k\right) U\left(\tau, \frac{1}{2}A_i\right) \right. \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_k\right) U\left(\tau, \overline{\alpha}A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_k\right) \\
&\quad \left. \times U\left(\tau, \frac{1}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_j\right) \right]_{i,j,k} \\
&= \left[U\left(\tau, \frac{1}{2}A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \frac{1}{2}A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j \left[U\left(\tau, \frac{\alpha}{2}A_k\right)\right]_k \\
&= \frac{1}{2}\overline{\alpha}\frac{\overline{\alpha}}{2} + \frac{1}{2}\overline{\alpha}\frac{\alpha}{2} = \frac{\overline{\alpha}^2 + \alpha\overline{\alpha}}{4} = \frac{\overline{\alpha}}{4}.
\end{aligned}
\tag{8.26}
$$

Analogously, from (8.20) we obtain:

$$[V_2(\tau)]_{i,j,k} = \frac{\alpha\overline{\alpha} + \alpha}{4}, \quad j < k < i. \tag{8.27}$$

From formula (8.11), with account of formulas (8.26) and (8.27), we obtain:

$$[V(\tau)]_{i,j,k} = \frac{\overline{\alpha} + \alpha\overline{\alpha} + \alpha}{8} = \frac{1}{6}, \quad j < k < i. \tag{8.28}$$

From here, due to **Remark 8.3**, we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i < k < j. \tag{8.29}$$

Uniting formulas (8.16),(8.17),(8.20),(8.21),(8.24),(8.25),(8.28) and (8.29), we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i, j, k = 1, ..., m. \tag{8.30}$$

Let us calculate the coefficients $[V(\tau)]_{i,j,k,l}$ $(i, j, k, l = 1, ..., m)$ corresponding to the fourth order members in formula (8.7). In the case when $i = j = k = l$, due to formulas (8.4) and (8.5), we obtain:

$$\begin{aligned}
[V_1(\tau)]_{i,i,i,i} &= [U(\tau, A_i)]_{i,i,i,i} = \frac{1}{24}, \\
[V_2(\tau)]_{i,i,i,i} &= [U(\tau, A_i)]_{i,i,i,i} = \frac{1}{24}.
\end{aligned}$$

From here, according to formula (8.12), we obtain:

$$[V(\tau)]_{i,i,i,i} = \frac{1}{24}. \tag{8.31}$$

Let us consider the case when only two of the indices i, j, k and l differ from each other. Let $i_1 = \min(i, j, k, l)$ and $i_2 = \max(i, j, k, l)$, then from formula (8.12), with account of (8.4) and (8.5), the representation is valid:

$$\begin{aligned}
[V(\tau)]_{i,j,k,l} = \frac{1}{2} \Bigg(&\left[U\left(\tau, \frac{\alpha}{2} A_{i_1}\right) U\left(\tau, \frac{1}{2} A_{i_2}\right) U(\tau, \overline{\alpha} A_{i_1}) \right. \\
&\left. U\left(\tau, \frac{1}{2} A_{i_2}\right) U\left(\tau, \frac{\alpha}{2} A_{i_1}\right) \right]_{i,j,k,l} \\
&+ \left[U\left(\tau, \frac{\alpha}{2} A_{i_2}\right) U\left(\tau, \frac{1}{2} A_{i_1}\right) U(\tau, \overline{\alpha} A_{i_2}) \right. \\
&\left. + U\left(\tau, \frac{1}{2} A_{i_1}\right) U\left(\tau, \frac{\alpha}{2} A_{i_2}\right) \right]_{i,j,k,l} \Bigg).
\end{aligned}$$

From here, due to (8.13), we obtain:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \tag{8.32}$$

for any indices i, j, k and l, where only two of them differ from each other.

Let us consider the case when only two of the indices i, j, k and l coincide with each other. On the whole, we have six cases, namely:

Case 1. $(i, j, k, l) = (i, j, k, i)$,
Case 2. $(i, j, k, l) = (i, j, i, k)$,
Case 3. $(i, j, k, l) = (i, i, j, k)$,
Case 4. $(i, j, k, l) = (i, j, k, j)$,
Case 5. $(i, j, k, l) = (i, j, j, k)$,
Case 6. $(i, j, k, l) = (i, j, k, k)$.

Comparing i, j and k indices we get six different subcases for each case. Let us consider **Case 1** and calculate its corresponding coefficients. The coefficients, corresponding to five other cases, can be calculated analogously.

Let us consider the subcases of **Case 1**:

Subcase 1.1. $i < j < k$. Due to formula (8.4) we have:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k,i} &= \left[U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{1}{2}A_k\right)\right. \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) U\left(\tau, \overline{\alpha}A_i\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) \\
&\quad \left.\times U\left(\tau, \frac{1}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_{i,j,k,i} \\
&= \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}\overline{\alpha} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}\frac{\alpha}{2} \\
&\quad + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}\frac{\alpha}{2} + \overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2}\frac{\alpha}{2} \\
&= \frac{\alpha^3 + 2\alpha^2\overline{\alpha} + \overline{\alpha}^2\alpha}{8} = \frac{\alpha^2\left(\alpha + \overline{\alpha}\right) + \overline{\alpha}\alpha\left(\alpha + \overline{\alpha}\right)}{8} \\
&= \frac{\alpha^2 + \overline{\alpha}\alpha}{8}. \qquad (8.33)
\end{aligned}$$

Analogously we obtain

$$[V_2(\tau)]_{i,j,k,i} = \frac{\overline{\alpha}^2}{8}. \qquad (8.34)$$

From formula (8.12), with account of (8.33) and (8.34), we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{\alpha^2 + \overline{\alpha}\alpha + \overline{\alpha}^2}{16} = \frac{1}{24}, \quad i < j < k. \qquad (8.35)$$

Subcase 1.2. $k < j < i$. From formula (8.35), due to **Remark 8.2**, we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \quad k < j < i. \qquad (8.36)$$

Subcase 1.3. $j < k < i$. According to formula (8.4), we have:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k,i} &= \left[U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_k\right) U\left(\tau, \frac{1}{2}A_i\right)\right. \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_k\right) U\left(\tau, \overline{\alpha}A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_k\right) \\
&\quad \left.\times U\left(\tau, \frac{1}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_j\right)\right]_{i,j,k,i} \\
&= \frac{1}{2}\overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2} = \frac{\overline{\alpha}^2}{8}. \qquad (8.37)
\end{aligned}$$

Analogously we obtain:

$$[V_2(\tau)]_{i,j,k,i} = \frac{\overline{\alpha}\alpha + \alpha^2}{8}. \qquad (8.38)$$

From formula (8.12), with account of (8.37) and (8.38), we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{\overline{\alpha}^2 + \overline{\alpha}\alpha + \alpha^2}{16} = \frac{1}{24}, \quad j < k < i. \qquad (8.39)$$

Subcase 1.4. $i < k < j$. From formula (8.39), due to **Remark 8.2**, we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \quad i < k < j. \qquad (8.40)$$

Subcase 1.5. $j < i < k$. According to formula (8.4), we have:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k,i} &= \left[U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{1}{2}A_k\right)\right. \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right) U\left(\tau, \overline{\alpha}A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right) \\
&\quad \left.\times U\left(\tau, \frac{1}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right)\right]_{i,j,k,i} \\
&= \frac{\alpha}{2}\overline{\alpha}\frac{1}{2}\frac{\alpha}{2} + \frac{\overline{\alpha}}{2}\overline{\alpha}\frac{1}{2}\frac{\alpha}{2} = \frac{\alpha^2\overline{\alpha} + \overline{\alpha}^2\alpha}{8} \\
&= \frac{\alpha\overline{\alpha}\left(\alpha + \overline{\alpha}\right)}{8} = \frac{1}{24}. \qquad (8.41)
\end{aligned}$$

Analogously, from (8.20), we obtain:

$$[V_2(\tau)]_{i,j,k,i} = \frac{1}{24}. \qquad (8.42)$$

From formula (8.12), with account of (8.41) and (8.42), we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \quad j < i < k. \qquad (8.43)$$

Subcase 1.6. $k < i < j$. From formula (8.43), due to **Remark 8.2**, we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \quad k < i < j. \tag{8.44}$$

Uniting formulas (8.35),(8.36),(8.39),(8.40),(8.43) and (8.44), we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \tag{8.45}$$

for any indices i, j and k different from each other. Analogously, for other five cases, we obtain:

$$\begin{aligned} [V(\tau)]_{i,j,i,k} &= [V(\tau)]_{i,i,j,k} = [V(\tau)]_{i,j,k,j} \\ &= [V(\tau)]_{i,j,j,k} = [V(\tau)]_{i,j,k,k} = \frac{1}{24}, \end{aligned} \tag{8.46}$$

for any indices i, j and k different from each other.

Uniting formulas (8.45) and (8.46), we obtain:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \tag{8.47}$$

for any indices i, j, k and l, where only two of them coincide with each other.

Now let us consider the case when the indices i, j, k and l are different. It is obvious that comparing i, j, k and l indices we get twenty four different cases. Let us consider one of them and calculate its corresponding coefficients (the coefficients corresponding to other cases can be calculated analogously).

Let $i < j < k < l$, then according to formula (8.4), we obtain:

$$\begin{aligned} [V_1(\tau)]_{i,j,k,l} &= \Big[U\left(\tau, \frac{\alpha}{2} A_i\right) U\left(\tau, \frac{\alpha}{2} A_j\right) U\left(\tau, \frac{\alpha}{2} A_k\right) \\ &\quad \times U\left(\tau, \frac{1}{2} A_l\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_j\right) \\ &\quad \times U\left(\tau, \overline{\alpha} A_i\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right) \\ &\quad \times U\left(\tau, \frac{1}{2} A_l\right) U\left(\tau, \frac{\alpha}{2} A_k\right) \\ &\quad \times U\left(\tau, \frac{\alpha}{2} A_j\right) U\left(\tau, \frac{\alpha}{2} A_i\right) \Big]_{i,j,k,l} \\ &= \frac{\alpha}{2}\frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2} \\ &\quad + \frac{\alpha}{2}\frac{\alpha}{2}\frac{\overline{\alpha}}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{\overline{\alpha}}{2}\frac{1}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{\overline{\alpha}}{2} + \overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2} \\ &= \frac{\overline{\alpha}^2\alpha + \overline{\alpha}\alpha^2 + \alpha^3 + \overline{\alpha}^3}{8} \frac{\overline{\alpha}}{2} \\ &= \frac{\overline{\alpha}^2 + \alpha^2}{8} = \frac{1}{24}. \end{aligned} \tag{8.48}$$

Analogously, from (8.20), we obtain:

$$[V_2(\tau)]_{i,j,k,l} = \frac{1}{24}, \quad i < j < k < l. \tag{8.49}$$

From formula (8.12), with account of formulas (8.48) and (8.49), we obtain:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \quad i < j < k < l.$$

Analogously we can show that this equality is valid for other twenty three cases. Therefore we have:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \tag{8.50}$$

for any indices i, j, k and l, which differ from each other.

Uniting formulas (8.31), (8.32), (8.47) and (8.50), we obtain:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \quad i, j, k, l = 1, ..., m. \tag{8.51}$$

From equality (8.7), with account of formulas (8.14), (8.15), (8.30) and (8.51), we obtain:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \sum_{i,j=1}^{m} A_i A_j - \frac{1}{6}\tau^3 \sum_{i,j,k=1}^{m} A_i A_j A_k \\ &\quad + \frac{1}{24}\tau^4 \sum_{i,j,k,l=1}^{m} A_i A_j A_k A_l + \widetilde{R}_5(\tau) \\ &= I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \left(\sum_{i=1}^{m} A_i \right)^2 \\ &\quad - \frac{1}{6}\tau^3 \left(\sum_{i=1}^{m} A_i \right)^3 + \frac{1}{24}\tau^4 \left(\sum_{i=1}^{m} A_i \right)^4 + \widetilde{R}_5(\tau) \\ &= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + \widetilde{R}_5(\tau). \end{aligned} \tag{8.52}$$

According to formula (1.7), we have:

$$U(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_5(\tau, A). \tag{8.53}$$

According to the second inequality of condition *(b)* of the Theorem 3.1 for $R_5(\tau, A)$, the following estimation is valid:

$$\begin{aligned} \|R_5(\tau, A)\varphi\| &\leq ce^{\omega\tau}\tau^5 \left\|A^5\varphi\right\| \\ &\leq ce^{\omega\tau}\tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \end{aligned} \tag{8.54}$$

According to equalities (8.52) and (8.53), we have:

$$U(\tau, A) - V(\tau) = R_5(\tau, A) - \widetilde{R}_5(\tau).$$

From here, with account of inequalities (8.8) and (8.54), the following estimation can be obtained:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le ce^{\omega_2 \tau}\tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \tag{8.55}$$

From equalities (3.2) and (8.2), with account of inequalities (8.6) and (8.55), we obtain:

$$\begin{aligned}
\|u(t_k) - u_k(t_k)\| &= \left\|\left[U(t_k, A) - V^k(\tau)\right]\varphi\right\| \\
&= \left\|\left[U^k(\tau, A) - V^k(\tau)\right]\varphi\right\| \\
&= \left\|\sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau, A) - V(\tau)\right]\, U((i-1)\tau, A)\varphi\right\| \\
&\le \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \\
&\quad \times \|[U(\tau, A) - V(\tau)]\, U((i-1)\tau, A)\varphi\| \\
&\le \sum_{i=1}^{k} e^{\omega_1 (k-i)\tau} ce^{\omega_2 \tau}\tau^5 \|U((i-1)\tau, A)\varphi\|_{A^5} \\
&\le ce^{\omega_0 t_k}\tau^5 \sum_{i=1}^{k} \|U((i-1)\tau, A)\varphi\|_{A^5} \\
&\le kce^{\omega_0 t_k}\tau^5 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^5} \\
&\le ce^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^5}. \qquad \square
\end{aligned}$$

2. Relation between two-dimensional and multi-dimensional decomposition formulas

In this section we propose a method by means of which in our opinion it is available on the basis of two-dimensional decomposition formula to construct a multi-dimensional decomposition formula with the same precision order. Let the two-dimensional decomposition formula has the following form:

$$V^{(2)}(\tau; A_1, A_2) = \sum_{i=1}^{q} \sigma_i \prod_{j=1}^{m_i} U\left(\tau, \alpha_j^{(i)} A_1\right) U\left(\tau, \beta_j^{(i)} A_2\right), \tag{8.56}$$

where parameters σ_i, $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ satisfy the following conditions (weights σ_i

are real numbers, and $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are generally complex numbers):

$$\sum_{i=1}^{q}\sigma_i = 1, \tag{8.57}$$

$$\sum_{i=1}^{q}\sigma_i\sum_{j=1}^{m_i}\alpha_j^{(i)} = \sum_{i=1}^{q}\sigma_i\sum_{j=1}^{m_i}\beta_j^{(i)} = 1. \tag{8.58}$$

In the formula (8.56) we mean, that $U(\tau,\gamma A_l) = I \ \ (l = 1,2)$, when $\gamma = 0$.

For the given method it is necessary that the parameters $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ additionally satisfy the following conditions:

$$\sum_{j=1}^{m_i}\alpha_j^{(i)} = \sum_{j=1}^{m_i}\beta_j^{(i)}, \quad i = 1,...,q. \tag{8.59}$$

At the first step of the method the formula (8.56) is written in such a form that one can clearly see its generalization for the multi-dimensional case. For this reason the formula (8.56) is written in the following form:

$$\begin{aligned} V^{(2)}(\tau;A_1,A_2) &= \sum_{i=1}^{q}\sigma_i\prod_{j=1}^{m_i}U\left(\tau,\mu_{1,j}^{(i)}A_1\right)U\left(\tau,\mu_{1,j}^{(i)}A_2\right) \\ &\quad\times U\left(\tau,\mu_{2,j}^{(i)}A_2\right)U\left(\tau,\mu_{2,j}^{(i)}A_1\right) \\ &= \sum_{i=1}^{q}\sigma_i\prod_{j=1}^{m_i}\left(\prod_{l=1}^{2}U\left(\tau,\mu_{1,j}^{(i)}A_l\right)\right) \\ &\quad\times\left(\prod_{l=1}^{2}U\left(\tau,\mu_{2,j}^{(i)}A_{3-l}\right)\right). \end{aligned} \tag{8.60}$$

where

$$\begin{aligned} \mu_{1,j}^{(i)} &= \beta_j^{(i)} + \sum_{k=1}^{j}\left(\alpha_k^{(i)} - \beta_k^{(i)}\right), \\ \mu_{2,j}^{(i)} &= \sum_{k=1}^{j}\left(\beta_k^{(i)} - \alpha_k^{(i)}\right). \end{aligned}$$

For the formula (8.60) to be the equivalent to the formula (8.56), it is necessary to fulfill the following equalities:

$$\begin{aligned} \mu_{1,j}^{(i)} + \mu_{2,j}^{(i)} &= \beta_j^{(i)}, \\ \mu_{2,j}^{(i)} + \mu_{1,j+1}^{(i)} &= \alpha_{j+1}^{(i)}, \\ \mu_{1,m_i}^{(i)} &= \beta_{m_i}^{(i)}, \\ \mu_{2,m_i}^{(i)} &= 0. \end{aligned}$$

It is easy to check that these equalities are fulfilled if the equalities (8.59) are fulfilled.

Let us construct the following decomposition formula on the basis of the formula (8.60):

$$
\begin{aligned}
V^{(m)}(\tau; A_1, ..., A_m) &= \sum_{i=1}^{q} \sigma_i \prod_{j=1}^{m_i} \left(\prod_{l=1}^{m} U\left(\tau, \mu_{1,j}^{(i)} A_l\right) \right) \\
&\times \left(\prod_{l=1}^{m} U\left(\tau, \mu_{2,j}^{(i)} A_{m-l+1}\right) \right). \qquad (8.61)
\end{aligned}
$$

Naturally the operators $A_3, ..., A_m$ ($m > 2$) have to satisfy the same conditions as operators A_1 and A_2. In our opinion, the formula (8.61) constructed for m summands ($A = A_1 + A_2 + ... + A_m$) will be of the same order as the decomposition formula (8.60) constructed for two summands ($A = A_1 + A_2$).

In the present work, using this method there are constructed third and fourth order precision multi-dimensional decomposition formulas.

To illustrate the method, let us consider the following case of Streng formula in detail $\left(V(\tau; A_1, A_2) = U\left(\tau, \frac{1}{2}A_1\right) U(\tau, A_2) U\left(\tau, \frac{1}{2}A_1\right)\right)$. We write it in the form as (8.60):

$$
V^{(2)}(\tau; A_1, A_2) = U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right).
$$

Hence, for a multi-dimensional case we obtain the following formula:

$$
\begin{aligned}
V^{(m)}(\tau; A_1, ..., A_m) &= U\left(\tau, \frac{1}{2}A_1\right) ... U\left(\tau, \frac{1}{2}A_{m-1}\right) U\left(\tau, \frac{1}{2}A_m\right) \\
&\times U\left(\tau, \frac{1}{2}A_m\right) U\left(\tau, \frac{1}{2}A_{m-1}\right) ... U\left(\tau, \frac{1}{2}A_1\right) \\
&= U\left(\tau, \frac{1}{2}A_1\right) ... U\left(\tau, \frac{1}{2}A_{m-1}\right) U(\tau, A_m) \\
&\times U\left(\tau, \frac{1}{2}A_{m-1}\right) ... U\left(\tau, \frac{1}{2}A_1\right).
\end{aligned}
$$

The given method has not been proven yet, though below we prove the theorem which partially justifies this method.

Theorem 8.4 *Let the decomposition formula (8.61) has the precision order p (≥ 2) at $m = p$. Then the decomposition formula (8.61) will have the same precision order for any m (≥ 2).*

Proof. As following to the condition of the Theorem 8.1, the decomposition formula (8.61) has the precision order p at $p = m$, therefore the equalities are valid:

$$\begin{aligned}\left[V^{(p)}\left(\tau;A_1,...,A_p\right)\right]_i &= 1, \quad i=1,...,p, &(8.62)\\ \left[V^{(p)}\left(\tau,A_1,...,A_p\right)\right]_{i_1,...,i_s} &= \frac{1}{s!}, &(8.63)\\ i_1,...,i_s &= 1,...,p, \quad s=2,...,p\end{aligned}$$

Therefore it follows that, for any $m \leq p$, the following equalities are valid:

$$\begin{aligned}\left[V^{(m)}\left(\tau;A_1,...,A_m\right)\right]_i &= 1, \quad i=1,...,m, &(8.64)\\ \left[V^{(m)}\left(\tau,A_1,...,A_m\right)\right]_{i_1,...,i_s} &= \frac{1}{s!}, &(8.65)\\ i_1,...,i_s &= 1,...,m, \quad s=2,...,p.\end{aligned}$$

It means that the decomposition formula (8.61) has the order p for any $m \leq p$. Now let us show that equalities (8.64) and (8.65) are valid for any $m > p$. Validity of equalities (8.64) can be easily checked, as, according to formula (8.58), we have:

$$\begin{aligned}\left[V^{(m)}\left(\tau;A_1,...A_m\right)\right]_i &= \sum_{i=1}^{q}\sigma_i\sum_{j=1}^{m_i}\left(\mu_{1,j}^{(i)}+\mu_{2,j}^{(i)}\right)\\ &= \sum_{i=1}^{q}\sigma_i\sum_{j=1}^{m_i}\beta_j^{(i)}=1. &(8.66)\end{aligned}$$

Let us prove the validity of equalities (8.65) for any $m > p$. Coefficients $\left[V^{(m)}\left(\tau,A_1,...A_m\right)\right]_{i_1,...,i_s}$ can be obtained from the decomposition of only those semigroups which are generated by the operators $\left(-A_{j_1}\right),...,\left(-A_{j_r}\right)$, where $(j_1,..,j_r)$ is a system of different indices from $(i_1,...,i_s)$ sorted ascending (for example, if $s=5$ and $(i_1,i_2,i_3,i_4,i_5)=(3,3,1,2,1)$, then $r=3$ and $(j_1,j_2,j_3)=(1,2,3)$). From the decompositions of other semigroups, there will participate only first summands (terms with identity operators). Therefore we have:

$$\left[V^{(m)}\left(\tau,A_1,...,A_m\right)\right]_{i_1,...,i_s}=\left[V^{(r)}\left(\tau,A_{j_1},...,A_{j_r}\right)\right]_{i_1,...,i_s}. \quad (8.67)$$

As $r \leq s \leq p$ in the right-hand side of equality (8.67), therefore, according to (8.65) we have:

$$\left[V^{(r)}\left(\tau,A_{j_1},...,A_{j_r}\right)\right]_{i_1,...,i_s}=\frac{1}{s!}, \quad s=2,...,p. \quad (8.68)$$

From (8.67) and (8.68) we obtain:

$$\begin{aligned}\left[V^{(m)}\left(\tau,A_1,...,A_m\right)\right]_{i_1,...,i_s} &= \frac{1}{s!}, &(8.69)\\ i_1,...,i_s &= 1,...,m, \quad s=2,...,p, \quad m>p.\end{aligned}$$

From (8.65), (8.66) and (8.69) it follows that decomposition formula (8.61) has a precision order p for any $m \geq 2$. □

From this theorem it follows that if formula (8.56) has second order precision, then decomposition formula (8.61) will automatically have second order precision (obviously, according to conditions (8.57) and (8.58), decomposition formula (8.61) will always have first order precision).

Below, on basis of the above-described method, we will construct a generalization of third and fourth order precision Schatzman decomposition formulas for any number m (≥ 2) of summands. In case of two summands, these formulas have the following form (see [12]):

$$\begin{aligned} V_1^{(2)}(\tau; A_1, A_2) &= \frac{2}{3}\left[U\left(\tau, \frac{1}{2}A_1\right) U(\tau, A_2)\, U\left(\tau, \frac{1}{2}A_1\right)\right. \\ &\quad \left. + U\left(\tau, \frac{1}{2}A_2\right) U(\tau, A_1)\, U\left(\tau, \frac{1}{2}A_2\right)\right] \\ &\quad -\frac{1}{6}\left[U(\tau, A_1)\, U(\tau, A_2)\right. \\ &\quad \left. + U(\tau, A_2)\, U(\tau, A_1)\right]. \end{aligned} \tag{8.70}$$

$$\begin{aligned} V_2^{(2)}(\tau; A_1, A_2) &= \frac{4}{3}U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) \\ &\quad \times U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) \\ &\quad -\frac{1}{3}U\left(\tau, \frac{1}{2}A_1\right) U(\tau, A_2)\, U\left(\tau, \frac{1}{2}A_1\right). \end{aligned} \tag{8.71}$$

Decomposition formula (8.70) has third order precision, and decomposition formula (8.71) has fourth order precision. Generalization of these formulas for any number m (≥ 2) of summands will be written as follows:

$$\begin{aligned} V_1^{(m)}(\tau; A_1, ..., A_m) &= \frac{2}{3}\left[U\left(\tau, \frac{1}{2}A_1\right) ...U\left(\tau, \frac{1}{2}A_{m-1}\right) U(\tau, A_m)\right. \\ &\quad \times U\left(\tau, \frac{1}{2}A_{m-1}\right) ...U\left(\tau, \frac{1}{2}A_1\right) \\ &\quad + U\left(\tau, \frac{1}{2}A_{m-1}\right) ...U\left(\tau, \frac{1}{2}A_2\right) U(\tau, A_1) \\ &\quad \left. \times\, U\left(\tau, \frac{1}{2}A_2\right) ...U\left(\tau, \frac{1}{2}A_m\right)\right] \\ &\quad -\frac{1}{6}\left[U(\tau, A_1) ...U(\tau, A_m)\right. \\ &\quad \left. + U(\tau, A_m) ...U(\tau, A_1)\right]. \end{aligned} \tag{8.72}$$

$$
\begin{aligned}
V_2^{(m)}(\tau; A_1, ..., A_m) \;=\; & \frac{4}{3}U\left(\tau, \frac{1}{4}A_1\right)...U\left(\tau, \frac{1}{4}A_{m-1}\right) \\
& \times U\left(\tau, \frac{1}{2}A_m\right)U\left(\tau, \frac{1}{4}A_{m-1}\right)...U\left(\tau, \frac{1}{4}A_2\right) \\
& U\left(\tau, \frac{1}{2}A_1\right)U\left(\tau, \frac{1}{4}A_2\right)....U\left(\tau, \frac{1}{4}A_{m-1}\right) \\
& \times U\left(\tau, \frac{1}{2}A_m\right)U\left(\tau, \frac{1}{4}A_{m-1}\right)...U\left(\tau, \frac{1}{4}A_1\right) \\
& -\frac{1}{3}U\left(\tau, \frac{1}{2}A_1\right)...U\left(\tau, \frac{1}{2}A_{m-1}\right)U(\tau, A_m) \\
& \times U\left(\tau, \frac{1}{2}A_{m-1}\right)...U\left(\tau, \frac{1}{2}A_1\right) \qquad (8.73)
\end{aligned}
$$

As a result of some calculations, we have obtained that decomposition formula (8.72) has third order precision for $m = 3$ summands, and decomposition formula (8.73) has fourth order precision for $m = 4$ summands. From here, due to Theorem 8.1 it follows that decomposition formulas (8.72) and (8.73) have respectively third and fourth order precision for any number m (≥ 2) of summands.

§9. The fourth order accuracy sequential type decomposition scheme for evolution problem

Let us consider the problem (3.1). Let $A = A_1 + A_2$, where $A_i \ \ (i = 1, 2)$ are closed operators, densely defined in X.

Together with problem (3.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\overline{\alpha}}{4} A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(2)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_2 v_k^{(2)}(t) &= 0, \quad v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k), \\
\frac{dv_k^{(3)}(t)}{dt} + \frac{1}{4} A_1 v_k^{(3)}(t) &= 0, \quad v_k^{(3)}(t_{k-1}) = v_k^{(2)}(t_k), \\
\frac{dv_k^{(4)}(t)}{dt} + \frac{\alpha}{2} A_2 v_k^{(4)}(t) &= 0, \quad v_k^{(4)}(t_{k-1}) = v_k^{(3)}(t_k), \\
\frac{dv_k^{(5)}(t)}{dt} + \frac{\alpha}{2} A_1 v_k^{(5)}(t) &= 0, \quad v_k^{(5)}(t_{k-1}) = v_k^{(4)}(t_k), \\
\frac{dv_k^{(6)}(t)}{dt} + \frac{\alpha}{2} A_2 v_k^{(6)}(t) &= 0, \quad v_k^{(6)}(t_{k-1}) = v_k^{(5)}(t_k), \\
\frac{dv_k^{(7)}(t)}{dt} + \frac{1}{4} A_1 v_k^{(7)}(t) &= 0, \quad v_k^{(7)}(t_{k-1}) = v_k^{(6)}(t_k), \\
\frac{dv_k^{(8)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_2 v_k^{(8)}(t) &= 0, \quad v_k^{(8)}(t_{k-1}) = v_k^{(7)}(t_k), \\
\frac{dv_k^{(9)}(t)}{dt} + \frac{\overline{\alpha}}{4} A_1 v_k^{(9)}(t) &= 0, \quad v_k^{(9)}(t_{k-1}) = v_k^{(8)}(t_k),
\end{aligned}$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Suppose that the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j), \ j = 1, 2$ generate strongly continuous semigroups.

$u_k(t), \ k = 1, 2, ..,$ is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = v_k^{(9)}(t). \tag{9.1}$$

We declare function $u_k(t)$ as an approximated solution of problem (3.1) on each interval $[t_{k-1}, t_k]$.

The following theorem takes place.

Theorem 9.1. *Let the conditions (a) and (b) of Theorem 1.1 be fulfilled and $U(s, A)\varphi \in D(A^5)$ for each fixed $s \geq 0$. Then the following estimation holds:*

$$\|u(t_k) - u_k(t_k)\| \leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [0, t_k]} \|U(s, A)\varphi\|_{A^5},$$

where c and ω_0 are positive constants.

Proof. From formula (9.1) we obtain:

$$u_k(t_k) = V^k(\tau)\,\varphi, \tag{9.2}$$

where

$$\begin{aligned} V(\tau) \;=\;& U\left(\tau, \frac{\overline{\alpha}}{4}A_1\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right) \\ & \times U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \frac{\overline{\alpha}}{4}A_1\right). \end{aligned}$$

Remark 9.2. *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem 1.1. In this case, for the solving operator, the following estimation holds:*

$$\left\| V^k(\tau) \right\| \le e^{\omega_1 t_k}, \tag{9.3}$$

where ω_1 is a positive constant.

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1, 2)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (1.7), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $(\tau^2 A_i A_j)$, $(-\tau^3 A_i A_j A_k)$ and $(-\tau^4 A_i A_j A_k A_l)$ $(i, j, k, l = 1, 2)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$, $[T(\tau)]_{i,j,k}$ and $[T(\tau)]_{i,j,k,l}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (1.7) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) \;=\;& I - \tau \sum_{i=1}^{2} [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^{2} [V(\tau)]_{i,j} A_i A_j \\ & -\tau^3 \sum_{i,j,k=1}^{2} [V(\tau)]_{i,j,k} A_i A_j A_k \\ & +\tau^4 \sum_{i,j,k,l=1}^{2} [V(\tau)]_{i,j,k,l} A_i A_j A_k A_l + \widetilde{R}_5(\tau). \end{aligned} \tag{9.4}$$

According to the first inequality of the condition *(b)* of the Theorem, for $\widetilde{R}_5(\tau)$, the following estimation holds:

$$\left\| \widetilde{R}_5(\tau)\,\varphi \right\| \le c e^{\omega_0 \tau} \tau^5 \left\| \varphi \right\|_{A^4}, \quad \varphi \in D\left(A^5\right), \tag{9.5}$$

where c and ω_0 are positive constants.

Let us calculate the coefficients $[V(\tau)]_i$ corresponding to the first order members in formula (9.4). It is obvious that the members, corresponding to these coefficients, are obtained from the decomposition of only those factors (semigroups) of the operator $V(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identical operators) will participate.

On the whole, we have two cases: $i = 1$ and $i = 2$. Let us consider the case $i = 1$. We obviously have:

$$[V(\tau)]_1 = [U(\tau, A_1)]_1 = 1. \tag{9.6}$$

Analogously for $i = 2$ we have:

$$[V(\tau)]_2 = [U(\tau, A_2)]_2 = 1. \tag{9.7}$$

By combining formulas (9.6) and (9.7), we will obtain:

$$[V(\tau)]_i = 1, \quad i = 1, 2. \tag{9.8}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i, j = 1, 2)$ corresponding to the second order members included in formula (9.8). On the whole we have two cases: $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2)$. Let us consider the case $(i, j) = (1, 1)$. We obviously have:

$$[V(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}. \tag{9.9}$$

Analogously for $(i, j) = (2, 2)$ we have:

$$[V(\tau)]_{2,2} = [U(\tau, A_2)]_{2,2} = \frac{1}{2}. \tag{9.10}$$

Let us consider the case $(i, j) = (1, 2)$, we obviously have:

$$\begin{aligned}
[V(\tau)]_{1,2} &= \left[U\left(\tau, \frac{\overline{\alpha}}{4}A_1\right)\right]_1 \left(2\left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 + 2\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2\right) \\
&\quad + \left[U\left(\tau, \frac{1}{4}A_1\right)\right]_1 \left(\left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 + 2\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2\right) \\
&\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \left(\left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 + \left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2\right) \\
&\quad + \left[U\left(\tau, \frac{1}{4}A_1\right)\right]_1 \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 \\
&= \frac{\overline{\alpha}(\overline{\alpha}+\alpha)}{4} + \frac{\overline{\alpha}+2\alpha}{8} + \frac{\alpha(\overline{\alpha}+\alpha)}{4} + \frac{\overline{\alpha}}{8} \\
&= \frac{2\overline{\alpha}+1+\alpha+2\alpha+\overline{\alpha}}{8} = \frac{1}{2}.
\end{aligned} \tag{9.11}$$

For $(i,j)=(2,1)$ we have:

$$[V(\tau)]_{2,1}=\frac{1}{2}. \tag{9.12}$$

Here we used the identity $\alpha+\overline{\alpha}=1$.

By combining formulas (9.9) - (9.12), we will obtain:

$$[V(\tau)]_{i,j}=\frac{1}{2}, \quad i,j=1,2. \tag{9.13}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i,j,k=1,2)$ corresponding to the third order members in formula (9.4). On the whole we have eight cases: $(i,j,k)=(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,1)$, $(2,2,2)$. Let us consider the case $(i,j,k)=(1,1,1)$. We obviously have:

$$[V(\tau)]_{1,1,1}=[U(\tau,A_1)]_{1,1,1}=\frac{1}{6}. \tag{9.14}$$

Analogously for $(i,j)=(2,2,2)$ we have:

$$[V(\tau)]_{2,2,2}=[U(\tau,A_2)]_{2,2,2}=\frac{1}{6}. \tag{9.15}$$

Thus Let us calculate the case $(i,j,k)=(1,1,2)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,1,2} \;=\;& \left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_{1,1}\left(2\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+2\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_{1,1}\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+2\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\\
&\times\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+2\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\\
&\times\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\\
&\times\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
=\;&\frac{\overline{\alpha}^2\left(\overline{\alpha}+\alpha\right)}{32}+\frac{\overline{\alpha}+2\alpha}{64}+\frac{\alpha^2\left(\overline{\alpha}+\alpha\right)}{16}+\frac{\overline{\alpha}}{64}+\frac{\overline{\alpha}\left(\overline{\alpha}+2\alpha\right)}{32}\\
&+\frac{\overline{\alpha}\alpha\left(\overline{\alpha}+\alpha\right)}{16}+\frac{\overline{\alpha}^2}{32}+\frac{\alpha\left(\overline{\alpha}+\alpha\right)}{16}+\frac{\overline{\alpha}}{32}+\frac{\alpha\overline{\alpha}}{16}=\frac{1}{6}.
\end{aligned}
\tag{9.16}
$$

For $(i,j,k)=(2,1,1)$ we have:

$$
[V(\tau)]_{2,1,1}=\frac{1}{6}
\tag{9.17}
$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus Let us calculate the case $(i,j,k)=(1,2,2)$. We have:

$$
[V(\tau)]_{1,2,2}=\frac{1}{6}.
\tag{9.18}
$$

For $(i,j,k)=(2,1,1)$ we have:

$$
[V(\tau)]_{2,1,1}=\frac{1}{6}
\tag{9.19}
$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus Let us calculate the case $(i,j,k)=(1,2,1)$. We have:

$$
\begin{aligned}
[V(\tau)]_{1,2,1}\;=\;&\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&\times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\right.\\
&+\left.\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right.\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\\
&\times\left(\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
= \; & \frac{\overline{\alpha}^2\left(2+2\alpha+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}\alpha\left(1+2\alpha+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}\alpha\left(1+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}^3}{32} \\
& +\frac{\alpha\left(1+2\alpha+\overline{\alpha}\right)}{32}+\frac{\alpha\left(1+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}^2}{32}+\frac{\alpha^2\left(1+\overline{\alpha}\right)}{16}+\frac{\alpha\overline{\alpha}^2}{16}+\frac{\overline{\alpha}^2}{32} \\
= \; & \frac{4+2\overline{\alpha}-\frac{2}{3}+2\alpha}{32}=\frac{6-\frac{2}{3}}{32}=\frac{1}{6}. \qquad (9.20)
\end{aligned}
$$

For $(i,j,k)=(2,1,2)$ we have:

$$
\begin{aligned}
\left[V(\tau)\right]_{2,1,2} &= \left[U\left(\tau,\alpha A_2\right)\right]_2\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2 \\
&= \alpha\frac{1}{2}\overline{\alpha}=\frac{1}{6}. \qquad (9.21)
\end{aligned}
$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

By combining formulas (9.14) - (9.21), we will obtain:

$$
\left[V(\tau)\right]_{i,j,k}=\frac{1}{6},\quad i,j,k=1,2. \qquad (9.22)
$$

Analogously we can show that

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \quad i,j,k,l = 1,2. \tag{9.23}$$

From equality (9.4), taking into account formulas (9.8), (9.13), (9.22) and (9.23), we will obtain:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{2} A_i + \frac{1}{2}\tau^2 \sum_{i,j=1}^{2} A_i A_j - \frac{1}{6}\tau^3 \sum_{i,j,k=1}^{2} A_i A_j A_k \\ &\quad + \frac{1}{24}\tau^4 \sum_{i,j,k,l=1}^{2} A_i A_j A_k A_l + \widetilde{R}_5(\tau) \\ &= I - \tau \sum_{i=1}^{2} A_i + \frac{1}{2}\tau^2 \left(\sum_{i=1}^{2} A_i\right)^2 - \frac{1}{6}\tau^3 \left(\sum_{i=1}^{2} A_i\right)^3 \\ &\quad + \frac{1}{24}\tau^4 \left(\sum_{i=1}^{2} A_i\right)^4 + \widetilde{R}_5(\tau) \\ &= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + \widetilde{R}_5(\tau). \end{aligned} \tag{9.24}$$

According to formula (1.7) we have:

$$U(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_5(\tau, A). \tag{9.25}$$

According to condition *(b)* of the second inequality of the Theorem, for $R_5(\tau, A)$, the following estimation holds:

$$\|R_5(\tau, A)\varphi\| \le c e^{\omega\tau} \tau^5 \left\|A^5 \varphi\right\| \le c e^{\omega\tau} \tau^5 \|\varphi\|_{A^5}. \tag{9.26}$$

According to equalities (9.24) and (9.25) we have:

$$U(\tau, A) - V(\tau) = R_5(\tau, A) - \widetilde{R}_5(\tau).$$

From here, taking into account inequalities (9.5) and (9.26), we will obtain the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le c e^{\omega_2 \tau} \tau^5 \|\varphi\|_{A^5}. \tag{9.27}$$

From equalities (3.2) and (9.2), taking into account inequalities (9.3) and

(9.27), we will obtain:

$$
\begin{aligned}
\|u(t_k)-u_k(t_k)\| &= \left\|\left[U(t_k,A)-V^k(\tau)\right]\varphi\right\| = \left\|\left[U^k(\tau,A)-V^k(\tau)\right]\varphi\right\| \\
&= \left\|\left[\sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau,A)-V(\tau)\right]U((i-1)\tau,A)\right]\varphi\right\| \\
&\leq \sum_{i=1}^{k}\|V(\tau)\|^{k-i}\left\|\left[U(\tau,A)-V(\tau)\right]U((i-1)\tau,A)\varphi\right\| \\
&\leq \sum_{i=1}^{k} e^{\omega_1(k-i)\tau} c e^{\omega_2\tau}\tau^5 \left\|U((i-1)\tau,A)\varphi\right\|_{A^5} \\
&\leq c e^{\omega_0 t_k}\tau^5 \sum_{i=1}^{k}\left\|U((i-1)\tau,A)\varphi\right\|_{A^5} \\
&\leq k c e^{\omega_0 t_k}\tau^4 \sup_{s\in[o,t_k]}\left\|U(s,A)\varphi\right\|_{A^5} \\
&\leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[o,t_k]}\left\|U(s,A)\varphi\right\|_{A^5}. \qquad \square
\end{aligned}
$$

Remark 9.3. *The fourth order of accurate decomposition formula in case of Multidimensional problem has the following form:*

$$
\begin{aligned}
V^{(m)}(\tau) &= U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)...U\left(\tau,\frac{\overline{\alpha}}{4}A_{m-1}\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_m\right)U\left(\tau,\frac{\overline{\alpha}}{4}A_{m-1}\right)... \\
&\times U\left(\tau,\frac{\overline{\alpha}}{4}A_2\right)U\left(\tau,\frac{1}{4}A_1\right)U\left(\tau,\frac{\alpha}{4}A_2\right)... \\
&\times U\left(\tau,\frac{\alpha}{4}A_{m-1}\right)U\left(\tau,\frac{\alpha}{2}A_m\right)U\left(\tau,\frac{\alpha}{4}A_{m-1}\right)... \\
&\times U\left(\tau,\frac{\alpha}{4}A_2\right)U\left(\tau,\frac{\alpha}{2}A_1\right)U\left(\tau,\frac{\alpha}{4}A_2\right)... \\
&\times U\left(\tau,\frac{\alpha}{4}A_{m-1}\right)U\left(\tau,\frac{\alpha}{2}A_m\right)U\left(\tau,\frac{\alpha}{4}A_{m-1}\right)... \\
&\times U\left(\tau,\frac{\alpha}{4}A_2\right)U\left(\tau,\frac{1}{4}A_1\right)U\left(\tau,\frac{\overline{\alpha}}{4}A_2\right)... \\
&\times U\left(\tau,\frac{\overline{\alpha}}{4}A_{m-1}\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_m\right)U\left(\tau,\frac{\overline{\alpha}}{4}A_{m-1}\right)...U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right).
\end{aligned}
$$

Chapter III

The Third Order Accuracy Decomposition Schemes for an Evolution Problem with Variable Operator

§10. Differential splitting

Let us consider Cauchy problem in the Banach space X :

$$\frac{du(t)}{dt} + A(t)\,u(t) = 0, \qquad t > 0, \quad u(0) = \varphi, \tag{10.1}$$

where φ is a given element from $D(A)$ and operator $A(t)$ satisfies the following conditions:

(a) The domain of the operator $A(t)$ do not depend on t and is everywhere dense in X;

(b) For every fixed $t_1, t_2, s \in [0;T]$, the following inequality is valid:

$$\left\| (A(t_1) - A(t_2))\, A^{-1}(s) \right\| \le c_1 \left| t_1 - t_2 \right|^q, \quad q \in (0;1], \quad c_1 = const > 0;$$

(c) For any complex number z, $\quad Re(z) \ge 0$, there exists operator $(zI + A(t))^{-1}$ and the following inequality is valid:

$$\left\| (zI + A(t))^{-1} \right\| \le \frac{c_2}{1 + |z|}, \quad c_2 = const > 0.$$

Then the solution of the problem (10.1) is given by the following formula (see [39],[42])::

$$u(t) = U(t, 0; A)\varphi,$$

Where $U(t, 0; A)$ is a solving operator of the problem (10.1).

Let $A(t) = b(t)\,A_0 = b(t)\,(A_1 + A_2)$, where $A_i \quad (i = 1, 2)$ are compactly defined, closed linear operators in X, the function $b(t) \ge b_0 > 0$ satisfies the condition of Helder.

Let us introduce difference net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Along with problem (10.1) we consider two sequences of the following problems on each interval $[t_{k-1}, t_k]$:

$$\frac{dv_k^1(t)}{dt} + \alpha b(t) A_1 v_k^1(t) = 0, \quad \frac{dw_k^1(t)}{dt} + \alpha b(t) A_2 w_k^1(t) = 0,$$

$$v_k^1(t_{k-1}) = u_{k-1}(t_{k-1}), \quad w_k^1(t_{k-1}) = u_{k-1}(t_{k-1}),$$

$$\frac{dv_k^2(t)}{dt} + b(t) A_2 v_k^2(t) = 0, \quad \frac{dw_k^2(t)}{dt} + b(t) A_1 w_k^2(t) = 0,$$

$$v_k^2(t_{k-1}) = v_k^1(t_k), \quad w_k^2(t_{k-1}) = w_k^1(t_k),$$

$$\frac{dv_k^3(t)}{dt} + \overline{\alpha} b(t) A_1 v_k^3(t) = 0, \quad \frac{dw_k^3(t)}{dt} + \overline{\alpha} b(t) A_2 w_k^3(t) = 0,$$

$$v_k^3(t_{k-1}) = v_k^2(t_k), \quad w_k^3(t_{k-1}) = w_k^2(t_k),$$

Here α is a numerical complex parameter with $Re(\alpha) > 0$, $u_0(0) = \varphi$. Suppose that $U(t_1, t_2; \gamma A_j)$, $\gamma = 1, \alpha, \overline{\alpha}$ $(j = 1, 2)$ operators exist. On each $[t_{k-1}, t_k]$ $(k = 1, 2, ...)$ interval $u_k(t)$ are defined as follows:

$$u_k(t) = \frac{1}{2}[v_k^3(t) + w_k^3(t)].$$

We consider the function $u_k(t)$ as an approximate solution of the problem (10.1) on the interval $[t_{k-1}, t_k]$.

Theorem 10.1. *Let the following conditions be satisfied:*

(a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ $\left(i = \sqrt{-1}\right)$;

(b) *The solving operators* $U(t, t_0; \gamma b(\cdot) A_j)$, $\gamma = 1, \alpha, \overline{\alpha}$ $(j = 0, 1, 2)$ *of the problems*

$$\frac{dv(t)}{dt} + \gamma b(t) A_j v(t) = 0, \qquad t \geq t_0 \geq 0, \qquad v(t_0) = \varphi \in D(A_j),$$

exist and the following inequalities hold true:

$$\|U(t, t_0; \gamma b(\cdot) A_j)\| \leq e^{\omega(t-t_0)},$$

$$\|U(t, t_0; b(\cdot) A_0)\| \leq M e^{\omega(t-t_0)}, \quad M, \omega = const > 0;$$

(c) *The function* $b(t) \geq b_0 > 0$ *satisfies the condition of Helder;*

(d) $U(s_1, s_2; b(\cdot) A_0) \varphi \in D(A_0^4)$ *for every fixed* $s_1, s_2 \geq 0$.

Then the following estimation holds:

$$\|u(t_k) - u_k(t_k)\| \leq c e^{\omega_0 t_k} t_k \tau^3 \sup_{s_1, s_2 \in [0, t_k]} \|U(s_1, s_2; b(\cdot) A_0) \varphi\|_{A_0^4},$$

where c, ω_0 *are positive constants.*

Let us prove the auxiliary Lemma on which the proof of the Theorem is based.

Lemma 10.2 *If the conditions (a),(b) and (c) of the Theorem are satisfied, then the following estimation holds:*

$$\left\| \left(U\left(t_i,t_{i-1};b\left(\cdot\right)A_0\right) - V\left(t_i,t_{i-1}\right)\right)\varphi \right\| \leq c e^{\omega_0 \tau}\tau^4 \left\|\varphi\right\|_{A_0^4},$$

where

$$V\left(t_i,t_{i-1}\right) = \frac{1}{2}\left[U\left(t_i,t_{i-1};\alpha b\left(\cdot\right)A_1\right)U\left(t_i,t_{i-1};b\left(\cdot\right)A_2\right)U\left(t_i,t_{i-1};\overline{\alpha} b\left(\cdot\right)A_1\right)\right.$$

$$\left. + U\left(t_i,t_{i-1};\alpha b\left(\cdot\right)A_2\right)U\left(t_i,t_{i-1};b\left(\cdot\right)A_1\right)U\left(t_i,t_{i-1};\overline{\alpha} b\left(\cdot\right)A_2\right)\right].$$

Here c, ω_0 are positive constants.

Proof. The following formula is true:

$$U\left(t_i,t_{i-1};A\right) = I - \int\limits_{t_{i-1}}^{t_i} A\left(s_1\right)U\left(t_i,s_1;A\right)ds_1,$$

Hence we obtain the following expansion:

$$U\left(t_i,t_{i-1};A\right) = I - \int\limits_{t_{i-1}}^{t_i} A\left(s_1\right)ds_1 + \int\limits_{t_{i-1}}^{t_i} A\left(s_1\right)\int\limits_{t_{i-1}}^{s_1} A\left(s_2\right)ds_2 ds_1 + ...$$

$$+\left(-1\right)^{k-1}\int\limits_{t_{i-1}}^{t_i} A\left(s_1\right)\int\limits_{t_{i-1}}^{s_1} A\left(s_2\right)...\int\limits_{t_{i-1}}^{s_{k-2}} A\left(s_{k-1}\right)ds_{k-1}...ds_2 ds_1 + R_k\left(t_i,t_{i-1},A\right), \tag{10.2}$$

where

$$R_k\left(t_i,t_{i-1},A\right) = \left(-1\right)^k \int\limits_{t_{i-1}}^{t_i} A\left(s_1\right)\int\limits_{t_{i-1}}^{s_1} A\left(s_2\right)...\int\limits_{t_{i-1}}^{s_{k-1}} U\left(t_i,s_k;A\right)A\left(s_k\right)ds_k...ds_2 ds_1. \tag{10.3}$$

Let us consider the first addend of the operator $V\left(t_i,t_{i-1}\right)$ and decompose its all multipliers from the right to left according to the formula (10.2) so that each residual member is of the fourth order. We shall get:

$$U\left(t_i,t_{i-1};\alpha b\left(\cdot\right)A_1\right)U\left(t_i,t_{i-1};b\left(\cdot\right)A_2\right)U\left(t_i,t_{i-1};\overline{\alpha} b\left(\cdot\right)A_1\right) =$$

$$= I - \left[\left(\alpha\int\limits_{t_{i-1}}^{t_i} b\left(s_1\right)ds_1 + \overline{\alpha}\int\limits_{t_{i-1}}^{t_i} b\left(s_1\right)ds_1\right)A_1 + \int\limits_{t_{i-1}}^{t_i} b\left(s_1\right)ds_1 A_2\right]$$

$$+\left[\left(\alpha^2\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b\left(s_1\right)b\left(s_2\right)ds_1 ds_2 + \alpha\overline{\alpha}\int\limits_{t_{i-1}}^{t_i} b\left(s_1\right)ds_1\int\limits_{t_{i-1}}^{t_i} b\left(s_1\right)ds_1\right.\right.$$

$$
+\overline{\alpha}^2 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 \Bigg) A_1^2 + \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 A_2^2
$$

$$
+ \alpha \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 A_1 A_2 + \overline{\alpha} \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 A_2 A_1 \Bigg]
$$

$$
- \Bigg[\Bigg(\alpha^3 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} \int\limits_{t_{i-1}}^{s_2} b(s_1)\, b(s_2)\, b(s_3)\, ds_1 ds_2 ds_3
$$

$$
+\alpha^2 \overline{\alpha} \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1
$$

$$
+\alpha \overline{\alpha}^2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2
$$

$$
+ \overline{\alpha}^3 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} \int\limits_{t_{i-1}}^{s_2} b(s_1)\, b(s_2)\, b(s_3)\, ds_1 ds_2 ds_3 \Bigg) A_1^3
$$

$$
+\alpha^2 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 A_1^2 A_2
$$

$$
+\alpha \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 A_1 A_2^2
$$

$$
+\alpha \overline{\alpha} \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 A_1 A_2 A_1
$$

$$
+\overline{\alpha} \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 A_2^2 A_1
$$

$$
+\overline{\alpha}^2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\, ds_1 \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} b(s_1)\, b(s_2)\, ds_1 ds_2 A_2 A_1^2
$$

$$
+ \int\limits_{t_{i-1}}^{t_i} \int\limits_{t_{i-1}}^{s_1} \int\limits_{t_{i-1}}^{s_2} b(s_2)\, b(s_2)\, b(s_3)\, ds_1 ds_2 ds_3 A_2^3 + R_{4,1}(t_{i-1}, t_i), \qquad (10.4)
$$

where for the residual member the following estimation holds:

$$\|R_{4,1}(t_{i-1},t_i)\varphi\| \leq ce^{\omega_0\tau}\tau^4\|\varphi\|_{A_0^4}. \tag{10.5}$$

Analogously for the second addend of the operator $V(t_i,t_{i-1})$ we get:

$$U(t_i,t_{i-1};\alpha b(\cdot)A_2)\,U(t_i,t_{i-1};b(\cdot)A_1)\,U(t_i,t_{i-1};\overline{\alpha}b(\cdot)A_2)=$$

$$=I-\left[\left(\alpha\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1+\overline{\alpha}\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1\right)A_2+\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1A_1\right]$$

$$+\Bigg[\Bigg(\alpha^2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_1ds_2+\alpha\overline{\alpha}\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1$$

$$+\overline{\alpha}^2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_1ds_2\Bigg)A_2^2+\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_1ds_2A_1^2$$

$$+\alpha\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1A_2A_1+\overline{\alpha}\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1A_1A_2\Bigg]$$

$$-\Bigg[\Bigg(\alpha^3\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2}b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3$$

$$+\alpha^2\overline{\alpha}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_2ds_2\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1$$

$$+\alpha\overline{\alpha}^2\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_1ds_2$$

$$+\overline{\alpha}^3\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2}b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\Bigg)A_2^3$$

$$+\alpha^2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_1ds_2\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1A_2^2A_1$$

$$+\alpha\int_{t_{i-1}}^{t_i}b(s_1)\,ds_1\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}b(s_1)\,b(s_2)\,ds_1ds_2A_2A_1^2$$

$$+\alpha\overline{\alpha}\int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 A_2A_1A_2$$

$$+\overline{\alpha}\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 A_1^2A_2$$

$$+\overline{\alpha}^2\int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 A_1A_2^2$$

$$+\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1}\int\limits_{t_{i-1}}^{s_2} b(s_2)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 A_1^3 + R_{4,2}(t_{i-1},t_i)\,, \tag{10.6}$$

where for the residual member the following estimation holds:

$$\|R_{4,2}(t_{i-1},t_i)\,\varphi\| \le ce^{\omega_0\tau}\tau^4\,\|\varphi\|_{A_0^4}\,. \tag{10.7}$$

From (10.4) and (10.6) we obtain:

$$V(t_i,t_{i-1}) = I - \int\limits_{t_{i-1}}^{t_i} b(s_1)(A_1+A_2)\,ds_1$$

$$+\frac{1}{2}\Bigg[\Bigg(\left(\alpha^2+\overline{\alpha}^2+1\right)\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2$$

$$+\alpha\overline{\alpha}\int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1\Bigg)\left(A_1^2+A_2^2\right)$$

$$+(\alpha+\overline{\alpha})\int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1\,(A_1A_2+A_2A_1)\Bigg]$$

$$-\frac{1}{2}\Bigg[\Bigg(\left(\alpha^3+\overline{\alpha}^3+1\right)\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1}\int\limits_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3$$

$$+\left(\alpha^2\overline{\alpha}+\alpha\overline{\alpha}^2\right)\int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2\Bigg)\left(A_1^3+A_2^3\right)$$

$$+\left(\alpha^2+\overline{\alpha}\right)\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1\left(A_1^2A_2+A_1A_2^2\right)$$

$$+\alpha\overline{\alpha}\int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \left(A_1A_2A_1 + A_2A_1A_2\right)$$

$$+\left(\alpha+\overline{\alpha}^2\right)\int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 \left(A_2A_1^2 + A_2^2A_1\right)\Bigg]$$

$$+R_4(t_i,t_{i-1}) = I - \int_{t_{i-1}}^{t_i} b(s_1)\left(A_1+A_2\right)ds_1$$

$$+\left[\left(\frac{\left(\alpha^2+2\alpha\overline{\alpha}+\overline{\alpha}^2+1\right)}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 + I_1\right)\left(A_1^2+A_2^2\right)\right.$$

$$\left.+\left(\frac{2\alpha+2\overline{\alpha}}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 + I_2\right)\left(A_1A_2+A_2A_1\right)\right]$$

$$-\left[\left(\frac{\alpha^3+3\alpha^2\overline{\alpha}+3\alpha\overline{\alpha}^2+\overline{\alpha}^3+1}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\right.\right.$$

$$\left.+I_3\right)\left(A_1^3+A_2^3\right)$$

$$+\left(\frac{3\alpha^2+3\overline{\alpha}}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_4\right)\left(A_1^2A_2+A_2^2A_1\right)$$

$$+\left(3\alpha\overline{\alpha}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_5\right)\left(A_1A_2A_1+A_2A_1A_2\right)$$

$$\left.+\left(\frac{3\alpha+3\overline{\alpha}^2}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_6\right)\left(A_2A_1^2+A_1^2A_2\right)\right]$$

$$+R_4(t_i,t_{i-1}) = I - \int_{t_{i-1}}^{t_i}\left(b(s_1)\,A_1 + b(s_1)\,A_2\right)ds_1$$

$$+\left[\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 + I_1\right)\left(A_1^2+A_2^2\right)\right.$$

$$\left.+\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 + I_2\right)\left(A_1A_2+A_2A_1\right)\right]$$

$$
\begin{aligned}
&-\left[\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_3\right)\left(A_1^3 + A_2^3\right)\right. \\
&+\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_4\right)\left(A_1^2A_2 + A_2^2A_1\right) \\
&+\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_5\right)\left(A_1A_2A_1 + A_2A_1A_2\right) \\
&\left.+\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3 + I_6\right)\left(A_2A_1^2 + A_1^2A_2\right)\right] \\
&+R_4(t_i,t_{i-1}), \qquad (10.8)
\end{aligned}
$$

where

$$R_4(t_i,t_{i-1}) = \frac{1}{2}\left(R_{4,1}(t_{i-1},t_i) + R_{4,2}(t_{i-1},t_i)\right),$$

and where

$$I_1 = \frac{\alpha\overline{\alpha}}{2}\left(\int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 - 2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2\right),$$

$$I_2 = \frac{1}{2}\left(\int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int_{t_{i-1}}^{t_i} b(s_1)\,ds_1 - 2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2\right),$$

$$
\begin{aligned}
I_3 = \frac{1}{2}\Bigg[&\left(\alpha^2\overline{\alpha} + \alpha\overline{\alpha}^2\right)\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 \int_{t_{i-1}}^{t_i} b(s_1)\,ds_1\right. \\
&\left.-3\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
I_4 = \frac{1}{2}\Bigg[&\left(\alpha^2 + \overline{\alpha}\right)\left(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 \int_{t_{i-1}}^{t_i} b(s_1)\,ds_1\right. \\
&\left.-3\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\right)\Bigg],
\end{aligned}
$$

$$I_5 = \frac{1}{2}\left[\alpha\overline{\alpha}\left(\int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1\right.\right.$$

$$\left.\left.-6\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1}\int\limits_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\right)\right],$$

$$I_6 = \frac{1}{2}\left[\left(\alpha^2+\overline{\alpha}^2\right)\left(\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2 \int\limits_{t_{i-1}}^{t_i} b(s_1)\,ds_1\right.\right.$$

$$\left.\left.-3\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1}\int\limits_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\right)\right].$$

From (10.5) and (10.7) follows the following estimation:

$$\left\|R_4(t_i,t_{i-1})\,\varphi\right\| \le ce^{\omega_0\tau}\tau^4\left\|\varphi\right\|_{A_0^4}. \tag{10.9}$$

Clearly for the $U(t_i,t_{i-1};A)$ we have:

$$U(t_i,t_{i-1};A) = I - \int\limits_{t_{i-1}}^{t_i} b(s_1)\,(A_1+A_2)\,ds_1$$

$$+\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} \left(b(s_1)\,A_1+b(s_1)\,A_2\right)\left(b(s_2)\,A_1+b(s_2)\,A_2\right)ds_1ds_2$$

$$-\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1}\int\limits_{t_{i-1}}^{s_2} \left(b(s_1)\,A_1+b(s_1)\,A_2\right)\left(b(s_2)\,A_1+b(s_2)\,A_2\right)$$

$$\times\left(b(s_3)\,A_1+b(s_3)\,A_2\right)ds_1ds_2ds_3 + R_4(t_i,t_{i-1},A)$$

$$= I - \int\limits_{t_{i-1}}^{t_i} \left(b(s_1)\,A_1+b(s_1)\,A_2\right)ds_1$$

$$+\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2\left(A_1^2+A_2^2\right)$$

$$+\int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{i-1}}^{s_1} b(s_1)\,b(s_2)\,ds_1ds_2\left(A_1A_2+A_2A_1\right)$$

$$-\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\left(A_1^3+A_2^3\right)$$

$$-\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\left(A_1^2A_2+A_2^2A_1\right)$$

$$-\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\left(A_1A_2^2+A_2A_1^2\right)$$

$$-\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s_1}\int_{t_{i-1}}^{s_2} b(s_1)\,b(s_2)\,b(s_3)\,ds_1ds_2ds_3\left(A_1A_2A_1+A_2A_1A_2\right)+R_4\left(t_i,t_{i-1},A\right), \tag{10.10}$$

where for the residual member the following estimation holds:

$$\left\|R_4\left(t_i,t_{i-1},A\right)\varphi\right\|\le ce^{\omega_0\tau}\tau^4\left\|\varphi\right\|_{A_0^4}. \tag{10.11}$$

From (10.8) and (10.10) we obtain:

$$U\left(t_i,t_{i-1};A\right)-V\left(t_i,t_{i-1}\right)=I_1\left(A_1^2+A_2^2\right)+I_2\left(A_1A_2+A_2A_1\right)$$
$$+I_3\left(A_1^3+A_2^3\right)+I_4\left(A_1^2A_2+A_2^2A_1\right)+I_5\left(A_1A_2A_1+A_2A_1A_2\right)$$
$$+I_6\left(A_1A_2^2+A_2A_1^2\right)+R_4\left(t_i,t_{i-1}\right)+R_4\left(t_i,t_{i-1},A\right). \tag{10.12}$$

Let us consider the following integral and transform it using integration by part:

$$\int_a^b\int_a^{s_1} f(s_1)\,\varphi(s_2)\,ds_2ds_1=\int_a^{s_1} f(s_2)\,ds_2\Bigg|_a^b\int_a^{s_1}\varphi(s2)\,ds_2\Bigg|_a^b$$

$$-\int_a^b\int_a^{s_1}\varphi(s_1)\,f(s_2)\,ds_2ds_1=\int_a^b f(s_1)\,ds_1\int_a^b\varphi(s_1)\,ds_1-\int_a^b\int_a^{s_1}\varphi(s_1)\,f(s_2)\,ds_2ds_1.$$

We receive:

$$\int_a^b f(s_1)\,ds_1\int_a^b\varphi(s_1)\,ds_1=\int_a^b\int_a^{s_1} f(s_1)\,\varphi(s_2)\,ds_2ds_1+\int_a^b\int_a^{s_1}\varphi(s_1)\,f(s_2)\,ds_2ds_1. \tag{10.13}$$

According to the formula (10.13) it follows that:

$$I_1=I_2=0. \tag{10.14}$$

Let us consider the following integral and transform it using integration by part:

$$\int_a^b \int_a^{s_1} \int_a^{s_2} f(s_1)\varphi(s_2)\psi(s_3)\,ds_3 ds_2 ds_1 = \int_a^{s_1} f(s_2)\,ds_2 \Bigg|_a^b \int_a^{s_1} \varphi(s_2)\int_a^{s_2}\psi(s_3)\,ds_3 ds_2\Bigg|_a^b$$

$$-\int_a^b \int_a^{s_1} f(s_2)\,ds_2 \varphi(s_1)\int_a^{s_1}\psi(s_2)\,ds_2 ds_1 = \int_a^b f(s_1)\,ds_1 \int_a^b \left(\varphi(s_1)\int_b^{s_1}\psi(s_2)\,ds_2\right) ds_1$$

$$-\int_a^b \varphi(s_1)\left(\int_a^{s_1} f(s_2)\,ds_2 \int_a^{s_1}\psi(s_2)\,ds_2\right) ds_1.$$

Hence according to the equality (10.13) we get:

$$\int_a^b \int_a^{s_1} \int_a^{s_2} f(s_1)\varphi(s_2)\psi(s_3)\,ds_3 ds_2 ds_1 = \int_a^b f(s_1)\,ds_1 \int_a^b \left(\varphi(s_1)\int_a^{s_1}\psi(s_2)\,ds_2\right) ds_1$$

$$-\int_a^b \int_a^{s_1} \int_a^{s_2} \varphi(s_1) f(s_2)\psi(s_3)\,ds_3 ds_2 ds_1 - \int_a^b \int_a^{s_1} \int_a^{s_2} \varphi(s_1)\psi(s_2) f(s_3)\,ds_3 ds_2 ds_1.$$

From this we obtain the following formula:

$$\int_a^b f(s_1)\,ds_1 \int_a^b \left(\varphi(s_1)\int_a^{s_1}\psi(s_2)\,ds_2\right) ds_1 = \int_a^b \int_a^{s_1} \int_a^{s_2} f(s_1)\varphi(s_2)\psi(s_3)\,ds_3 ds_2 ds_1$$

$$+\int_a^b \int_a^{s_1} \int_a^{s_2} \varphi(s_1) f(s_2)\psi(s_3)\,ds_3 ds_2 ds_1 + \int_a^b \int_a^{s_1} \int_a^{s_2} \varphi(s_1)\psi(s_2) f(s_3)\,ds_3 ds_2 ds_1. \tag{10.15}$$

From the formula (10.15) it follows that:

$$I_3 = I_4 = I_6 = 0. \tag{10.16}$$

Finally let us consider the following integral and transform it according to the formula (10.13):

$$\int_a^b f(s_1)\,ds_1 \int_a^b \left(\varphi(s_1)\int_a^{s_1}\psi(s_2)\,ds_2\right) ds_1 = \int_a^b f(s_1)\,ds_1 \int_a^b \varphi(s_1)\,ds_1 \int_a^b \psi(s_1)\,ds_1$$

$$-\int_a^b f(s_1)\,ds_1 \int_a^b \psi(s_1)\int_a^{s_1}\varphi(s_2)\,ds_2 ds_1.$$

Hence according to the formula (10.15) we obtain:

$$\int_a^b f(s_1)\,ds_1 \int_a^b \varphi(s_1) \int_a^{s_1} \psi(s_2)\,ds_2 ds_1 = \int_a^b f(s_1)\,ds_1 \int_a^b \varphi(s_1)\,ds_1 \int_a^b \psi(s_1)\,ds_1$$

$$-\int_a^b \int_a^{s_1} \int_a^{s_2} f(s_1)\,\psi(s_2)\,\varphi(s_3)\,ds_3 ds_2 ds_1 - \int_a^b \int_a^{s_1} \int_a^{s_2} \psi(s_1)\,f(s_2)\,\varphi(s_3)\,ds_3 ds_2 ds_1$$

$$-\int_a^b \int_a^{s_1} \int_a^{s_2} \psi(s_1)\,\varphi(s_2)\,f(s_3)\,ds_3 ds_2 ds_1.$$

From this and formula (10.15) we get:

$$\int_a^b f(s_1)\,ds_1 \int_a^b \varphi(s_1)\,ds_1 \int_a^b \psi(s_1)\,ds_1 = \int_a^b \int_a^{s_1} \int_a^{s_2} f(s_1)\,\varphi(s_2)\,\psi(s_3)\,ds_3 ds_2 ds_1$$

$$+\int_a^b \int_a^{s_1} \int_a^{s_2} \varphi(s_1)\,f(s_2)\,\psi(s_3)\,ds_3 ds_2 ds_1 + \int_a^b \int_a^{s_1} \int_a^{s_2} \varphi(s_1)\,\psi(s_2)\,f(s_3)\,ds_3 ds_2 ds_1$$

$$+\int_a^b \int_a^{s_1} \int_a^{s_2} f(s_1)\,\psi(s_2)\,\varphi(s_3)\,ds_3 ds_2 ds_1 + \int_a^b \int_a^{s_1} \int_a^{s_2} \psi(s_1)\,f(s_2)\,\varphi(s_3)\,ds_3 ds_2 ds_1$$

$$+\int_a^b \int_a^{s_1} \int_a^{s_2} \psi(s_1)\,\varphi(s_2)\,f(s_3)\,ds_3 ds_2 ds_1. \tag{10.17}$$

from the formula (10.17) it follows that:

$$I_5 = 0 \tag{10.18}$$

From the equality (10.12) according to the formulas (10.14), (10.16), (10.18) and the inequalities (10.9) and (10.11) we obtain the sought estimation. □

Let us return to the proof of the Theorem 10.1.

Solution of the problem (10.1) in the point $t = t_k$ can be written as follows:

$$u(t_k) = U(t_k, t_0; A)\,\varphi. \tag{10.19}$$

Solution of the decomposed problem can be written as follows:

$$u_k(t_k) = L(t_k, t_0)\,\varphi, \tag{10.20}$$

where

$$L(t_i, t_j) = V(t_i, t_{i-1})\,V(t_{i-1}, t_{i-2})\,...V(t_{j+1}, t_j)\,, \quad i > j.$$

Let us estimate the operator $L\left(t_i, t_j\right) \quad (i > j)$:

$$\|L\left(t_i, t_j\right)\| \leq \|V\left(t_i, t_{i-1}\right)\| \, \|V\left(t_{i-1}, t_{i-2}\right)\| \ldots \|V\left(t_{j+1}, t_j\right)\|$$

$$\leq e^{\omega_0 \tau} e^{\omega_0 \tau} \ldots e^{\omega_0 \tau} \leq e^{\omega_0 (i-j)\tau}. \tag{10.21}$$

From the equalities (10.19) and (10.20) according to the inequality (10.21) and Lemma we obtain:

$$\|u(t_k) - u_k(t_k)\| = \|\left[L\left(t_k, 0\right) - U\left(t_k, 0; A\right)\right]\varphi\|$$

$$= \sum_{i=1}^{k} \|L\left(t_k, t_i\right)\left[L\left(t_i, t_{i-1}\right) - U\left(t_i, t_{i-1}; A\right)\right] U\left(t_{i-1}, 0; A\right)\varphi\|$$

$$\leq \sum_{i=1}^{k} \|L\left(t_k, t_i\right)\| \, \|\left[L\left(t_i, t_{i-1}\right) - U\left(t_i, t_{i-1}; A\right)\right] U\left(t_{i-1}, 0; A\right)\varphi\|$$

$$\leq \sum_{i=1}^{k} c e^{\omega_0 (k-i)\tau} c e^{\omega_0 \tau} \left\|U\left(t_{i-1}, 0; A\right)\varphi\right\|_{A_0^4}$$

$$\leq c e^{\omega_0 t_k} t_k \tau \sup_{s_1, s_2 \in [0, t_k]} \|U(s_1, s_2; A)\varphi\|_{A_0^4}. \qquad \square$$

§11. Rational splitting

1. Construction of decomposition algorithm

Let us consider Cauchy problem in the Banach space X :

$$\frac{du\left(t\right)}{dt}+A\left(t\right)u\left(t\right)=0,\qquad t>0,\quad u(0)=\varphi, \tag{11.1}$$

where φ is a given element from $D\left(A\right)$ and operator $A\left(t\right)$ satisfies conditions of the previous paragraph.

In the previous paragraph we have built the following decomposition formula which is locally of the fourth order of accuracy.

$$V\left(t_k,t_{k-1}\right)=\frac{1}{2}\left[U\left(t_k,t_{k-1};\alpha b\left(\cdot\right)A_1\right)U\left(t_k,t_{k-1};b\left(\cdot\right)A_2\right)U\left(t_k,t_{k-1};\overline{\alpha}b\left(\cdot\right)A_1\right)\right.$$

$$\left.+U\left(t_k,t_{k-1};\alpha b\left(\cdot\right)A_2\right)U\left(t_k,t_{k-1};b\left(\cdot\right)A_1\right)U\left(t_k,t_{k-1};\overline{\alpha}b\left(\cdot\right)A_2\right)\right], \tag{11.2}$$

where $\alpha=\frac{1}{2}\pm i\frac{1}{2\sqrt{3}}\quad\left(i=\sqrt{-1}\right)$.

In this paragraph we have shown, that:

$$U\left(t_k,t_{k-1},b\left(\cdot\right)A_0\right)-V\left(t_k,t_{k-1}\right)=O_p\left(\tau^4\right),$$

where $O_p\left(\tau^4\right)$ is an operator, the norm of which is of the fourth order with respect to τ (more precisely, in case of unbounded operator $\left\|O_p\left(\tau^4\right)\varphi\right\|=O\left(\tau^4\right)$ for any φ from the domain of $O_p\left(\tau^4\right)$). In the present work the following approximation formulas of the fourth order accuracy will be built for the solving operator of the problem (10.1), using rational approximations:

$$W_1\left(t_k,t_{k-1},b\left(\cdot\right)A_0\right)=a_kI+b_k\left(I+\lambda_k\tau A_0\right)^{-1}+c_k\left(I+\lambda_k\tau A_0\right)^{-2}, \tag{11.3}$$

$$W_2\left(t_k,t_{k-1},b\left(\cdot\right)A_0\right)=\left(I-\lambda_{0,k}\tau A_0\right)\left(I+\lambda_{1,k}\tau A_0\right)^{-1}\left(I+\lambda_{2,k}\tau A_0\right)^{-1}, \tag{11.4}$$

where

$$\lambda_k=\frac{1}{2}\frac{\gamma_{2,k}}{\gamma_{1,k}}+\frac{1}{2\sqrt{3}}\frac{\sqrt{3\gamma_{2,k}-2\gamma_{1,k}\gamma_{3,k}}}{\gamma_{1,k}},$$

$$a_k=1-\frac{2\gamma_{1,k}}{\lambda_k}+\frac{\gamma_{2,k}}{2\lambda_k^2},\quad b_k=\frac{3\gamma_{1,k}}{\lambda_k}-\frac{\gamma_{2,k}}{\lambda_k^2},\quad c_k=\frac{\gamma_{2,k}}{2\lambda_k^2}-\frac{\gamma_{1,k}}{\lambda_k};$$

$$\lambda_{0,k}=\frac{6\gamma_{1,k}^3-6\gamma_{1,k}\gamma_{2,k}+\gamma_{3,k}}{6\gamma_{1,k}^2-3\gamma_{2,k}},$$

$$\lambda_{s,k}=\frac{1}{2}\left(d_k+(-1)^s i\sqrt{4e_k-d_k^2}\right),\ s=1,2,$$

$$d_k=\frac{3\gamma_{1,k}\gamma_{2,k}-\gamma_{3,k}}{6\gamma_{1,k}^2-3\gamma_{2,k}},\quad e_k=\frac{3\gamma_{2,k}^2-2\gamma_{1,k}\gamma_{3,k}}{2\left(6\gamma_{1,k}^2-3\gamma_{2,k}\right)},$$

and where

$$\gamma_{1,k} = \frac{3b\left(t_k\right) + b\left(t_{k-1/3}\right)}{4}, \quad \gamma_{2,k} = b^2\left(t_{k-1/2}\right), \quad \gamma_{3,k} = b^3\left(t_{k-1/2}\right).$$

Let us note, that the parameters a_k, b_k, c_k and λ_k satisfy the following system of equations:

$$a_k + b_k + c_k = 1,$$
$$b_k + mc_k = \frac{\gamma_{m,k}}{(m-1)!\lambda_k^m}, \quad m = 2, 3, 4$$

and the parameters $\lambda_{0,k}, \lambda_{1,k}$ and $\lambda_{2,k}$ - the following system of equations:

$$\begin{cases} \lambda_{0,k} + \lambda_{1,k} + \lambda_{2,k} = \gamma_{1,k}, \\ \lambda_{0,k}\left(\lambda_{1,k} + \lambda_{2,k}\right) + \lambda_{1,k}^2 + \lambda_{2,k}^2 + \lambda_{1,k}\lambda_{2,k} = \gamma_{2,k}/2, \\ \lambda_{0,k}\left(\lambda_{1,k}^2 + \lambda_{2,k}^2 + \lambda_{1,k}\lambda_{2,k}\right) + \lambda_{1,k}^3 + \lambda_{2,k}^3 + \lambda_{1,k}^2\lambda_{2,k} + \lambda_{1,k}\lambda_{2,k}^2 = \gamma_{3,k}/6. \end{cases}$$

These equations can be received by equalizing the coefficients of the operators I, A_0, A_0^2 and A_0^3 obtained after decomposing the (1.4) and (1.5) rational approximations with the corresponding coefficients in the decomposition of the solving operator of the problem (11.1).

In the present work we shall also show that for the both formulas we have:

$$U\left(t_k, t_{k-1}, b\left(\cdot\right) A_0\right) - W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_0\right) = O_p\left(\tau^4\right), \quad l = 1, 2.$$

According to the formulas (11.3),(11.4) and (11.5) we can build the following decomposition formulas:

$$V_l\left(t_k, t_{k-1}\right) = \frac{1}{2}\left[W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_1\right) W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_2\right) W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_1\right)\right.$$
$$\left. + W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_2\right) W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_1\right) W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_2\right)\right], \quad l = 1, 2. \tag{11.5}$$

Below we shall show that both formulas are of the fourth order of accuracy,

$$U\left(t_k, t_{k-1}, b\left(\cdot\right) A_0\right) - V_l\left(t_k, t_{k-1}\right) = O_p\left(\tau^4\right), \quad l = 1, 2.$$

According to the formulas (11.6) in the present work the third order accuracy decomposition schemes will be built for the solution of the problem (1.1).

According to the formula (11.2) we have:

$$u(t_k) = U\left(t_k, t_{k-1}, b\left(\cdot\right) A_0\right) u\left(t_{k-1}\right). \tag{11.6}$$

On the basis of the formula (11.7) let us construct the following scheme:

$${}^l u_k = V_l\left(t_k, t_{k-1}\right)\left({}^l u_{k-1}\right) \quad {}^l u_0 = \varphi, \quad l = 1, 2, \tag{11.7}$$

where

$$V_l(t_k, t_{k-1}) = \frac{1}{2}\left[W_l\left(t_k, t_{k-1}, \alpha b\left(\cdot\right) A_1\right) W_l\left(t_k, t_{k-1}, b\left(\cdot\right) A_2\right) W_l\left(t_k, t_{k-1}, \overline{\alpha} b\left(\cdot\right) A_1\right)\right.$$

$$+W_l\left(t_k,t_{k-1},\alpha b\left(\cdot\right)A_2\right)W_l\left(t_k,t_{k-1},b\left(\cdot\right)A_1\right)W_l\left(t_k,t_{k-1},\overline{\alpha}b\left(\cdot\right)A_2\right)\right].$$

Let us realize the scheme (11.8) by the following algorithm:

$$\begin{aligned} {}^l v_{k-2/3} &= W_l\left(t_k,t_{k-1},\overline{\alpha}b\left(\cdot\right)A_1\right)\left({}^l u_{k-1}\right), \\ {}^l w_{k-2/3} &= W_l\left(t_k,t_{k-1},\overline{\alpha}b\left(\cdot\right)A_2\right)\left({}^l u_{k-1}\right), \end{aligned}$$

$$\begin{aligned} {}^l v_{k-1/3} &= W_l\left(t_k,t_{k-1},b\left(\cdot\right)A_2\right)\left({}^l v_{k-2/3}\right), \\ {}^l w_{k-1/3} &= W_l\left(t_k,t_{k-1},b\left(\cdot\right)A_1\right)\left({}^l w_{k-2/3}\right), \end{aligned}$$

$$\begin{aligned} {}^l v_k &= W_l\left(t_k,t_{k-1},\alpha b\left(\cdot\right)A_2\right)\left({}^l v_{k-1/3}\right), \\ {}^l w_k &= W_l\left(t_k,t_{k-1},\alpha b\left(\cdot\right)A_2\right)\left({}^l w_{k-1/3}\right), \end{aligned}$$

$${}^l u_k = \frac{1}{2}\left[{}^l v_k + {}^l w_k\right], \quad {}^l u_0 = \varphi,$$

2. Error estimation of approximation solution

The following theorem takes place:

Theorem 11.1. *Let the following conditions be satisfied:*

(a) *There exists such* $\tau_0 > 0$, *that for any* $0 \le \tau \le \tau_0$ *there exist operators* $\left(I+\gamma\lambda_k\tau A_j\right)^{-1}$, $j = 1,2$, $\gamma = 1,\alpha,\overline{\alpha}$ *and they are bounded, besides the following inequalities are true:*

$$\left\|W_l\left(\tau,\gamma A_j\right)\right\| \le e^{\omega\tau}, \quad \omega = const > 0, \quad l = 1,2;$$

(b) *There exist the solving operator* $U(t,t_0;\gamma b\left(\cdot\right)A_j)$, $\gamma = 1,\alpha,\overline{\alpha}$ $(j = 0,1,2)$ *of the following problem:*

$$\frac{dv\left(t\right)}{dt} + \gamma b\left(t\right)A_j v\left(t\right) = 0, \quad t \ge t_0 \ge 0, \quad v\left(t_0\right) = \varphi \in D\left(A_j\right),$$

and the following inequality is true:

$$\left\|U(t,t_0;\gamma b\left(\cdot\right)A_j)\right\| \le e^{\omega(t-t_0)},$$

$$\left\|U(t,t_0;b\left(\cdot\right)A_0)\right\| \le Me^{\omega(t-t_0)}, \quad M,\omega = const > 0;$$

(c) $b\left(t\right) \ge b_0 > 0$ *and* $b\left(t\right) \in C^3[0;\infty)$;

(d) $U\left(s_1,s_2;b\left(\cdot\right)A_0\right)\varphi \in D\left(A_0^4\right)$ *for any fixed* $s_1,s_2 \ge 0$.

Then the following estimation holds:

$$\left\|u(t_k) - {}^l u_k\right\| \le ce^{\omega_0 t_k}t_k\tau^3 \sup_{s_1,s_2\in[0,t_k]}\left\|U(s_1,s_2,A)\varphi\right\|_{A_0^4}, \quad l = 1,2,$$

where c,ω_0 *are positive constants.*

Let us prove the theorem in case of $l = 1$ (in case of $l = 2$ the theorem can be proved analogously). Let us prove the auxiliary lemmas on which the prove of the theorem is based.

Lemma 11.2. *If the condition (a) of the theorem is satisfied, then for the operator* $W_1(t_i, t_{i-1}, b(\cdot) A_0)$ *the following decomposition is true:*

$$\begin{aligned} W_1(t_i, t_{i-1}, b(\cdot) A_0) &= \sum_{j=0}^{k-1} (-1)^j \frac{t^j}{j!} \gamma_{j,i} A_0^j \\ &\quad + R_{W_1,k}(t_i, t_{i-1}, \overline{\alpha} b(\cdot) A_0), \\ k &= 1, 2, 3, 4, \end{aligned} \tag{11.8}$$

where

$$\gamma_{0,i} = 1, \quad \gamma_{1,i} = \frac{3b(t_i) + b\left(t_{i-1/3}\right)}{4}, \quad \gamma_{2,i} = b^2\left(t_{i-1/2}\right), \quad \gamma_{3,i} = b^3\left(t_{i-1/2}\right),$$

and where for the residual member the following estimation holds:

$$\left\| R_{W_1,k}(t_i, t_{i-1}, \overline{\alpha} b(\cdot) A_0) \varphi \right\| \le c e^{\omega_0 t} \tau^k \left\| A_0^k \varphi \right\|, \quad \varphi \in D\left(A_0^k\right). \tag{11.9}$$

Proof. Clearly we have:

$$(I + \gamma A)^{-1} = I - I + (I + \gamma A)^{-1} = I - (I + \gamma A)^{-1}(I + \gamma A - I) =$$

$$= I - \gamma A (I + A)^{-1}.$$

From this for any natural k we can get the following expansion:

$$(I + \gamma A)^{-1} = \sum_{i=0}^{k-1} (-1)^i \gamma^i A^i + \gamma^k A^k (I + \gamma A)^{-1}. \tag{11.10}$$

Let us decompose the rational approximation $W_1(t_i, t_{i-1}, b(\cdot) A_0)$ according to the formula (11.11) up to the first order, we get:

$$W_1(t_i, t_{i-1}, b(\cdot) A_0) = a_i I + b_i (I + \lambda_i \tau A_0)^{-1} + c_i (I + \lambda_i \tau A_0)^{-2}$$

$$= a_i I + b_i I - (b_i + c_i) \lambda_i \tau A_0 (I + \lambda_i \tau A_0)^{-1} + c_i I - c_i \lambda_i \tau A_0 (I + \lambda_i \tau A_0)^{-2}$$

$$= (a_i + b_i + c_i) I + R_{W_1,1}(t_i, t_{i-1}, b(\cdot) A_0), \tag{11.11}$$

where

$$R_{W_1,1}(t_i, t_{i-1}, b(\cdot) A_0) = -(b_i + c_i) \lambda_i \tau A_0 (I + \lambda_i \tau A_0)^{-1} - c_i \lambda_i \tau A_0 (I + \lambda_i \tau A_0)^{-2}.$$

According to the condition (a) of the theorem we have:

$$\left\| R_{W_1,1}(t_i, t_{i-1}, b(\cdot) A_0) \varphi \right\| \le c\tau \left\| A_0 \varphi \right\|, \quad c = const > 0. \tag{11.12}$$

If we insert the values of the parameters a_i, b_i and c_i in (11.12), we get:

$$W_1(\tau, A) = I + R_{W_1,1}\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right). \tag{11.13}$$

Let us decompose the rational approximation $W_1\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right)$ according to the formula (11.11) up to the second order, we get:

$$a_i I + b_i\left(I + \lambda_i \tau A_0\right)^{-1} + c_i\left(I + \lambda_i \tau A_0\right)^{-2} =$$

$$= a_i I + b_i\left[I - \lambda_i \tau A_0 + \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1}\right] +$$

$$+c_i\left(I + \lambda_i \tau A_0\right)^{-1}\left[I - \lambda_i \tau A_0 + \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1}\right] =$$

$$= a_i I + b_i I - b_i \lambda_i \tau A + b_i \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1} +$$

$$+c_i\left(I + \lambda_i \tau A_0\right)^{-1} - c_i \lambda_i \tau A_0\left(I + \lambda \tau A\right)^{-1} + c_i \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1} -$$

$$= \left(a_i + b_i\right) I - b_i \lambda_i \tau A_0 + b_i \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1} +$$

$$+c_i\left[I - \lambda_i \tau A_0 + \lambda_i^2 \tau^2 A^2\left(I + \lambda_i \tau A_0\right)^{-1}\right] -$$

$$-c_i \lambda_i \tau\left[I - \lambda_i \tau A_0\left(I + \lambda_i \tau A_0\right)^{-1}\right] A_0 + \lambda_i^2 \tau^2\left(I + \lambda_i \tau A_0\right)^{-2} A_0^2 =$$

$$= \left(a_i + b_i\right) I - b_i \lambda_i \tau A_0 + b_i \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1} +$$

$$+c_i I - c_i \lambda_i \tau A_0 + c_i \lambda_i^2 \tau^2\left(I + \lambda_i \tau A_0\right)^{-1} A_0^2 -$$

$$-c_i \lambda_i \tau A_0 + c_i \lambda_i^2 \tau^2\left(I + \lambda_i \tau A_0\right)^{-1} A_0^2 + \lambda_i^2 \tau^2\left(I + \lambda_i \tau A_0\right)^{-2} A_0^2 =$$

$$= \left(a_i + b_i + c_i\right) I - \left(b_i + 2c_i\right) \lambda_i \tau A_0 + R_{W_1,2}\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right), \tag{11.14}$$

where

$$\begin{aligned} R_{W_1,2}\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right) &= \left(b_i + 2c_i\right) \lambda_i^2 \tau^2 A_0^2\left(I + \lambda_i \tau A_0\right)^{-1} \\ &\quad + \lambda_i^2 \tau^2\left(I + \lambda_i \tau A_0\right)^{-2} A_0^2. \end{aligned}$$

According to the condition (a) of the theorem we have:

$$\left\| R_{W_1,2}\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right) \varphi \right\| \leq c\tau^2 \left\| A_0^2 \varphi \right\|. \tag{11.15}$$

If we insert the values of the parameters a_i, b_i and c_i in (11.15), we get:

$$\begin{aligned} W_1(t_i, t_{i-1,} b\left(\cdot\right) A_0) &= I - \tau \frac{3b\left(t_i\right) + b\left(t_{i-1/3}\right)}{4} A_0 \\ &\quad + R_{W_1,2}\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right). \end{aligned} \tag{11.16}$$

Let us decompose the rational approximation $W_1\left(t_i, t_{i-1}, b\left(\cdot\right) A_0\right)$ according to the formula (11.11) up to the third order, we get:

$$a_i I + b_i\left(I + \lambda_i \tau A_0\right)^{-1} + c_i\left(I + \lambda_i \tau A_0\right)^{-2}$$

$$= a_i I + b_i\left[I - \lambda_i \tau A_0 + \lambda_i^2 \tau^2 A_0^2 - \lambda_i^3 \tau^3 A_0^3\left(I + \lambda_i \tau A_0\right)^{-1}\right]$$

$$+c_i\left(I+\lambda_i\tau A_0\right)^{-1}\left[I-\lambda_i\tau A_0+\lambda_i^2\tau^2A_0^2-\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\right]$$

$$\begin{aligned}
&= a_iI+b_iI-b_i\lambda_i\tau A_0+b_i\lambda_i^2\tau^2A_0^2-b_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\\
&\quad +c_i\left(I+\lambda_i\tau A_0\right)^{-1}-c_i\lambda_i\tau A_0\left(I+\lambda_i\tau A_0\right)^{-1}\\
&\quad +c_i\lambda_i^2\tau^2A_0^2\left(I+\lambda_i\tau A_0\right)^{-1}-c_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-2}\\
&=\left(a_i+b_i\right)I-b_i\lambda_i\tau A_0+b_i\lambda_i^2\tau^2A_0^2-b_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\\
&\quad +c_i\left[I-\lambda_i\tau A_0+\lambda_i^2\tau^2A_0^2-\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\right]\\
&\quad -c_i\lambda_i\tau\left[I-\lambda_i\tau A_0+_0\lambda_i^2\tau^2A_0^2\left(I+\lambda_i\tau A_0\right)^{-1}\right]A_0\\
&\quad +\lambda_i^2\tau^2\left[I-\lambda_i\tau A_0\left(I+\lambda_i\tau A_0\right)^{-1}\right]A_0^2-c_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-2}\\
&=\left(a_i+b_i\right)I-b_i\lambda_i\tau A_0+b_i\lambda_i^2\tau^2A_i^2-b_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\\
&\quad +c_iI-c_i\lambda_i\tau A_0+c_i\lambda_i^2\tau^2A_0^2-c_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\\
&\quad -c_i\lambda_i\tau A_0+c_i\lambda_i^2\tau^2A_0^2-c_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}\\
&\quad +c_i\lambda_i^2\tau^2A_0^2-c_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}-c_i\lambda_i^3\tau^3A_0^3\left(I+\lambda_i\tau A_0\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \left(a_i+b_i+c_i\right)I-\left(b_i+2c_i\right)\lambda_i\tau A_0+\left(b_i+3c_i\right)\lambda_i^2\tau^2A_0^2\\
&\quad +R_{W_1,3}\left(t_i,t_{i-1},b\left(\cdot\right)A_0\right),
\end{aligned}\tag{11.17}$$

where

$$\begin{aligned}
R_{W_1,3}\left(t_i,t_{i-1},b\left(\cdot\right)A_0\right) &= -\left(b_i+3c_i\right)\lambda_i^3\tau^3\left(I+\lambda_i\tau A_0\right)^{-1}A_0^3\\
&\quad -c_i\lambda_i^3\tau^3\left(I+\lambda_i\tau A_0\right)^{-2}A_0^3,
\end{aligned}$$

According to the condition (a) of the theorem we have:

$$\left\|R_{W_1,3}\left(t_i,t_{i-1},b\left(\cdot\right)A_0\right)\varphi\right\|\le c\tau^3\left\|A_0^3\varphi\right\|.\tag{11.18}$$

If we insert the values of the parameters a_i, b_i and c_i in (11.18), we get:

$$\begin{aligned}
W_1\left(t_i,t_{i-1},b\left(\cdot\right)A_0\right) &= I-\tau\frac{3b\left(t_i\right)+b\left(t_{i-1/3}\right)}{4}A_0+\\
&\quad +\frac{1}{2}\tau^2b^2\left(t_{i-1/2}\right)A_0^2+R_{W_1,3}\left(t_i,t_{i-1},b\left(\cdot\right)A_0\right).
\end{aligned}\tag{11.19}$$

And finally let us decompose the rational approximation $W_1\left(t_i,t_{i-1},b\left(\cdot\right)A_0\right)$ according to the formula (11.11) up to the fourth order, we get:

$$a_iI+b_i\left(I+\lambda_i\tau A_0\right)^{-1}+c_i\left(I+\lambda_i\tau A_0\right)^{-2}=a_iI$$

$$+b_i\left[I-\lambda_i\tau A_0+\lambda_i^2\tau^2A_0^2-\lambda_i^3\tau^3A_0^3+\lambda_i^4\tau^4A_0^4\left(I+\lambda_i\tau A_0\right)^{-1}\right]+$$

$$+c_i\left(I+\lambda_i\tau A_0\right)^{-1}\left[I-\lambda_i\tau A_0+\lambda_i^2\tau^2A_0^2-\lambda_i^3\tau^3A_0^3+\lambda_i^4\tau^4A_0^4\left(I+\lambda_i\tau A_0\right)^{-1}\right]=$$

$$
\begin{gathered}
= a_i I + b_i I - b_i\lambda_i\tau A_0 + b_i\lambda_i^2\tau^2 A_0^2 - b_i\lambda_i^3\tau^3 A_0^3 + b_i\lambda_i^4\tau^4 A_0^4 (I+\lambda_i\tau A_0)^{-1} + \\
+c_i (I+\lambda_i\tau A_0)^{-1} - c_i\lambda_i\tau A_0 (I+\lambda_i\tau A_0)^{-1} + c_i\lambda_i^2\tau^2 A_0^2 (I+\lambda_i\tau A_0)^{-1} - \\
-c_i\lambda_i^3\tau^3 A_0^3 (I+\lambda_i\tau A_0)^{-1} + c_i\lambda_i^4\tau^4 A_0^4 (I+\lambda_i\tau A_0)^{-2} = \\
= (a_i+b_i) I - b_i\lambda_i\tau A_0 + b_i\lambda_i^2\tau^2 A_0^2 - b_i\lambda_i^3\tau^3 A_0^3 + b_i\lambda_i^4\tau^4 A_0^4 (I+\lambda_i\tau A_0)^{-1} + \\
+c_i \left[I - \lambda_i\tau A_0 + \lambda_i^2\tau^2 A_0^2 - \lambda_i^3\tau^3 A_0^3 + \lambda_i^4\tau^4 A_0^4 (I+\lambda_i\tau A_0)^{-1}\right] - \\
-c_i\lambda_i\tau \left[I - \lambda_i\tau A_0 + \lambda_i^2\tau^2 A_0^2 - \lambda_i^3\tau^3 A_0^3 (I+\lambda_i\tau A_0)^{-1}\right] A_0 + \\
+\lambda_i^2\tau^2 \left[I - \lambda_i\tau A_0 + \lambda_i^2\tau^2 A_0^2 (I+\lambda_i\tau A_0)^{-1}\right] A_0^2 - \\
-\lambda_i^3\tau^3 \left[I - \lambda_i\tau A_0 (I+\lambda_i\tau A_0)^{-1}\right] A_0^3 + \lambda_i^4\tau^4 (I+\lambda_i\tau A_0)^{-2} A_0^4 = \\
= (a_i+b_i) I - b_i\lambda_i\tau A_0 + b_i\lambda_i^2\tau^2 A_0^2 - b_i\lambda_i^3\tau^3 A_0^3 + b_i\lambda_i^4\tau^4 A_0^4 (I+\lambda_i\tau A_0)^{-1} + \\
+c_i I - c_i\lambda_i\tau A_0 + c_i\lambda_i^2\tau^2 A_0^2 - c_i\lambda_i^3\tau^3 A_0^3 + c_i\lambda_i^4\tau^4 A_0^4 (I+\lambda_i\tau A_0)^{-1} - \\
-c_i\lambda_i\tau A_0 + c_i\lambda_i^2\tau^2 A_0^2 - c_i\lambda_0^3\tau^3 A_0^3 + c_i\lambda_i^4\tau^4 A_0^3 (I+\lambda_i\tau A_0)^{-1} A_0 + \\
+c_i\lambda_i^2\tau^2 A_0^2 - c_i\lambda_i^3\tau^3 A_0^3 + c_i\lambda_i^4\tau^4 A_0^2 (I+\lambda_i\tau A_0)^{-1} A_0^2 - c_i\lambda_i^3\tau^3 A_0^3 + \\
+c_i\lambda_i^4\tau^4 A_0 (I+\lambda_i\tau A_0)^{-1} A_0^3 + c_i\lambda_i^4\tau^4 (I+\lambda_i\tau A_0)^{-2} A_0^4 = (a_i+b_i+c_i) I -
\end{gathered}
$$

$$
\begin{aligned}
&- (b_i+2c_i)\lambda_i\tau A_0 + (b_i+3c_i)\lambda_i^2\tau^2 A_0^2 - (b_i+4c_i)\lambda_i^3\tau^3 A_0^3 \\
&+R_{W_1,4}(t_i,t_{i-1},b(\cdot)A_0), \qquad (11.20)
\end{aligned}
$$

where

$$
\begin{aligned}
R_{W_1,4}(t_i,t_{i-1},b(\cdot)A_0) &= (b_i+4c_i)\lambda_i^4\tau^4 (I+\lambda_i\tau A_0)^{-1} A_0^4 \\
&\quad +c_i\lambda_i^4\tau^4 (I+\lambda_i\tau A_0)^{-2} A_0^4.
\end{aligned}
$$

According to the condition (a) of the theorem we have:

$$
\|R_{W_1,4}(t_i,t_{i-1},b(\cdot)A_0)\varphi\| \le c\tau^4 \left\|A^4\varphi\right\|. \qquad (11.21)
$$

If we insert the values of the parameters a_i, b_i and c_i in (11.21), we get:

$$
W_1(t_i,t_{i-1},b(\cdot)A_0) = I - \frac{3b(t_i)+b\left(t_{i-1/3}\right)}{4}A_0 +
$$

$$
+\frac{1}{2}\tau^2 b^2\left(t_{i-1/2}\right)A_0^2 - \frac{1}{6}\tau^3 b^3\left(t_{i-1/2}\right)A_0^3 + R_{W_1,4}(t_i,t_{i-1},b(\cdot)A_0). \qquad (11.22)
$$

Uniting the formulas (11.14),(11.17),(11.20) and (11.23) we get the formula (11.9), and uniting the inequalities (11.13), (11.16), (11.19) and (11.22) we obtain the estimation (11.10). □

Lemma 11.3. *If the condition (a) of the theorem 11.1 is satisfied, then the following estimation holds:*

$$
\|U(t_i,t_{i-1};b(\cdot)A_0) - V_1(t_i,t_{i-1})\| \le ce^{\omega_0\tau}\tau^4 \|\varphi\|_{A_0^4}, \qquad (11.23)
$$

where

$$V_1(t_i,t_{i-1})=\frac{1}{2}\left[W_1(t_i,t_{i-1};\alpha b(\cdot)A_1)W_1(t_i,t_{i-1};b(\cdot)A_2)W_1(t_i,t_{i-1};\overline{\alpha}b(\cdot)A_1)+\right.$$

$$\left.+W_1(t_i,t_{i-1};\alpha b(\cdot)A_2)W_1(t_i,t_{i-1};b(\cdot)A_1)W_1(t_i,t_{i-1};\overline{\alpha}b(\cdot)A_2)\right],$$

here c and ω_0 *are positive constants.*

Proof. The following formula is true:

$$U(t_i,t_{i-1};A)=I-\int\limits_{t_{i-1}}^{t_i}A(s_1)U(t_2,s_1;A)\,ds_1,$$

hence we get the following decomposition:

$$U(t_i,t_{i-1};A)=I-\int\limits_{t_{i-1}}^{t_i}A(s_1)\,ds_1+\int\limits_{t_{i-1}}^{t_i}A(s_1)\int\limits_{t_{i-1}}^{s_1}A(s_2)\,ds_2ds_1+...+$$

$$+(-1)^k\int\limits_{t_{i-1}}^{t_i}A(s_1)\int\limits_{t_{i-1}}^{s_1}A(s_2)...\int\limits_{t_{i-1}}^{s_{k-1}}A(s_k)\,ds_k...ds_2ds_1+R_k(t_i,t_{i-1},A),\tag{11.24}$$

where

$$R_k(t_2,t_1,A)=(-1)^k\int\limits_{t_{i-1}}^{t_i}A(s_1)\int\limits_{t_{i-1}}^{s_1}A(s_2)...\int\limits_{t_{i-1}}^{s_{k-1}}U(t_2,s_k;A)A(s_k)\,ds_k...ds_2ds_1.$$

From the equality (10.18) according to the formulas (10.13) and (10.14)

$$U(t_i,t_{i-1};A)=I-\int\limits_{t_{i-1}}^{t_i}b(s)\,dsA_0+\frac{1}{2}\left(\int\limits_{t_{i-1}}^{t_i}b(s)\,ds\right)^2A_0^2-$$

$$-\frac{1}{6}\left(\int\limits_{t_{i-1}}^{t_i}b(s)\,ds\right)^3A_0^3+R_4(t_i,t_{i-1},A).\tag{11.25}$$

If $b(t) \in C^3[0;\infty)$, then the following inequality is true:

$$\begin{aligned}
\left\| \int\limits_{t_{i-1}}^{t_i} b(s)\,ds - \tau \frac{3b(t_i) + b\left(t_{i-1/3}\right)}{4} \right\|_C &\le c\tau^4, \\
\left\| \left(\int\limits_{t_{i-1}}^{t_i} b(s)\,ds \right)^2 - \tau^2 b^2\left(t_{i-1/2}\right) \right\|_C &\le c\tau^4, \\
\left\| \left(\int\limits_{t_{i-1}}^{t_i} b(s)\,ds \right)^3 - \tau^3 b^3\left(t_{i-1/2}\right) \right\|_C &\le c\tau^4.
\end{aligned}$$

According to this inequality from the formula (11.25) we get the following equality:

$$U(t_i, t_{i-1}; A) = I - \tau \frac{3b(t_i) + b\left(t_{i-1/3}\right)}{4} A_0 + \frac{1}{2}\tau^2 b^2\left(t_{i-1/2}\right) A_0^2 -$$

$$-\frac{1}{6}\tau^3 b^3\left(t_{i-1/2}\right) A_0^3 + \widetilde{R}_4(t_i, t_{i-1}, A). \tag{11.26}$$

where for the residual member $\widetilde{R}_4(t_i, t_{i-1}, A)$ the following inequality is true

$$\left\| \widetilde{R}_4(t_i, t_{i-1}, A)\varphi \right\| \le c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A_0^4}. \tag{11.27}$$

Let us decompose all rational approximations in the operator $V_1(t_i, t_{i-1})$ according to the formula (11.9) from the right to left, so that each residual member is of the fourth order. We shall get:

$$V_1(t_i, t_{i-1}) = I - \tau \frac{3b(t_i) + b\left(t_{i-1/3}\right)}{4} A_0 + \frac{1}{2}\tau^2 b^2\left(t_{i-1/2}\right) A_0^2 -$$

$$-\frac{1}{6}\tau^3 b^3\left(t_{i-1/2}\right) A_0^3 + R_{V_1,4}(t_i, t_{i-1}), \tag{11.28}$$

where for the residual member $R_{V_1,4}(t_i, t_{i-1})$ the following inequality is true:

$$\left\| R_{V_1,4}(t_i, t_{i-1})\varphi \right\| \le c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A_0^4}. \tag{11.29}$$

From the equalities (11.27) and (11.29) according to the inequalities (11.28) and (11.30) we can get the sought estimation. □

Let us return to the proof of the Theorem 11.1.

Let us introduce the solution of the decomposed problem as follows:

$${}^1u_k = L_1(t_k, t_0)\varphi, \tag{11.30}$$

where

$$L_1\left(t_i,t_j\right)=V_1\left(t_i,t_{i-1}\right)V_1\left(t_{i-1},t_{i-2}\right)...V_1\left(t_{j+1},t_j\right), \quad i>j.$$

Let us estimate the operator $L_1\left(t_i,t_j\right) \quad (i>j)$:

$$\left\|L_1\left(t_i,t_j\right)\right\| \leq \left\|V_1\left(t_i,t_{i-1}\right)\right\| \left\|V_1\left(t_{i-1},t_{i-2}\right)\right\| ... \left\|V_1\left(t_{j+1},t_j\right)\right\| \leq$$

$$\leq e^{\omega_0\tau}e^{\omega_0\tau}...e^{\omega_0\tau} \leq e^{\omega_0(i-j)\tau}. \tag{11.31}$$

From the equalities (10.2) and (11.31) according to the inequalities (11.24) and (11.32) we get:

$$\left\|u(t_k)-{}^1u_k\right\| = \left\|\left[L_1\left(t_k,0\right)-U\left(t_k,0;b\left(\cdot\right)A_0\right)\right]\varphi\right\| =$$

$$=\sum_{i=1}^{k}\left\|L_1\left(t_k,t_i\right)\left[V_1\left(t_i,t_{i-1}\right)-U\left(t_i,t_{i-1};b\left(\cdot\right)A_0\right)\right]U\left(t_{i-1},0;b\left(\cdot\right)A_0\right)\varphi\right\| \leq$$

$$\leq\sum_{i=1}^{k}\left\|L_1\left(t_k,t_i\right)\right\| \left\|\left[V_1\left(t_i,t_{i-1}\right)-U\left(t_i,t_{i-1};b\left(\cdot\right)A_0\right)\right]U\left(t_{i-1},0;b\left(\cdot\right)A_0\right)\varphi\right\| \leq$$

$$\leq\sum_{i=1}^{k}ce^{\omega_0(k-i)\tau}ce^{\omega_0\tau}\left\|U\left(t_{i-1},0;b\left(\cdot\right)A_0\right)\varphi\right\|_{A_0^4} \leq$$

$$\leq ce^{\omega_0t_k}t_k\tau\sup_{s_1,s_2\in[0,t_k]}\left\|U(s_1,s_2;A)\varphi\right\|_{A_0^4}. \qquad \square$$

Appendix

In the appendix there are given results of numerical calculations for heat transfer equation. These calculations are carried out using existing first and second order and constructed in this work third order accuracy decomposition schemes. Comparative analysis of numerical calculations for different order decomposition schemes is carried out.

$$\begin{aligned}
\frac{\partial u(t,x,y)}{\partial t} &= a(x,y)\frac{\partial^2 u(t,x,y)}{\partial x^2} + b(x,y)\frac{\partial^2 u(t,x,y)}{\partial y^2} + f(t,x,y),\\
t &> 0, \quad (x,y) \in [0,1]\times[0,1],\\
u(0,x,y) &= \varphi(x,y),\\
u(t,0,y) &= u(t,1,y) = u(t,x,0) = u(t,x,1) = 0.
\end{aligned}$$

There are calculated the following test problems:

Test 1.

$$\begin{aligned}
f(t,x,y) &= 0,\\
\varphi(x,y) &= \sin(\pi x)\sin(\pi y),\\
a(x,y) &= b(x,y) = 1.
\end{aligned}$$

The solution of this problem is

$$u(t,x,y) = e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y).$$

The interesting point of this test is that with increase of t the solution decreases very rapidly (tends to machine zero) and for this reason it is very difficult to catch the solution behavior. The suggested third order scheme makes possible to achieve good precision, what is confirmed by calculations (see tables: 1-4).

In the tables 4-5 there are shown results of calculations of test 1 according to semi-discrete analog of first order accuracy averaged differential scheme (see [28]):

$$\begin{aligned}
\frac{v_k - v_{k-1}}{\tau} + 2A_1 v_k &= 0, \quad v_{k-1} = u_{k-1},\\
\frac{w_k - w_{k-1}}{\tau} + 2A_2 v_k &= 0, \quad w_{k-1} = u_{k-1},
\end{aligned}$$

$$u_k = \frac{1}{2}(v_k + w_k), \quad k = 1,2,..., \quad u_0 = \varphi. \tag{1}$$

u_k is given as an approximate solution at the point $t = t_k = k\tau$.

As it can be seen from the above-mentioned tables, this scheme cannot catch the behavior of the problem in test 1.

In the tables 6-7 there are shown results of calculations of test 1 according to semi-discrete analog of second order accuracy averaged differential scheme (see [28]), which is constructed on the basis of Crank-Nickolson scheme:

$$
\begin{aligned}
\frac{v_k^1 - v_{k-1}^1}{\tau} + \frac{1}{2} A_1 \left(v_k^1 + v_{k-1}^1 \right) &= 0, \quad v_{k-1}^1 = u_{k-1}, \\
\frac{v_k^2 - v_{k-1}^2}{\tau} + \frac{1}{2} A_2 \left(v_k^2 + v_{k-1}^2 \right) &= 0, \quad v_{k-1}^2 = v_k^1, \\
\frac{w_k^1 - w_{k-1}^1}{\tau} + \frac{1}{2} A_2 \left(w_k^1 + w_{k-1}^1 \right) &= 0, \quad w_{k-1}^1 = u_{k-1}, \\
\frac{w_k^2 - w_{k-1}^2}{\tau} + \frac{1}{2} A_2 \left(w_k^2 + w_{k-1}^2 \right) &= 0, \quad w_{k-1}^2 = w_k^1,
\end{aligned}
$$

$$
u_k = \frac{1}{2} \left(v_k^2 + w_k^2 \right), \quad k = 1, 2, \ldots, \quad u_0 = \varphi. \tag{2}
$$

u_k is given as an approximate solution at the point $t = t_k$.

As it can be seen from the mentioned tables, this scheme catches behavior of problem in test 1 with satisfying precision, but the results are much worse than those of calculations by scheme (2.6)(see table 1-3).

In the tables 8-9 there are shown results of calculations of test 1 according to semi-discrete analog of second order accuracy symmetrized differential scheme (see [3], [4]), which is constructed on the basis of Crank-Nickolson scheme:

$$
\begin{aligned}
\frac{v_k^1 - v_{k-1}^1}{\tau} + \frac{1}{4} A_1 \left(v_k^1 + v_{k-1}^1 \right) &= 0, \quad v_{k-1}^1 = u_{k-1}, \\
\frac{v_k^2 - v_{k-1}^2}{\tau} + \frac{1}{2} A_2 \left(v_k^2 + v_{k-1}^2 \right) &= 0, \quad v_{k-1}^2 = v_k^1, \\
\frac{v_k^3 - v_{k-1}^3}{\tau} + \frac{1}{4} A_1 \left(v_k^3 + v_{k-1}^3 \right) &= 0, \quad v_{k-1}^3 = v_k^2
\end{aligned}
$$

$$
u_k = v_k^3, \quad k = 1, 2, \ldots, \quad u_0 = \varphi. \tag{3}
$$

As an approximate solution at point $t = t_k$ is taken u_k.

As it can be seen from the mentioned tables, this scheme, as well as the previous one, catches behavior of problem in test 1 with satisfying precision. The results are a bit better than results in tables 6-7, but much worse than than those of calculations by scheme (2.6) (see tables 1-3).

To fully represent a comparison of the above-mentioned schemes it is important to construct the solution which is exact with regard to spatial coordinates.

For the considered tests the solutions of semi-discrete split problems (1),

(2) and (3) are given by the following formulas:

$$u_k = \left(\frac{1}{1+2\pi^2\tau}\right)^k \sin(\pi x)\sin(\pi y),$$

$$u_k = \left(\frac{1-0.5\pi^2\tau}{1+0.5\pi^2\tau}\right)^{2k} \sin(\pi x)\sin(\pi y),$$

$$u_k = \left(\frac{(1-0.5\pi^2\tau)(1-0.25\pi^2\tau)^2}{(1+0.5\pi^2\tau)(1+0.25\pi^2\tau)^2}\right)^k \sin(\pi x)\sin(\pi y).$$

Solution of the problem (2.6) is given by the following formula:

$$u_k = \left(\frac{\left(1-\frac{1}{3}\pi^2\tau\right)\left(1-\frac{1}{3}\pi^2\tau+\frac{1}{27}\pi^4\tau^2\right)}{\left(1+\frac{2}{3}\pi^2\tau+\frac{1}{6}\pi^4\tau^2\right)\left(1+\frac{2}{3}\pi^2\tau+\frac{11}{54}\pi^4\tau^2+\frac{1}{27}\pi^6\tau^3+\frac{1}{324}\pi^8\tau^4\right)}\right)^k \times \sin(\pi x)\sin(\pi y).$$

By comparing these solutions to exact solution we can see that coefficients of $\sin(\pi x)\sin(\pi y)$ approximate $e^{-2\pi^2 t}$ respectively by first, second, second and third order precision with regard to time step and this is shown on tables 10-17.

Test 2.

$$f(t,x,y) = \left(\pi^2\left(2+m_2^2 a(x,y)+m_3^2 b(x,y)\right)\sin(m_1\pi t)+m_1\pi\cos(m_1\pi t)\right) \times e^{2\pi^2 t}\sin(m_2\pi x)\sin(m_3\pi y),$$

$$\varphi(x,y) = 0,$$

$$a(x,y) = 2+\sin(\pi x)\sin(\pi y),$$

$$b(x,y) = 2+0.5\sin(\pi x)\sin(\pi y).$$

Solution of the problem is $u(t,x,y) = e^{2\pi^2 t}\sin(m_1\pi t)\sin(m_2\pi x)\sin(m_3\pi y)$.

The interesting point of this test is that increase of parameters $m1, m2$ and $m3$ yields to rapid sign-changing oscillation of solution. In addition, on expense of parameters changing we can regulate oscillation frequency according to time and spatial coordinates. As the algorithm provides high order accuracy with regards to time coordinate, therefore it is natural that oscillation with regards to t is much higher. It is obvious that multiplier $e^{2\pi^2 t}$ with increase of t yields to rapid increase of oscillation amplitude. This factor along with oscillation makes essentially difficult to catch solution behavior and for this reason it is necessary to use high order accuracy schemes. This can be confirmed by tables 18-21.

Test 3.

$$f(t,x,y) = e^{m\pi t}\pi\left(m+\pi\left(a(x,y)+b(x,y)\right)\right)\sin(\pi x)\sin(\pi y),$$

$$\varphi(x,y) = 0,$$

$$a(x,y) = 2+\sin(\pi x)\sin(\pi y),$$

$$b(x,y) = 2+0.5\sin(\pi x)\sin(\pi y).$$

Solution of the problem is $u(t,x,y) = e^{m\pi t}\sin(\pi x)\sin(\pi y)$.

On Fig.1, there is given a dependance of the relative error of the approximated solution on the logarithm of time step (on the horizontal axis it is logarithm of time step, and on the vertical axis it is relative error of the approximated solution). On Fig.2, there is given a dependance of the absolute error of the approximated solution on the logarithm of time step (On the horizontal axis it is logarithm of time step, and on the vertical axis it is absolute error of the approximated solution). On the both figures the calculations are carried out for the following values of the time step: $\tau_k = 1/N_k$, $N_k = \left[10 * 1.2^k\right]$, $k = 0, 1, ..., 30$, and the spatial step is constant $h_x = h_y = 0.001$. On the both figures there are given three cases: $m = 1$, $m = 3$ and $m = 5$. Our aim was to find the convergence rate of the method by means of the numerical experiment. If the method is of third order, then, starting from some value of τ, the graph of the function (logarithm of solution error) have to approach to the straight line, the tangent of which equals three. On the both figures it is clearly seen that, starting from $\tau = 0.01$ ($Log(\tau) = -2$), the graph approaches to the straight line, the tangent of which equals to three with the sufficient accuracy, and this verifies the theoretical result proved in the article.

Let us also note that, for the approximation of the second derivatives according to the spatial variables, there is used classical difference formulas. It is obvious that $u_1, u_2, ..., u_k$ are complex functions, but their complex parts are of $O(\tau^3)$ order.

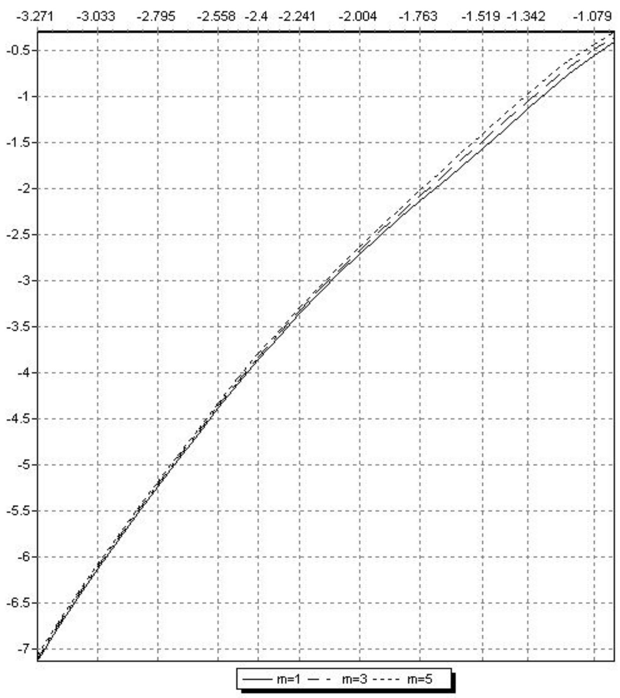

Fig 1. Dependence of the relative error on the time step

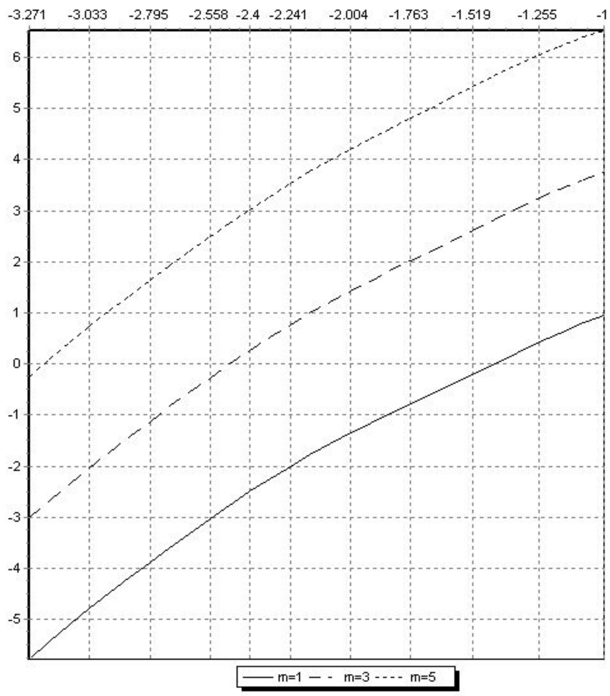

Fig 2. Dependence of the absolute error on the time step

In the tables below u is an exact solution, and $\tilde{u}$ is an approximate solution.

Third order of accuracy decomposition scheme - (2.6) **Table 1**

(x,y)=(0.5,0.5) tau=1/64, h=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert(u-\tilde{u})/u\rvert$
0.125	8479.62 E-05	8480.53 E-05	9.1 E-06	0.11 E-03
0.250	719.03 E-05	719.19 E-05	1.6 E-06	0.21 E-03
0.375	609.71 E-06	609.91 E-06	2.0 E-07	0.32 E-03
0.500	517.01 E-07	517.23 E-07	2.2 E-08	0.43 E-03
0.625	438.40 E-08	438.64 E-08	2.4 E-09	0.53 E-03
0.750	371.75 E-09	371.99 E-09	2.4 E-10	0.64 E-03
0.875	315.23 E-10	315.46 E-10	2.3 E-11	0.75 E-03
1.000	267.30 E-11	267.53 E-11	2.3 E-12	0.85 E-03

Third order of accuracy decomposition scheme - (2.6) **Table 2**

(x,y)=(0.5,0.5) tau=1/100, h=1/128				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert(u-\tilde{u})/u\rvert$
0.1	13891.54 E-05	13891.11 E-05	4.3 E-06	0.31 E-04
0.2	1929.75 E-05	1929.63 E-05	1.2 E-06	0.62 E-04
0.3	2680.72 E-06	2680.50 E-06	2.2 E-07	0.82 E-04
0.4	3723.93 E-07	3723.47 E-07	4.6 E-08	0.12 E-03
0.5	5173.11 E-08	5172.32 E-08	7.9 E-09	0.15 E-03
0.6	718.62 E-08	718.49 E-08	1.3 E-09	0.18 E-03
0.7	998.28 E-09	998.07 E-09	2.1 E-10	0.21 E-03
0.8	1386.77 E-10	1386.43 E-10	3.4 E-11	0.25 E-03
0.9	1926.43 E-11	1925.90 E-11	5.3 E-12	0.28 E-03
1.0	2676.11E-12	2675.29 E-12	8.2 E-13	0.31 E-03

Third order of accuracy decomposition scheme - (2.6) **Table 3**

(x,y)=(0.5,0.5) tau=1/100, h=1/142				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert(u-\tilde{u})/u\rvert$
0.1	13891.28 E-05	13891.11 E-05	1.7 E-06	0.12 E-04
0.2	19296.77 E-06	19296.30 E-06	4.7 E-07	0.24 E-04
0.3	26805.69 E-07	26804.71 E-07	9.8 E-08	0.37 E-04
0.4	3723.65 E-07	3723.47 E-07	1.8 E-08	0.48 E-04
0.5	5172.63 E-08	5172.32 E-08	3.1 E-09	0.60 E-04
0.6	7185.45 E-09	7184.90 E-09	5.5 E-10	0.77 E-04
0.7	9981.51 E-10	9980.66 E-10	8.5 E-11	0.85 E-04
0.8	1386.56 E-10	1386.43 E-10	1.3 E-11	0.94 E-04
0.9	1926.11 E-11	1925.90 E-11	2.1 E-12	0.11 E-03
1.0	2675.61 E-12	2675.29 E-12	3.2 E-13	0.12 E-03

First order of accuracy averaged decomposition scheme – (1) **Table 4**

(x,y)=(0.5,0.5) tau=1/64, h=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.125	1.16 E-01	0.85 E-01	3.1 E-02	0.36
0.250	0.14 E-01	0.72 E-02	6.8 E-03	0.94
0.375	0.16 E-02	0.61 E-03	9.9 E-04	1.6
0.500	0.18 E-03	0.52 E-04	1.3 E-04	2.5
0.625	0.21 E-04	0.44 E-05	1.7 E-05	3.8
0.750	0.25 E-05	0.37 E-06	2.1 E-06	5.8
0.875	0.29 E-06	0.32 E-07	2.6 E-07	8.1
1.000	0.34 E-07	0.27 E-08	3.1 E-08	11

First order of accuracy averaged decomposition scheme – (1) **Table 5**

(x,y)=(0.5,0.5) tau=1/100, h=1/128				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	1.65 E-01	1.39 E-01	2.6 E-02	0.19
0.2	2.72 E-02	1.93 E-02	7.9 E-03	0.41
0.3	0.45 E-02	0.27 E-02	1.8 E-03	0.67
0.4	0.74 E-03	0.37 E-03	3.7 E-04	1.0
0.5	0.12 E-03	0.52 E-04	6.8 E-05	1.3
0.6	0.20 E-04	0.72 E-05	1.3 E-05	1.8
0.7	0.33 E-05	0.10 E-05	2.3 E-06	2.3
0.8	0.55 E-06	0.14 E-06	4.1 E-07	2.9
0.9	0.91 E-07	0.19 E-07	7.2 E-08	3.8
1.0	0.15 E-07	0.27 E-08	1.2 E-08	4.6

Second order of accuracy averaged decomposition scheme - (2) **Table 6**

(x,y)=(0.5,0.5) tau=1/64, h=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.125	84.41 E-03	84.80 E-03	3.9 E-04	0.46 E-02
0.250	71.24 E-04	71.92 E-04	6.8 E-05	0.95 E-02
0.375	60.14 E-05	60.99 E-05	8.5 E-06	0.14 E-01
0.500	50.76 E-06	51.72 E-06	9.6 E-07	0.19 E-01
0.625	4.28 E-06	4.39 E-06	1.1 E-07	0.25 E-01
0.750	3.62 E-07	3.72 E-07	1.0 E-08	0.27 E-01
0.875	3.05 E-08	3.15 E-08	1.0 E-09	0.32 E-01
1.000	2.58 E-09	2.68 E-09	1.0 E-10	0.37 E-01

Second order of accuracy averaged decomposition scheme - (2) **Table 7**

(x,y)=(0.5,0.5) tau=1/100, h=1/128				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	138.70 E-03	138.91 E-03	2.1 E-04	0.15 E-02
0.2	192.38 E-04	192.96 E-04	5.8 E-05	0.30 E-02
0.3	26.68 E-04	26.81 E-04	1.3 E-05	0.49 E-02
0.4	37.01 E-05	37.23 E-05	2.2 E-06	0.59 E-02
0.5	51.34 E-06	51.72 E-06	3.8 E-07	0.74 E-02
0.6	71.20 E-07	71.85 E-07	6.5 E-08	0.91 E-02
0.7	9.88 E-07	9.98 E-07	1.0 E-08	0.10 E-01
0.8	13.69 E-08	13.86 E-08	1.7 E-09	0.13 E-01
0.9	18.99 E-09	19.26 E-09	2.7 E-10	0.14 E-01
1.0	26.34 E-10	26.75 E-10	4.1 E-11	0.15 E-01

Second order of accuracy symetrised decomposition scheme - (3) **Table 8**

(x,y)=(0.5,0.5) tau=1/64, h=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.125	84.56 E-03	84.80 E-03	2.4 E-04	0.28 E-02
0.250	71.51 E-04	71.92 E-04	4.1 E-05	0.57 E-02
0.375	60.47 E-05	60.99 E-05	5.2 E-06	0.85 E-02
0.500	51.13 E-06	51.72 E-06	5.9 E-07	0.11 E-01
0.625	43.24 E-07	43.86 E-07	6.2 E-08	0.14 E-01
0.750	36.57 E-08	37.20 E-08	6.3 E-09	0.17 E-01
0.875	30.93 E-09	31.55 E-09	6.2 E-10	0.20 E-01
1.000	26.16 E-10	26.75 E-10	5.9 E-11	0.23 E-01

Second order of accuracy symetrised decomposition scheme - (3) **Table 9**

(x,y)=(0.5,0.5) tau=1/100, h=1/128				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	138.79 E-03	138.91 E-03	1.2 E-04	0.86 E-03
0.2	192.61 E-04	192.96 E-04	3.5 E-05	0.18 E-02
0.3	267.32 E-05	268.05 E-05	7.3 E-06	0.27 E-02
0.4	37.10 E-05	37.23 E-05	1.3 E-06	0.35 E-02
0.5	51.49 E-06	51.72 E-06	2.3 E-07	0.45 E-02
0.6	71.46 E-07	71.85 E-07	3.9 E-08	0.54 E-02
0.7	99.18E-07	99.80 E-07	6.2 E-09	0.62 E-02
0.8	13.76E-08	13.86 E-08	1.0 E-09	0.72 E-02
0.9	19.10E-09	19.26 E-09	1.6 E-10	0.83 E-02
1.0	26.51E-10	26.75 E-10	2.4 E-11	0.90 E-02

First order of accuracy averaged decomposition scheme - (1) **Table 10**

(x,y)=(0.5,0.5) tau=1/10				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	0.34 E 00	0.14 E 00	2.0 E-01	1.4 E 00
0.2	0.11 E 00	0.19 E-01	9.1 E-02	4.8 E 00
0.3	0.38 E-01	0.27 E-02	3.5 E-02	1.3 E 01
0.4	0.13 E-01	0.37 E-03	1.3 E-02	3.4 E 01
0.5	0.43 E-02	0.52 E-04	4.2 E-03	8.1 E 01
0.6	0.15 E-02	0.72 E-05	1.5 E-03	2.1 E 02
0.7	0.49 E-03	0.10 E-05	4.9 E-04	4.9 E 02
0.8	0.16 E-03	0.14 E-06	1.6 E-04	1.1 E 03
0.9	0.55 E-04	0.19 E-07	5.5 E-05	2.9 E 03
1.0	0.18 E-05	0. 27 E-08	1.8 E-05	6.7 E 03

Second order of accuracy averaged decomposition scheme - (2) **Table 11**

(x,y)=(0.5,0.5) tau=1/10				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	1.24 E-01	1.39 E-01	1.5 E-02	0.11
0.2	1.53 E-02	1.93 E-02	4.0 E-03	0.21
0.3	1.90 E-03	2.68 E-03	7.8 E-04	0.29
0.4	0.23 E-03	0.37 E-03	1.4 E-04	0.38
0.5	0.29 E-04	0.52 E-04	2.3 E-05	0.44
0.6	0.36 E-05	0.72 E-05	3.6 E-06	0.50
0.7	0.45 E-06	0.10 E-05	5.5 E-07	0.55
0.8	0.55 E-07	0.14 E-06	8.5 E-08	0.61
0.9	0.68 E-08	0.19 E-07	1.2 E-08	0.64
1.0	0.80 E-09	0.27 E-08	1.9 E-09	0.70

Second order of accuracy symetrized decomposition scheme - (3) **Table 12**

(x,y)=(0.5,0.5) tau=1/10				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	1.15 E-01	1.39 E-01	2.4 E-02	0.17
0.2	1.32 E-02	1.93 E-02	6.1 E-03	0.32
0.3	0.15 E-02	0.27 E-02	1.2 E-03	0.44
0.4	0.18 E-03	0.37 E-03	1.9 E-04	0.51
0.5	0.20 E-04	0.52 E-04	3.2 E-05	0.62
0.6	0.23 E-05	0.72 E-05	4.9 E-06	0.68
0.7	0.27 E-06	0.10 E-05	7.3 E-07	0.73
0.8	0.31 E-07	0.14 E-06	1.1 E-07	0.78
0.9	0.3.5 E-09	0.19 E-07	1.6 E-08	0.82
1.0	0.44 E-09	0.27 E-08	2.3 E-09	0.85

Third order of accuracy decomposition scheme - (2.6) **Table 13**

(x,y)=(0.5,0.5) tau=1/10				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert(u-\tilde{u})/u\rvert$
0.1	13.76 E-02	13.89 E-02	1.3 E-03	0.94 E-02
0.2	18.92 E-03	19.30 E-03	3.8 E-04	0.20 E-01
0.3	26.02 E-04	26.80 E-04	7.8E-05	0.30 E-01
0.4	3.58 E-04	3.72 E-04	1.4 E-05	0.38 E-01
0.5	4.92 E-05	5.17 E-05	2.5 E-06	0.48 E-01
0.6	6.77 E-06	7.18 E-06	4.1 E-07	0.57 E-01
0.7	9.32 E-07	1.00 E-07	6.8 E-08	0.68 E-01
0.8	1.28 E-07	1.39 E-07	1.1 E-08	0.80 E-01
0.9	1.76E-08	1.93E-08	1.7 E-09	0.88 E-01
1.0	2.41E-09	2.67E-09	2.6 E-10	0.97 E-01

First order of accuracy averaged decomposition scheme - (1) **Table 14**

(x,y)=(0.5,0.5) tau=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert(u-\tilde{u})/u\rvert$
0.1	1.65 E-01	1.39 E-01	2.6 E-02	0.19
0.2	2.72 E-02	1.93 E-02	7.9 E-03	0.41
0.3	0.45 E-02	0.27 E-02	1.8 E-03	0.67
0.4	0.74 E-03	0.37 E-03	3.7 E-04	1.0
0.5	0.12 E-03	0.52 E-04	6.8 E-05	1.3
0.6	0.20 E-04	0.72 E-05	1.3 E-05	1.8
0.7	0.33 E-05	0.10 E-05	2.3 E-06	2.3
0.8	0.55 E-06	0.14 E-06	4.1 E-07	2.9
0.9	0.91 E-07	0.19 E-07	7.2 E-08	3.8
1.0	0.15 E-07	0.27 E-08	1.2 E-08	4.6

Second order of accuracy symetrised decomposition scheme - (3) **Table 15**

(x,y)=(0.5,0.5) tau=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert(u-\tilde{u})/u\rvert$
0.1	138.69 E-03	138.91 E-03	2.2 E-04	0.16 E-02
0.2	192.34 E-04	192.96 E-04	6.2 E-05	0.32 E-02
0.3	26.68 E-04	26.81 E-04	1.3 E-05	0.48 E-02
0.4	37.00 E-05	37.23 E-05	2.3 E-06	0.62 E-02
0.5	51.31 E-06	51.72 E-06	4.1 E-07	0.79 E-02
0.6	71.16 E-07	71.85 E-07	6.9 E-08	0.10 E-01
0.7	9.87 E-08	99.81 E-08	1.1 E-08	0.11 E-01
0.8	13.68E-08	13.86 E-08	1.8 E-09	0.13 E-01
0.9	18.99E-09	19.26 E-09	2.7 E-10	0.14 E-01
1.0	26.33E-10	26.75 E-10	4.0 E-11	0.15 E-01

Second order of accuracy averaged decomposition scheme - (2) **Table 16**

(x,y)=(0.5,0.5) tau=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	138.77 E-03	138.91 E-03	1.4 E-04	0.10 E-02
0.2	192.58 E-04	192.96 E-04	3.8 E-05	0.20 E-02
0.3	267.24 E-05	268.05 E-05	8.1 E-06	0.30 E-02
0.4	37.09 E-05	37.23 E-05	1.4 E-06	0.38 E-02
0.5	51.46 E-06	51.72 E-06	2.6 E-07	0.50 E-02
0.6	71.42 E-07	71.85 E-07	4.3 E-08	0.60 E-02
0.7	99.11 E-08	99.81 E-08	7.0 E-09	0.70 E-02
0.8	13.75 E-08	13.86 E-08	1.1 E-09	0.79 E-02
0.9	19.09 E-09	19.26 E-09	1.7 E-10	0.88 E-02
1.0	26.49 E-10	26.75 E-10	2.6 E-11	0.97 E-02

Third order of accuracy decomposition scheme - (2.6) **Table 17**

(x,y)=(0.5,0.5) tau=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.1	13890.95 E-05	13891.11 E-05	1.6 E-06	0.12 E-05
0.2	19295.86 E-06	19296.29 E-06	4.3 E-07	0.22 E-05
0.3	26803.80 E-07	26804.71 E-07	9.1 E-08	0.34 E-05
0.4	3723.30 E-07	3723.47 E-07	1.7 E-08	0.46 E-05
0.5	5172.02 E-08	5172.32 E-08	3.0 E-09	0.58 E-05
0.6	7184.41 E-09	7184.90 E-09	4.9 E-10	0.68 E-05
0.7	9979.87 E-10	9980.66 E-10	7.9 E-11	0.79 E-05
0.8	1386.30 E-10	1386.43 E-10	1.3 E-11	0.94 E-05
0.9	1925.70 E-11	1925.90 E-11	2.0 E-12	0.10 E-04
1.0	2672.99 E-12	2673.29 E-12	3.0 E-13	0.11 E-04

Third order of accuracy decomposition scheme - (2.6) **Table 18**

(m1, m2,m3)=(17,1,1) (x,y)=(0.5,0.5) tau=1/500, h=1/64				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.114	-183.78 E-02	-183.68 E-02	1.0 E-03	0.54 E-03
0.116	-86.83 E-02	-86.73 E-02	1.0 E-03	0.11 E-02
0.118	19.26 E-02	19.36 E-02	1.0 E-03	0.50 E-02
0.120	1338.09 E-03	1339.00 E-03	9.1 E-04	0.68 E-03
…	…	…	…	…
0.232	-170.69 E-01	-170.58 E-01	1.1 E-02	0.60 E-03
0.234	-70.12 E-01	-70.02 E-01	1.0 E-02	0.14 E-03
0.236	396.56 E-02	397.53 E-02	9.7 E-03	0.25 E-03
0.238	1579.16 E-02	1580.09 E-02	9.3 E-03	0.59 E-03
…	…	…	…	…
0.350	-1566.80 E-01	-1565.86 E-01	9.4 E-02	0.60 E-03
0.352	-52.42 E 00	-52.32 E 00	1.0 E-01	0.20 E-02
0.354	611.23 E-01	612.21 E-01	9.8 E-02	0.16 E-02
0.356	183.16 E 00	183.26 E 00	1.0 E-01	0.56 E-03
…	…	…	…	…
0.468	-141.75 E 01	141.65 E 01	1.0 E 00	0.74 E-03
0.470	-33.69 E 01	-33.59 E 01	1.0 E 00	0.31 E-02
0.472	836.98 E 00	837.97 E 00	9.9 E-01	0.12 E-02
0.474	2096.06 E 00	2097.00 E 00	9.4 E-01	0.45 E-03
…	…	…	…	…
0.586	-125.84 E 02	-125.73 E 02	1.1 E 01	0.85 E-03
0.588	-13.90 E 02	-13.80 E 02	1.0 E 01	0.76 E-02
0.590	107.42 E 02	107.52 E 02	1.0 E 01	0.94 E-03
0.592	2372.59 E 01	2373.54 E 01	9.5 E 00	0.40 E-03
…	…	…	…	…
0.702	-214.68 E 03	-214.57 E 03	1.1 E 02	0.52 E-03
0.704	-108.93 E 03	-108.82 E 03	1.1 E 02	0.10 E-02
0.706	6.98 E 03	7.09 E 03	1.1 E 02	0.15 E-01
0.708	132.32 E 03	132.42 E 03	1.0 E 02	0.77 E-03
…	…	…	…	…
0.750	1901.19 E 03	1900.89 E 03	3.0 E 02	0.16 E-03

Third order of accuracy decomposition scheme - (2.6) **Table 19**

(m1,m2,m3)=(17,1,1) (x,y)=(0.5,0.5) tau=1/1000, h=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.116	-867.65 E-03	-867.31 E-03	3.4 E-04	0.39 E-03
0.117	-348.23 E-03	-347.90 E-03	3.3 E-04	0.94 E-03
0.118	193.26 E-03	193.57 E-03	3.1 E-04	0.16 E-02
0.119	755.91 E-03	756.21 E-03	3.0 E-04	0.40 E-03
…	…	…	…	…
0.234	-700.54 E-02	-700.20 E-02	3.4 E-03	0.49 E-03
0.235	-162.76 E 00	-162.43 E-02	3.3 E-03	0.20 E-02
0.236	397.21 E 00	397.53 E-02	3.2 E-03	0.80 E-03
0.237	978.42 E 00	978.72 E-02	3.0 E-03	0.31 E-03
…	…	…	…	…
0.351	-1056.88 E-01	-1056.52 E-01	3.6 E-02	0.34 E-03
0.352	-523.53 E-01	-523.18 E-01	3.5 E-02	0.66 E-03
0.353	33.03 E-01	33.36 E-01	3.3 E-02	0.99 E-03
0.354	611.89 E-02	612.21 E-02	3.2 E-02	0.52 E-03
…	…	…	…	…
0.469	-888.65 E 00	-888.29 E 00	3.6 E-01	0.40 E-03
0.470	-336.25 E 00	-335.90 E 00	3.5 E-01	0.10 E-02
0.471	239.51 E 00	239.84 E 00	3.3 E-01	0.14 E-02
0.472	837.65 E 00	837.97 E 00	3.2 E-01	0.38 E-02
…	…	…	…	…
0.587	-710.25 E 01	-709.88 E 01	3.7 E 00	0.51 E-02
0.588	-138.36 E 01	-138.01 E 01	3.5 E 00	0.25 E-02
0.589	457.02 E 01	457.35 E 01	3.3 E 00	0.74 E-03
0.590	1074.85 E 01	1075.18 E 01	3.3 E 00	0.31 E-03
…	…	…	…	…
0.704	-1088.55 E 02	-1088.18 E 02	3.7 E 01	0.34 E-03
0.705	-521.30 E 02	-520.94 E 02	3.6 E 01	0.70 E-03
0.706	70.52 E 02	70.87 E 02	3.5 E 01	0.50 E-02
0.707	685.93 E 02	686.27 E 02	3.4 E 01	0.49 E-03
…	…	…	…	…
0.750	1901.01 E 03	1900.89 E 03	1.2 E 02	0.63 E-04

Third order of accuracy decomposition scheme - (2.6) **Table 20**

(m1,m2,m3)=(101,1,1) (x,y)=(0.5,0.5) tau=1/1000, h=1/100				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.602	8436.00 E 01	8436.47 E 01	4.7 E 00	0.56 E-04
0.603	442.96 E 02	443.07 E 02	1.1 E 01	0.26 E-03
0.604	-19.10 E 02	-18.93 E 02	1.7 E 01	0.90 E-03
0.605	-497.81 E 02	-497.60 E 02	2.1 E 01	0.43 E-02
…	…	…	…	…
0.750	-19008.69 E 02	-19008.90 E 02	2.1 E 01	0.11 E-04

Third order of accuracy decomposition scheme - (2.6) **Table 21**

(m1,m2,m3)=(101,3,3) (x,y)=(0.5,0.5) tau=1/1000, h=1/128				
t	$\tilde{u}$	u	$\lvert u-\tilde{u}\rvert$	$\lvert (u-\tilde{u})/u\rvert$
0.652	-174.49 E 03	-174.18 E 03	3.1 E 02	0.18 E-02
0.653	-58.65 E 03	-58.29 E 03	3.6 E 02	0.62 E-02
0.654	67.85 E 03	68.23 E 03	3.8 E 02	0.56 E-02
0.655	192.50 E 03	192.86 E 03	3.6 E 02	0.19 E-02
…	…	…	…	…
0.750	-190.25 E 04	-190.09 E 04	1.6 E 03	0.84 E-03

References

[1] Kh. A. Alibekov, P.E. Sobolevskii *On the stability of difference scheme for parabolic equation*, Dokl. Akad. Nauk USSR, **232** (1977), No. 2, pp. 737-740 (Russian).

[2] V. B. Andreev, *On difference schemes with a splitting operator for general p-dimensional parabolic equations of second order with mixed derivatives*, SSSR. Comput. Math. Math. Phys., **7** (1967), No. 2, pp. 312-321.

[3] G. A. Baker, *An implicit, numerical method for solving the two-dimensional heat equation*, Quart. Appl. Math., **17** (1959/1960), No. 4, pp. 440-443.

[4] G. A. Baker, T. A. Oliphant, *An implicit, numerical method for solving the two-dimensional heat equation*, Quart. Appl. Math., **17** (1959/1960), No. 4, pp. 361-373.

[5] G. A. Baker Jr, P. Graves-Morris, *Pade Approximations*, M.: Mir, 1986 (Russian).

[6] G. Birkhoff, R. S. Varga, *Implicit alternating direction methods*, Trans. Amer. Math. Soc., **92** (1959), No. 1, pp. 13-24.

[7] G. Birkhoff, R. S. Varga, D. Young, *Alternating direction implicit methods*, Advances in Computers., Academic Press, New York, **3** (1962), pp. 189-273.

[8] P. R. Chernoff, *Note on product formulas for operators semigroups*, J. Functional Anal., **2** (1968), pp. 238-242.

[9] P. R. Chernoff, *Semigroup product formulas and addition of unbounded operators*, Bull. Amer. Mat. Soc., **76** (1970), pp. 395-398.

[10] E. G. Diakonov, *Difference schemes with a splitting operator for nonstationary equations*, Dokl. Akad. Nauk SSSR., **144** (1962), No. 1, pp. 29-32.

[11] E. G. Diakonov, *Difference schemes with splitting operator for higher-dimensional non-stationary problems*, SSSR. Comput. Math. Math. Phys., **2** (1962), No. 4, pp. 549-568.

[12] B. O. Dia, M. Schatzman, *Comutatuers semi-groupes holomorphes et applications aux directions alternees*, M2AN, **30** (1996), No. 3, pp. 343-383.

[13] J. Douglas, *On numerical integration of by impilicit methods*, SIAM, **9** (1955), No. 1, pp. 42-65.

[14] J. Douglas, H. Rachford, *On the numerical solution of heat condition problems in two and three space variables*, Trans. Amer. Math. Soc., **82** (1956), No. 2, pp. 421-439.

[15] M. Dryja, *Stability in* ${W_2}^2$ *of schemes with splitting operators*, SSSR. Comput. Math. Math. Phys., **7** (1967), No. 2, pp. 296-302.

[16] N. Dunford, J.T Schwartz, *Linear Operators. General Theory*, Mir, 1962.

[17] G. Fairweather, A. R. Gourlay, A. R. Mitchell, *Some high accuracy difference schemes with a splitting operator for equations of parabolic and elliptic type*, Numer. Math., **10** (1967), No. 2, pp. 56-66.

[18] I. V. Fryazinov, *Increased precision order economical schemes for the solution of parabolic type multi-dimensional equations*, SSSR. Comput. Math. Math. Phys., **9** (1969), No. 6, pp. 1319-1326.

[19] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High-degree precision decomposition method for an evolution problem*, Tbilisi, Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics, **14** (1999), No. 3, pp. 45-48.

[20] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *Sequention-Parallel method of high degree precision for Cauchy abstract problem solution* , Minsk, Comput. Methods in Appl. Math., **1** (2001), No. 2, pp. 173-187.

[21] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High degree precision decomposition formulas of semigroup approximation*, Tbilisi, Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics, **16** (2001), No. 1-3, pp. 89-92.

[22] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High Degree Precision Decompostion Method for the Nonhomogenous Evolution Problem With an Operator Under a Split Form*, Tbilisi, Bulletin of TICMI, **5** (2001), pp. 13-18.

[23] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *Differential Scheme of High Degree Precision Decomposition of Nonhomogenous Evolution Problem*, Tbilisi, Applied Mathematics and Informatics, **6** (2001), No. 1, pp. 45-80.

[24] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *High degree precision decomposition method for the evolution problem with an operator under a split form*, Paris, M2AN Math. Model. Numer. Anal., **36** (2002), No. 4, pp. 693-704.

[25] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *The Third Order of Accuracy Operator Split of Evolution Problem Using Pade Approximation*, Tbilisi, Applied Mathematics and Informatics, **9** (2004), No. 1, pp. 16-36.

[26] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *The Fourth Order Accuracy Decomposition Scheme for an Evolution Problem*, Paris, M2AN Math. Model. Numer. Anal., **38** (2004), No. 4, pp. 707-722.

[27] Z. G. Gegechkori, J. L. Rogava, M. A. Tsiklauri, *Fourth Order of Accuracy Kranc-Nickolson Type Decomposition Scheme for Evolution Problem*, Tbilisi, Applied Mathematics and Informatics, **9** (2004), No. 2, pp. ??.

[28] D. G. Gordeziani, *On application of local one-dimensional method for solving parabolic type multi-dimensional problems of 2m-degree*, Proceeding of Science Academy of GSSR, **3** (1965), No. 39, pp. 535-542.

[29] D. G. Gordeziani, A. A. Samarskii, *Some problems of plates and shells thermo elasticity and method of summary approximation*, Complex analysis and it's applications (1978), pp. 173-186.

[30] D.G. Gordeziani, H.V. Meladze, *On modeling multi-dimensional quasi-linear equation of parabolic type by one-dimensional ones*, Proceeding of Science Academy of GSSR., **60** (1970), No. 3, pp. 537-540.

[31] D.G. Gordeziani, H.V. Meladze, *On modeling of third boundary value problem for the multi-dimensional parabolic equations of arbitrary area by the one-dimensional equations*, SSSR. Comput. Math. Math. Phys., **14** (1974), No. 1, pp. 246-250.

[32] A. R. Gourlay, A. R. Mitchell, *Intermediate boundary corrections for split operator methods in three dimensions*, Nordisk Tidskr. Informations-Behandling, **7** (1967), No. 1, pp. 31-38.

[33] N. N. Ianenko, *On Economic Implicit Schemes (Fractional steps method)*, Dokl. Akad. Nauk SSSR., **134** (1960), No. 5, pp. 84-86.

[34] N. N. Ianenko, *Fractional steps method of solving for multi-dimensional problems of mathematical physics*, Novosibirsk, Nauka, 1967.

[35] N. N. Ianenko, G. V. Demidov, *The method of weak approximation as a constructive method for building up a solution of the Cauchy problem*, Izdat. "Nauka", Sibirsk. Otdel., Novosibirsk, Certain Problems Numer. Appl. Math. (1966), pp. 60-83.

[36] T. Ichinose, S. Takanobu, *The norm estimate of the difference between the Kac operator and the Schrodinger emigroup*, Nagoya Math. J., **149** (1998), pp. 53-81.

[37] T. Ichinose, H. Tamura, *The Norm Convergence of the Trotter-Kato Product Formula with Error Bound*, Commun. Math. Phys., **217** (2001), pp. 489-502.

[38] V. P. Ilin, *On the splitting of difference parabolic and elliptic equations*, Sibirsk. Mat. Zh, **6** (1965), No. 1, pp. 1425-1428.

[39] K. Iosida, *Functional analysis*, Springer-Verlag, 1965.

[40] T. Kato, *The theory of perturbations of linear operators*, M.:Mir, 1972.

[41] A. N. Konovalov, *The fractional step method for solving the Cauchy problem for an n-dimensional oscillation equation*, Dokl. Akad. Nauk SSSR., **147** (1962), No. 1, pp. 25-27.

[42] S. G. Krein, *Linear equations in Banach space*, M.: Nauka, 1971.

[43] A. M. Kuzyk, V. L. Makarov, *Estimation of an exactitude of summarized approximation of a solution of Cauchy abstract problem*, Dokl. Akad. Nauk USSR, **275** (1984), No. 2, pp. 297-301.

[44] G. I. Marchuk, *Split methods*, M.: Nauka, 1988.

[45] G. I. Marchuk, N. N. Ianenko, *The solution of a multi-dimensional kinetic equation by the splitting method*, Dokl. Akad. Nauk SSSR., **157** (1964), No. 6, pp. 1291-1292.

[46] G. I. Marchuk, U. M. Sultangazin, *On a proof of the splitting method for the equation of radiation transfer*, SSSR. Comput. Math. Math. Phys., **5** (1965), No. 5, pp. 852-863.

[47] D. Peaceman, H. Rachford, *The numerical solution of parabolic and elliptic differential equations*, SIAM, **3** (1955), No. 1, pp. 28-41.

[48] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, New York-London, Academic Press [Harcourt Brace Jovanovich, Publishers], (1975).

[49] J. L. Rogava, *On the error estimation of Trotter type formulas in the case of self-Andjoint operator*, Functional analysis and its aplication, **27** (1993), No. 3, pp. 84-86.

[50] J. L. Rogava, *Semi-discrete schemes for operator differential equations*, Tbilisi, Georgian Technical University press, 1995.

[51] J. L. Rogava, M. A. Tsiklauri, *Third Order of Accuracy Sequential Type Decomposition Schemes for Two and Multidimensional Evolution Problems*, Tbilisi, Applied Mathematics and Informatics, **10** (2005), No. 1, pp. 72-87.

[52] J. L. Rogava, M. A. Tsiklauri, *Fourth Order of Accuracy Sequential Type Decomposition Schemes for Evolution Problems* , Tbilisi, Applied Mathematics and Informatics, **10** (2005), No. 2, pp. 90-101.

[53] J. L. Rogava, M. A. Tsiklauri, *An Automatically Stable and Order Three Split Rational Approximation of a Semigroup* , Georgian Mathematical Journal, **13** (2006), No. 2, pp. 343-362.

[54] A. A. Samarskii, *On an economical difference method for the solution of a multi-dimensional parabolic equation in an arbitrary region*, SSSR. Comput. Math. Math. Phys., **2** (1962), No. 5, pp. 787-811.

[55] A. A. Samarskii, *On the convergence of the method of fractional steps for the heat equation*, SSSR. Comput. Math. Math. Phys., **2** (1962), No. 6, pp. 1117-1121.

[56] A. A. Samarskii, *Locally homogeneous difference schemes for higher-dimensional equations of hyperbolic type in an arbitrary region.*, SSSR. Comput. Math. Math. Phys., **4** (1962), pp. 638-648.

[57] A. A. Samarskii, P. N. Vabishchevich, *Additive schemes for mathematical physics problems*, M., Nauka, 1999.

[58] Q. Sheng, *Solving linear partial differential equation by exponential splitting*, IMA J. Numerical Anal., **9** 1989, pp. 199-212.

[59] R. Temam, *Sur la stabilite et la convergence de la methode des pas fractionnaires*, Ann. Mat. Pura Appl., **4** (1968), No. 79, pp. 191-379.

[60] H. Trotter, *On the product of semigroup of operators*, Proc. Amer. Mat. Soc., **10** (1959), pp. 545-551.

[61] M. A. Tsiklauri, *High Order of Accuracy Decomposition Method for an Evolution Problem with Variable Operator*, Tbilisi, Proceedings of I. Vekua Institute of Applied Mathematics, **50-51** (2000-2001), pp. 103-117.

[62] M. A. Tsiklauri, *High Order of Accuracy Decomposition of the Evolution Problem with Variable Operator Using Resolvent Polinoms*, Tbilisi, Proceedings of I. Vekua Institute of Applied Mathematics, **50-51** (2000-2001), pp. 118-128.

[63] M. Zlamal, *Finite elements methods for parabolic equation*, Math. Comp., **28** (1974), No. 126, pp. 393-404.

რეზიუმე

ბანახის სივრცეში აგებულია მესამე და მეოთხე რიგის სიზუსტის დეკომპოზიციის სქემები ევოლუციური ამოცანისათვის ოპერატორით $A = A_1 + A_2 + \ldots + A_m \quad (m \geq 2)$, რომელიც წარმოქმნის ძლიერად უწყვეტ ნახევარჯგუფს $U(t, A) = \exp(-tA)$. ეს სქემები ეფუძნება შემდეგ დეკომპოზიციის ფორმულებს:

$$V_1(t) = \frac{1}{2}\left[T(t,\overline{\alpha})\overline{T}(t,\alpha) + \overline{T}(t,\alpha)T(t,\overline{\alpha})\right],$$

$$V_2(t) = T\left(t,\frac{\alpha}{2}\right)\overline{T}\left(t,\frac{\alpha}{2}\right)T\left(t,\frac{\overline{\alpha}}{2}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{2}\right),$$

$$V_3(t) = \frac{1}{2}\left[T\left(t,\frac{\alpha}{2}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{2}\right)T\left(t,\frac{\overline{\alpha}}{2}\right)\overline{T}\left(t,\frac{\alpha}{2}\right) + \overline{T}\left(t,\frac{\alpha}{2}\right)T\left(t,\frac{\overline{\alpha}}{2}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{2}\right)T\left(t,\frac{\alpha}{2}\right)\right],$$

$$V_4(t) = T\left(t,\frac{\overline{\alpha}}{4}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{4}\right)T\left(t,\frac{\alpha}{4}\right)\overline{T}\left(t,\frac{\alpha}{4}\right)T\left(t,\frac{\alpha}{4}\right)\overline{T}\left(t,\frac{\alpha}{4}\right)T\left(t,\frac{\overline{\alpha}}{4}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{4}\right).$$

სადაც $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$,

$$T(t,\alpha) = U(t,\alpha A_1)U(t,\alpha A_2)\ldots U(t,\alpha A_m)$$

$$\overline{T}(t,\alpha) = U(t,\alpha A_m)U(t,\alpha A_{m-1})\ldots U(t,\alpha A_1).$$

ვგულისხმობთ, რომ $(-\gamma A_j)$, $\gamma = 1, \alpha, \overline{\alpha}$, $j = 1,\ldots m$, ოპერატორები წარმოქმნიან ძლიერად უწყვეტ ნახევარჯგუფებს და $\left\|U(t,\gamma A_j)\right\| \leq e^{\omega t}$, $\omega = const > 0$.

ზემოთ მოყვანილი ფორმულებისთვის სამართლიანია შემდეგი შეფასებები:

$$\left\|\left(U(t_k, A) - \left[V_j(\tau)\right]^k\right)\varphi\right\| = O\left(\tau^3\right), \quad \varphi \in D\left(A^4\right), \quad j = 1,2,$$

$$\left\|\left(U(t_k, A) - \left[V_j(\tau)\right]^k\right)\varphi\right\| = O\left(\tau^4\right), \quad \varphi \in D\left(A^5\right), \quad j = 3,4,$$

სადაც $t_k = k\tau$, $\tau > 0$ დროითი ბიჯია.

ერთგვაროვანი ევოლუციური ამოცანის შემთხვევაში ცხადია ის წესი, რომლის მიხედვითაც აიგება ზემოთ მოყვანილი ფორმულების შესაბამისი დეკომპოზიციის სქემები. მაგალითად $V_1(t)$-ს $m = 2$ შემთხვევისათვის შეესაბამება შემდეგი დეკომპოზიციის სქემა:

$$\frac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) = 0, \quad v_k^1(t_{k-1}) = u_{k-1}(t_{k-1}),$$
$$\frac{dv_k^2(t)}{dt} + A_2 v_k^2(t) = 0, \quad v_k^2(t_{k-1}) = v_k^1(t_k),$$
$$\frac{dv_k^3(t)}{dt} + \overline{\alpha} A_1 v_k^3(t) = 0, \quad v_k^3(t_{k-1}) = v_k^2(t_k),$$
$$\frac{dw_k^1(t)}{dt} + \alpha A_1 w_k^1(t) = 0, \quad w_k^1(t_{k-1}) = u_{k-1}(t_{k-1}),$$
$$\frac{dw_k^2(t)}{dt} + A_2 w_k^2(t) = 0, \quad w_k^2(t_{k-1}) = w_k^1(t_k),$$
$$\frac{dw_k^3(t)}{dt} + \overline{\alpha} A_1 w_k^3(t) = 0, \quad w_k^3(t_{k-1}) = w_k^2(t_k),$$
$$u_k(t) = \frac{1}{2}\left[v_k^3(t) + w_k^3(t)\right], \quad t \in [t_{k-1}, t_k], \quad k = 1,2,\ldots,$$

სადაც $u_0(0)$ არის ევოლუციური ამოცანის $u(t)$ ზუსტი ამონახსნის საწყისი მნიშვნელობა. $u(t)$-ს მიახლოებით მნიშვნელობად $t = t_k$ წერტილში ვაცხადებთ $u_k(t_k)$-ს. ზემოთ მოყვანილი სქემის სიზუსტე არის $O(\tau^3)$.

ეს შეფასებები რჩება ძალაში, თუ $U(t, A)$-ს შევცვლით შესაბამისად მესამე და მეოთხე რიგის სიზუსტის რაციონალური აპროქსიმაციებით:

$$W(t, A) = \left(I - \frac{1}{3}tA\right)(I + \lambda tA)^{-1}\left(I + \overline{\lambda} tA\right)^{-1}, \quad \lambda = \frac{1}{3} \pm i\frac{1}{3\sqrt{2}},$$
$$W(t, A) = \left(I - \frac{\alpha}{2}tA\right)\left(I + \frac{\overline{\alpha}}{2}tA\right)^{-1}\left(I - \frac{\overline{\alpha}}{2}tA\right)\left(I + \frac{\alpha}{2}tA\right)^{-1}.$$

ნაჩვენებია აგებული დეკომპოზიციის სქემების მდგრადობა და მიახლოებითი ამონახსნის ცდომილებისათვის მიღებულია ცხადი აპრიორული შეფასებები. წარმოდგენილ ნაშრომში ასევე განხილულია შემთხვევა, როდესაც ძირითადი ოპერატორი დამოკიდებულია t-ზე, კერძოდ წარმოადგენს t-ზე დამოკიდებული სკალარული ფუნქციისა და მუდმივი ოპერატორის ნამრავლს.