COMPLEX PARTIAL DIFFERENTIAL EQUATIONS IN A MANNER OF I. N. VEKUA

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Dedicated to the memory of Ilya N. Vekua

Abstract. Some classes of complex partial differential equations of arbitrary order in one complex variable are reduced to singular integral equations via potential operators related to the leading term of the equation. This motivates the study of model equations. Particular cases are polyanalytic and polyharmonic equations. As an example some boundary value problems for the inhomogeneous biharmonic equation are investigated. In order to be explicit the problems are solved for the unit disc.

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Contents

1. Introduction

In the way as I. N. Vekua has treated the generalized Beltrami equation any kind of higher order complex partial differential equation can be reduced to a singular integral equation to which the Fredholm theory applies. The reduction is managed by certain potential operators for the leading term of the equation. This demands to handle model equations beforehand, i.e. equations the differential operator of which consists just of a leading term. They are compositions of polyanalytic and of polyharmonic operators of appropriate orders. Decomposing these model equations in a system of a polyanalytic or polyantianalytic and a polyharmonic equation leads to certain boundary value problems for the model equation. Naturally the boundary value problems attained this way for the model equations can be supplemented by other ones not stated in accordance with the mentioned decomposition of the equation.

The statement of boundary value problems even for the particular cases of polyanalytic and of polyharmonic equations is far from being obvious. Some of these problems are going along with the decomposition of these equations in ones of lower orders. Others do not. Exemplarily this is illuminated here by studying the biharmonic equation which was treated by I. N. Vekua in one of his last papers published in 1976. For the polyanalytic operator the particular Schwarz problem is solved explicitly in case of the unit disc [8, 29]. For the related general linear equation this Schwarz problem is treated in [5], see also [3, 4], in the manner indicated here in general.

2. The Beltrami Equation

In [50] I. N. Vekua is treating the generalized Beltrami equation

$$w_{\overline{z}} + q_1 w_z + q_2 \overline{w_z} + aw + b\overline{w} + c = 0 \tag{1}$$

in a plane domain D with

$$|\, q_1(z)| + |\, q_2(z)| \le q_0 < 1, \; a, b, c \in L_p(D; \mathbb{C}), \; 1 < p,$$

by using the Pompeiu operator

$$Tf(z) = -\frac{1}{\pi} \int\limits_D f(\zeta) \frac{d\xi d\eta}{\zeta - z}.$$

The latter has weak derivatives with respect to $\overline{z}$ and z satisfying

$$\partial_{\overline{z}} Tf = f, \;\; \partial_z Tf(z) = \Pi f(z)$$

with the Ahlfors - Beurling operator

$$\Pi f(z) = -\frac{1}{\pi} \int\limits_D f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2}.$$

The properties of the Pompeiu and the Ahlfors - Beurling operators are well studied in [50]. Using the representation

$$w = \varphi + T\rho, \;\; \varphi_{\overline{z}} = 0, \;\; w_{\overline{z}} = \rho, \tag{2}$$

for functions being weakly differentiable with respect to $\overline{z}$ with derivatives in $L_1(D; \mathbb{C})$ equation (1) becomes

$$\rho + q_1 \Pi\rho + q_2 \overline{\Pi\rho} + aT\rho + b\,\overline{T\rho} + q_1\varphi' + q_2\overline{\varphi'} + a\varphi + b\overline{\varphi} + c = 0. \tag{3}$$

This is a singular integral equation consisting of a contracting operator $q_1\Pi\rho + q_2\overline{\Pi\rho}$ in a proper $L_p(D;\mathbb{C})$ space, $2 < p$ with $p-2$ small enough, see [50], and a compact operator $aT\rho + b\overline{T\rho}$ in $L_p(D;\mathbb{C})$, $1 < p$. Hence the Fredholm theory applies. But in (3) besides ρ also the analytic function φ is unknown. It can be determined through boundary value problems. Describing one for w will lead via (2) to one for φ. This is expressed through the one for w and the one for $T\rho$, which is a function continuous in $\mathbb{C}$.

Thus the analytic function φ splits into a known one and one expressed via an area integral operator acting on ρ. Hence (3) is lead to a singular integral equation where just the compact operator is perturbed while the contractive operator is not changed at all.

Riemann and Riemann-Hilbert boundary value problems are investigated for (1) e.g. in [7, 19, 22, 24, 27, 33, 34, 37, 44, 48, 50, 53, 54]. Basic boundary value problems for the related model equation

$$w_{\overline{z}} = f$$

are studied for the unit disc in [7, 8, 9, 11], for the upper half plane in [35], in a quarter plane in [1], for a ring domain in [49]. For analytic functions fundamental investigations were done in by N. I. Muskhelishvili [46] and F. D. Gakhov [36], see also [38], for generalized analytic functions by I. N. Vekua [50], see also [7, 39, 41, 42, 43, 48, 54]. For higher order equations see [2], and for systems in several complex variables [14, 16, 17, 28, 45].

3. Higher Order Equations

An arbitrary higher order complex partial differential equation has the form

$$\partial_z^m \partial_{\overline{z}}^n w + \sum_{\substack{\mu+\nu=m+n,\\ (\mu,\nu)\neq(m,n)}} \Big[q_{\mu\nu}\, \partial_z^\mu \partial_{\overline{z}}^\nu w + \widehat{q}_{\mu\nu}\, \overline{\partial_z^\mu \partial_{\overline{z}}^\nu w} \Big] \tag{4}$$
$$+ \sum_{\mu+\nu<m+n} \Big[a_{\mu\nu}\, \partial_z^\mu \partial_{\overline{z}}^\nu w + b_{\mu\nu}\, \overline{\partial_z^\mu \partial_{\overline{z}}^\nu w} \Big] + c = 0.$$

In case

$$\sum_{\substack{\mu+\nu=m+n,\\ (\mu,\nu)\neq(m,n)}} \{ |\, q_{\mu\nu}(z)| + |\, \widehat{q}_{\mu\nu}(z)| \} \le q_0 < 1, \;\; a_{\mu\nu},\, b_{\mu\nu},\, c \in L_p(D;\mathbb{C}) \tag{5}$$

it can be treated in the same way as I. N. Vekua did with (1).

Higher order Pompeiu operators [6,7,23] are given by the respective Cauchy-Poisson kernels

$$K_{m,n}(z) = \begin{cases} \dfrac{(-1)^m(-m)!}{(n-1)!\,\pi} z^{m-1}\overline{z}^{\,n-1}, \; m \le 0, \\[2mm] \dfrac{(-1)^n(-n)!}{(m-1)!\,\pi} z^{m-1}\overline{z}^{\,n-1}, \; n \le 0, \\[2mm] \dfrac{z^{m-1}}{(m-1)!} \dfrac{\overline{z}^{\,n-1}}{(n-1)!\,\pi} \Big[\log|\,z|^2 - \displaystyle\sum_{\mu=1}^{m-1} \frac{1}{\mu} - \sum_{\nu=1}^{n-1} \frac{1}{\nu} \Big], \quad 0 < m,\, n \end{cases}$$

for $0 \le m+n, \; 0 < m^2 + n^2$ as

$$T_{m,n} f(z) = \int_D K_{m,n}(z-\zeta) f(\zeta) d\xi d\eta, \; f \in L_1(D;\mathbb{C}).$$

Defining $T_{0,0}f = f$, $f \in L_1(D;\mathbb{C})$, the differential properties of these operators are

$$\partial_z^\mu \partial_{\overline{z}}^\nu T_{m,n} f = T_{m-\mu, n-\nu} f \ \text{ for } \mu + \nu \le m + n$$

in the weak sense. If $0 < m + n$ then $T_{m,n}$ are weakly singular having the same properties as $T_{0,1} = T$, $T_{1,0} = \overline{T}$. For $m + n = 0 < m^2 + n^2$ the operators are strongly singular of Calderon-Zygmund type to be interpreted as Cauchy principal integrals. Their properties [23] are the same as $T_{-1,1} = \Pi$, $T_{1,-1} = \overline{\Pi}$ in particular

$$\|T_{k,-k}\|_{L_2} = 1 \ \text{ for } k \in \mathbb{Z}.$$

Using the representation

$$w = \varphi + T_{m,n}\rho, \ \ \partial_z^m \partial_{\overline{z}}^n \varphi = 0, \ \ \partial_z^m \partial_{\overline{z}}^n w = \rho,$$

equation (4) is transformed into the singular integral equation

$$\begin{aligned}
\rho &+ \sum_{\substack{\mu+\nu=m+n,\\ (\mu,\nu)\neq(m,n)}} \Big[q_{\mu,\nu} T_{m-\mu,n-\nu}\rho + \widehat{q}_{\mu,\nu}\overline{T_{m-\mu,n-\nu}\rho}\Big] \\
&+ \sum_{\mu+\nu<m+n} \Big[a_{\mu,\nu} T_{m-\mu,n-\nu}\rho + b_{\mu,\nu}\overline{T_{m-\mu,n-\nu}\rho}\Big] \\
&+ \sum_{\substack{\mu+\nu=m+n,\\ (\mu,\nu)\neq(m,n)}} \Big[q_{\mu,\nu}\partial_z^\mu \partial_{\overline{z}}^\nu \varphi + \widehat{q}_{\mu,\nu}\overline{\partial_z^\mu \partial_{\overline{z}}^\nu \varphi}\Big] \\
&+ \sum_{\mu+\nu<m+n} \Big[a_{\mu,\nu}\partial_z^\mu \partial_{\overline{z}}^\nu \varphi + b_{\mu,\nu}\overline{\partial_z^\mu \partial_{\overline{z}}^\nu \varphi}\Big] + c = 0.
\end{aligned} \tag{6}$$

Because of (5) the first sum determines a contraction in $L_p(D;\mathbb{C})$ for $2 < p$ with $p-2$ small enough while the second sum gives a compact operator in $L_p(D;\mathbb{C})$. Having determined φ by proper boundary conditions on w so that φ will be expressed by some area integral operator acting on ρ as in the case of the Beltrami equation only the compact operator in (6) will be perturbed. A particular case of (5) with $m = 0$ i.e. for the polyanalytic operator in the leading part prescribing Schwarz boundary values is considered in the PhD thesis [3], see also [4,5,29].

The reduction of (5) to (6) makes it necessary to study the related model equation first.

4. Model Equations

For treating the model equation

$$\partial_z^m \partial_{\overline{z}}^n w = f, \ f \in L_1(D;\mathbb{C}) \tag{7}$$

a fundamental solution to the differential operator is appropriate. It can be obtained from the fundamental solution $-\dfrac{1}{\pi z}$ of the Cauchy-Riemann operator ∂_z by integration. Iteratively it is seen that

$$-\frac{1}{\pi}\frac{\overline{z}^{\,n-1}}{(n-1)!z} \tag{8}$$

is a fundamental solution to $\partial_{\overline{z}}^n$ as well as $\log|z|^2$ is one for the Laplacian $\partial_z\partial_{\overline{z}}$ as

$$-\frac{1}{\pi}\frac{z^{m-1}}{(m-1)!}\frac{\overline{z}^{\,n-1}}{(n-1)!}\Big[\log|z|^2 - \sum_{\mu=1}^{m-1}\frac{1}{\mu} - \sum_{\nu=1}^{n-1}\frac{1}{\nu}\Big] \tag{9}$$

is one for $\partial_z^m \partial_{\overline{z}}^n$.

Differentiating (8) with respect to z leads to the kernel function $K_{m,n}(z)$ for $m < 0$, (9) is $K_{m,n}(z)$ for positive m and n.

Equation (7) can be rewritten for $m \le n$ as the system

$$\partial_{\overline{z}}^{n-m} w = \omega,\ (\partial_z \partial_{\overline{z}})^m \omega = f$$

of a polyanalytic and polyharmonic equation. For the first type the Schwarz problem is a well posed boundary value problem. It can be explicitly solved for the unit disk [28] and in principal also for other regular domains, see e.g. [7,39]. But other problems are available also, see e.g. [8,20,26].

Boundary value problems for the polyharmonic equation are treated in [12,13,18, 21,25]. To illustrate the variety of available boundary value problems for this equation a particular case is investigated, see [10].

5. The Biharmonic Equation

In one of his last papers [52] published in 1976 I. N. Vekua has solved the Dirichlet problem

$$(\partial_z \partial_{\overline{z}})^2 = 0 \text{ in } D,\ w = \gamma_0,\ \partial_\nu w = \gamma_1 \text{ on } \partial D,$$

$$\gamma_0 \in C^{2+\alpha}(\partial D; \mathbb{C}),\ \gamma_1 \in C^{1+\alpha}(\partial D; \mathbb{C})$$

for a regular domain D and $0 < \alpha$, where ∂_ν denotes the outward normal derivative on ∂D. Using the Goursat representation

$$w = \overline{z}\varphi + z\overline{\varphi} + \psi + \overline{\psi},\quad \varphi_{\overline{z}} = 0,\quad \psi_{\overline{z}} = 0,$$

he is constructing φ and ψ in an approximative manner by quadratures.

Another method is based on the biharmonic Green-Almansi function $G_2(z,\zeta)$ [10,13,30,40]. It has the properties

- $G_2(\cdot,\zeta)$ is biharmonic in $D \setminus \{\zeta\}$, $\zeta \in D$
- $G_2(z,\zeta) + |\,\zeta - z|^2 \log|\,\zeta - z|^2$ is biharmonic in $z \in D,\ \zeta \in D$
- $G_2(z,\zeta) = 0,\ \partial_{\nu_z} G_2(z,\zeta) = 0$ for $z \in \partial D,\ \zeta \in D$
- $G_2(z,\zeta) = G_2(\zeta,z)$ for $z,\zeta \in D,\ z \ne \zeta$.

Using the Gauss theorem [7, 50] the representation formula

$$w(z) = -\frac{1}{4\pi} \int\limits_{\partial D} \Big[w(\zeta)\partial_{\nu_\zeta} \partial_\zeta \partial_{\overline{\zeta}} G_2(z,\zeta) - \partial_\nu w(\zeta) \partial_\zeta \partial_{\overline{\zeta}} G_2(z,\zeta) \Big] ds_\zeta \tag{10}$$

$$-\frac{1}{\pi} \int\limits_D (\partial_\zeta \partial_{\overline{\zeta}})^2 w(\zeta) G_2(z,\zeta) d\xi d\eta$$

follows providing a solution to the Dirichlet problem

$$(\partial_z \partial_{\overline{z}})^2 w = f \text{ in } D,\ f \in L_1(D; \mathbb{C}), \tag{11}$$

$$w = \gamma_0,\ \partial_\nu w = \gamma_1 \text{ on } \partial D,\ \gamma_0 \in C^{2+\alpha}(\partial D; \mathbb{C}),\ \gamma_1 \in C^{1+\alpha}(\partial D; \mathbb{C}).$$

For a verification in the case $D = \mathbb{D} = \{|z| < 1\}$ see [12,18,30,32].

This Dirichlet problem is not in accordance with the decomposability of the biharmonic equation (11) in a system of two Poisson equations

$$\partial_z \partial_{\overline{z}} w = \omega,\ \partial_z \partial_{\overline{z}} \omega = f.$$

For the Poisson equation several basic boundary value problems are available [7,21,30,32], e.g. the Dirichlet, the Neumann, the Robin boundary value problems. It will be explained how these problems lead to respective problems for (11) by just using Dirichlet and Neumann conditions. In order to be explicit D is chosen to be the unit disk $\mathbb{D}$. For this domain continuity rather then Hölder continuity of the boundary data is sufficient, see [46].

Dirichlet-Dirichlet problem Find the solution to the problem

$$(\partial_z\partial_{\overline{z}})^2 w = f \text{ in } \mathbb{D},\ f \in L_1(\mathbb{D};\mathbb{C}),$$

$$w = \gamma_0,\ \partial_z\partial_{\overline{z}}\, w = \gamma_2 \text{ on } \partial\mathbb{D},\ \gamma_0, \gamma_2 \in C(\partial\mathbb{D};\mathbb{C}).$$

Iterating the Poisson formulas for for the solutions to the two Dirichlet problems

$$\partial_z\partial_{\overline{z}}\, w = \omega \text{ in } \mathbb{D},\ \ w = \gamma_0 \text{ on } \partial\mathbb{D},$$

$$\partial_z\partial_{\overline{z}}\, \omega = f \text{ in } \mathbb{D},\ \ \omega = \gamma_2 \text{ on } \partial\mathbb{D},$$

in the form

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_0(\zeta) g_1(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} \omega(\zeta)\, G_1(z,\zeta) d\xi d\eta,$$

$$\omega(\zeta) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\widetilde{\zeta}) g_1(\zeta,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} - \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\widetilde{\zeta})\, G_1(\zeta,\widetilde{\zeta}) d\widetilde{\xi} d\widetilde{\eta},$$

with the Poisson kernel

$$g_1(z,\zeta) = \frac{1}{1 - z\overline{\zeta}} + \frac{1}{1 - \overline{z}\zeta} - 1$$

and the harmonic Green function

$$G_1(z,\zeta) = \log\left|\frac{1 - z\overline{\zeta}}{\zeta - z}\right|^2$$

gives the solution to the Dirichlet-Dirichlet problem as

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \Big[\gamma_0(\zeta) g_1(z,\zeta) + \gamma_2(z,\zeta)\widehat{g}_2(z,\zeta)\Big]\frac{d\zeta}{\zeta} \\ &- \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)\, \widehat{G}_2(z,\zeta) d\xi d\eta. \end{aligned} \tag{12}$$

Here

$$\widehat{g}_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\widetilde{\zeta}) g_1(\widetilde{\zeta},\zeta) d\widetilde{\xi} d\widetilde{\eta} \tag{13}$$

is the primitive of the Poisson kernel with respect to the Laplace operator, vanishing at the boundary of $\mathbb{D}$

$$\partial_z\partial_{\overline{z}}\, \widehat{g}_2(z,\zeta) = g_1(z,\zeta) \text{ in } \mathbb{D},\ \ \widehat{g}_2(z,\zeta) = 0 \text{ on } \partial\mathbb{D} \text{ for } \zeta \in \mathbb{D}$$

and

$$\widehat{G}_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\widetilde{\zeta})\, G_1(\widetilde{\zeta},\zeta) d\widetilde{\xi} d\widetilde{\eta}$$

is the convolution of the harmonic Green function with itself satisfying for any $\zeta \in \mathbb{D}$

$$\partial_z\partial_{\overline{z}}\, \widehat{G}_2(z,\zeta) = G_1(z,\zeta) \text{ in } \mathbb{D},\ \ \widehat{G}_2(z,\zeta) = 0 \text{ on } \partial\mathbb{D}.$$

It is a biharmonic Green function satisfying the same conditions as $G_2(z,\zeta)$ up to the third one. Its boundary behavior instead is

$$\widehat{G}_2(z,\zeta) = 0,\ \ \partial_z\partial_{\overline{z}}\, \widehat{G}_2(z,\zeta) = 0 \text{ for } z \in \partial\mathbb{D},\ \zeta \in \mathbb{D}.$$

Neumann-Neumann problem. Find the solution to the problem

$$(\partial_z\partial_{\overline{z}})^2 w = f \text{ in } \mathbb{D},\ f \in L_1(\mathbb{D};\mathbb{C}),$$
$$\partial_\nu w = \gamma_1,\ \ \partial_\nu\partial_z\partial_{\overline{z}}\, w = \gamma_3 \text{ on } \partial\mathbb{D},\ \gamma_1,\gamma_3 \in C(\partial\mathbb{D};\mathbb{C}),$$
$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta} = c_0,\ \ \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_{\zeta\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta} = c_2,\ \ c_0, c_2 \in \mathbb{C}.$$

Proceeding as before on the basis of the Neumann formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \partial_\nu w(\zeta) N_1(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta$$

with the harmonic Neumann function

$$N_1(z,\zeta) = -\log|(\zeta - z)(1 - z\overline{\zeta})|^2$$

the solution to the Neumann-Neumann problem is

$$w(z) = c_0 - c_2(1-|z|^2) + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\Big[\gamma_1(\zeta)N_1(z,\zeta) + \gamma_3(\zeta)N_2(z,\zeta)\Big]\frac{d\zeta}{\zeta} \tag{14}$$
$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)N_2(z,\zeta)d\xi d\eta$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_1(\zeta)\frac{d\zeta}{\zeta} = 2c_2 - \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)(1-|\,\zeta|^2)d\xi d\eta,$$
$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_3(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)d\xi d\eta.$$

Here the biharmonic Neumann function is the convolution of the harmonic one with itself

$$N_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} N_1(z,\widetilde{\zeta})N_1(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}.$$

It satisfies for any $\zeta \in \mathbb{D}$

$$\partial_z\partial_{\overline{z}}\, N_2(z,\zeta) = N_1(z,\zeta) \text{ in } \mathbb{D},\ \ \partial_{\nu_z} N_2(z,\zeta) = 2(1-|\zeta|^2) \text{ on } \partial\mathbb{D}.$$

Its properties differ from those of G_2 only in the boundary behavior which is for $\zeta \in \mathbb{D}$

$$\partial_{\nu_z} N_2(z,\zeta) = 2(1-|\zeta|^2),\ \ \partial_{\nu_z}\partial_z\partial_{\overline{z}}\, N_2(z,\zeta) = 2.$$

Moreover the normalization conditions

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} N_2(z,\zeta)\frac{dz}{z} = 0,\ \ \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \partial_z\partial_{\overline{z}}\, N_2(z,\zeta)\frac{dz}{z} = 0$$

hold.

Dirichlet-Neumann problem. Find the solution to the problem

$$(\partial_z\partial_{\overline{z}})^2 w = f \text{ in } \mathbb{D},\ f \in L_1(\mathbb{D};\mathbb{C}),$$
$$w = \gamma_0,\ \ \partial_\nu\partial_z\partial_{\overline{z}}\, w = \gamma_3 \text{ on } \partial\mathbb{D},\ \gamma_0,\gamma_3 \in C(\partial\mathbb{D};\mathbb{C}),$$
$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_{\zeta\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta} = c_2,\ \ c_2 \in \mathbb{C}.$$

Composing the respective Green and Neumann representation formulas shows

$$w(z) = -c_2(1-|z|^2) + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \Big[\gamma_0(\zeta)g_1(z,\zeta) + \frac{1}{2}\,H_2(z,\zeta)\gamma_3(\zeta)\Big]\frac{d\zeta}{\zeta} \tag{15}$$
$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)H_2(z,\zeta)d\xi d\eta$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_3(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)d\xi d\eta$$

with the hybrid biharmonic Green-Neumann function

$$H_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\widetilde{\zeta})N_1(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}.$$

It satisfies

$$\partial_z\partial_{\overline{z}}\,H_2(z,\zeta) = N_1(z,\zeta) \text{ in } \mathbb{D},\ \ H_2(z,\zeta) = 0 \text{ on } \partial\mathbb{D}\ \text{ for any } \zeta\in\mathbb{D}$$

and

$$\partial_\zeta\partial_{\overline{\zeta}}\,H_2(z,\zeta) = G_1(z,\zeta) \text{ in } \mathbb{D},\ \ \partial_{\nu_\zeta}H_2(z,\zeta) = 2(1-|z|^2) \text{ on } \partial\mathbb{D}\ \text{ for any } z\in\mathbb{D}.$$

Moreover the normalization condition

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} H_2(z,\zeta)\frac{d\zeta}{\zeta} = 0$$

holds.

As a function of z but also of ζ it satisfies the same first two conditions of $G_2(z,\zeta)$. It obviously is not symmetric and its boundary behavior is

$$H_2(z,\zeta) = 0,\ \ \partial_{\nu_z}\partial_z\partial_{\overline{z}}\,H_2(z,\zeta) = 2 \text{ on } \partial\mathbb{D}\ \text{ for any } \zeta\in\mathbb{D}$$

and

$$\partial_{\nu_\zeta}H_2(z,\zeta) = 2(1-|z|^2),\ \ \partial_\zeta\partial_{\overline{\zeta}}\,H_2(z,\zeta) = 0 \text{ on } \partial\mathbb{D}\ \text{ for any } z\in\mathbb{D}.$$

This hybrid biharmonic Green-Neumann function serves also to solve the next problem.

Neumann-Dirichlet problem. Find the solution to the problem

$$(\partial_z\partial_{\overline{z}})^2 w = f \text{ in } \mathbb{D},\ f\in L_1(\mathbb{D};\mathbb{C}),$$
$$\partial_\nu w = \gamma_1,\ \partial_z\partial_{\overline{z}}\,w = \gamma_2 \text{ on } \partial\mathbb{D},\ \ \gamma_1,\gamma_2\in C(\partial\mathbb{D};\mathbb{C}),$$
$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta} = c_0,\ \ c_0\in\mathbb{C}.$$

The solution is

$$\begin{aligned} w(z) &= c_0 + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\Big[\gamma_1(\zeta)N_1(z,\zeta) - \gamma_2(\zeta)\partial_{\nu_\zeta}H_2(\zeta,z)\Big]\frac{d\zeta}{\zeta} \\ &- \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)H_2(\zeta,z)d\xi d\eta, \end{aligned} \tag{16}$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}[\gamma_1(\zeta)+2\gamma_2(\zeta)]\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int\limits_{\mathbb{D}}f(\zeta)(1-|\zeta|^2)d\xi d\eta.$$

These considerations are not restricted to the unit disk. They hold in the same way for any regular domain.

6. Biharmonic Green Function for the Unit Disk

The biharmonic Green functions from the preceding section can be calculated explicitly for the unit disk $\mathbb{D}$. They are

$$G_2(z,\zeta)=|\zeta-z|^2\log\Big|\frac{1-z\overline{\zeta}}{\zeta-z}\Big|^2-(1-|z|^2)(1-|\zeta|^2),$$

$$\widehat{G}_2(z,\zeta)=|\zeta-z|^2\log\Big|\frac{1-z\overline{\zeta}}{\zeta-z}\Big|^2+(1-|z|^2)(1-|\zeta|^2)\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}+\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big],$$

$$N_2(z,\zeta)=|\zeta-z|^2[4-\log|(\zeta-z)(1-z\overline{\zeta})|^2]-4\sum_{k=2}^{\infty}\frac{1}{k^2}[(z\overline{\zeta})^k+(\overline{z}\zeta)^k]$$

$$-2[z\overline{\zeta}+\overline{z}\zeta]\log|1-z\overline{\zeta}|^2-(1+|z|^2)(1+|\zeta|^2)\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}+\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big],$$

$$\begin{aligned}H_2(z,\zeta)&=-|\zeta-z|^2\log|\zeta-z|^2\\&-(1-|z|^2)\Big[4+\frac{1-z\overline{\zeta}}{z\overline{\zeta}}\log(1-z\overline{\zeta})+\frac{1-\overline{z}\zeta}{\overline{z}\zeta}\log(1-\overline{z}\zeta)\Big]\\&-\frac{(\zeta-z)(1-z\overline{\zeta})}{z}\log(1-z\overline{\zeta})-\frac{(\overline{\zeta-z})(1-\overline{z}\,\zeta)}{\overline{z}}\log(1-\overline{z}\zeta).\end{aligned}$$

Other ones can be determined, see [13].

For higher order polharmonic operators there exist a variety of Green functions. The respective functions $\widehat{G}_n$ and N_n [30,31] are iteratively defined. But their evaluation seems involved and is not yet done. The same holds for the higher order Poisson kernels

$$\widehat{g}_n(z,\zeta)=-\frac{1}{\pi}\int\limits_{\mathbb{D}}G_1(z,\widetilde{\zeta})\,\widehat{g}_{n-1}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta},\quad \widehat{g}_1(z,\zeta)=g_1(z,\zeta),$$

see [15,18]. Only the Green-Almansi function is known explicitly, it is, see [12, 51]

$$\begin{aligned}G_n(z,\zeta)&=\frac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\log\Big|\frac{1-z\overline{\zeta}}{\zeta-z}\Big|^2\\&+\sum_{\nu=1}^{n-1}\frac{(-1)^\nu}{\nu}|\zeta-z|^{2(n-1-\nu)}(1-|z|^2)^\nu(1-|\zeta|^2)^\nu.\end{aligned}$$

For the upper half plane, see [35].

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